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A SOLUTION TECHNIQUE FOR A CLASS OF OPTIMAL CONTROL PROBLEMS IN DISTRIBUTIVE SYSTEMS

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A,B,C	Infinite matrices defined by Equations (3.14), (3.24) and (3.30), respectively.
D	The infinite column vector (1,1,...,1,...) .
D(F)	The domain of the transformation F .
F	A linear (bounded) transformation defined by Equation (3.42) for Section (3.2) and by Equation (3.61) for Section 3.3.
F_1, F_2, F_3	Linear (bounded) transformations defined by Equations (3.17), (3.26), (3.33) respectively.
$\hat{F}_1$	A linear (bounded) transformation defined by Equation (4.8).
$F_i^\dagger$	The pseudoinverse of the transformation F_i .
G	Green's function.
H_i	A separable Hilbert function space.
I	The identity operator.
J	A performance index.
K	The kernel of an integral transformation.
$L_2(0,b)$	The usual L_2 Hilbert space of square integrable functions defined on the closed interval [0,b] .
$\overline{L_2}(t_0,t_1)$	A Hilbert function space defined on page 45
M(F)	The orthogonal complement of the null space of the transformation F .
N(F)	The null space of the transformation F .
P,Q	Self-adjoint positive-definite matrix operators defined on page 77
$\overline{P}$, $\overline{Q}$	Bounded positive definite measurable functions defined on $\overline{\Delta} = [t_0,t_1] \times [0,b]$.
R^m	The m-dimensional Euclidean space.

R_n	The n^{th} Fourier coefficient of the system state x with respect to the orthogonal complete basis $\{\omega_n\}$ of $L_2(o,b)$.
$R(F)$	The range of the linear transformation F .
S	A subspace of $\overline{L}_2(t_o,t_1)$ defined by Equation (4.13) .
T_n	The n^{th} Fourier coefficient of the system state x with respect to the orthonormal complete basis $\{\varphi_n\}$ of $L_2(0,b)$.
$\mathcal{F},\mathcal{G},\mathcal{H}$	Hilbert function spaces defined on page 40
$\mathcal{L}$	Spatial differential operator.
$\mathcal{S}$	The cartesian product space $\overline{L}_2(t_o,t_1) \times \mathcal{G} \times \mathcal{H}$.
f	Spatially distributed control input function.
g_i	Spatially discreted control input function.
h_1,h_2	Boundary control input functions.
$\overline{h}_1,\overline{h}_2$	Elements in $\mathcal{H}$ defined by Equations (3.36) and (3.37), respectively.
k^2	Coefficient of diffusivity of the diffusion system.
ℓ_2	The usual ℓ_2 Hilbert space of square summable infinite tuplets of scalars.
q	A (boundary) control input defined by Equation (3.52) .
t	The temporal variable.
t_o,t_1	Initial and terminal times, respectively.
u	$(u_1,\ldots,u_n,\ldots)$ is the image of the spatially distributed control input f in the Hilbert space $\overline{L}_2(t_o,t_1)$.
u_n	The n^{th} Fourier coefficient of the spatially distributed control input function f with respect to $\{\varphi_n\}$, i.e., $u_n(t) = \langle f(t,\alpha),\varphi_n(\alpha) \rangle$
x	The state trajectory of the distributed parameter system defined on $\Delta = [t_o,t_1] \times \overline{\Omega}$.
x^o,x^1	The initial and terminal states of the distributed parameter system defined on $\overline{\Omega}$.

x_f	The system state trajectory under the spatially distributed control input f .
z	The image of the system state x in the space $\bar{L}_2(t_o,t_1)$.
z^o, z^1	The images of the x^o, x^1, respectively, in the ℓ_2 space.
z_f	The image of x_f in $\bar{L}_2(t_o,t_1)$.
Γ	Admissible set of control inputs.
Δ	The cartesian product $[t_o,t_1] \times \Omega$.
Λ, Φ, Ψ	Infinite diagonal matrices defined by Equations (4.34), (3.16), (3.60), respectively.
Ω	The spatial domain; an open connected set in the m-dimensional Euclidean space R^m .
$\partial\Omega$	The boundary of Ω .
α	A point in the spatial domain, i.e., $\alpha \in \bar{\Omega}$.
η	An infinite tuplet of scalars .
κ	A scalar.
λ_n	An eigenvalue - A Lagrangian Multiplier .
ξ_n^1	The n^{th} Fourier coefficient of the terminal state x^1 with respect to $\{\varphi_n\}$, i.e., $\xi_n^1 = < x^1, \varphi_n >$.
τ	The closed time interval $[t_o,t_1]$.
$\{\varphi_n\}, \{\sqrt{1/b}, \psi_n\}$	Ortonormal complete bases for $L_2(0,b)$ defined by Equations (3.9) and (4.3) respectively.
$\{\omega_n\}$	An orthogonal complete basis for $L_2(0,b)$ defined by Equation (3.53).
$\| \ \|$	Absolute value.
$\| \| \ \| \|$	Norm.
$< , >$	Inner product
$[,]$	The usual inner product in ℓ_2 .

(,],[,) An open-closed and a closed-open intervals, respectively.

1. The asterisk on capital letter is used to denote the adjoint.

2. The asterisk on small letters is used to denote the optimum element.

3. The bar on the capital Greek letters is used to denote the closure of the set.

CHAPTER 1

INTRODUCTION

1.1. Introduction

During the last decade, many aspects of control theory have advanced rapidly. This is largely due to the fact that control engineers have been called upon to deal with increasingly complex system problems. The increase in system complexity has been accompanied by more stringent demands on system performance. Thus the need for a comprehensive theory of optimal control systems has been recognized by both engineers and mathematicians.

Presently, the majority of results developed within the domain of optimal control theory has been for systems with lumped parameters. The dynamic behavior of such systems is describable by a system of ordinary differential equations,

$$\dot{x} = f(x,u) \ . \tag{1.1}$$

Here f is the vector-valued function $f(x,u) = \mathrm{col}[f_1(x,u),\ldots,f_n(x,u)]$, of the tuplets $x = (x_1,\ldots,x_n)$ and $u = (u_1,\ldots,u_m)$ defined on the cartesian product space $X^n x\Gamma = \{(x,u):x \in X^n, u \in \Gamma\}$, where X^n is an n-dimensional state space and Γ is an arbitrary **topological** Hausdorff space of the **admissible** values of the control parameter u. For convenience, the independent variable t is suppressed in Equation (1.1) and the dot denotes time derivatives. The typical optimal control problem is that of choosing from the class of admissible controls a function

$u(t)$ $(t_o \leq t \leq t_1)$, for which the control trajectory $x(t)$ for Equation (1.1) traverses a path from the point $x_o = x_o(t_o)$ to some manifold M_s whose dimensions should not exceed $(n-1)$, and such that an integral of the form

$$J = \int_{t_o}^{t_1} f_o(x,u)dt \tag{1.2}$$

along the path should be a minimum. Here $f_o(x,u)$ is a known function of its arguments, defined on the same set as the function $f(x,u)$.

In practice, however, systems with distributed parameters occur much more often than systems with lumped parameters or systems which can be reduced to the case of lumped parameters. Such systems, for instance, include a large number of production-line industrial processes. In particular, the heating of metals in through-passage furnaces, the drying of strip and friable materials, continuous etching and deposition of coatings, distillation and other chemical processes are examples. As in lumped parameter systems, a not uncommon distributed parameter systems design problem consists of determining the optimal (spatial) distribution of a certain number of parameters which will give the best possible performance in a definite sense. It is, therefore, of great importance to develop an optimal control theory for systems with distributed parameters, and even more so if lumped parameter systems may be considered as a special case.

Since one of the fundamental prequisites for the analytic design of control systems is the establishment of an adequate mathematical model for the physical system to be controlled, the basic ingredients of such a model for distributed-parameter systems will be presented here.

1.1.1. <u>The Mathematical Model.</u>

The independent variables of a distributed parameter system usually consists of a temporal variable t and a finite tuplet of spatial variables $(\alpha_1,\ldots,\alpha_m)$. The range of values for the temporal variable is denoted by τ and Ω will denote the subset of the m-dimensional Euclidean space R^n for which the spatial variables have significance. Ω will be assumed to be a connected open set, the closure of which will be denoted by $\overline{\Omega}$ and the boundary by $\partial\Omega$. It may happen that the spatial region Ω be dependent of t; such dependence is denoted by: Ω_t, $t \in \tau$. The set

$$\Delta = \{(t,\alpha):t \in \tau\ ,\ \alpha = (\alpha_1,\ldots,\alpha_m)\in \Omega_t\}$$

is then the region of interest for the independent variables.

The dependent variables (usually taken as the state variables) consist of a finite collection $\{x_i:i = 1,\ldots,n\}$ of scalar valued functions defined on the set Δ. It is important to notice that at any time $t \in \tau$, the state of the system is given by a vector-valued <u>function</u> $x(\alpha)$ defined on Ω, in contrast with the lumped-parameter case where the state at any instant $t \in \tau$ is given by a finite tuplet of <u>scalars</u>. The spatial state function space $X(\Omega)$ will be defined as the set of all possible functions which $x(\alpha)$ may assume at any time $t \in \tau$. An example of $X(\Omega)$ is $L_2^n(\Omega)$-the cartesian product of n copies of the $L_2(\Omega)$ Hilbert space.

The control action is described by a vector-valued function $u = \mathrm{col}[u_1, \ldots, u_r]$ defined over all or certain subsets of Δ and has values in some admissible set Γ which may, in general, be any topological Hausdorff space. In particular, an important case is when Γ is a closed region of some r-dimensional Euclidean space R^r. It may prove useful, in many cases, to differentiate between two types of control inputs:

(1) Distributed inputs which act on the interior of the spatial domain Ω, and

(2) Boundary inputs which act on all or certain subsets of the boundary of Ω, namely $\partial\Omega$.

The dynamic behavior of many (deterministic) distributed parameter systems can be described by a family of partial differential equations defined on the interior of Δ, together with some initial (with respect to time) and boundary (with respect to the spatial domain Ω) conditions. Any trajectory of the state of the system should satisfy the partial differential equations in the interior of Δ and, at the same time, fulfill (in the sense of the limit) the initial and boundary conditions. Such a mathematical model is considered to be well-posed if corresponding to every control input $u \in \Gamma$, there exists a unique stable trajectory satisfying all the above conditions. By "stable", it is meant (roughly) that small changes in any of the given conditions must cause a correspondingly small change in the trajectory. The existence and uniqueness requirements mean that among the given conditions there are none that are incompatible and that these conditions are sufficient to determine a unique trajectory. The stability requirement is necessary for the following reason. In the given conditions for a specific system, especially if they are obtained from experiment, there is always some error, and it is necessary that a small error in the given conditions causes only a small inaccuracy in determining the trajectory. This requirement expresses the determinate nature of the system under consideration.

It is worth remarking here, that the inverse of a partial differential operator (like the ordinary differential operator) is an integral operator, the kernel of which is called the Green's function of the operator (see [21]). Therefore, an alternate representation of a distributed parameter system may be in the form of a family of integral equations.

As an illustration of the mathematical model described above, two examples, which have been presented by a number of authors and have provided the motivation for the study, will be discussed here. (See [11], [54], [55])

Example 1: Consider the continuous furnace of Figure 1. A continuous strip of homogeneous material is fed with flow rate 'v' into the furnace by a variable speed transport mechanism. The temperature of regions I and II of the furnace is denoted $h_1(t,\beta)$ and $h_2(t,\beta)$ respectively. The spatial domain for the variables (α,β) is given by

$$\Omega = \{(\alpha,\beta) : \alpha \in [0,d],\ \beta \in [0,1]\} .$$

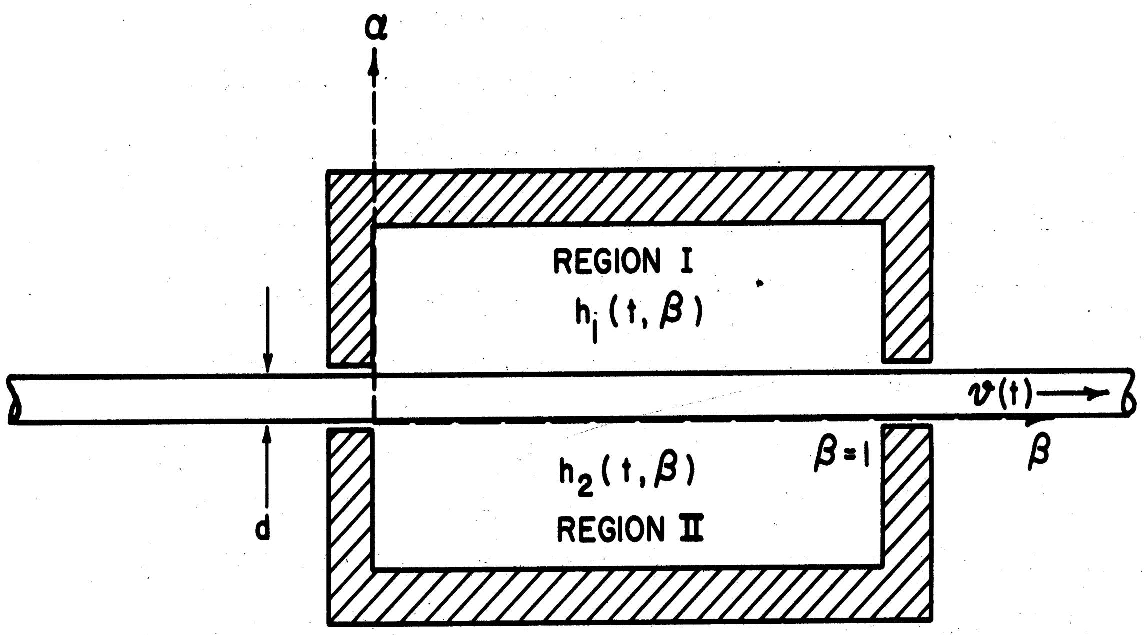

Figure 1. A Continuous Furnace.

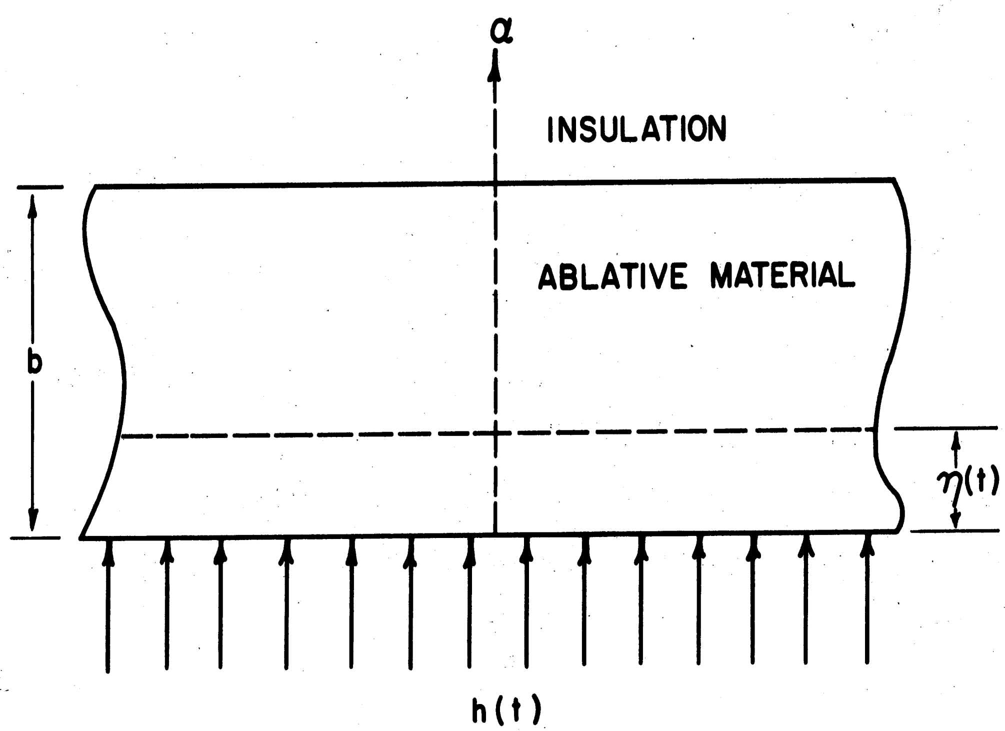

Figure 2. An Ablative Surface.

The temperatures h_1, h_2 and the flow rate v are the manipulatable controls of the systems.

Consider first the case where the material is thin and the temperature distribution are spatially uniform in the two regions of the furnace (i.e., $h_1 = h_2 = h$), then the temperature of the material, $x(t,\beta)$, which represents the state of the system, can be approximately described by the equation

$$x_t(t,\beta) = \mu\, x_{\beta\beta}(t,\beta) + v(t)x_\beta(t,\beta) + \sigma[x(t,\beta) - h(t,\beta)] \qquad (1.3)$$

where μ is the coefficient of diffusivity, σ a constant proportional to the surface conductivity, and subscripts are used to denote the obvious partial derivatives. On the other hand, if the material is thick and stationary and if h_1 and h_2 are independent of β, then the equation governing the temperature distribution interior to the strip and in the α direction is given by

$$x_t(t,\alpha) = \quad x_{\alpha\alpha}(t,\alpha) \qquad (1.4)$$

with the boundary conditions:

$$x(t,0) = h_1(t), \; x(t,d) = h_2(t) \; . \qquad (1.5)$$

In both cases, initial conditions must be satisfied to complete the formulation of the mathematical model.

Example 2: In many aerodynamic re-entry vehicles, ablative shields are necessary to protect the vehicle from damage caused by aerodynamic heating. In such cases, the velocity and attitude of the vehicle must be closely controlled so that the ablation rate does not exceed a certain maximum allowable value at any time during the re-entry flight.

