Z. Wahrscheinlichkeitstheorie verw. Geb. 23, 114–120 (1972)
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Excision of a Strong Markov Process

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0. Introduction and Definitions

Let $(\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, P^x)$ be a strong Markov process on a locally compact space $(E_\Delta, \mathscr{E}_\Delta)$ with countable base, where Δ denotes the usual adjoined absorbing point and $\mathscr{E}_\Delta$ the Borel sets of E_Δ. The definitions and notation follow those of [1]. In particular, $\mathscr{F}_t$ is complete in $\mathscr{F}$ relative to the family P^x, and $\zeta(\omega)=\inf\{t\colon X_t(\omega)=\Delta\}$ is the "lifetime". We assume also that for all $\omega\in\Omega$, $X_t(\omega)=X(t)=\omega(t)$ is right continuous for all $0\leqq t$ and has left limits on $0<t<\zeta(\omega)$.

Now let A_0 and B_0 in $\mathscr{E}_\Delta$ be fixed, with (i) $\bar{A}_0\cap\bar{B}_0=\emptyset$, where $\bar{A}$ denotes the closure of A, and (ii) $\Delta\in B_0$.

The purpose of this paper is to construct a strong Markov process Y from X by excising or "splicing out" the round trip excursions from A_0 to B_0 back to A_0. With $A_0=\{\alpha\}$ and $B_0=\bar{B}_0$ this operation arose in [3] in analyzing the local time of a reflected stable process. The proof that the excised process Y was again a Hunt process raised unanticipated difficulties not resolved in [3], and led to the present paper.

A second purpose of this paper is to give a concrete example of a "non-Markovian" time change of a process which nonetheless preserves the strong Markov property. A different type of such example was given in [4] in the form $t\to t+L$, where L is an exact coterminal time, e.g., the last exit from a set in $\mathscr{E}_\Delta$ prior to ζ.

Our first observation is

Lemma 0.1. *Intervals of excursion from A_0 to B_0 back to A_0 cannot accumulate before ζ.*

Proof. This is clear from the existence of left-hand limits up to ζ and the assumption (i).

Recalling that the passage times

$$D_C=\inf\{t\geqq 0\colon X(t)\in C\}$$

for $C\in\mathscr{E}_\Delta$ are $\mathscr{F}_t$-stopping times [1], we next define successive random times T'_n, L_n, and T^*_n such that the excised excursions occur in the intervals $[L_n, T'_n)$, $n\geqq 1$.

Definition 0.1. With $\inf(\emptyset)=\infty$ and $\sup(\emptyset)=0$, let $T'_0=D_{A_0}$, $T^*_1=D_{B_0}$, and $L_1=\sup\{t<T^*_1, X_t\in A_0\}$. Define inductively for $n\geqq 1$

$$T'_n=T^*_n+T'_0\circ\theta_{T^*_n},$$

$$T^*_{n+1}=T'_n+T^*_1\circ\theta_{T'_n},$$

and

$$L_{n+1}=T_n'+L_1\circ\theta_{T_n'}.$$

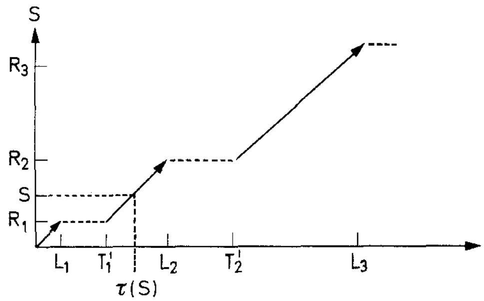

Fig. 1. Excised time S and given time $\tau(S)$

Note that T_n^* and T_n' are stopping times but L_n is not. However, $L_n\in\mathscr{F}_{T_n^*}$. In the following definition, if s is the time parameter of Y then $N(s,\omega)$ is the number of excursions excised by time s and $\tau(s)$ is corresponding time of X (see Fig. 1).