In Figure 2, a one dimensional version of the ablation problem is depicted. One surface $(\alpha = b)$ of the ablative slab is insulated and the other $(\alpha = 0)$ is subjected to the normalized heat input 'h' . Let $x(t_o,\alpha)$ denote the slab temperature at the time of initial re-entry t_o and t_1 the time at which $x(t,0)$ reaches the melting point x_m . The diffusion equation

$$x\ (t,\alpha) = \mu\, x_{\alpha\alpha}(t,\alpha) \qquad (1.4)$$

with the initial and boundary conditions

$$\begin{aligned} x(t_o,\alpha) &= x^o(\alpha) && \alpha \in [0,b], \\ x_\alpha(t,0) &= (1/\kappa)h(t) && t \in [t_o,t_1], \text{ and} \\ x_\alpha(t,b) &= 0\ , && t \in [t_o,t_1], \end{aligned} \qquad (1.6)$$

where κ is the thermal conductivity of the slab, describe the slab temperature during the "premelt" time interval. At time t_1 , the slab surface begins to melt and it is assumed that the material is immediately removed by aerodynamic forces. If $\eta(t)$ denotes the depth of erosion of this process at time $t \geq t_1$, then the slab temperature is still governed by the diffusion equation with the modified initial conditions

$$\eta(t_1) = 0,\ x(t_1 - 0,\alpha) = x(t_1 + 0,\alpha), \qquad (1.7)$$

and boundary conditions

$$x(t,\eta(t)) = x_m ,$$
$$\zeta L\eta_t(t) - \kappa\, x_\alpha(t,\eta(t)) = h(t) , \qquad (1.8)$$
$$x\,(t,b) = 0$$

where κ , ζ and L represent the thermal conductivity, the density and the latent heat of melting of the slab respectively.

1.1.2. The Performance Index and the Optimal Control Problem.

The second step following the establishment of a suitable mathematical model for the physical system to be controlled is to choose a realistic performance index, i.e., an adequate analytic statement of the purpose of control. In its most general form, the performance index may be described by the functional

$$J = \int_\Omega G_0(t_1;\alpha;x_1(t_1,\alpha),\ldots,x_n(t_1,\alpha))d\Omega + \int_{t_0}^{t_1}\int_\Omega G_1(t,\alpha;x_1,\ldots,x_n;u_1,\ldots,u_r)d\Omega dt \qquad (1.9)$$

where G_0 and G_1 are specified scalar functions of their arguments, and t_1 is the terminal time.[1] The first integral in Equation (1.9) represents a terminal error measure, while the second integral represents an error measure defined over the entire time interval. In terms of this general performance index, the optimal control problem is that of choosing

[1] Here the terminal time is defined as the first instant of time $t > t_0$ when the motion enters a specified set $S \subset \Gamma(\Omega) \times \tau$ where $\tau = \{t : t > t_0\}$.

from the class of admissible controls a function $u(t,\alpha)$, $(t_o \leq t \leq t$, $\alpha \in \Omega)$ for which the control trajectory $x(t,\alpha)$ for the given system traverses a path from the point $x_o = x(t_o,\alpha)$ to some terminal set $S \subset X(\Omega) x \tau$, $(\tau = \{t : t \geq t_o\})$, and such that the performance index (1.9) takes on the least possible value.

This general optimal control problem can be reduced to types analogous to those studied in the case of lumped-parameter systems. A few examples will be given here for illustration.

(1) Optimum Terminal Control: In this problem, it is required to drive the system from the initial state $x(t_o,\alpha)$ as close as possible to a desired terminal set $X^*(\Omega) \subset X(\Omega)$ at a specified terminal time t_1 . Here, $S = X(\Omega) x \{t_1\}$, $G_1 = 0$, and $\int_\Omega G_o \, d\Omega$ represents the distance from the given set $X^*(\Omega)$.

(2) Minimum Time Control: In this problem, it is required to drive the system from the initial state $x(t_o,\alpha)$ to a desired state $x^*(\alpha)$ in the shortest possible time. Here, $S = \{x^*(\alpha)\} x \tau$, $G_o = 0$, and $\int_\Omega G_1 d\Omega = 1$.

(3) Minimum Energy Control: In this problem it is required to drive the system from the given initial state $x(t_o,\alpha)$ to a desired state $x^*(\alpha)$ at a specified time t_1 with the expenditure of the least possible amount of energy. Here, $S = X(\Omega) x \{t_1\}$, $G_o = 0$, and G_1 is a non-negative function of u only.

With these remarks as background on the statement of the problem a review of the literature that has appeared in this area is presented in the following section.

1.2. Recent Contributions

The first serious study of optimal control problems for the distributive parameter systems was undertaken by A. G. Butkovskii and A. Y. Lerner (see [8], [9]) in 1960. In these two similar papers, three general optimal control problems are formulated. The controlled processes considered were those describable by systems of first order partial differential equations in two independent variables and the control inputs were subjected to certain constraints. During the period of 1961 to 1963, Butkovskii continued this work in a series of papers (see [10], [11], [12], [13], [14]). The major results of his work is contained in [10] and [14] which shall now be summarized.

In [10], Butkovskii studied processes whose dynamic behavior can be described by the system of nonlinear integral equations

$$x_i = x_i(p) = \int_{\Delta} K_i(p,s,x(s),u(s))ds, \quad (i=1,\ldots,n) \tag{1.10}$$

where Δ is an m-dimensional region of the Euclidean space R^m, and $x(p) = \text{col}[x_1(p),\ldots,x_n(p)]$, $p \in \Delta$, is a vector-valued function representing the state of the system. The function $u(s) = \text{col}[u_1(s),\ldots,u_r(s)]$, $s \in \Delta' \subseteq \Delta$, is the control vector which, like the process itself, may be distributed in time and space, and $K = \text{col}[K_1,\ldots,K_n]$ is a vector-valued function of the four variables p, s, x and u. The components

K_i's are assumed to belong to the L_2 space and have continuous partial derivatives $\partial K_i/\partial x_j$, $(i=1,\ldots,n)$, $(j=1,\ldots,n)$ almost everywhere on Δ. The r components of the control vector u are assumed to be measurable, bounded and square integrable functions of some subspace Δ' of Δ, and have values in some admissible set Ω which may, in general, be any topological Hausdorff space. The optimum problem discussed in this paper can be stated as follows: On a set of states $x = x(p)$ and controls $u = u(s)$ related by the integral Equation (1.10), let q (in Butkovskii paper, he implied the condition $q < r$) functionals having continuous gradients be defined by the relations:

$$\begin{aligned} I_i &= I_i[x(p)], && (i=1,\ldots,\ell) \\ I_i &= I_i[x(p),u(s)], && (i=\ell+1,\ldots,q). \end{aligned} \tag{1.11}$$

It is required to find a control $u \in \Omega$ such that

$$I_i = 0 \quad (i=1,\ldots,p-1,p+1,\ldots,q)$$

with the function I_p taking its minimum value.

Butkovskii made use of the Lagrange multipler rule to prove a theorem which he called, "The maximum principle for optimum systems with distributed parameters". He then suggested how this theorem could be used to obtain a system of equations which must be satisfied by any optimal control for the problem outlined above. In [12], the validity of the Lagrange multipler rule for the minimization of a particular class of functionals subjected to a set of equality constraints, all defined on a general Banach space, was proved and used to generalize the theorem, which was the main result in [10], to cases when the process is described by operational equations in a Banach space.

In [14], Butkovskii considered an optimal problem for the class of linear systems (with distributed parameters) described by the linear integral equation

$$x(t,\alpha) = \int_0^t K(\alpha, t-\tau)u(\tau)d\tau \qquad (1.12)$$

where $x(t,\alpha)$ is the state of the system at $(t,\alpha) \in [0,T]x[0,b]$ and K is the system Green's function. The system control u is a function only of the variable t and must satisfy the constraint $|u(t)| \leq L$, $0 \leq t \leq T$ for scalar L. In this paper, the minimum-time optimal problem was reduced to the L-problem of the theory of moments (see [3]). Applying the well-known results of this latter problem, the optimal control was expressed as the limit of a sequence of controls $\{u_n\}$. This method is easily generalized to the case where there are several controlling inputs to the system. For example, Equations (1.12) may take the form

$$x(t,\alpha) = \int_0^t \sum_{i=1}^r K_i(\alpha,t,\tau)u_i(\tau)d\tau \qquad (1.13)$$

where $|u_i(t)| \leq L_i$, $i=1,\ldots,r$, $0 \leq t \leq T$.

In addition to the scientific contribution of Butkovskii, his efforts have served to stimulate other researchers in this area. In 1962, J. V. Egorov[17] studied problems of existence and uniqueness of optimal controls associated with a particular linear diffusion system for various performance indices. In 1964, A. I. Egorov[16] considered a certain type of the optimal control problem in which the process is described by a system of second-order partial differential equations in two independent variables of the form

$$x_{i_{\alpha t}} = f_i(\alpha,t;x_1,\ldots,x_n;x_{1_\alpha},\ldots,x_{n_\alpha};x_{1_t},\ldots,x_{n_t};u_1,\ldots,u_r) \quad (1.14)$$

which hold on the domain: $0 < \alpha < b$, $0 < t \leq T$, $(i=1,\ldots,n)$, and the boundary conditions (Goursat conditions)

$$x_i(0,\alpha) = \varphi_i(\alpha),\ x_i(t,0) = \psi_i(t),\ (i=1,\ldots n) \quad (1.15)$$

where, as before, $x = \text{col}[x_1,\ldots,x_n]$ is the state of the system, $u = \text{col}[u_1,\ldots,u_r]$ denotes the control, and the subscripts in Equation (1.14) denote the obvious partial derivatives. The class of admissible controls is taken as the set of piecewise-continuous and bounded (vector-valued) functions, u defined in the region $\Delta = \{(t,\alpha): 0 \leq t \leq T,\ 0 \leq \alpha \leq b\}$ with values in some convex region Γ (open or closed) of r-dimensional Euclidian space R^r. Conditions are imposed on the functions $\{f_i,\varphi_i,\psi_i\}$ to assure the existence of a unique solution corresponding to every admissible control. The optimal control problem considered is to find an admissible control which minimizes (or maximizes) the functional

$$J = \sum_{i=1}^{n} A_i x_i(T,b) \quad (1.16)$$

where $A_i (i=1,\ldots,n)$ is a given set of real numbers.

Guided by the work of Rozonoer[43] on Pontryagin Maximum Principle, Egorov obtained a necessary condition which the optimal control must satisfy. It was also shown that this conditions is sufficient in the local sense if, instead of Equation (1.14), the system is described by the second order linear partial differential Equation

$$x_{i_{t\alpha}} = \sum_{k=1}^{n} [c_{ik}x_{k_t}+d_{ik}x_{k_\alpha}+g_{ik}x_k] + f_i(u), \quad (i=1,\ldots,n) \tag{1.17}$$

where the coefficients $\{c_{ik}\}$ $\{d_{ik}\}$, $\{g_{ik}\}$ are independent of x and u and defined along with x and u on Δ.

While A. I. Egorov was working on the above problem T. K. Sirazentdinov[47] was working independently on a similar one.[1] Sirazentdinov considered processes which are governed by a single quasi-linear first order partial differential equation in more than two independent variables, i.e., equations of the form

$$\frac{\partial x}{\partial t} = f_o(t;\alpha_1,\ldots,\alpha_m;x;u_1,\ldots,u_r) + \sum_{k=1}^{m} f_k(t;\alpha_1,\ldots,\alpha_m;x;u_1,\ldots,u_r)\frac{\partial x}{\partial \alpha_k} \tag{1.18}$$

with the constraints

$$\varphi_k(u_1,\ldots,u_r) \leq 0 \quad (k=1,\ldots,q< r) \tag{1.19}$$

The performance index considered was of the form

$$\int_o^T \int_\Omega G \, d\Omega dt \tag{1.20}$$

[1] It is interesting to note that although the Sirazentdinov paper appeared first, the paper by A. I. Egorov was submitted just one day (April 12, 1963) prior to the submission of the Sirazentdinov paper.

where Ω , the spatial domain on which the system is defined, is an open connected subset of a m-dimensional Euclidian space R^m , T is the terminal time of the process (the initial time being 0), and the function G is of the form

$$G(t,\alpha,x,u) = G_o(t;\alpha_1,\ldots,\alpha_m;x;u_1,\ldots,u_r) + \sum_{k=1}^{m} G_k(t;\alpha_1,\ldots,\alpha_m;x;u_1,\ldots,u_t)\frac{\partial x}{\partial \alpha_k} \tag{1.21}$$

Making use of Rozonoer's work [43], Sirazentdinov obtained a necessary condition of optimality which proved to be sufficient when Equations (1.18) and (1.20) above have, respectively, the following linear forms

$$\frac{\partial x(t,\alpha)}{\partial t} = a_o(t,\alpha)x(t,\alpha) + \sum_{k=1}^{n} a_k(t,\alpha)\frac{\partial x(t,\alpha)}{\partial \alpha_k} + \varphi(t,\alpha,u) \tag{1.22}$$

and G degenerated to the form

$$G(t,\alpha,x,u) = b_o(t,\alpha)x(t,\alpha) + \sum_{k=1}^{n} b_k(t,\alpha)\frac{\partial x(t,\alpha)}{\partial \alpha_k} + \psi(t,\alpha,u) \tag{1.23}$$

where α and u in the above two equations denote the tuplets $(\alpha_1,\ldots,\alpha_m)$ and $(u_1,\ldots,u_r)$ respectively.

Another important series of articles have been contributed by P. K. C. Wang who has (since 1962) studied various aspects of control of distributed parameter systems (see [52], [53], [54], [55], [56]). In [52] and [53] the stability of distributed-parameter systems with time delays (which are governed by a set of partial differential-difference equations) was studied. The notions of controllability and observability were extended, in [54], to distributed parameter systems. In the optimal control area, Wang (see [54], [55]) considered systems describable by a set of partial differential equations of the form

$$\frac{\partial x_i(t,\alpha)}{\partial t} = \mathcal{L}_1(x_1(t,\alpha),\ldots,x_n(t,\alpha);u_1(t,\alpha),\ldots,u_r(t,\alpha)) \qquad (1.24)$$

defined on the domain $\Delta = \{(t,\alpha), 0 < t \leq T, \alpha \in \Omega\}$ $(i=1,\ldots,n)$ where $\alpha = (\alpha_1,\ldots,\alpha_n)$ denotes the spatial coordinates, Ω is defined as in Equation (1.20), and $\mathcal{L}_1$ is a specified spatial differential (or differential-integral) operator. It is assumed that the problem is well-posed in the sense that Equation (1.24) has a unique solution for any admissible control, and that the initial state function $(x_1(0,\alpha),\ldots, x_n(0,\alpha))$ is sufficiently smooth so that the solution corresponding to sufficiently small time increment δ can be written as

$$x_i(\delta,\alpha) \approx x_i(0,\alpha) + \delta\mathcal{L}_i(x_1(0,\alpha),\ldots,x_n(0,\alpha);u_1(0,\alpha),\ldots, u_r(0,\alpha)) + O(\delta)$$

where $O(\delta)$ is an infinitesimal quantity of higher order than δ. The performance index considered is of the form

$$J = \int_\Omega G_o(T;\alpha;x_1,\ldots,x_n)d\Omega + \int_o^T \int_\Omega G_1(t;\alpha;x_1,\ldots,x_n;u_1,\ldots,u_r)d\Omega dt \qquad (1.25)$$

where G_o and G_1 are specified scalar functions of their arguments.

The dynamic programming technique was used to derive the usual functional equations associated with such a problem. These equations are analogous to that for the lumped-parameter systems but require the introduction of the notion of a functional partial (variational) derivative.

In recent months, several authors have refined, extended, or modified the problems considered by Butkovskii, Egorov, and Wang. The articles [31], [32], and [45] list several related contributions in their respective bibliographies. In addition, attention is called to the recent article [4] by Axelband, where the author proceeds in the same spirit as the present thesis although there is little overlap in results.

In this paper by Axelband, function space techniques are used to study the class of linear distributed parameter systems described by the partial differential equation

$$\frac{\partial x(t,\alpha)}{\partial t} = (\mathcal{L} x)(t,\alpha) + u(t,\alpha) \tag{1.26}$$

defined on $\Delta = \{(\alpha,t): 0 < t \leq T, \alpha \in \Omega\}$ where Ω is a simply connected open region in a m-dimensional Euclidian space R^m. $x(t,\alpha)$ is the state of the system, $u(t,\alpha)$ is the control and $\mathcal{L}$ is a linear partial differential operator with respect to α. It is assumed that the initial and boundary conditions are such that the problem is well-posed in the sense of Hadamard[1] and that the solution can be expressed in the form

[1] See [36] .

$$x(t,\alpha) = G_1[u(t',\alpha')] + G_2[h(t',\partial\Omega)] + G_3[x(0,\alpha)] \qquad (1.27)$$

almost everywhere in the interior of Δ , where $h(t',\partial\Omega)$ is the force acting on the boundary of Ω , $\partial\Omega$, $x(0,\alpha)$ is the given initial conditions and $G_i(i=1,2,3)$ are linear continuous operators. The performance index considered was of the form

$$J(u) = ||x* - x(u)||^2 + \gamma^2||u||^2 \qquad (1.28)$$

where $x* = x*(t,\alpha_1,\ldots,\alpha_m)$ is the desired state of the system at time $t \in \tau$, $x(u) = x(t,\alpha_1,\ldots,\alpha_m)$ is the state of the system at time t due to the control u , and γ is a given scalar. $|| \;\; ||$ represents the norm of the defined Hilbert function space. The solution of this problem was expressed implicitly as the solution of an operator equation. The article emphasizes the developement of an iterative procedure for the solution of this problem.

In short, most previous efforts have been directed towards studying particular classes of distributed parameter systems for which one of the following optimization techniques is used to solve certain optimal problems.

(1) Classical Variational Methods

(2) Pontryagin Maximum Principle

(3) Dynamic Programming

(4) Function Space Approach

The preponderate majority of work has used one of the first three techniques. Although the last technique has received scant attention from the engineering community, it offers powerful tools for solving a large variety of optimization problems in linear systems.

It should be noted that no encompassing optimization theory has been established for the distributed parameter systems. This is due, in part, to the fact that the theory of partial differential equations is at present less fully developed than that of ordinary differential equations. However, present extensive efforts on the part of mathematicians in this direction lead one to suspect that new results will soon be having their effect on the optimization theory for the distributed parameter systems.

1.3. Research Objectives

This study is concerned with the class of linear distributed parameter systems having separable Green's function. A new technique, based on a functional analysis approach, is introduced to obtain explicit solutions to a class of optimization problems. To make the development concise, the problems considered are all related to a diffusion system with one spatial coordinate and all function spaces involved are assumed to be (separable) real Hilbert spaces.

The first optimization problem solved by the developed technique is the following. For a specific initial state $x^o(\alpha)$ and a fixed state $x(t_1,\alpha) = x^1(\alpha)$ find the control which achieves the transfer $(t_o,x^o(\alpha))\rightarrow(t_1,x^1(\alpha))$ with minimum norm. Here, the control consists of three kinds of input: spatially distributed, spatially discreted and boundary inputs. This problem includes various combinations of minimum energy problems, some of which have received attention by some previous authors (see [54], [55]).

The second problem considered consists of finding the control which achieves the transfer $(t_o, x^o(\alpha)) \rightarrow (t_1, x^1(\alpha))$ while minimizing the functional

$$J(f) = \frac{1}{2} \int_{t_o}^{t_1} \int_{\Omega} \overline{P}(t,\alpha)|f(t,\alpha)|^2 d\alpha dt + \frac{1}{2} \int_{t_o}^{t_1} \int_{\Omega} \overline{Q}(t,\alpha)|x_f(t,\alpha)|^2 d\alpha dt$$

where $\Omega(\alpha \epsilon \Omega)$ is the spatial domain, f is the spatially distributed input, x_f is the system response associated with the input f and $\overline{P}$, $\overline{Q}$ are bounded strictly positive measurable functions.

The last problem in this dissertation, to which the developed technique is applied, differs from the second one in that f has to satisfy the inequality constraint

$$\int_{t_o}^{t_1} \int_{\Omega} |f(t,\alpha)|^2 d\alpha dt \leq \kappa^2$$

where κ^2 is a given scalar and the functional to be minimized is of the form

$$J(f) = \int_{\Omega} [x^1(\alpha) - x_f(t_1,\alpha)]^2 d\alpha$$

where $x^1(\alpha)$ is the specified terminal state and $x_f(t_1,\alpha)$ is the terminal state under the input f .

CHAPTER 2

SOME MATHEMATICAL TOOLS

In this chapter, certain mathematical tools that are extensively used in this thesis, are discussed. A brief review of the separation of variables method of solving partial differential equations is given in Section 2.1. Section 2.2 summarizes several specific results concerning the abstract theory of linear system minimum energy control problems.

2.1. Partial Differential Equations and the Separation of Variables Techniques

It has been pointed out that partial differential equations play a fundamental part in the mathematical description of distributed parameter systems. The reader is assumed to be familiar with such material as may be found in any good introductory text book on partial differential equations (see [18], [36], [49]). Since the examples presented in later sections use extensively the method of separation of variables as a solution technique, a brief review of the pertinent aspects of this method will now be given. Here the n^{th} order linear partial differential equation is taken as a typical example.

Consider the homogeneous equation:

$$\mathcal{L} x = Ax_t + Bx_{tt} \; . \tag{2.1}$$

where A and B are constants, the subscript t denotes the partial derivatives with respect to t, and $\mathcal{L}$ denotes the n^{th} order partial differential operator defined by

$$[\mathcal{L}x](t,\alpha) = P_o(\alpha)\,\frac{\partial^n x(t,\alpha)}{\partial\alpha^n} + P_1(\alpha)\,\frac{\partial^{n-1}(t,\alpha)}{\partial\alpha^{n-1}} + \ldots$$

$$+ P_n(\alpha)x(t,\alpha)$$

defined on the domain $\Delta = \{(t,\alpha):t>0\ ,\ a<\alpha<b\}$. Equation (2.1) together with the initial conditions,

$$\begin{aligned} x(0,\alpha) &= h(\alpha), \quad a \leq \alpha \leq b\ , \\ x_t(0,\alpha) &= g(\alpha), \quad a \leq \alpha \leq b\ , \end{aligned} \tag{2.2}$$

and the homogeneous boundary conditions,

$$\psi_i(x) = 0\ , \quad (i=1,\ldots,n) \tag{2.3}$$

define the behavior of a distributive system. Here the $\psi_i(x)$'s are linearly independent functions in the 2n variables

$$x(t,a),\ \frac{\partial x(t,a)}{\partial\alpha},\ldots,\ \frac{\partial^{n-1}x(t,a)}{\partial\alpha^{n-1}}$$

$$x(t,b),\frac{\partial x(t,b)}{\partial\alpha},\ldots,\ \frac{\partial^{n-1}x(t,b)}{\partial\alpha^{n-1}}\ .$$

In the separation of variables method, it is assumed that the solution of the system equations has the form

$$x(t,\alpha) = T(t)S(\alpha)\ . \tag{2.4}$$

Substituting Equation (2.4) in Equations (2.1) and (2.3) produces the ordinary differential equations

$$B\ddot{T} + A\dot{T} + \lambda T = 0 \ , \tag{2.5}$$

$$\mathcal{L}S + \lambda S = 0 \tag{2.6}$$

and the boundary conditions

$$\psi_i(S) = 0 \ , \ (i=1,\ldots,n) \tag{2.7}$$

where λ is a constant and the dot denotes differentiation with respect to t. The general solution of Equation (2.5) is given by

$$T = C_1 e^{m_1 t} + C_2 e^{m_2 t} \tag{2.8}$$

where m_1 and m_2 are the roots (which, for convenience, are assumed to be distinct) of the quadratic equation in m :

$$Bm^2 + Am + \lambda = 0 \ ,$$

and C_1 and C_2 are arbitrary constants.

Equations (2.6) and (2.7) constitute what is known as "Sturm-Liouville Problem" which is discussed in Appendix A . It suffices here to mention that, in general, Equations (2.6) and (2.7) will yield nontrivial solutions ($S(\alpha) \neq 0$) for only a denumerable set of values of λ which are called the eigenvalues (or the characteristic values) of the problem and are denoted by

$$\lambda_1, \ \lambda_2, \ldots, \ \lambda_j, \ldots \ . \tag{2.9}$$

The functions $S_j(\alpha)$ associated with the λ_j's are called the eigenfunctions (or characteristic functions). It may happen that more than one eigenfunction is associated with some eigenvalue λ ; in this case λ is repeated a suitable number of times in (2.9). It is clear, however, that not more than n eigenfunctions can be associated with a given eigenvalue.

Let S_j, T_j, $C_1^{(j)}$, $C_2^{(j)}$, $m_1^{(j)}$, $m_2^{(j)}$ denote the values of these quantities associated with λ_n where $n=1,2,\ldots,$. Thus $T_n(t)S_n(\alpha)$ is a solution of Equations (2.1) and (2.3). Now, if it is <u>assumed</u> that: (1) the infinite sum $\sum_{j=1}^{\infty} T_j(t)S_j(\alpha)$ is also a solution[1], i.e.,

$$x(t,\alpha) = \sum_{j=1}^{\infty} T_j(t)S_j(\alpha) \tag{2.10}$$

also satisfies Equations (2.1) and (2.3), and (2) the infinite series $\sum_{j=1}^{\infty} T_j(t)S_j(\alpha)$ can be differentiated term by term with respect to t to get $x_t(t,\alpha)$, then the initial conditions (2.2) imply that

$$\sum_{j=1}^{\infty} (C_1^{(j)} + C_2^{(j)})\, S_j(\alpha) \,, \tag{2.11}$$

and

$$\sum_{j=1}^{\infty} (m_1^{(j)}\, C_1^{(j)} + m_2^{(j)}\, C_2^{(j)})\, S_j(\alpha) = g(\alpha) \,. \tag{2.12}$$

It is clear that if $h(\alpha)$ and $g(\alpha)$ can be expanded in series of the form

$$h(\alpha) = \sum_{j=1}^{\infty} \gamma_j S_j, \quad g(\alpha) = \sum_{j=1}^{\infty} \delta_j S_j \tag{2.13}$$

then $C_1^{(j)}$ and $C_2^{(j)}$ can be determined from Equations (2.11) and

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[1] This is always true for a finite sum, since Equations (2.1) and (2.3) are homogeneous.

(2.12) for $j = 1,2,\ldots$, and the problem is, thus, completely solved.

To summarize: the first main problem is to solve Equations (2.6) and (2.7) and to find $\lambda_1, \lambda_2, \ldots$; the second is to find expansions of the form (2.13). If these are solved, it still remains to show that the function $x(t,\alpha)$ obtained in this way is really a solution of Equations (2.1), (2.2) and (2.3).

From the above discussion, it is clear that this method will certainly lead to a solution of the problem if the following two conditions are satisfied: (1) the functions $h(\alpha)$ and $g(\alpha)$ defined by Equation (2.2) have series expansions (2.13) with respect to the eigenfunctions $S_j(\alpha)$ which converge to $h(\alpha)$ and $g(\alpha)$, and (2) the coefficients $C_1^{(j)}$ and $C_2^{(j)}$ defined by Equations (2.11) and (2.12) are such as to guarantee the pointwise convergence of the series (2.10) and justify differentiating it twice with respect to t and α term by term.

However, it can be shown [48, Sec. 9.7] that whether or not these conditions are met, every time the problem has a solution, the solution can be found in the form of the series (2.10) by the method outlined above. In other words, whenever physical reasoning can establish the existence of a solution, formal series manipulations can be used to arrive at the correct result although the intermediate steps may be hard to justify mathematically.

After this brief review of the separation of variables method which is one of the most widely used methods of obtaining explicit solutions to certain partial differential equations, attention now will be turned to the discussion of some optimization techniques used in this thesis.

2.2. Some Optimization Tools for Minimum Energy Problems

In recent years considerable effort has been devoted to the development of the abstract theory of linear system minimum energy control problems. Several authors have contributed to the development of minimum energy techniques. Among these the work of Kalman [23], Balakrishnan [5], Kuo [29], Votruba [50], Beutler [6], and Porter [37], [38], [39], [40] is most significant, the last four references being closest to the spirit of the present work.

This thesis is concerned with the application of several specific results which will now be summarized for later usage. All the theorems of subsection 2.2.1 assume that the linear bounded transformation considered has a closed range (or onto). In subsection 2.2.2 these results are extended to the case where the range is dense rather than closed.

2.2.1. Linear Bounded Transformations with Closed Range.

The simplest problem that is fundamental to the discussion is the following:

Problem I [40, Sec. 4.3]: Let $T:H \to B$ be a bounded linear transformation from the Hilbert function space H onto the Banach space B. For fixed $\xi \in B$, find the element $u_\xi \in H$ satisfying

$$\xi = Tu$$

while minimizing the performance index

$$J(u) = ||u||$$

The solution of Problem I is given by the following theorem

<u>Theorem I</u> [40, Sec. 4.3]: Problem I has the unique solution $u_\xi = T^\dagger \xi$ where $T^\dagger$ is the transformation from the Banach space B into the Hilbert space H defined by[1]

$$T^\dagger = T_M^{-1} . \tag{2.14}$$

Here M is the orthogonal complement of the null space $N(T)$ of T (which is denoted by $M = N(T)^\perp$) and T_M denotes the restriction of T to M .

A clear understanding of the geometry of this relatively simple problem is helpful in visualizing methods of solution of the more complicated ones, hence an outline of the proof of Theorem I will be given here.

<u>Proof</u>: It can be shown that for every element in B , there exists a <u>unique</u> preimage in M , and thus T_M , the restriction of T to M , is one-to-one and onto. Therefore, T_M^{-1} exists and, like T_M , is bounded and linear. Now, since H can be expressed as the direct sum of the null space of T and its orthogonal complement, any element $u \in H$ can be decomposed in a unique manner as

$$u = u_1 + u_2 , \quad u_1 \in N(T) , \quad u_2 \in M$$

and thus

$$T(u) = T(u_1) + T(u_2) = T(u_2)$$

[1] $T^\dagger$ is called the pseudo-inverse of T .

Therefore, if u_2 is a preimage for ξ under T so will also u be and the minimum norm property of u_2 follows from the fact that u_1 and u_2 are orthogonal and hence

$$||u||^2 = ||u_1||^2 + ||u_2||^2 > ||u_2||^2, \quad u_1 \neq 0$$

It is, therefore, clear from the above that the solution of Problem I amounts to the problem of locating the subspace M. One way of doing this is by determining a suitable basis for M. When T has a finite range, such a basis is easy to find by well-known systematic techniques (see [29] and [40]). However, when T has an infinite range, another form of $T^\dagger$ which proves to be useful for computational purposes is given by

$$T^\dagger = T^*(TT^*)^{-1} \tag{2.15}$$

where T^* denotes the adjoint of T. The equivalence of the two expressions (2.14) and (2.15) is shown in [40, concluding remarks of Section 4.3].

Before going to Problem II, it is worth mentioning that Votruba [47] has proved the existence and uniqueness of $T^\dagger$ when both H and B are Hilbert spaces and the range $R(T)$ of T is closed in B. In this setting the definition of $T^\dagger$ is extended to mean that when $\xi \notin R(T)$, $T^\dagger \xi$ is the "best approximate solution" in the sense that $||Tu - \xi||$ is minimized by $u = T^\dagger \xi$, and $||T^\dagger \xi|| < ||u_o||$ for any other $u_o \in H$ which minimizes $||Tu - \xi||$; this extension can be given an obvious interpretation in terms of the orthogonal projection of B on $R(T)$.

The second problem to be considered here can be stated as follows:

Problem II: Let F and T denote bounded linear transformations defined on H_1 with values in H_2 and B respectively. The function spaces H_1 and H_2 are Hilbert spaces, B is a Banach space and T is assumed, as before, to be onto. For an arbitrary $\xi \in B$, determine the element which minimizes the functional

$$J(u) = ||Fu||^2 + ||u||^2 \tag{2.16}$$

over the set $T^{-1}(\xi) \subset H_1$. Here $T^{-1}(\xi)$ denotes the set of all pre-images of ξ under T.

To reduce this problem to the form of Problem I, the notion of the graph of a transformation proves to be useful. Before proceeding, the definition of the graph of a function together with some of its properties are given here.

Definition: If F is a mapping defined from H_1 into H_2, the set $H(F) = \{(u,v): v = Fu,\ u \in H_1\}$ is called the graph of F.

Lemma [40, Sec. 4.4]: If F is a linear bounded (or merely closed) transformation from H_1 into H_2, then

(1) $H(F)$ is a closed subspace of the cartesian product space $H_1 \times H_2$ and thus a Hilbert space in its own right when equipped with the inner product

$$< (u_1,v_1),(u_2,v_2) >_{H(F)} = < u_1,u_2 >_{H_1} + < v_1,v_2 >_{H_2} \tag{2.17}$$

where u_1 and u_2 are elements in H_1 and $v_i = Fu_i (i=1,2)$.

(2) Let G be the transformation from $H(F)$ into B defined by: $G(u,Fu) = Tu$ where T is as defined above, then G is a bounded linear transformation from $H(F)$ onto B .

With these tools at hand, the solution of Problem II is given by the following theorem:

Theorem II [40, Sec. 4.4]: Problem II has a unique solution u_ξ for each $\xi \in B$. Specifically, u_ξ is the abscissa of the vector $G^\dagger(\xi)$ in $H(F)$. It is clear from this theorem that Problem II reduces to Problem I with $G:H(F) \to B$ replacing $T:H_1 \to B$ and thus the orthogonal complement of the null space of G in $H(F)$ plays the same role in Problem II as that of the subspace M in Problem I. Following this line of thought, it can be shown that u_ξ is the unique element in $T^{-1}(\xi) \cap S$, where the subspace S of H_1 is defined by

$$S = (I + F^*F)^{-1}M , \tag{2.18}$$

where M is as defined in Problem I. In other words, the function $T^\dagger$ here is the inverse of the restriction of T to the subspace S defined by Equation (2.18).

The last problem to be discussed here may be stated as follows:

Problem III: Let F be a bounded linear transformation from H_1 into H_2, T a bounded linear transformation from H_1 onto B , and let $\hat{u}$, $\hat{y}$ and ξ be given vectors in H_1 , H_2 and B respectively. Here H_1 , H_2 and B are as defined above. Find an element $u \in H_1$ satisfying $Tu = \xi$ which minimizes

$$J(u) = ||u - \hat{u}||^2 + ||Fu - \hat{y}||^2$$

The solution of this problem is given by the following theorem:

Theorem III [40, Sec. 4.4]: Problem III always has a unique solution u_ξ which is characterized by the conditions:

(1) If $\hat{y} = F\hat{u}$, this solution is determined by
$(u_\xi, Fu_\xi) = G^\dagger(\xi - T\hat{u}) + (\hat{u}, \hat{y})$

(2) If $\hat{y} \neq Fu$, let P_H be the orthogonal projection of $H_1 \times H_2$ onto the graph $H(F)$ of F , then u_ξ satisfies:
$(u_\xi, Fu_\xi) = G^\dagger(\xi - T\bar{u}) + (\bar{u}, F\bar{u})$
where $(\bar{u}, F\bar{u}) = P_H(\hat{u}, \hat{y})$ can be computed by the formula
$\bar{u} = (I + F^*F)^{-1}[\hat{u} + F^*\hat{y}]$.

It is clear that Problem III contains Problem II ($\hat{u} = \hat{y} = 0$) as well as Problem I ($\hat{u} = \hat{y} = 0$, $F = 0$). Other modifications and extensions of these results are readily apparent. For example both Problems II and III can be raised to the generality of Votruba. It is also possible to carry these results over to suitable classes of Banach spaces (see [40]) although the computational aspects of the solution lose many of their nice (that is linear) properties.

2.2.2. Linear Bounded Transformations with Non-Closed Range.

In all the pseudo-inverse theorems stated in subsection 2.2.1. the linear bounded transformation considered is assumed to be onto (or closed). This is always true when the range is finite dimensional. However, for transformations with infinite rank, this is not necessarily the case and the range may be dense rather than closed. To cover such cases, the following definition of a Generalized Pseudo-Inverse, which coincides with Votruba's definition when the range is closed, is introduced by Beutler [6] .