Definition 0.2[1]. Using $\sum_{k=1}^{0}(\cdot)=0$, for $0\leqq s<\infty$ define

$$N(s,\omega)=\begin{cases}0; & 0\leqq s<L_1.\\ n; & s\in\left[L_1+\sum_1^{n-1}(L_{k+1}-T_k'),\, L_1+\sum_1^{n}(L_{k+1}-T_k')\right), \qquad T_n'<\infty\\ \infty; & \text{there is no such } n.\end{cases}$$

Setting $S_n(\omega)=\sum_1^n(T_k'-L_k)$, let

$$\tau(s)=\begin{cases}s+S_{N(s)}(\omega); & N(s)<\infty\\ \infty; & N(s)=\infty.\end{cases}$$

The time change is plotted for a fixed ω with $0<L_1$ in Fig. 1[2]. We set $R_n=T_n'-S_n$ so that S_n represents the total deleted time prior to T_n', and R_n is the total included time before T_n'.

The following relations are consequences of the definitions:

$$\{\tau(0)=0\}=\{0<L_1\}, \tag{0.1}$$

$$\begin{aligned}\{R_n\leqq t<R_{n+1}\}&=\{T_n'\leqq\tau(t)<L_{n+1}\}\\ &=\{T_n'\leqq\tau(t)<T_{n+1}'\},\end{aligned} \tag{0.2}$$

$$\tau(s+t)=\tau(s)+\tau(t)\circ\theta_{\tau(s)}. \tag{0.3}$$

We can now define the excised process and its probability function.

[1] In definitions by cases, we use the semicolon to separate the functions from their domains of application.

[2] We are indebted to P.A. Meyer for this figure, and for a number of suggestions concerning our statement of the problem.

Definition 0.3. For $\Lambda \in \mathscr{F}$ and $y \in E_\Delta$,

$$Q^y(\Lambda) = \begin{cases} P^y(\Lambda \mid \tau(0)=0); & P^y\{\tau(0)=0\}>0 \\ I_\Lambda(\omega_x); & \text{otherwise} \end{cases}$$

where $\omega_x(t)=x$ for all $t \geqq 0$ and I_Λ is the indicator function of Λ.

$$Y_t(\omega)=Y(t)=\begin{cases} X(\tau(t)); & P^{X(0)}\{\tau(0)=0\}>0 \\ X(0); & P^{X(0)}\{\tau(0)=0\}=0, \end{cases}$$

$$\zeta_Y(\omega)=\inf\{t\colon Y(t)=\Delta\}.$$

Note that $Q^y = P^y$ if y is regular for A_0. We cannot assert, however, that Q^y is $\mathscr{E}_\Delta$-measurable. If one wishes to obtain this property it seems to be necessary to assume

(iii) A_0 and B_0 are open or closed,

(iv) X is a standard process.

We record here two propositions related to these extra assumptions.

Proposition 0.1. *Under assumptions* (i) *through* (iv), $Q^y\{Y(t)\in A\}$ *is* $\mathscr{E}_\Delta$*-measurable for each* $A \in \mathscr{E}_\Delta$.

Proof. If A_0 and B_0 are both open, then all of the variables in Definition 0.1 are $\mathscr{F}^0$-measurable, where $\mathscr{F}^0$ is the minimal, pre-completed σ-field, and thus setting $B_1=\{x\colon P^x\{\tau(0)=0\}=0\}-\{\Delta\}$ we have $B_1 \in \mathscr{E}_\Delta$. If A_0 (or B_0) is closed, we may find, by [1, I, Corollary 10.17], a decreasing sequence of open sets whose passage times converge to that of A_0 (or B_0, respectively) P^x almost surely for each x. Using these limits in place of D_{A_0} and D_{B_0}, Q^y and B_1 will be unchanged, and we can define a new process equal to $X(\tau(t))$ for all t a.s. Q^y for each y. Since the new process is $\mathscr{F}^0$-measurable, the proof is complete.

Proposition 0.2. *With the notations and assumptions of Proposition 0.1, for* $y \in E_\Delta - B_1$,

$$Q^y\{Y(t)\in E_\Delta - B_1 \text{ for } 0 \leqq t < \infty\}=1.$$

Proof. Since Q^y is absolutely continuous with respect to P^y and X has the strong Markov property at each T_n', it is enough to show that $P^y\{D_{B_1}<L_1\}=0$ for all y in $E_\Delta - B_1$. By [1, I, Corollary 10.17] there is an increasing sequence of compact sets $K_n \subset B_1$ such that $D_{K_n} \downarrow D_{B_1}$, P^y a.s. Thus we have

$$\begin{aligned} P^y\{D_{B_1}<L_1\} &= \lim_{n\to\infty} P^y\{D_{K_n}<\infty, D_{K_n}<L_1\} \\ &\leqq \lim_{n\to\infty} E^y[D_{K_n}<\infty, P^{X(D_{K_n})}\{\tau(0)=0\}] \\ &=0, \end{aligned}$$

since $X(D_{K_n}) \in B_1$ on $\{D_{K_n}<\infty\}$.

In view of Proposition 0.2, we may pose

Definition 0.4.

$$\theta_t^Y(\omega)=\begin{cases} \theta_{\tau(t)}(\omega); & X(0)\in E_\Delta - B_1 \\ \omega; & \text{otherwise}. \end{cases}$$

Under (i)–(iv), we then have (see (0.3)) $Y_t \circ \theta_s^Y = Y_{s+t}$ for all $s, t \geqq 0$, except on a $Q^{(\cdot)}$-null set.

We give three final remarks before stating the theorem. First, under assumptions (i)–(iv), it is possible that $T_n' = L_{n+1}$ a.s. for all n with $T_n' < \infty$. In this case, one has $Q^y\{\zeta_Y = D_{A_0}\} = 1$ for $y \in E - B_1$. To exclude such a degenerate case, and at the same time to insure that $P^y = Q^y$ for all $y \in A_0$, it suffices to assume

(v) Every point of A_0 is regular for A_0.

Indeed, this implies that $T_n' < L_{n+1}$ a.s. on $\{T_n' < \infty\}$ for each n. Second, we have $Q^y\{Y(0) = y\} = 1$ for all $y \in E_\Delta$. However, the set $\{Y(0) \in B_1\}$ is clearly foreign to the excised process, and is retained only to avoid reducing the original state space. Third, $Y(t)$ is right-continuous, with left limits on $0 < t < \zeta_Y$.

1. Statement and Proof of the Theorem

We let $\mathscr{F}_Y^0(t)$ denote the σ-field on Ω generated by $Y(s)$, $0 \leqq s \leqq t$, and let $\mathscr{F}_Y(t)$ be the usual completion of $\mathscr{F}_Y^0(t)$ in $\mathscr{F}_Y = \mathscr{F}_Y(\infty)$ with respect to $\{Q^y, y \in E_\Delta\}$.

Theorem. *If A_0 and B_0 satisfy conditions* (i) *and* (ii), *then* $Y = (\Omega, \mathscr{F}_Y, \mathscr{F}_Y(t), Y(t), Q^y)$ *is a strong Markov process on $(E_\Delta, \hat{\mathscr{E}}_\Delta)$ where $\hat{\mathscr{E}}_\Delta$ is the collection of universally measurable sets of E_Δ.*

Corollary. *If* (i)–(iv) *are assumed, and if $A_0 = \{\alpha\}$ and α is regular for $\{\alpha\}$, then $Y = (\Omega, \mathscr{F}_Y, \mathscr{F}_Y(t), Y(t), \theta_t^Y, Q^y)$ is a standard process. If, moreover, α is recurrent (i.e. for each N, $X(t) = \alpha$ for some $t > N$ a.s.) then Y is a Hunt process.*

Proof of the Theorem. In view of [1, I., Thm. 7.3] it suffices to prove the strong Markov property for $\mathscr{F}_Y^0(t+)$-stopping times T_Y ($\{T_Y < t\} \in \mathscr{F}_Y^0(t)$ for all t). In particular, this will imply the right-continuity of $\mathscr{F}_Y(t)$. The key to the proof is the introduction of a sequence T_n, $n \geqq 0$, of $\mathscr{F}$-stopping times such that $T_n = \tau(T_Y)$ on $\{N(T_Y) = n\}$, and we begin with T_0.