<u>Definition</u>: Let T be a linear bounded transformation with domain $D(T) \subset H_1$ and range $R(T) \subset H_2$ where H_1 and H_2 are Hilbert function spaces. $T^\dagger$ is a Generalized Pseudo-Inverse (abbreviated by GPI) if

(a) The domain $D(T^\dagger)$ of $T^\dagger$ is dense in H_2 ,

(b) For every $y \in D(T^\dagger)$,

$$\inf_{x \in D(T)} ||Tx - y|| \tag{2.19}$$

is attained by

$$x^* = T^\dagger y$$

(c) Whenever an $x' \in D(T)$ also attains the infimum for given $y \in D(T^\dagger)$ then $||x^*|| < ||x'||$ unless $x^* = x'$.

Such a definition does not preclude the existence of various GPI's defined on different dense sets in H_2 . However, the following uniqueness theorem holds (see [6]).

<u>Theorem</u>: Let T have GPI's $T_1^\dagger$ and $T_2^\dagger$ and let

$$D = D(T_1^\dagger) \cap D(T_2^\dagger),$$

then

$$T_1^\dagger y = T_2^\dagger y , \quad \text{for all } y \in D . \tag{2.20}$$

In particular if a GPI $T^\dagger$ is defined on all of H_2 , any other GPI must be a restriction of $T^\dagger$.

Proof: For $y \in D$, both $x_1 = T_1^\dagger y$ and $x_2 = T_2^\dagger y$ attain the infimum (2.19). If Equation (2.20) is not true, there is some $y \in D$ for which $x_1 \neq x_2$. But since both $T_1^\dagger$ and $T_2^\dagger$ are GPI's, $||x_1|| < ||x_2||$ and $||x_2|| < ||x_1||$ which is clearly impossible. This proves the first statement of the theorem; the second statement is a direct consequence of the first.

It follows from this theorem that when the range $R(T)$ is dense, one can use the theorems of subsection 2.2.1. to define $T^\dagger$ on all of the range space and then restrict it to the range of T. It is in this sense that these theorems are applied to the problems discussed in this thesis.[1]

It is interesting to notice that $T^\dagger$ is bounded if and only if $R(T)$ is closed. Here the "only if" part is proved, referring the reader to [40] for the "if" part proof.

Lemma: Let $T:H_1 \rightarrow H_2$ be a linear bounded transformation. If $R(T)$ is not closed, $T^\dagger$ is unbounded.

Proof: This lemma will be proved by contradiction. Assume that $T^\dagger$ is bounded, then there exists some constant $C > 0$ such that

$$||T^\dagger(Tx)|| \leq C||Tx|| , \qquad \text{for every } x \in D(T) .$$

Let y be a limit point of $R(T)$. By definition, it follows that

$$y = \lim_n \{Tx_n\}$$

---

[1] Note that if the range of T is dense, then $N(T^*) = [\overline{R(T)}]_\perp$ is vacuous and therefore T^* is one-to-one. Since $R(T^*) \subset M = [N(T)]^\perp$ and T_M is one-to-one, it follows that TT^* is one-to-one and thus $(TT^*)^{-1}$ makes sense as a densely defined, not necessarily bounded operator. In this sense the identification $T^\dagger = T^*(TT^*)^{-1}$ is valid without the assumption that T is onto.

for some Cauchy sequence $\{x_n\}$ contained in $D(T)$. Since both T and $T^\dagger$ are linear, the equality

$$||T^\dagger(T(x_n-x_m))|| = ||T^\dagger Tx_n-T^\dagger Tx_m||$$

holds. Also, by the assumed boundedness of $T^\dagger$,

$$||T^\dagger(T(x_n-x_m))|| \leq C||T(x_n-x_m)|| .$$

It thus follows that

$$||T^\dagger Tx_n-T^\dagger Tx_m|| \leq C||Tx_n-Tx_m|| ,$$

which means that $\{T^\dagger Tx_n\}$ is a Cauchy sequence in $D(T)$, i.e. $T^\dagger Tx_n \to x \in H_1$. But since every linear bounded transformation is closed[1] (see [39, page 100]), it follows that $x \in D(T)$ and $Tx = \lim Tx_n = y$, i.e. $y \in R(T)$. Therefore, $R(T)$ is closed which contradicts the assumption that $R(T)$ is not closed. Therefore $T^\dagger$ cannot be bounded.[2]

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[1] A linear Transformation S is called closed if it has the property that for every sequence $\{g_n\}$ of elements of $D(S)$ such that $g_n \to g$ and $Sg_n \to h$, the limit element g also belongs to $D(S)$ and $Sg = h$.

[2] Note that the boundedness of T is not used in the proof and thus the lemma is true for any closed transformation.

CHAPTER 3

MINIMUM ENERGY PROBLEMS

3.1. Introduction

In this chapter several forms of the first minimum energy control problem (that is problem I of Section 2.2) are considered. In each case, the application is a natural consequence of the behavior of linear distributed parameter systems. To obtain explicit solutions an expansion technique (see Section 2.1) is used in conjunction with the results of Section 2.2. The conciseness of such procedures makes evident the utility of an abstract approach to linear system theory.

The gist of the method to be used can be summarized as follows: The dynamic behavior of any linear distributed parameters systems under the effect of a forcing function f may be depicted as a (bounded) linear mapping $T:H_1 \to H_2$ and H_1 and H_2 are the multivariable Hilbert function spaces to which the forcing function f and the state of the system x belong respectively. Instead of solving the optimization problem in these spaces, two other infinite dimensional vector-valued univariable function spaces $\overline{H}_1$, $\overline{H}_2$ are introduced which are isometrically isomorphic, or more briefly, congruent,[1] (with isometries R , S) to the original function spaces H_1 and H_2 respectively. The linear

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[1] Two Hilbert spaces H and $\overline{H}$ are said to be congruent if there exists a linear transformation U with domain H and range $\overline{H}$, such that U^{-1} exists and $< Ux_1,Ux_2 >_{\overline{H}} = < x_1,x_2 >_H$ for all $x_1,x_2 \in H$.

transformation $\bar{T}:\bar{H}_1 \to \bar{H}_2$ is then defined by the relation

$$T = S^{-1}\,\bar{T}R$$

or equivalently,

$$\bar{T} = STR^{-1} .$$

Such a transformation may be symbolically represented as shown in Figure 3, from which the above two relations are quite obvious.

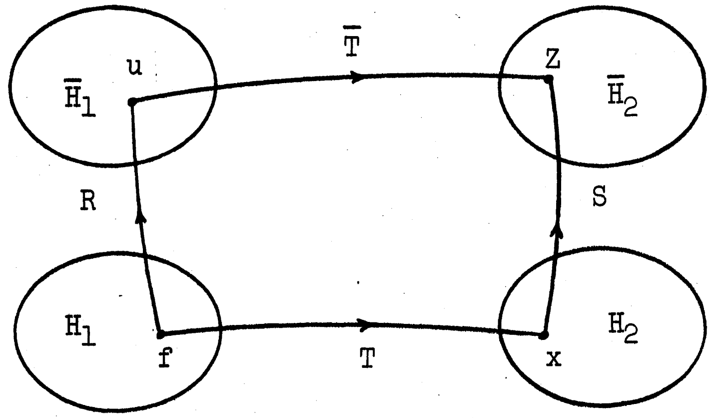

Figure 3. A Conceptual Model for Distributed Parameter Systems

By definition of an isometry, it is clear that the preimage of the element with minimum norm in any set contained in $\bar{H}_1$ will also be the element with minimum norm in the corresponding preimage set in H_1. It follows from this observation that instead of solving the minimum energy control problem (or abstractly the minimum norm problem) in the original multi-variable function spaces H_1 and H_2, one can solve an equivalent problem in the univariable function spaces $\overline{H}_1$ and $\overline{H}_2$. The optimum element in H_1 is obviously the preimage of $u^* \in \overline{H}_1$ under R where u^* denotes the optimal element in $\overline{H}_1$. The principle advantage of this (apparently complicated) procedure is that it provides a systematic technique to get explicit expressions for the solution of the optimization problem - a result of considerable importance to the practicing engineer. Such an approach of solving problems through transforming the original function spaces to others which are handier to work with is familiar to most engineers. Fourier and Laplace transforms are examples of this fruitful approach.

The class of linear distributed parameter systems considered in this thesis are those whose solution can be expressed as

$$x(t,\alpha) = (G_1 f)(t,\alpha) + (G_2 h)(t,\alpha) + (G_3 x^o)(t,\alpha)$$

almost everywhere in the interior of the domain $\Delta = \{(t,\alpha): t \in \tau, \alpha \in \Omega\}$. Here, x (which may be a vector-valued function) is the state of the system, f is the distributed input, h is the boundary input, x^o is the initial state, and the $G_i (i=1,2,3)$ are (bounded) linear integral operators with separable kernels; all function spaces involved are assumed to be Hilbert spaces. The first term in the above equation, $G_1 f$, represents the forced response of the system due to the distributed input f , the second term, $G_2 h$, is the forced response to the boundary input h , and the last term, $G_3 x^o$ is the free response to the initial state. It is thus clear that this equation simply states that the superposition principle applies to the system under consideration.

As a typical example, a one-dimensional diffusion equation is considered in the following sections to illustrate the proposed technique. Attention is called once more to the examples of physical systems described by such an equation which have been given in Section 1.1.

3.2. Minimum Energy Control of a Diffusion System - First Example

In this section, the minimum energy control problem (Problem I, Section 2.2) is solved in its most general form for systems describable by the diffusion equation

$$\frac{\partial x(t,\alpha)}{\partial t} = k^2 \frac{\partial^2 x(t,\alpha)}{\partial \alpha^2} + f^1(t,\alpha) \tag{3.1}$$

defined on the domain $\Delta = \{(t,\alpha): t \in [t_o,t_1],\ 0 < \alpha < b\}$, together with the initial condition

$$x(t_o,\alpha) = x^o(\alpha) \ , \qquad \alpha \in [0,b] \tag{3.2}$$

and the boundary conditions

$$x(t,o) = h_1(t) \ , \ x(t,b) = h_2(t) \ , \ t \in [t_o,t_1] \tag{3.3}$$

with

$$h_1(t_o) = x^o(0); \ h_2(t_o) = x^o(b) \ .$$

The force f^1 consists of two parts; a distributed force f defined on the domain Δ , and a finite number of forces $\{g_1,\dots,g_m\}$ concentrated at fixed spatial positions $0 < \alpha_1 < \alpha_2 < \dots < \alpha_m < b$. Accordingly, f^1 may be expressed mathematically as

$$f^1(t,\alpha) = f(t,\alpha) + \sum_{i=1}^{m} \delta(\alpha-\alpha_i)g_i(t) \;, \quad (t,\alpha) \in \Delta$$

where $\delta(\alpha-\alpha_i)$ is the Dirac delta function defined by

$$\int_{\alpha_i-\epsilon}^{\alpha_i+\epsilon} \varphi(\alpha)\delta(\alpha-\alpha_i)d\alpha = \varphi(\alpha_i)$$

for every continuous φ and every $\epsilon > 0$.

To state precisely the optimization problem, the following three basic real function spaces are introduced[1]:

(1) the Hilbert function space $\mathcal{F}$ of all functions measurable on Δ and being finite with respect to the norm induced by the inner product

$$< x,y > = \int_{t_1}^{t_o} \int_{o}^{b} x(t,\alpha)y(t,\alpha)d\alpha dt \;, \quad x,y \in \mathcal{F}.$$

(2) The Hilbert function space $\mathcal{G} = [L_2(t_o,t_1)]^m$ (the cartesian product of m copies of the $L_2(t_o,t_1)$ space) equipped with the usual inner product

$$< p,q > = \int_{t_o}^{t_1} [\sum_{i=1}^{m} p_i(t)q_i(t)]dt \qquad p,q \in \mathcal{G}$$

where the tuplets $p = (p_1,\ldots,p_m)$ and $q = (q_1,\ldots,q_m)$ are elements in $\mathcal{G}$.

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1 These basic function spaces may be modified by inclusion of positive definite weighting functions in the respective inner products. However, since the following discussion depends only on the abstract Hilbert space properties of these spaces and not on their concrete nature, the additional clutter of such complications can be avoided without loss of generality.

(3) The Hilbert function space $\mathcal{H} = L_2(t_o,t_1) x \cdot L_2(t_o,t_1)$ equipped with the inner product[1]

$$< r,s > = \frac{1}{2}\int_{t_o}^{t_1} [r_1(t)s_1(t) + r_2(t)s_2(t)]dt$$

where the tuplets $r = (r_1,r_2)$ and $s = (s_1,s_2)$ are elements in $\mathcal{H}$.

The optimization problem to be considered in this section is the following: Find the controls $f^* \in \mathcal{F}$, $g^* = (g_1^*,g_2^*,\ldots,g^*) \in \mathcal{G}$, $h^* = (h_1^*,h_2^*) \in \mathcal{H}$ which carry the system from the given initial state $x^o(\alpha)$ to a specified final state $x^1(\alpha)$, an element in the solution space, at a specified time t_1 , while minimizing the functional

$$J(f,g,h) = \frac{1}{2}\int_{t_o}^{t_1} [\int_{o}^{b} |f(t,\alpha)|^2 d\alpha + \sum_{i=1}^{m} |g_i(t)|^2 + \sum_{i=1}^{2} |h_i(t)|^2]dt \ . \qquad (3.4)$$

Making use of the obvious linearity of the system under consideration, the above optimization problem can be solved in stages. Indeed, the solution of Equations (3.1) to (3.3) can be put in the form

$$x(t,\alpha) = \sum_{i=1}^{3} x_i(t,\alpha) \ , \qquad (t,\alpha) \in \Delta$$

where x_1 is the response (solution) of "System I" which is defined by the equation set

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[1] The reason for introducing the scale factor (1/2) in the definition of this inner product will be evident in subsection 3.2.4. (See Equation (3.43)).

$$\frac{\partial x_1(t,\alpha)}{\partial t} = k^2 \frac{\partial^2 x_1(t,\alpha)}{\partial \alpha^2} + f(t,\alpha) , \qquad (t,\alpha) \in \Delta ,$$

$$\begin{aligned} x_1(t_o,\alpha) &= 0 , & \alpha &\in [0,b] , \\ x_1(t,0) &= 0 , & t &\in [t_o,t_1] , \\ x_1(t,b) &= 0 , & t &\in [t_o,t_1] . \end{aligned} \tag{3.5}$$

The function x_2 is the response (solution) of "System II" defined by

$$\frac{\partial x_2(t,\alpha)}{\partial t} = k^2 \frac{\partial^2 x_2(t,\alpha)}{\partial \alpha^2} + \sum_{i=1}^{m} g_i(t)\delta(\alpha-\alpha_i), \qquad (t,\alpha) \in \Delta ,$$

$$\begin{aligned} x_2(t_o,\alpha) &= 0 , & \alpha &\in [0,b] , \\ x_2(t,0) &= 0 , & t &\in [t_o,t_1] \\ x_3(t,b) &= 0 , & t &\in [t_o,t_1] . \end{aligned} \tag{3.6}$$

Finally, x_3 is the response of "System III" defined by

$$\frac{\partial x_3(t,\alpha)}{\partial t} = k^2 \frac{\partial^2 x_3(t,\alpha)}{\partial \alpha^2} , \qquad (t,\alpha) \in \Delta$$

$$\begin{aligned} x_3(t_o,\alpha) &= x^o(\alpha) , & \alpha &\in [0,b] , \\ x_3(t,0) &= h_1(t) , & t &\in [t_o,t_1] , \\ x_3(t,b) &= h_2(t) , & t &\in [t_o,t_1] . \end{aligned} \tag{3.7}$$

Using the separation of variables technique, the response x_i ($i=1,2,3$) of any of the above three systems takes the form of the infinite series

$$x_i(t,\alpha) = \sum_{n=1}^{\infty} T_{in}(t)\varphi_n(\alpha) , \quad i=1,2,3, \ (t,\alpha) \in \Delta \tag{3.8}$$

Here $\{\varphi_n\}$ are the (normalized) eigenfunctions of the accompanying Sturm-Liouville equation, namely,

$$\frac{d^2\varphi(\alpha)}{d\alpha^2} + \lambda\varphi(\alpha) = 0 \qquad\qquad 0 < \alpha < b$$

with the homogeneous boundary conditions

$$\varphi(0) = \varphi(b) = 0 \; .$$

It can be easily shown (see [49]) that the eigenfunctions $\{\varphi_n\}$ are given by,

$$\varphi_n(\alpha) = \sqrt{2/b}\,\sin\,(n\pi/b)\alpha \;, \qquad n=1,2,\ldots, \qquad \alpha \in [0,b]. \quad (3.9)$$

and that they constitute a complete orthonormal basis for the Hilbert function space $L_2(0,b)$. Also, the functions T_{in} in Equation (3.8) are the Fourier coefficients of x_i with respect to $\{\varphi_n\}$, namely,

$$T_{in}(t) = \langle x_i, \varphi_n \rangle_{L_2(0,b)} = \int_0^b x_i(t,\alpha)\varphi_n(\alpha)d\alpha \;,\; t \in [t_o,t_1] \; . \qquad (3.10)$$

In the following, the optimization problem will be first solved for each of the above three systems. These solutions will then be utilized to solve the problem of the original system defined by Equations (3.1) to (3.3). For convenience, the subscript i in x_i and T_i will be suppressed whenever no confusion on the part of the reader is apt to occur.

3.2.1. <u>Minimum Energy Control of System I.</u>

For system I above, defined by Equation (3.5), the controls g and h are identically zero and thus the performance index of Equation (3.4) is reduced to

$$J(f) = \frac{1}{2} \int_{t_o}^{t_1} \int_{o}^{b} |f(t,\alpha)|^2 d\alpha dt$$

and the problem is to find the element $f^* \in \Gamma(x^1)$ which minimizes $J(f)$. Here $\Gamma(x^1)$ is defined as

$$\Gamma(x^1) = \{f: f \in \mathcal{F} \ , \ x_f(t_1,\alpha) = x^1(\alpha)\}$$

where x_f is the response of System I to the control input function f and $x^1(\alpha)$ is the specified state of the system to be attained at the specified time t_1.