By definition of $\mathscr{F}_Y^0(t)$, there is a Borel function $f_t\ (x_1, x_2, \ldots)$ on the usual infinite product space, and a sequence $s_1, s_2, \ldots$ with $s_n \leqq t$, such that

$$I_{\{T_Y < t\}} = f_t(Y(s_1), Y(s_2), \ldots).$$

To simplify the notation, we suppress the dependence of the s_n on t, and assume that f_t takes only the values 0 and 1. Note that we have

$$T_Y(\omega) = \inf\{s\colon 0 = \prod_{\substack{r<s \\ r \text{ rational}}} (1 - f_r(Y(s_1), Y(s_2), \ldots))\}.$$

Definition 1.1. Let $f_t(\omega) = f_t(X(s_1), X(s_2), \ldots)$ and

$$T_0 = \inf\{s\colon 0 = \prod_{\substack{r<s \\ r \text{ rational}}} (1 - f_r(X(s_1), X(s_2), \ldots))\},$$

where the countable family $\{s_i, s_i \leqq r\}$ appearing in f_r depends on r.

Lemma 1.1. *T_0 is an $\mathscr{F}_t$-stopping time. Moreover,*

$$\{N(T_Y) = 0\} = \{N(T_0) = 0\} \subset \{T_0 = T_Y\}.$$

Proof. The first assertion follows from

$$\{T_0 < t\} = \bigcup_{\substack{r<t \\ r \text{ rational}}} \{f_r(\omega) = 1\} \in \mathscr{F}_t.$$

Now if $N(T_Y)=0$, then for some $\varepsilon>0$, $X_s=Y_s$ for $0\leqq s<T_Y+\varepsilon$, $T_0=T_Y$, and $N(T_0)=0$. Conversely if $N(T_0)=0$, there must be a hit of A_0 after time T_0 and before D_{B_0}. Hence $X(s)=Y(s)$ up to $T_0+\varepsilon$ for some $\varepsilon>0$, $T_Y=T_0$, and $N(T_Y)=0$.

To define T_n, $n\geqq 1$, we first introduce a sequence of mappings φ_n: $\Omega\to\Omega$ which "splice out" the first n round trips from A_0 to B_0 back to A_0, if such exist.

Definition 1.2. For all $\omega\in\Omega$, let $\varphi_0\,\omega=\omega$ and for $n\geqq 1$

$$(\varphi_n\,\omega)(t)=\begin{cases}\omega(\tau(t)); & N(t)<n\\ \omega(t+S_n); & N(t)\geqq n\end{cases}$$

where $\omega(\infty)=\Delta$ and S_n is as in Definition 0.2. Then letting $\Lambda_n=\{T_0\circ\varphi_n+S_{n-1}\geqq L_n\}$,

$$T_n(\omega)=\begin{cases}T_0(\varphi_n\,\omega)+S_n(\omega); & \omega\in\Lambda_n\\ \infty; & \omega\in\Lambda_n^c.\end{cases}$$

As before we have for $n\geqq 1$

Lemma 1.2. *T_n is an $\mathscr{F}_t$-stopping time, and*

$$\{N(T_Y)=n\}=\{N(T_0\circ\varphi_n)=n\}\subset\{T_n=\tau(T_Y)\}.$$

Proof. By adding $T_n'-L_n$ to both sides of the inequality defining Λ_n we observe that $T_n'\leqq T_n$. An examination of the definition shows that S_n is $\mathscr{F}_{T_n'}$-measurable, and that the definition of Λ_n depends on $\omega(s)$ only for $s\leqq L_n-S_{n-1}+S_n=T_n'$. Hence $\Lambda_n\in\mathscr{F}_{T_n'}$.

Since $\{T_n<t\}=\Lambda_n\cap\{T_n'<t\}\cap\{T_0\circ\varphi_n+S_n<t\}$, it suffices to show that the last set is in $\mathscr{F}_t$. But this follows from the same argument—the dependence on $\omega(s)$ is only up through $(t-S_n)+S_n=t$.

Now suppose that t is any time and $N(t,\omega)=n$. Then for some $\varepsilon>0$

$$X(\varphi_n\,\omega(s))=Y(s),\qquad 0\leqq s<t+\varepsilon.$$

Consequently if $N(T_Y)=n$, then

$$f_r(Y(s_1),Y(s_2),\ldots)=f_r(X(\varphi_n\,\omega(s_1)),X(\varphi_n\,\omega(s_2)),\ldots)$$

for all $r<T_Y+\varepsilon$, some $\varepsilon>0$, and $T_Y=T_0\circ\varphi_n$. If $N(T_0\circ\varphi_n)=n$, the same argument applies. Since $T_n=T_0\circ\varphi_n+S_n=\tau(T_Y)$ on $\{N(T_Y)=n\}$, the proof is complete.