It is clear from the definition of the Hilbert space $\mathcal{F}$ that for every $f \in \mathcal{F}$ and for almost every $t \in [t_o,t_1]$, the function $f(t,\alpha)$ is an element of $L_2(0,b)$. It thus follows that the Fourier series expansion

$$f(t,\alpha) = \sum_{n=1}^{\infty} u_n(t)\varphi_n(\alpha) \ , \qquad (t,\alpha) \in \Delta$$

holds[1], where $\{\varphi_n\}$ is the complete orthonormal basis given by Equation (3.9) and $\{u_n\}$ are the Fourier coefficients of the function f given by

$$u_n(t) = \langle f,\varphi_n \rangle_{L_2(0,b)} = \int_{o}^{b} f(t,\alpha)\varphi_n(\alpha)d\alpha \ , \quad t \in [t_o,t_1] \tag{3.11}$$

By Parseval's identity (see [51, Section 6.6]), the equality

$$\sum_{n=1}^{\infty} u_n^2(t) = \int_{o}^{b} |f(t,\alpha)|^2 d\alpha \tag{3.12}$$

[1] Notice that the equality sign here means that at any $t \in [t_o,t_1]$, the infinite series $\sum_{n=1}^{\infty} u_n(t)\varphi_n(\alpha)$ converges to $f(t,\alpha)$ in the metric for $L_2(0,b)$.

is satisfied almost everywhere in $[t_o, t_1]$. It directly follows from Equation (3.12) and the definition of the space $\mathcal{J}$ that u_n is an element in $L_2(t_o, t_1)$, $n=1,2,\ldots,$ and the tuplet $u = (u_1,\ldots,u_n,\ldots)$ is an element of the Hilbert function space $\bar{L}_2(t_o, t_1)$ which is defined as the set of all tuplets which are finite with respect to the norm induced by the inner product

$$< u,v > = \int_{t_o}^{t_1} \{ \sum_{i=1}^{\infty} u_n(t)v_n(t)\} dt = \int_{t_o}^{t_1} [u(t),v(t)]dt$$

where the tuplets $u = (u_1,\ldots,u_n,\ldots)$ and $v = (v_1,\ldots,v_n,\ldots)$ are elements in $\bar{L}_2(t_o, t_1)$, and $[\ ,\ ]$ represents the usual inner product in ℓ_2.

By substituting Equations (3.8) and (3.11) in the system partial differential equation $(3.5)_1$ it can be easily shown that $T_n(t)$ is the solution of the first order ordinary differential equation given by

$$\dot{T}_n(t) = -(nk\pi/b)^2 T_n(t) + u_n(t) , \qquad n = 1,2,\ldots, \qquad t \in (t_o, t_1]$$

with the initial conditions $T_n(t_o) = 0$. This infinite system of differential equations can be put in the following matrix form

$$\dot{z}(t) = Az(t) + u(t) , \ t \in [t_o, t_1] ; \ z(t_o) = 0 , \tag{3.13}$$

if z denotes the tuplet $z(t) = (T_1(t),\ldots,T_n(t),\ldots)$, $t \in [t_o, t_1]$, and A is the infinite diagonal matrix defined by

$$A = \text{diag}[-(k\pi/b)^2,\ldots,-(nk\pi/b)^2,\ldots] . \tag{3.14}$$

The solution of Equation (3.13) can be put in the form

$$z(t) = \int_{t_o}^{t} \Phi(t-s)u(s)ds \ , \qquad t \in [t_o,t_1] \tag{3.15}$$

where the infinite transition matrix $\Phi(t)$ is given by

$$\Phi(t) = \mathrm{diag}[\exp\{-(k\pi/b)^2t\},\ldots,\exp\{-(nk\pi/b)^2t\} ,\ldots] \tag{3.16}$$

Let F_1 be the linear transformation defined on $\overline{L}_2(t_o,t_1)$ and with values in ℓ_2 defined by[1]

$$F_1u = \int_{t_o}^{t_1} \Phi(t -s)u(s)ds \ . \tag{3.17}$$

The minimum energy control problem for System I can thus be stated as follows: From the set $\Gamma(z^1) = \{u: u \in \overline{L}_2(t_o,t_1) \ , F_1u = z^1\}$ find the element u^* with minimum norm, i.e., find u^* which minimizes the functional

$$J(u) = ||u||^2 = \int_{t_o}^{t_1} [\sum_{n=1}^{\infty} |u_n(t)|^2]dt = \int_{t_o}^{t_1} \int_{o}^{b} |f(t,\alpha)|^2 d\alpha dt \ .$$

Here z^1 is the tuplet $(\xi_1^1,\ldots,\xi_n^1,\ldots)$ whose components are the Fourier coefficients of x^1 with respect to $\{\varphi_n\}$, i.e.

$$\xi_n^1 = < x^1,\varphi_n > = \int_{o}^{b} x^1(\alpha)\varphi_n(\alpha)d\alpha \ .$$

It follows from Theorem I of Section 2.2 that the optimum element is given by[2]

---

[1] Note that the components of F_1u are the Fourier coefficients of $x(t_1,\alpha) \in L_2(0,b)$.

[2] Here Theorem I is applied in the sense of subsection 2.2.2, since the range of F_1 is dense in ℓ_2 (see Appendix B).

$$u^*(t) = (F_1^{\dagger} z^1)(t) = [F_1^* (F_1F_1^*)^{-1}z^1](t) \tag{3.18}$$

The adjoint of F_1 can be determined as follows: let $\eta = (\eta_1, \ldots, \eta, \ldots)$ be an element in ℓ_2, then

$$\begin{aligned} < F_1u, \eta > &= \sum_{n=1}^{\infty} [\int_{t_o}^{t_1} \exp\{-(nk\pi/b)^2(t_1-s)\}u_n(s)ds]\eta_n \\ &= \sum_{n=1}^{\infty} [\int_{t_o}^{t1} \exp\{-(nk\pi/b)^2(t_1-s)\}\eta_n u_n(s)ds] \\ &= \int_{t_o}^{t_1} [\sum_{n=1}^{\infty} \exp\{-(nk\pi/b)^2(t_1-s)\}\eta_n u_n(s)ds] \\ &= < u, F_1^*\eta >_{L_2(t_o,t_1)} \end{aligned} \tag{3.19}$$

Here the interchange of the integral and summation signs are justified through the use of "Lebesgue's Theorem on Dominated Convergence"[1] (see

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[1] Lebesgue's Theorem on Dominated Convergence: Let a sequence of measurable functions $f_1, \ldots, f_k, \ldots$ converging almost everywhere to a function f, be defined on a set E. If there exists a function H summable on E such that for all k and almost all t

$$|f\ (t)| \leq H(t) ,$$

then

$$\lim_{k\to\infty} \int_E f_k(t)dt = \int_E f(t)dt .$$

In Equation (3.19) choose

$$f_k(t) = \sum_{n=1}^{k} \eta_n u_n(t)\exp\{-(nk\pi\ b)^2(t_1-t)\} ,$$

and note that

$$|f_k(t)| \leq ||\eta|| \ . \ ||f(t,\alpha)||$$

holds by the Cauchy-Schwartz inequality.

[35, page 161]). By inspection of Equation (3.19), it follows that F_1^* is computed by the rule.

$$(F_1^* \eta)_n(t) = \exp\{-(nk\pi/b)^2(t_1-t)\}\eta_n, \quad t \in [t_o,t_1]$$

and thus $F_1^* : \ell_2 \to \overline{L}_2(t_o,t_1)$ may be identified with the left multiplication on ℓ_2 by the infinite time-varying diagonal matrix given by

$$\begin{aligned} F_1^* &= \mathrm{diag}[\exp\{-(nk\pi/b)^2(t_1-t)\},\ldots,\exp\{-(nk\pi/b)^2(t_1-t)\},\ldots] \\ &= \Phi(t_1-t) \end{aligned} \tag{3.20}$$

Here the same symbol is used to denote the transformation and its corresponding matrix representation.

It follows from above that

$$\begin{aligned} F_1F_1^* &= \int_{t_o}^{t_1} \Phi(t_1-s)\Phi(t_1-s)ds \\ &= \mathrm{diag}[\gamma_1,\ldots,\gamma_n,\ldots], \end{aligned} \tag{3.21}$$

where

$$\gamma_n = [2(nk\pi/b)^2]^{-1}[1-\exp\{-2(nk\pi/b)^2(t_1-t_o)\}] ,$$

and thus the inverse of $F_1F_1^*$ is readily computed as

$$(F_1F_1^*)^{-1} = \mathrm{diag}\ [\gamma_1^{-1},\ldots,\gamma_n^{-1},\ldots] .$$

By direct substitution in Equation (3.18) it follows that the expression

$$F_1^\dagger = \mathrm{diag}\ [\delta_1,\ldots,\delta_n,\ldots]$$

where

$$\delta_n(t) = 2(nk\pi/b)^2 \exp\{-(nk\pi/b)^2(t_1-t)\}[1-\exp\{-2(nk\pi/b)^2(t_1-t_o)\}]^{-1}$$

makes the operator $F_1^{\dagger}$ well-defined.

In short, the optimum control u^* has values computed by

$$u^*(t) = (F_1^{\dagger} z^1)(t) \qquad t \in [t_o, t_1] ,$$

and thus the optimum distributed input $f^*(t,\alpha)$ is given by

$$\begin{aligned} f^*(t,\alpha) &= \sum_{n=1}^{\infty} [(F_1^{\dagger} z^1)(t)]_n \cdot \varphi_n(\alpha) \\ &= \sum_{n=1}^{\infty} \delta_n(t)\varphi_n(\alpha) \int_o^b x^1(\alpha)\varphi_n(\alpha)d\alpha \end{aligned} \qquad (3.23)$$

where δ_n and φ_n are given by Equations (3.22) and (3.9) respectively.[1]

As a concluding remark, it is obvious that the above technique is also applicable to cases where the controls g and h , instead of being identically zero are prescribed non-vanishing elements in the corresponding function spaces.

3.2.2. Minimum Energy Control of the System II.

In this subsection, the minimum energy control problem is considered for System II defined by Equation (3.6). In this example, the controls f and h are identically zero. Consequently, the performance index of Equation (3.4) is reduced to

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[1] Note that corresponding to every $f \in \mathcal{F}$ there exists a (generalized) solution for the system equations (see Appendix C).

$$J(g) = \frac{1}{2}\int_{t_o}^{t_1} [\sum_{i=1}^{m} |g_i(t)|^2]dt \; .$$

In the present case, by integrating Equation (3.10) twice by parts, it follows that

$$T_n(t) = \sqrt{\frac{2b}{n\pi}}\,[x(t,0)-(-1)^n x(t,b)] - \left(\frac{b}{n\pi}\right)^2 \int_0^b \frac{\partial^2 x(\alpha)}{\partial\alpha^2}\varphi_n(\alpha)d\alpha \; , \quad t \in [t_o,t_1].$$

Substitution of Equations $(3.6)_{1,3,4}$ in the above equation gives

$$T_n(t) = -\left(\frac{b}{nk\pi}\right)^2 \int_0^b \;[\frac{\partial x(t,\alpha)}{\partial t} - \sum_{i=1}^{m} g_i(t)\delta(\alpha-\alpha_i)]\varphi_n(\alpha)d\alpha$$

$$= -(b/nk\pi)^2\dot{T}_n(t) + (b/nk\pi)^2 \sum_{i=1}^{m} g_i(t)\varphi_n(\alpha_i) \; , \quad t \in [t_o,t_1];$$

the last equality following from a differentiation of Equation (3.10) with respect to t and noticing that

$$\int_0^b \varphi_n(\alpha)\delta(\alpha-\alpha_i)d\alpha = \varphi_n(\alpha_i) \; , \qquad 0 < \alpha_i < b \; .$$

It is thus concluded that the system response may be identified with the infinite tuplet of first order ordinary differential equations:

$$\dot{T}_n(t) = -(nk\pi/b)^2 T_n(t) + \sum_{i=1}^{m} g_i(t)\varphi_n(\alpha_i) \; , \quad t \in [t_o,t_1] \; .$$

with the initial conditions $T_n(t_o) = 0$, $n=1,2,\ldots$.

This infinite system of equations can be written in the matrix form as

$$\dot{z}(t) = Az(t) + Bg(t)$$

where, as before, z denotes the tuplet $z(t) = (T_1(t),\ldots,T_n(t),\ldots)$,

$t \in [t_o, t_1]$ and A is defined in Equation (3.14). Here, however, $g = (g_1, \ldots, g_m)$ is an element in the Hilbert function space $\mathcal{G}$ (defined on page 40), and B is the time-invariant $\infty \times m$ matrix defined by

$$B = \begin{bmatrix} \varphi_1(\alpha_1) & \varphi_1(\alpha_2) & \cdots & \varphi_1(\alpha_m) \\ - - - & - - - & - - - & - - - \\ \varphi_n(\alpha_1) & \varphi_n(\alpha_2) & \cdots & \varphi_n(\alpha_m) \\ - - - & - - - & - - - & - - - \\ - - - & - - - & - - - & - - - \end{bmatrix} \qquad (3.24)$$

The solution of the above matrix equation may be written as

$$z(t) = \int_{t_o}^{t} \Phi(t-s)Bg(s)ds \ , \qquad t \in [t_o, t_1] \qquad (3.25)$$

where the transition matrix Φ is the same as that of System I.

Let F_2 denote the (bounded) linear transformation $g \to z(t_1)$ from the Hilbert function space $\mathcal{G}$ into ℓ_2 , that is

$$F_2 g = \int_{t_o}^{t_1} \Phi(t_1 - s)Bg(s)ds \ . \qquad (3.26)$$

The present minimum energy problem can then be restated as follows: From the set $\Gamma(z^1) = \{g : g \in \mathcal{G} \ , \ F_2 g = z^1\}$, determine the element g^* which minimizes the performance index

$$J(g) = ||g||^2 = \int_{t_o}^{t_1} [\sum_{i=1}^{m} |g_i(t)|^2]dt \ .$$

Theorem I of Section 2.2 gives the direct answer of this problem, namely,

$$g^* = F_2^{\dagger} z^1 = F_2^*(F_2F_2^*)^{-1} z^1, \; z^1 \in R(F_2)$$

It can be easily shown that F_2^* may be identified with the left multiplication on ℓ_2 by the time-varying matrix $B^*\Phi^*(t_1-t)$:

$$B^*\Phi^*(t_1-t) = \begin{bmatrix} \varphi_1(\alpha_1) & \cdots & \varphi_n(\alpha_1) & \cdots \\ \vdots & & & \\ \varphi_1(\alpha_m) & & \varphi_n(\alpha_m) & \cdots \end{bmatrix} \begin{bmatrix} \exp\{-(k\pi/b)^2(t_1-t)\} & & & \\ & \ddots & & \\ & & \exp -(nk\pi/b)^2(t_1-t) & \\ & & & \ddots \end{bmatrix}$$

This matrix has dimensions $m \times \infty$ with typical element γ_{ij} given by

$$\gamma_{ij}(t) = \varphi_j(\alpha_i)\exp\{-(jk\pi/b)^2(t_1-t)\}, \quad i=1,\ldots,m; \quad j=1,2,\ldots .$$

The operator $F_2F_2^*$ on ℓ_2 is computed by the rule

$$F_2F_2 = \int_{t_o}^{t_1} \Phi(t_1-s)BB^*\Phi^*(t_1-s)ds ,$$

and by a direct evaluation it follows that the ij^{th} element of this infinite symmetric matrix is given by

$$(F_2F_2^*)_{ij} = \left(\frac{b}{k\pi}\right)^2 \frac{1-\exp\{-(i^2+j^2)(k\pi/b)^2(t_1-t_o)\}}{(i^2+j^2)} \sum_{\ell=1}^{m} \varphi_i(\alpha_\ell)\varphi_j(\alpha_\ell) . \quad (3.27)$$

At this stage, the problem of getting (exact) explicit expressions for the inverse of an infinite matrix arises and one has to resort to one of the available approximation techniques. This problem will be discussed in Chapter 5. It suffices here to mention that it is possible to compute the pointwise inverse of $F_2F_2^*$ to any degree of closeness and accordingly the optimum control element $g^* \in \Gamma(z^1)$ can be determined. Again, the same technique holds if the controls f and h , instead of being identically zero, are prescribed elements in their function spaces.

3.2.3. Minimum Energy Control of System III.

In this subsection, the minimum energy control problem is solved for System III (see Equation 3.7)). Here the controls f and g are identically zero and thus the performance index to be minimized is given by

$$J(h) = \frac{1}{2}\int_{t_o}^{t_1} \{|h_1(t)|^2 + |h_2(t)|^2\}dt$$

In the present case, the functions $\{T_n\}$ can be found as follows: Integrating Equation (3.10) by parts twice gives

$$T_n(t) = \sqrt{\frac{2b}{n\pi}}\,[x(t,0) - (-1)^n x(t,b)] - \left(\frac{b}{n\pi}\right)^2 \int_0^b \frac{\partial^2 x(t,\alpha)}{\partial\alpha^2}\,\varphi_n(\alpha)d\alpha \,,$$

$$t \in [t_o, t_1]$$

By substituting Equations $(2.7)_{1,3,4}$ in the above equation, it follows that

$$T_n(t) = \sqrt{\frac{2b}{n\pi}}\ [h_1(t) - (-1)^n h$$

However, differentiating Equation (3.10) with respect to t shows that

$$\dot{T}_n(t) = \int_0^b \frac{\partial x(t,\alpha)}{\partial t} \varphi_n(\alpha) d\alpha , \qquad t \in [t_o,t_1] ,$$

and hence it is concluded that

$$\dot{T}_n(t) = -(nk\pi/b)^2 T_n(t) + (2n^2k^4\pi^2/b^3)^{1/2} [h_1(t) - (-1)^n h_2(t)] \tag{3.28}$$

holds for all $t \in [t_o,t_1]$ and $n=1,2,\ldots$. The associated initial conditions are given by

$$T_n(t_o) = \langle x^o,\varphi_n\rangle = \int_0^b x^o(\alpha)\ \varphi_n(\alpha)\ d\alpha .$$

As in Systems I and II, the present system response may be identified with a first order matrix ordinary differential equation. Using the matrix A , defined by Equation (3.14) and the tuplet $z(t) = (T_1(t),\ldots,T_n(t),\ldots)$ the system response may be written as

$$\dot{z}(t) = As(t) + CN\overline{h}(t), \quad t \in [t_o,t_1]; \quad z(t_o) = z^o \tag{3.29}$$

Here z is an element in the Hilbert function space $\overline{L}_2(t_o,t_1)$, and $\overline{h}$ denotes the tuplet $\overline{h}(t) = (h_1(t)+h_2(t)\ ,\ h_1(t) - h_2(t))$ of the function space $\mathcal{H}$ (defined on page 41). The infinite matrix C is constant and diagonal being given by

$$C = \text{diag}[(2k^4\pi^2/b^3)^{1/2},\ldots,(2n^2k^4\pi^2/b^3)^{1/2},\ldots], \tag{3.30}$$

and N is the $\infty \times 2$ matrix given by

$$N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ \vdots & \vdots \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \\ \vdots & \vdots \end{bmatrix} \qquad (3.13)$$

The solution of the matrix Equation (3.29) may also be written in the form

$$z(t) = \Phi(t-t_o)z(t_o) + \int_{t_o}^{t} \Phi(t-s)CN\bar{h}(s)ds , \qquad (3.32)$$

or equivalently,

$$\hat{z}(t) = z(t) - \Phi(t-t_o)z(t_o) = \int_{t_o}^{t} \Phi(t-s)CN\bar{h}(s)ds , \quad t \in [t_o,t_1]$$

where $\Phi(t)$ is the transition matrix defined by Equation (3.16). If F_3 denotes the (bounded) linear transformation from the Hilbert function space $\mathcal{H}$ to the ℓ_2 space, defined by

$$F_3\bar{h} = \int_{t_o}^{t_1} \Phi(t_1-s)CN\bar{h}(s)\, ds, \qquad \bar{h} \in \mathcal{H} \qquad (3.33)$$

then the optimization problem is reduced to determining the element $\bar{h}^*$ from the set $\Gamma(z^1) = \{\bar{h}:\bar{h} \in \mathcal{H} , F_3\bar{h} = z^1\}$ with minimum norm. Here $\hat{z}^1 = z^1 - \Phi(t_1-t_o)z(t_o)$, and the norm is defined by

$$||\bar{h}||^2 = \frac{1}{2}\int_{t_o}^{t_1} \{[h_1(t) + h_2(t)]^2 + [h_1(t) - h_2(t)]^2\}dt$$

$$= \int_{t_o}^{t_1} \{|h_1(t)|^2 + |h_2(t)|^2\}dt .$$

Since the range of F_3 , $R(F_3)$, is dense in ℓ_2 , (see Appendix B), the application of Theorem I in the generalized sense (see subsection 2.2.2) gives the optimal element $\overline{h}^*$ as

$$\overline{h}^* = F_3^{\dagger} \hat{z}^1 = F_3^* (F_3F_3^*)^{-1}\hat{z}^1 \tag{3.34}$$

Here, F_3^* can be identified as

$$(F_3^* \eta)(t) = N^*C^*\Phi(t_1-t)\eta , \qquad t \in [t_o,t_1] , \eta \in \ell_2$$

and hence

$$F_3^*F_3^* = \int_{t_o}^{t_1} \Phi(t_1-s)CNN^*C^*\Phi^*(t_1-s)ds.$$

The explicit determination of the elements in the matrix representing $F_3F_3^*$ is a straight forward. First note that the matrix $\Phi(t_1-s)C = C^*\Phi(t_1-s)$ is diagonal with typical elements γ_n of the form

$$\gamma_n = (2n^2k^4\pi^2/b^3)^{1/2}\exp\{-(nk\pi/b)^2(t_1-s)\} .$$

The matrix NN^* has a regular structure, namely

$$NN^* = \begin{bmatrix} 1 & 0 & 1 & 0 \ldots & 1 & 0 & 1 \ldots \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \ldots \\ 1 & 0 & 1 & 0 \ldots & 1 & 0 & 1 \ldots \\ - & - & - & - & - & - & - \\ - & - & - & - & - & - & - \\ 1 & 0 & 1 & 0 \ldots & 1 & 0 & 1 \ldots \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \ldots \\ 1 & 0 & 1 & 0 \ldots & 1 & 0 & 1 \ldots \\ - & - & - & - & - & - & - \\ - & - & - & - & - & - & - \end{bmatrix}$$

Using these matrices, it is not difficult to see that

$$NN^*C^*\Phi(t_1-s) = \begin{bmatrix} \gamma_1(s) & 0 & \gamma_3(s) & 0 & \gamma_5(s) & \cdots \\ 0 & \gamma_2(s) & & \gamma_4(s) & 0 & \cdots \\ \gamma_1(s) & 0 & \gamma_3(s) & 0 & \gamma_5(s) & \cdots \\ - - - & - - - & - - - & - - - & - - - & - - - \\ & & \cdot & & & \\ - - - & - - - & - - - & - - - & - - - & - - - \end{bmatrix}$$

and $\Phi(t_1-s)CNN^*C^*\Phi^*(t_1-s)$ is an infinite symmetric matrix whose ij^{th} element is given by

$$\begin{aligned} c_{ij}(s) &= 0\,, \quad \text{if } (i+j) \text{ is odd} \\ &= \gamma_i(s)\gamma_j(s) \quad \text{if } (i+j) \text{ is even.} \end{aligned}$$

Accordingly, the ij^{th} element of the infinite matrix representing the operator $F_3F_3^*$ is given by

$$\begin{aligned} (F_3F_3^*)_{ij} &= 0\,, \quad \text{if } (i+j) \text{ is odd} \\ &= (2ijk^2/(i^2+j^2)b)[1 - \exp\{-(i^2+j^2)(k\pi/b)^2(t_1-t_o)\}]\,, \\ &\qquad \text{if } (i+j) \text{ is even} \end{aligned} \tag{3.35}$$

Again the problem of inverting $F_3F_3^*$ can be approximately solved by any of the techniques of Chapter 5 and thus the optimal element $\bar{h}^* = (\bar{h}_1^*, \bar{h}_2^*)$ can be computed by Equation (3.33). The optimal boundary controls $h_1^*(t)$ and $h_2^*(t)$ can be obtained from the following obvious relations

$$2h_1(t) = \bar{h}_1(t) + \bar{h}_2(t) \tag{3.36}$$

and

$$2h_2(t) = \bar{h}_1(t) - \bar{h}_2(t) . \qquad (3.37)$$

Having in hand the solution of the minimum energy control problem for the above three systems, the general minimum energy control problem posed at the opening of this section may now be solved.

3.2.4. The General Minimum Energy Control Problem.

In this subsection the system is described by Equations (3.4) to (3.6), and the performance index to be minimized is that of Equation (3.4), namely,

$$J(f,g,h) = \frac{1}{2}\int_{t_o}^{t_1} [\int_{o}^{b} |f(t,\alpha)|^2 d\alpha + \sum_{i=1}^{m} |g_i(t)|^2 + \sum_{i=1}^{2} |h_i(t)|^2]dt . \quad (3.4)$$

It has already been pointed out (see page 41) that the solution of the system equations can be put in the form

$$x(t,\alpha) = \sum_{i=1}^{3} x_i(t,\alpha) , \qquad (t,\alpha) \in \Delta \qquad (3.38)$$

where x_1, x_2, and x_3 are the responses of Systems I, II, and III respectively. Noticing that $x_i(t,\alpha) = \sum_{n=1}^{\infty} T_{in}(t)\varphi_n(\alpha)$, (i=1,2,3) , and putting $T_n(t) = \sum_{i=1}^{3} T_{in}(t)$, it follows that Equation (3.38) can be written in the form

$$x(t,\alpha) = \sum_{n=1}^{\infty} T_n(t)\varphi_n(\alpha) .$$

Let $z(t)$ be the infinite dimensional column vector given by

$$z(t) = (T_1(t),\ldots,T_n(t),\ldots) ,$$

then it is clear that $z(t)$ can be expressed as

$$z(t) = z_1(t) + z_2(t) + z_3(t) ,$$

where $z_1(t)$, $z_2(t)$, and $z_3(t)$ are given by Equations (3.15), (3.25), and (3.32) respectively, i.e.,

$$z(t) = \int_{t_o}^{t} \Phi(t-s)u(s)ds + \int_{t_o}^{t} \Phi(t-s)Bg(s)ds + \int_{t_o}^{t} \Phi(t-s)CN\bar{h}(s)ds + \Phi(t-t_o)z^o$$

or equivalently,

$$\hat{z}(t) \triangleq z(t) - \Phi(t-t_o)z^o = \int_{t_o}^{t} \Phi(t-s)u(s)ds + \int_{t_o}^{t} \Phi(t-s)Bg(s)ds + \int_{t_o}^{t} \Phi(t-s)CN\bar{h}(s)ds . \tag{3.39}$$

It thus follows that at $t = t_1$, with system inputs $u \in \bar{L}_2(t_o,t_1), g \in \mathcal{G}$, $h \in \mathcal{H}$,

$$\hat{z}(t_1) = F_1u + F_2g + F_3\bar{h} \tag{3.40}$$

where F_1, F_2, and F_3 are the bounded linear transformations defined by Equations (3.17), (3.26) and (3.33) respectively.

To bring the optimization problem under consideration into the form of Problem I of Section 2.2, Equation (3.40) suggests that a new function space $\mathcal{S}$ and a new transformation F be introduced. The function space $\mathcal{S}$ is defined as the cartesian product of the Hilbert function space $\bar{L}_2(t_o,t_1)$, $\mathcal{G}$, and $\mathcal{H}$, i.e., $\mathcal{S}$ is the set of all ordered triples $s = (u,g,h)$ such that $u \in \bar{L}_2(t_o,t_1)$, $g \in \mathcal{G}$, and $h \in \mathcal{H}$ equipped with the inner product

$$< s_1,s_2 >_{\mathcal{S}} = < u_1,u_2 >_{\bar{L}_2(t_o,t_1)} + < g_1,g_2 >_{\mathcal{G}} + < h_1,h_2 >_{\mathcal{H}} \tag{3.41}$$

where $s_1 = (u_1,g_1,h_1)$ and $s_2 = (u_2,g_2,h_2)$ are elements in $\mathcal{S}$. It can be easily shown (see [15, Section 6.4] that $\mathcal{S}$ is a Hilbert space in its own right.

The transformation $F: \mathcal{S} \to \ell_2$ is defined with the meaning of Equation (3.40) as the direct sum of the bounded linear transformation F_1, F_2, and F_3, i.e.,

$$F\, s = F_1 u + F_2 g + F_3 h , \qquad s = (u,g,h) \in \mathcal{S} \tag{3.42}$$

It is obvious that F is a bounded linear transformation from the Hilbert function space $\mathcal{S}$ into ℓ_2. Accordingly, the optimization problem can be restated as follows: From the set $\Gamma(z^1) = \{s : s \in \mathcal{S}, Fs = \hat{z}^1 \triangleq z^1 - \Phi(t_1-t_o)z^o\}$, find the element s^* with minimum norm. Here the identification of the norm as the performance index (3.4) is obvious from the following chain of equalities:

$$\begin{aligned} ||(u,g,\bar{h})||^2_{\mathcal{S}} &= ||u||^2_{\bar{L}_2(t_o,t_1)} + ||g||^2_{\mathcal{G}} + ||\bar{h}||^2_{\mathcal{H}} \\ &= \int_{t_o}^{t_1} [\sum_{n=1}^{\infty} |u_n(t)|^2 dt + \int_{t_o}^{t_1} \sum_{i=1}^{m} |g_i(t)|^2 dt \\ &\quad + (1/2) \int_{t_o}^{t_1} \sum_{i=1}^{2} |\bar{h}_i(t)|^2 dt \\ &= \int_{t_o}^{t_1} [\int_{o}^{b} |f(t,\alpha)|^2 d\alpha + \sum_{i=1}^{m} |g_i(t)|^2 + \sum_{i=1}^{2} |h\,(t)|^2] dt \end{aligned} \tag{3.43}$$

It thus follows that the optimal element is given by

$$s^* = F^{\dagger}\hat{z}^1 = F^*(FF^*)^{-1}\hat{z}^1 \ . \tag{3.44}$$

In short, the general problem differs from the particular cases in computational details, but not at all in concept.

The function F^* may be computed as follows: let $\eta \in \ell_2$ and $s = (u,g,h) \in \mathcal{S}$, then

$$\begin{aligned} < Fs,\eta >_{\ell_2} &= < F_1u + F_2g + F_3h,\eta >_{\ell_2} \\ &= < F_1u,\eta >_{\ell_2} + < F_2g,\eta >_{\ell_2} + < F_3h,\eta >_{\ell_2} \\ &= < u, F_1^*\eta >_{L_2(t_o,t_1)} + < g,F_2^*\eta >_{\mathcal{G}} + < h,F_3^*\eta >_{\mathcal{H}} \\ &= < (u,g,h), (F_1^*\eta,F_2^*\eta,F_3^*\eta) >_{\mathcal{S}} \\ &= < s,F^*\eta >_{\mathcal{S}} \end{aligned}$$

Therefore, $F^*\eta$ is identified as

$$F^*\eta = (F_1^*\eta,F_2^*\eta,F_3^*\eta) \ , \tag{3.45}$$

and thus, Equation (3.42) gives

$$FF^*\eta = F_1F_1^*\eta + F_2F_2^*\eta + F_3F_3^*\eta \ ,$$

or equivalently

$$FF^* = F_1F_1^* + F_2F_2^* + F_3F_3^* \ . \tag{3.46}$$

Substitution of Equations (3.21), (3.27) and (3.35) in Equation (3.46) results in expressing the ij^{th} element of the corresponding infinite matrix FF^* as

$$
\begin{aligned}
(FF^*)_{ij} &= \delta_{ij}[2(ik\pi/b)^2]^{-1}[1-\exp\{-2(ik\pi/b)^2(t_1-t_o)\}] \\
&\quad + (b/k\pi)^2[i^2+j^2]^{-1}[1-\exp\{-(i^2+j^2)(k\pi/b)^2(t_1-t_o)\}] \\
&\quad \sum_{\ell=1}^{m} \varphi_i(\alpha_\ell)\varphi_j(\alpha_\ell) \qquad (3.47) \\
&\quad + (ijk^2/b)[i^2+j^2]^{-1}[1+(-1)^{i+j}][1-\exp\{-(i^2+j^2)(k\pi/b)^2(t_1-t_o)\}]
\end{aligned}
$$

Here, $\delta_{ij} = 1$ for $i=j$ and equals zero otherwise. Applying any of the approximation techniques discussed in Chapter 5, the inverse matrix $(FF^*)^{-1}$ is computable. By direct substitution in Equation (3.47), the explicit expressions of the optimal controls f,g,h are determined and thus minimum energy control of this section is completely solved.

3.3. Minimum Energy Control of a Diffusion System - Second Example

In Section 3.2, the separation of variables technique has been used to obtain explicit solutions to the minimum energy control problem for the class of systems described by Equations (3.1) to (3.3). In this section, it will be shown (by a concrete example) that the same optimization methods produce explicit results whenever the systems Green's function $G(t,\alpha)$ can be expressed in the separable form[1]

---

[1] It is emphasized that the abstract theorems of Section 2.2 apply whether or not the Green's function is separable. The present discussion focuses on the use of separable kernels to obtain explicit results.

$$G(t,\alpha) = \sum_{n=1}^{\infty} R_n(t)\omega_n(\alpha) , \qquad (t,\alpha) \in \Delta$$

where R_n is a function of t only and, for convenience, the set $\{\omega_n\}$ is taken to be a complete orthonormal basis for the underlying Hilbert space in question.

The system to be considered here is described by the homogeneous diffusion equation

$$\frac{\partial x(t,\alpha)}{\partial t} = k^2 \frac{\partial^2 x(t,\alpha)}{\partial \alpha^2} , \qquad t \in [t_o,t_1] , \; 0 < \alpha < b \qquad (3.48)$$

with the initial condition

$$x(t_o,\alpha) = 0 , \qquad 0 \leq \alpha \leq b , \qquad (3.49)$$

and boundary conditions

$$\frac{\partial x(t,0)}{\partial \alpha} = \gamma^2[x(t,0) - v(t)] , \qquad t_o \leq t \leq t_1 , \qquad (3.50)$$

$$\frac{\partial x(t,b)}{\partial \alpha} = 0 \qquad t_o \leq t \leq t_1 . \qquad (3.51)$$

Here γ^2 is a constant and the time-dependent function $v(t)$ is related to the control input $q(t)$ by the first order ordinary differential equation

$$\frac{dv(t)}{dt} + \sigma^2 v(t) = q(t) , \qquad t_o \leq t \leq t_1 \qquad (3.52)$$

where σ^2 is a constant. Physically, this mathematical model represents the process of one-sided heating of metal in a furnace. Equation (3.48) represents the temperature distribution $x(t,\alpha)$ in the metal; Equation (3.50) states that the temperature gradient of the surface $\alpha = 0$ is proportional to the difference between the surface temperature $x(t,0)$ and the medium temperature $v(t)$ while Equation (3.51) states that the temperature gradient at the other surface $\alpha = b$ is zero. The delay action between the fuel flow $q(t)$ and the medium temperature $v(t)$ is described by the first order ordinary differential Equation (3.52).