To prove the theorem we need one more lemma, whose proof will be postponed to follow that of the theorem itself.

Lemma 1.3. *For each $\Lambda_0\in\mathscr{F}_Y^0(T_Y)$ and each $n\geqq 0$ there exists a $\Gamma_n\in\mathscr{F}_{T_n}$ such that*

$$\Gamma_n\cap\{N(T_0\circ\varphi_n)=n\}=\Lambda_0\cap\{N(T_Y)=n\}.$$

Granting this lemma, and replacing $\Lambda\in\mathscr{F}_Y(T_Y)$ by a P^y-equivalent $\Lambda_0\in\mathscr{F}_Y^0(T_Y)$, we have for $A\in\mathscr{E}_\Delta$,

$$\begin{aligned}&P^y(\Lambda,\tau(0)=0,N(T_Y)=n;\ Y(T_Y+t)\in A)\\ &\quad=P^y(\Gamma_n,\tau(0)=0,N(T_Y)\geqq n,\tau(0)\circ\theta_{T_n}(\omega)=0,X(T_n+\tau(t)\circ\theta_{T_n})\in A)\\ &\quad=E^y(Q^{X(T_n)}(X(\tau(t))\in A);\Gamma_n,\tau(0)=0,N(T_Y)=n)\\ &\quad=E^y(Q^{Y(T_Y)}(Y(t)\in A);\tau(0)=0,\Lambda,N(T_Y)=n)\end{aligned}$$

where the second equality follows from the strong Markov property for X and the definition of Q^y. Noting that

$$\bigcup_{n=0}^{\infty}\{N(T_Y)=n\}=\{T_Y<\zeta_Y\}$$

we have after a summation over n

$$Q^y(\Lambda, T_Y<\zeta_Y, Y(T_Y+t)\in A)=\hat{E}^y(Q^{Y(T_Y)}(Y(t)\in A); \Lambda, T_Y<\zeta_Y),$$

where $\hat{E}^y$ denotes the conditional expectation given $\tau(0)=0$. Since both sides are universally measurable, for any probability measure μ we can integrate to obtain

$$Q^\mu(\Lambda, T_Y<\zeta_Y, Y(T_Y+t)\in A)=\hat{E}^\mu(Q^{Y(T_Y)}(Y(t)\in A); \Lambda, T_Y<\zeta_Y).$$

Over $\{T_Y\geqq\zeta_Y\}$ the analogous result is obvious, completing the proof of the strong Markov property.

We return to the proof of Lemma 1.3. For each t, let $h_t(x_1, x_2, \ldots)$ be a Borel function such that

$$I_{\Lambda_0}(\omega)\, I_{\{T_Y<t\}}(\omega)=h_t(Y(s_1), Y(s_2), \ldots)$$

where as before the countable family $\{s_i, s_i\leqq t\}$ depends on t and h_t takes on only the values 0 and 1. Let $h_t(\omega)=h_t(X(\varphi_n\omega(s_1)), X(\varphi_n\omega(s_2)), \ldots)$, and introduce the functions

$$k_t=\begin{cases}0; & N(T_0\circ\varphi_n\omega)<n\\ 1-\prod\limits_{\substack{r<t\\ r\text{ rational}}}(1-h_r(\omega)\, I_{\{T_0\circ\varphi_n<r\}}); & \text{otherwise}.\end{cases}$$