Assuming, as in Section 3.2, that the solution of the above system is of the form $x(t,\alpha) = R(t)\omega(\alpha)$, Equation (3.50) reduces to

$$\frac{d\omega(0)}{d\alpha} - \gamma^2\omega(0) = -\gamma^2 \frac{v(t)}{R(t)}$$

which is a contradiction (disregarding the very special case when $v(t)/R(t)$ is a constant). Therefore, the assumed form of solution is not valid and one cannot use directly the separation of variables method to solve the problem. It should be noticed that the source of difficulty here is that the first boundary condition (Equation (3.50)) is inhomogeneous. If it is replaced by the homogeneous boundary condition

$$\frac{\partial x(t,0)}{\partial \alpha} - \gamma^2 x(t,0) = 0 , \qquad t \in [t_o,t_1] \qquad (3.50a)$$

the solution can be written as $x(t,\alpha) = \sum_{n=1}^{\infty} R_n(t)\omega_n(\alpha)$, where $\{\omega_n\}$ are the eigenfunctions of the following Sturm-Liouville problem:

$$\frac{d^2\omega(\alpha)}{d\alpha^2} + \lambda^2\omega(\alpha) = 0 \qquad 0 < \alpha < b$$

with the homogeneous boundary conditions

$$\frac{d\omega(0)}{d\alpha} - \gamma^2\omega(0) = 0 ,$$

$$\frac{d\omega(b)}{d\alpha} = 0 .$$

It can be shown (see [48, Section 23]) that the eigenfunctions of this problem are given by

$$\omega_n(\alpha) = \cos\{\beta_n (1 - \frac{\alpha}{\beta})\} , \qquad 0 \le \alpha \le b , \quad n=1,2,\ldots \qquad (3.53)$$

where the scalars $\beta_n = \lambda_n b$ are the positive (real) roots of the equation

$$\beta \tan \beta = \gamma^2 b$$

arranged in increasing order. The system $\{\omega_n(\alpha) = \cos \beta_n(1 - \frac{\alpha}{\beta})\}$ is thus a complete orthogonal basis for $L_2(0,b)$(see Appendix A).

Returning now to the original system defined by Equations (3.48) to (3.51) it is shown in Appendix D that the solution x is given by

$$x(t,\alpha) = \int_{t_o}^{t} G(t-s,\alpha)q(s)ds \qquad (3.54)$$

where the Green's function G is defined by

$$G(t,\alpha) = \frac{\cos\{(b\sigma/k)(1-\alpha/b)\}}{\cos(b\sigma/k) - (\sigma/k\gamma^2)} \exp\{-\sigma^2 t\}$$

$$+ 2(k/b)^2 \sum_{i=1}^{\infty} \frac{[\cos \beta_i(1-\alpha/b)]\exp\{-(k\beta_i/b)^2 t\}}{[\sigma^2-(k\beta_i/b)^2][b/\gamma^2+(1+b\gamma^2)/\beta_i^2]\cos \beta_i} \qquad (3.55)$$

To put Equation (3.55) in the form $\sum_{n=1}^{\infty} R_n(t)\omega_n(\alpha)$, where $\{\omega_n\}$ is the orthogonal complete system given by Equation (3.53) the function $\cos\{(b\sigma/k)(1-\alpha/b)\}$ is put in its Fourier series expansion with respect to $\{\omega_n\}$ as

$$\cos\{(b\sigma/k)(1-\alpha/b)\} = \sum_{n=1}^{\infty} A_n\omega_n(\alpha) \quad .$$

where

$$A_n = [\int_0^b \cos\{(b\sigma/k)(1-\alpha/b)\}\omega_n(\alpha)d\alpha][\int_0^b |\omega_n(\alpha)|^2 d\alpha]^{-1}$$

It is shown in Appendix E that

$$\int_0^b |\omega_n(\alpha)|^2 d\alpha = (b/2)[1 + (\sin 2\beta_n)/2\beta_n] \ ,$$

and thus A_n takes the form

$$A_n = (2/b)[1+(\sin 2\beta_n)/2\beta_n]^{-1}[\int_0^b \cos\{(b\sigma/k)(1-\alpha/b)\}\omega_n(\alpha)d\alpha] \ . \qquad (3.56)$$

It thus follows that

$$G(t,\alpha) = \sum_{n=1}^{\infty} [C_n\exp\{-\sigma^2 t\} + D_n\exp\{-(k\beta_n/b)^2 t\}]\omega_n(\alpha) \qquad (3.57)$$

where the scalars C_n, D_n are given by

$$C_n = A_n[\cos\{(b\sigma/k)\} - (\sigma/k\gamma^2)]^{-1} \ , \qquad (3.58)$$

$$D_n = 2(k/b)^2[\{\sigma^2-(k\beta_n/b)^2\}\{b/\gamma^2 + (1+b\gamma^2)/\beta_n^2\}\cos\beta_n]^{-1}$$

To determine the function x , these results may be substituted in Equation (3.54) giving

$$x(t,\alpha) = \int_{t_o}^{t} \sum_{n=1}^{\infty} [C_n \exp\{-\sigma^2(t-s)\} + D_n \exp\{-(k\beta_n/b)^2(t-s)\}]\omega_n(\alpha)q(s)ds$$

$$= \sum_{n=1}^{\infty} \omega_n(\alpha) \int_{t_o}^{t} [C_n \exp\{-\sigma^2(t-s)\} + D_n \exp\{-(k\beta_n/b)^2(t-s)\}]q(s)ds$$

Thus if the functions

$$R_n(t) = \int_{t_o}^{t} [C_n \exp\{-\sigma^2(t-s)\} + D_n \exp\{-(k\beta_n/b)^2(t-s)\}]q(s)ds \tag{3.59}$$

are defined, the desired expansion

$$x(t,\alpha) = \sum R_n(t)\omega_n(\alpha)$$

is achieved. Equation (3.59) can be put immediately in the matrix form

$$z(t) = \int_{t_o}^{t} \Psi(t-s)Dq(s)ds$$

where $z(t) = (R_1(t),\ldots,R_n(t),\ldots)$, Ψ is the time-varying infinite <u>diagonal</u> matrix with the n^{th} element given by

$$[\Psi]_n(t) = [C_n \exp\{-\sigma^2 t\} + D_n \exp\{-(k\beta_n/b)^2 t\}] \tag{3.60}$$

and D is the column vector $D = (1,1,\ldots)$. The mapping $q \to z(t_1)$ is a linear (bounded) transformation from the $L_2(t_o,t_1)$ space to the ℓ_2 space which we denote by F

$$Fq = \int_{t_o}^{t_1} \Psi(t_1-s)Dq(s)ds \; . \tag{3.61}$$

In terms of these definitions the optimization problem of finding the control input function $q(t) \in L_2(t_o,t_1)$ which brings the system to the state $x^1(\alpha)$ at time t_1 while minimizing the funtional

$$J(u) = \frac{1}{2}\int_{t_o}^{t_1} |q(t)|^2 dt ,$$

can be restated as follows: From the set $\Gamma(z^1) = \{q : q \in L_2(t_o,t_1) , Fq = z^1\}$ find the element q^* with minimum norm. Here $z^1 = (\xi_1^1,\ldots, \xi_n^1,\ldots)$ where ξ_n^1 is given by

$$\xi_n^1 = (2/b)[1 + (\sin 2\beta_n)/2\beta_n]^{-1} \int_0^b x^1(\alpha)\omega_n(\alpha)d\alpha$$

i.e. $\{\xi_n^1\}$ are the Fourier coefficients of the terminal state $x^1(\alpha)$ with respect to $\{\omega_n\}$.

In order to use Theorem I of Section 2.2 which specifies the solution of the problem, the transformation F^* must be found. It is apparent that this transformation can be identified with multiplication by the $(1 \times \infty)$ matrix given by

$$F^*\eta(t) = D^*\Psi(t_1-t)\eta , \qquad \eta \in \ell_2$$

The quantity $D^*\Psi(t_1-t)$ is the row vector

$$D^*\Psi(t_1-t) = [\gamma_1(t_1-t),\ldots,\gamma_n(t_1-t),\ldots]$$

where

$$\gamma_n(t) = C_n\exp\{-\sigma^2 t\} + D_n\exp\{-(k\beta_n/b)^2 t\} , \qquad n=1,2,\ldots .$$

It thus follows that

$$FF^* = \int_{t_o}^{t_1} \Psi(t_1-s)DD^*\Psi(t_1-s)ds$$

is the infinite symmetric matrix whose mn^{th} element is given by

$$\int_{t_o}^{t_1} \gamma_m(t_1-s)\gamma_n(t_1-s)ds \ .$$

Using a suitable approximation technique $(FF^*)^{-1}$ can be obtained to any desired degree of accuracy and thus the problem will be completely solved by substitution in the equation

$$q^* = F^*(FF^*)^{-1}z^1 \ .$$

3.4. Synthesis Problem of Feedback Loops

Where optimal controls are unique and the system in question is causal, it can usually be shown that there exists a closed loop feedback controller which will achieve the desired optimality. In the present framework, the linearity of F assures that this feedback controller is linear. Since it is of significant engineering interest to do so, the feedback law for the problem of minimum energy control will be located. In the following, it is assumed that both the state and the control input functions (if necessary) have been transformed to suitable univariable spaces and that the system response-input relation in these spaces is given by

$$z(t) = \Phi(t-t_o)z^o + \int_{t_o}^{t} \Phi(t-s)B(s)u(s)ds \ , \qquad t \in [t_o,t_1] \qquad (3.62)$$

Here $z(t)$, the instananeous state of the system, is an element of ℓ_2 , Φ denotes the infinite-dimensional transition matrix, $z^o = z(t_o)$ is the given initial state of the system, the control u is a (possibly finite) tuplet of time functions, and the time-varying matrix B has compatible dimensions.

The synthesis problem to be solved is that of constructing the feedback loop which generates the control $u^*(t)$ that brings the system at time t_1 to the specified state z^1 while minimizing the functional

$$J(u) = \frac{1}{2} \int_{t_o}^{t_1} \sum_i |u_i(t)|^2 dt \tag{3.63}$$

Defining the linear transformation F as

$$Fu = \int_{t_o}^{t_1} \Phi(t_1-s)B(s)u(s)ds$$

the open loop control $u^*(t)$ is given by

$$u^* = F^*(FF^*)^{-1}(z^1-\Phi(t_1-t_o)z^o) \tag{3.64}$$

where

$$(F^*\eta)(t) = B^*(t)\Phi^*(t_1-t)\eta\,, \qquad t \in [t_o,t_1], \eta \in \ell_2\,,$$

If this optimal control u^* is applied to the system, the response of the system at any time $t' \in (t_o,t_1)$ is given by the equation

$$z(t') = \Phi(t'-t_o)z^o + \int_{t_o}^{t'} \Phi(t'-s)B(s)B^*(s)\Phi^*(t'-s)(FF^*)^{-1}\hat{z}^1 ds$$

or equivalently

$$z(t') - \Phi(t'-t_o)z^o = [\int_{t_o}^{t'} \Phi(t'-s)B(s)B^*(s)\Phi^*(t'-s)ds][(FF^*)^{-1}\hat{z}^1] \tag{3.65}$$

where

$$\hat{z}^1 = z^1 - \Phi(t_1-t_o)z^o$$

Multiplying both sides of Equation (3.65) from the left by $\Phi(t_1-t')$, and noticing that $\Phi(t_1-t')\Phi(t'-s) = \Phi(t_1-s)$ for $t_1 < t' < s$, gives the following chain of equalities:

$$\Phi(t_1-t')z(t') - \Phi(t_1-t_o)z^o$$

$$= [\int_{t_o}^{t'} \Phi(t_1-s)B(s)B^*(s)\Phi^*(t_1-s)ds][(FF^*)^{-1}\hat{z}^1]$$

$$= [\int_{t_o}^{t_1} \Phi(t_1-s)B(s)B^*(s)\Phi^*(t_1-s)ds - \int_{t'}^{t_1} \Phi(t_1-s)B(s)B^*(s)\Phi^*(t_1-s)ds]$$

$$x[(FF^*)^{-1}\hat{z}^1]$$

$$= [FF^* - \int_{t'}^{t_1} \Phi(t_1-s)B(s)B^*(s)\Phi^*(t_1-s)ds][(FF^*)^{-1}\hat{z}^1]$$

$$= \hat{z}^1 - [\int_{t'}^{t_1} \Phi(t_1-s)B(s)B^*(s)\Phi^*(t_1-s)ds][(FF^*)^{-1}\hat{z}^1]$$

It thus follows that

$$(FF^*)^{-1}\hat{z}^1 = [\int_{t'}^{t} \Phi(t_1-s)B(s)B^*(s)\Phi^*(t_1-s)ds]^{-1}[z^1 - \Phi(t_1-t')z(t')]$$

Substitution of Equation (3.66) in Equation (3.64) and replacing t' by t give

$$u^*(t) = F^*[\int_{t}^{t_1} \Phi(t_1-s)B(s)B^*(s)\Phi^*(t_1-s)ds]^{-1}[z^1-\Phi(t_1-t)z(t)] \tag{3.67}$$

for all $t \in (t_o,t_1)$.

Equation (3.67) expresses the instantaneous value of the optimum control $u^*(t)$ in terms of the instantaneous value of the system $z(t)$ and thus is the required feedback control law. It is interesting to notice that this expression does not depend on the initial state of the system, and therefore indicates that the system will be brought to the state z^1 with the expenditure of minimum energy even in the case when some interfering disturbance, in the interval (t_o, t_1), forces the system state to deviate from its trajectory. In a block diagram form Equation (3.67) may be represented as shown in Figure 4.

As an example of the above result, consider the feedback control law for System I of Subsection 3.2.1. For this system, B is the identity and Φ is the diagonal matrix

$$\Phi(t) = \text{diag}[\exp\{-(k\pi/b)^2 t\},\ldots,\exp\{-(nk\pi/b)^2 t\},\ldots] .$$

Thus it follows that

$$\int_t^{t_1} \Phi(t_1-s)B(s)B^*(s)\Phi^*(t_1-s)ds = \int_t^{t_1} \Phi^2(t_1-s)ds$$

which is exactly the matrix:

$$\text{diag}\,[\gamma_1,\ldots,\gamma_n,\ldots] ,$$

where

$$\gamma_n = [2(nk\pi/b)^2]^{-1}[1 - \exp\{-2(nk\pi/b)^2(t_1-t)\}]$$

The inverse is readily computed, namely,

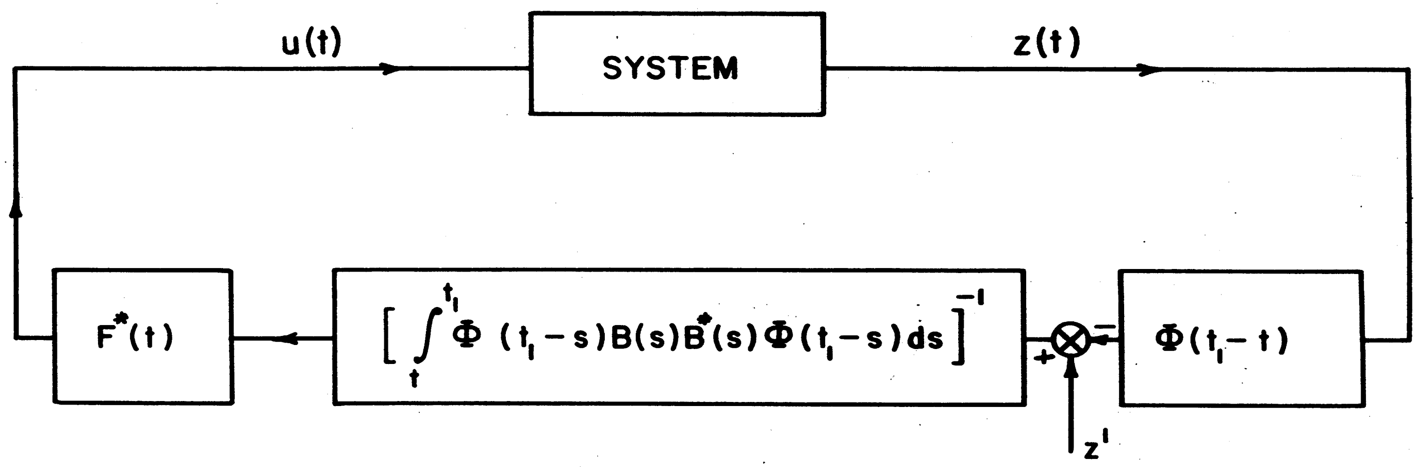

Figure 4. Block Diagram Representation of Equation (3.67)

$$[\int_{t}^{t_1} \Phi(t_1-s)B(s)B^*(s)\Phi^*(t_1-s)ds]^{-1} = \text{diag}[\gamma^{-1},\ldots,\gamma_n^{-1},\ldots] \tag{3.68}$$

Substituting Equations (3.20) and (3.68) in Equation (3.67) gives the following infinite system of equations:

$$u^*(t) = 2(nk\pi/b)^2\exp\{-(nk\pi/b)^2(t_1-t)\}[1-\exp\{-2(nk\pi/b)^2(t_1-t)\}]$$

$$\times \int_0^b [x^1(\alpha)-\exp\{-(nk\pi/b)^2(t_1-t)\}x(t,\alpha)]\varphi_n(\alpha)d\alpha ,$$

$$t \in (t_o,t_1]$$

where $\varphi_n = \sqrt{2/b}\, \sin \{(n\pi/b)\alpha\}$, n=1,2,..., and thus the optimal feedback control law is given by

$$f^*(t,\alpha) = \sum_{n=1}^{\infty} u_n^*(t)\varphi_n(\alpha) . \qquad (t,\alpha) \in \Delta \tag{3.69}$$

It is interesting to notice that for the special case when $x^1(\alpha) = 0$ $(x^o(\alpha) \neq 0)$, and $b = k = 1$, Equation (3.69) yields the same results derived by Wang [54, Equation 5.48] who used the technique of dynamic programming.

CHAPTER 4

GENERALIZED MINIMUM ENERGY PROBLEMS

In many distributed-parameter system problems, the performance index takes concrete forms which appear to be much more complicated than the functional discussed in Chapter III. For a broad class of problems, however, the seemingly more complicated physical problems can be reduced to a simpler abstract form by a judicious choice of function spaces.