One notes that k_t is non-decreasing in t, left-continuous, and takes on only the values 0 and 1. Moreover, it is 1 only if $T_0\circ\varphi_n<t$. To handle a technical detail, we require the rather cumbersome last restriction in the following definition. Let $\Gamma_t=\{k_t=1$, and for all rational $r'<r<t$, $(k_r-k_{r'})\, I_{\{T_0\circ\varphi_n<r'\}}=0\}$. We show first that Γ_t is non-decreasing in t, and second that $\Gamma_n=\lim\limits_{t\to\infty}\Gamma_t$ is the desired set. Suppose, indeed, that for $s<t$, $\Gamma_s-\Gamma_t\neq\emptyset$. Then for $\omega\in\Gamma_s-\Gamma_t$ we have $k_s=k_t=1$, and so $(k_r-k_{r'})\, I_{\{T_0\circ\varphi_n<r'\}}=1$ for some $r'<r<t$. Then $k_{r'}=0$ and hence $r'<s$. But since k_t is left-continuous, there is also an $r<s$ with $k_r=1$, contradicting $\omega\in\Gamma_s$. Similarly, if $\omega\in\Gamma_n\cap\{T_0\circ\varphi_n<t\}$ then for $s>t$, $\omega\in\Gamma_s$ implies $k_r=1$ for $t-\varepsilon<r<s$ for some $\varepsilon>0$, and hence $\omega\in\Gamma_t$. Thus $\Gamma_n\cap\{T_0\circ\varphi_n<t\}=\Gamma_t$.

To prove the lemma, we now show that

$$\Lambda_0\cap\{N(T_Y)=n, T_Y<t, N(t)=n\}=\Gamma_n\cap\{N(T_Y)=n, T_0\circ\varphi_n<t, N(t)=n\}$$

for each t. If ω is in the left hand side then $Y(s)=X(\varphi_n\omega(s))$, $0\leqq s<t$, and we have easily $k_t=1$ and $\omega\in\Gamma_t\subset\Gamma_n$. Together with $T_Y=T_0\circ\varphi_n$, this gives the inclusion from left to right. On the other hand, if $\omega\notin\Lambda_0$ but satisfies the other conditions on the left, then $h_r(\omega)=0$ for $r<t$, $k_t=0$, and $\omega\notin\Gamma_t$. Since $T_0\circ\varphi_n<t$, $\omega\notin\Gamma_u$ for

$u>t$, and so $\omega\notin\Gamma_n$. Finally, to prove that $\Gamma_n\in\mathscr{F}_{T_n}$, we have

$$\Gamma_n\cap\{T_n<t\}=\bigcup_{\substack{r<t\\ r\text{ rational}}}\Gamma_n\cap\{T_n'<t\}\cap\{T_0\circ\varphi_n\,\omega<r<t-S_n\}$$
$$=\bigcup_{\substack{r<t\\ r\text{ rational}}}\Gamma_r\cap\{T_n'<t\}\cap\{T_0\circ\varphi_n\,\omega<r<t-S_n\}$$

and simply argue as in Lemma 1.2 that the dependence on ω does not extend beyond $(t-S_n)+S_n=t$. This completes the proof of the theorem.

Proof of the Corollary. The measurability conditions and the translation operators were discussed in Proposition 0.1 ff., and we need only prove quasi-left continuity on $(0,\zeta_Y)$. For $T_Y\leqq\zeta_Y$ let T_n and $\{N(T_Y)=n\}$ correspond to T_Y as before, let R_j converge upward to T_Y, and let $T_{n,j}$ correspond in the same way to R_j. By [1, p. 36] we may assume that R_j and T_Y are $\mathscr{F}_Y^0(t+)$-stopping times, and we must show that

$$Q^\mu\{Y(R_j)\to Y(T_Y);\ T_Y<\zeta_Y\}=Q^\mu\{T_Y<\zeta_Y\}.$$

For $\omega\in\{N(T_Y)=n,\ T_Y<\zeta_Y\}$ either $T_n'<T_n<L_{n+1}$, or else $\tau(T_Y)=T_n'$. In the former case, $Y(R_j)\to Y(T_Y)$ follows from $X(T_{n,j})\to X(T_n)$. In the latter case, either $R_j=T_Y$ for all large j, or else $\tau(R_j)=T_{n-1,j}$ for all large j a.s., and

$$\lim_j Y(R_j)=\lim_j X(T_{n-1,j})=\alpha=X(T_n')=Y(T_Y),$$

since $A_0=\{\alpha\}$.

If α is recurrent, then $T_n'=\infty$ implies $L_n=\infty$ a.s. Since α is regular for $\{\alpha\}$ we have $\zeta=\infty$ a.s., and also since $X(T_n')=\alpha$, $\sum_{k=0}^{\infty}(L_{k+1}-T_k')=\infty$ a.s. If follows that $\zeta_Y=\infty$, Q^μ a.s. for each μ, and Y is a Hunt process. This completes the proof.

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(Received September 15, 1971)