In this chapter, two such instances are considered. Because of the strong analogies which have been established between the minimum energy problems for System I, II, and III and the general case, it suffices here to restrict attention to System I.

4.1. First Generalized Problem

The optimization problem to be solved in this section may be stated as follows: Let System I be defined by Equation (3.5), and let $\mathcal{F}$ denote the Hilbert space defined on page 40. The problem is then to find the element $f^* \in \mathcal{F}$ which carries the system from the given initial state $x^o(\alpha)$ at a specified time t_1, while minimizing the functional

$$J(f) = \frac{1}{2} \int_{t_o}^{t_1} \int_0^b \overline{P}(t,\alpha)|f(t,\alpha)|^2 d\alpha dt + \frac{1}{2} \int_{t_o}^{t_1} \int_0^b \overline{Q}(t,\alpha)|x_f(t,\alpha)|^2 d\alpha dt \qquad (4.1)$$

Here $\overline{P}(t,\alpha)$ and $\overline{Q}(t,\alpha)$ are bounded strictly positive measureable functions defined on $\overline{\Delta} = \{[t_o,t_1] \times [0,b]\}$, and x_f denotes the system response to the input f.

[1] These conditions are sufficient to ensure that $J(f)$ may be written as $J(f) = \|f\|^2 + \|x_f\|^2$ in the obvious respective Hilbert spaces.

In subsection 4.1.1, this problem is reduced to Problem II of Section 2.2. In subsections 4.1.2 and 4.1.3 two alternative techniques are used to compute the optimal control element $f^*(t,\alpha)$, while in subsection 4.1.4, a necessary condition which the optimal control has to satisfy is derived.

4.1.1. Abstract Formulation of the Problem.

In this subsection, the performance index given by Equation (4.1) shall be put in the norm form of Equation (2.16), namely

$$J(u) = \|Fu\|^2 + \|u\|^2$$

for the Hilbert function space $L_2(0,b)$, both the systems $\{\varphi_n = \sqrt{2/b}\,\sin(n\pi/b)\alpha\}$ and $\{\sqrt{1/b}\,,\ \psi_n = \sqrt{2/b}\,\cos(n\pi/b)\alpha\}$ are orthonormal bases. By expanding $f(t,\alpha)$ and $\overline{P}(t,\alpha)$ in their Fourier series with respect to $\{\varphi\}$ and $\{\sqrt{1/b},\psi_n\}$ respectively, it follows that the two expansions

$$f(t,\alpha) = \sum_{n=1}^{\infty} u_n(t)\,\varphi_n(\alpha),$$

$$u_n(t) = \langle f,\varphi_n \rangle = \sqrt{2/b}\int_0^b f(t,\alpha)\sin(n\pi/b)\alpha\,d\alpha,\quad n=1,2,\ldots \tag{4.2}$$

and

$$\overline{P}(t,\alpha) = c_o(t)/\sqrt{2} + \sum_{n=1}^{\infty} c_n(t)\psi_n(\alpha),$$

$$c_n(t) = \langle \overline{P},\ \psi_n \rangle = \sqrt{2/b}\int_0^b P(t,\alpha)\cos(n\pi/b)\alpha\,d\alpha,\quad n=0,1,2,.. \tag{4.3}$$

hold on $(t,\alpha)\in\Delta$.

It can be readily shown (see [48, page 127]) that the Fourier expansion of $\overline{P}(t,\alpha)f(t,\alpha)$ with respect to $\{\varphi_n\}$ is given by

$$\overline{P}(t,\alpha)f(t,\alpha) = \sum_{n=1}^{\infty} d_n(t)\varphi_n(\alpha),$$

$$d_n(t) = (1/\sqrt{2b}) \sum_{m=1}^{\infty} \{u_m(t)(c_{m-n}(t) - c_{m+n}(t))\}.$$

Here the functions $c_n(t)$ are defined for positive integers by Equation (4.3) and satisfy the relation: $c_{-n}(t) = c_n(t)$, $t \in [t_o,t_1]$. Using Parseval's Theorem[1] the following equality chain holds:

$$\int_0^b \overline{P}(t,\alpha)|f(t,\alpha)|^2 d\alpha = \int_0^b \{\overline{P}(t,\alpha)f(t,\alpha)\}\{f(t,\alpha)\}d\alpha$$

$$= \sum_{n=1}^{\infty} d_n(t)u_n(t)$$

$$= \sum_{n,m=1}^{\infty} p_{mn}(t)u_m(t)u_n(t), \quad t \in [t_o,t_1]$$

where

$$p_{mn}(t) = (1/\sqrt{2b})(c_{m-n}(t) - c_{m+n}(t)), \quad t \in [t_o,t_1]$$

Defining P as the infinite matrix whose mn^{th} element is the

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[1] Parseval's Theorem: Let $\{\varphi_n\}$ be an orthonormal basis in the (separable) Hilbert space H, then

$$\langle f,g \rangle_H = \sum_{n=1}^{\infty} f_n g_n, \text{ for all } f, g \in H \text{ where } f_n \text{ and } g_n (n=1,2,\ldots)$$

are the Fourier coefficients of f and g respectively given by $f_n = \langle f, \varphi_n \rangle$ and $g_n = \langle g,\varphi_n \rangle$.

function p_{mn} the above equality may be written as

$$\int_0^b \overline{P}(t,\alpha) \mid f(t,\alpha)|^2 \, d\alpha = [u(t), P(t)\, u(t)] \;, \; t\epsilon[t_o,t_1]$$

where $u(t)$ denotes the tuplet $u(t) = (u_1(t),\ldots,u_n(t),\ldots)$ and [,] denotes the usual inner product on ℓ_2. Noticing that $p_{mn} = p_{nm}$ and that $\overline{P}(t,\alpha)$ is a bounded strictly positive (measurable) function on $[t_o,t_1] \times [o,b]$ it immediately follows that P is a symmetric positive definite infinite matrix. Similarly, it can be shown that

$$\int_0^b \overline{Q}(t,\alpha)| \, x_f(t,\alpha)|^2 \, d\alpha = [z_f(t), Q(t) z_f(t)], \; t \in [t_o,t_1]$$

where $z_f(t)$ denotes the tuplet $z_f(t) = (T_{f1}(t),\ldots,T_{fn}(t),\ldots)$, T_{fn} being the Fourier coefficients of $x_f(t,\alpha)$ with respect to $\{\varphi_n\}$ and Q

is the positive definite symmetric infinite matrix corresponding to the weighting function $\overline{Q}(t,\alpha)$, $(t,\alpha)\epsilon\, \overline{\Delta}$. It thus follows that the performance index of Equation (4.1) may be put in the form

$$J(u) = \frac{1}{2}\int_{t_o}^{t_1}[u(s),P(s)u(s)]ds + \frac{1}{2}\int_{t_o}^{t_1}[z_f(s),Q(s)z_f(s)\,]\,ds. \tag{4.5}$$

Now let H_1 and H_2 denote the vector valued Hilbert function spaces consisting of the set of all tuplets which are finite with respect to the norm induced by the inner product

$$< u,v >_1 = \int_{t_o}^{t_1} [u(s),P(s)v(s)]ds, \qquad u,v \in H_1 \tag{4.6}$$

$$< y,z >_2 = \int_{t_o}^{t_1} [y(s),Q(s)z(s)]ds, \qquad y,z \in H_2 \tag{4.7}$$

respectively. Also, let $\hat{F}_1 : H_1 \to H_2$ be the (bounded) linear transformation defined by

$$\hat{F}_1 u = z(t) = \int_{t_o}^{t} \Phi(t-s)u(s)ds \tag{4.8}$$

where Φ is the system transition matrix given by Equation (3.16). It is clear that $(\hat{F}_1 u)(t_1) = F_1 u$ where F_1 is defined by Equation (3.17), namely,

$$F_1 u = \int_{t_o}^{t_1} \Phi(t_1-s)u(s)ds \tag{3.17}$$

In terms of the above definitions, the optimization problem may be stated as follows: Find the element $u \in H_1$ which minimizes the functional

$$J(u) = \frac{1}{2}[||u||_1^2 + ||z_f||_2^2] = \frac{1}{2}[||u||_1^2 + ||\hat{F}_1 u||_2^2] \tag{4.9}$$

and satisfies the terminal constraint

$$F_1 u = z^1 . \tag{4.10}$$

Here $z^1 = (\xi_1^1, \ldots, \xi_n^1, \ldots) \in \ell_2$ is the tuplet whose elements are the Fourier coefficients of x^1 with respect to $\{\varphi_n\}$, i.e.

$$\xi_n^1 = < x^1, \varphi_n > = \sqrt{2/b} \int_o^b x^1(\alpha)\sin(n\pi/n)\alpha d\alpha . \tag{4.11}$$

According to Theorem II of Section 2.2, the optimum element u^* is the unique preimage of z^1 under F_1 in the subspace $S \subset H_1$ defined by

$$S = (I + \hat{F}_1^{\circledast} \hat{F}_1)^{-1} M \tag{4.12}$$

where $M = N(F_1)^{\perp}$ is the orthogonal complement of the null space $N(F_1)$ of F_1. The unusual notation $\hat{F}_1^{\circledast}$ is adopted to remind the reader that the adjoint of $\hat{F}_1$ must now be computed with respect to the spaces H_1 and H_2. Thus this transformation is distinct from the adjoint of $\hat{F}_1$ when considered (as in Chapter 3) as an operator of the space $\bar{L}_2(t_o,t_1)$. Using $\hat{F}_1^*$ for the adjoint in this latter case, it is easily shown that $\hat{F}_1^{\circledast} = \hat{F}_1^* Q$. In fact, Equation (3.12) may be given the following equivalent form

$$S = (P + \hat{F}_1^* Q \hat{F}_1)^{-1} M \tag{4.13}$$

in the space $\bar{L}_2(t_o,t_1)$. In other words, the optimal control element u^* satisfies the integral operator equation

$$(P + \hat{F}_1^* Q \hat{F}_1)u^* = v \tag{4.14}$$

for some $v \in M$. Two methods for computing this optimal control element are introduced in the following subsections.

4.1.2 Computation of u* - First Method.

From above, it is clear that the optimal control element u^* is the unique element in the subspace S which satisfies the equality constraint

$$F_1 u = z^1, \quad u \in S.$$

It thus follows that if $\{h_n\}$ is a basis for S, then u^* may be expressed as

$$u^* = \sum_{n=1}^{\infty} \eta_n h_n, \tag{4.15}$$

where $\eta = (\eta_1,\ldots,\eta_n,\ldots)$ is the unique solution of the equation

$$F_1\left(\sum_{n=1}^{\infty} \eta_n h_n\right) = z^1 . \tag{4.16}$$

A convenient way for computing a basis for S is to solve the integral equation

$$[P + \hat{F}_1^* Q\, \hat{F}_1]\, h_n = g_n \;, \quad n=1,2,\ldots \tag{4.17}$$

where $\{g_n\}$ is the basis for M given by the rows of F_1^* (see Equation (3.20)), namely,

$$g_n(t) = (0,\ldots,0,\exp\{-(nk\pi\, b)^2(t_1-t)\},0,\ldots) \tag{4.18}$$

It is to be noted here that Equation (4.17) is generally unsolvable in a closed form and thus one has to resort to an approximation technique to determine the basis $\{h_n\}$. Assuming that $\{h_n\}$ has such been determined, Equation (4.16) may be put in the matrix form

$$F_1 J \eta = z^1 \tag{4.19}$$

where J is the time-varying infinite matrix formed by using the h_n as columns, i.e.,

$$J = \begin{bmatrix} \uparrow & & \uparrow & \cdot \\ h_1 & \cdots\cdots & h_n & \cdots\cdots \\ \downarrow & & \downarrow & \end{bmatrix} .$$

It is clear from the definition of F_1 (see Equation (3.17)) that $F_1 J$ is an operator on ℓ_2 which may be identified with the infinite matrix whose ij element is given by

$$[F_1 J]_{ij} = \int_{t_o}^{t_1} [g_i(s), h_j(s)]ds$$

where $[\ ,\ ]$ denotes the usual inner product in ℓ_2. By substituting Equation (4.18) in the above equation, it follows that

$$[F_1 J]_{ij} = \int_{t_o}^{t_1} [\exp\{-(ik\pi/b)^2(t_1-s)\}] h_{ij}(s)ds \qquad (4.20)$$

where h_{ij} is the ij element of the matrix J. Having computed the matrix F_1J, the tuplet η, which uniquely defines u, is thus given by

$$\eta = [F_1 J]^{-1} z^1$$

and the solution of the problem is completely specified.

4.1.3. <u>Computation of u* - Second Method.</u>

As mentioned before, one drawback of the above method is the difficulty of computing the basis $\{h_n\}$ in a closed form. Here, this difficulty is partially circumvented by manipulating the solution equations in terms of a basis for the subspace M instead of the subspace S.

It has been shown in subsection 4.1.1 that u* satisfies the integral equation

$$(P + \hat{F}_1^* Q \hat{F}_1) u^* = v \qquad (4.14)$$

for some $v \in M$. This equation is manipulated as follows: Since the rows of $\Phi(t_1-t)$ constitute a basis for M (see Equation (3.20)), every $v \in M$ has the form

$$v(t) = \Phi^*(t_1-t)\eta \qquad t \in [t_o, t_1] \qquad (4.21)$$

for some η. Also, it follows from the definition of $\hat{F}_1$, given by Equation (3.20), that F_1^* is computed by the rule

$$(\hat{F}_1^* z)(t) = \int_t^{t_1} \Phi^*(s-t)z(s)ds, \quad t \in [t_o,t_1]. \tag{4.22}$$

Noticing that $\hat{F}_1 u^* = z^*$, and substituting for v and $\hat{F}_1^*$ in Equation (4.14) their corresponding expressions given above (Equation (4.21) and (4.22) respectively), it follows that[1]

$$\begin{aligned}(Pu^*)(t) &= v(t) - (\hat{F}_1^* Qz^*)(t). \\ &= \Phi^*(t_1-t)\eta - \int_t^{t_1} \Phi^*(s-t)Q(s)z^*(s)ds \\ &= \Phi^*(t_1-t)[\eta - \int_t^{t_1} \Phi^*(s-t_1)Q(s)z^*(s)ds] \\ &= \Phi^*(t_1-t)\gamma(t), \quad t \in [t_o,t_1]\end{aligned} \tag{4.23}$$

where $\gamma(t)$ has the equivalent forms:

$$\begin{aligned}\gamma(t) &= \eta - \int_t^{t_1} \Phi^*(s-t_1)Q(s)z^*(s)ds \\ &= \eta - [\int_{t_o}^{t_1} \Phi^*(s-t\ )Q(s)z^*(s)ds - \int_{t_o}^{t} \Phi^*(s-t_1)Q(s)z^*(s)ds] \\ &= \gamma(t_o) + \int_{t_o}^{t} \Phi^*(s-t_1)Q(s)z^*(s)ds\end{aligned} \tag{4.24}$$

for all $t \in [t_o,t_1]$.

Setting

$$y = Pu^* , \tag{4.25}$$

it follows from Equations (4.23) and (4.24) that

$$\begin{aligned}y(t) &= \Phi^*(t\ -t)\gamma(t) \\ &= \Phi^*(t_1-t)\gamma(t_o) + \Phi^*(t_1-t) \int_{t_o}^{t} \Phi^*(s-t_1)Q(s)z^*(s)ds\end{aligned} \tag{4.26}$$

---

[1] Note that $\Phi^*(s-t) = \Phi^*(t_1-t)\ \Phi^*(s-t_1)$

$$= \Phi^*(t_o-t)\,\Phi^*(t_1-t_o)\gamma(t_o) + \int_{t_o}^{t} \Phi^*(s-t)Q(s)z^*(s)ds$$

$$= \Psi(t-t_o)y(t_o) + \int_{t_o}^{t} \Psi(t-s)Q(s)z^*(s)ds,\ t \in [t_o,t_1] \qquad (4.26)$$

where $y(t_o) = \Phi^*(t_1-t_o)\gamma(t_o)$, and $\Psi(t)$ denotes the transition matrix of the adjoint system

$$\dot{\Psi}(t) = -A^*\Psi(t);\quad \Psi(t_o) = I$$

where A^* is the adjoint of the infinite matrix A given by Equation (3.14), and I is the identity matrix. It is clear from the form of Equation (4.26) that y is the solution of the matrix first order ordinary differential equation

$$\dot{y}(t) = -A^*y(t) + Q(t)z^*(t),\quad t \in [t_o,t_1], \qquad (4.27)$$

with the initial condition $y(t_o) = P(t_o)u^*(t_o)$. It is interesting to notice that Equation (4.27) together with the equation of the original system (Equation (3.13)) may be put in the following matrix form

$$\begin{bmatrix} \dot{y}(t) \\ \dot{z}^*(t) \end{bmatrix} = \begin{bmatrix} -A^* & Q(t) \\ P^{-1}(t) & A \end{bmatrix} \begin{bmatrix} y(t) \\ z^*(t) \end{bmatrix},\quad t \in [t_o,t_1] \qquad (4.28)$$

with the initial conditions

$$\begin{bmatrix} y(t_o) \\ z^*(t_o) \end{bmatrix} = \begin{bmatrix} P(t_o)u^*(t_o) \\ z(t_o) \end{bmatrix} \qquad (4.29)$$

Equation (4.28) may be transformed[1] to a matrix second order ordinary differential equation in the single variable y as shown by the following

[1] Here it is assumed that $\dot{Q}(t)$ exists.

chain of equalities

$$\ddot{y}(t) = -A^*\dot{y}(t) + \dot{Q}(t)z^*(t) + Q(t)\dot{z}^*(t)$$

$$= -A^*\dot{y}(t) + \dot{Q}(t)z^*(t) + Q(t)[Az^*(t) + P^{-1}(t)y(t)]$$

$$= -A^*\dot{y}(t) + [\dot{Q}(t) + Q(t)A]Q^{-1}(t)[\dot{y}(t) + A^*y(t)] + Q(t)P^{-1}(t)y(t)$$

or equivalently,[1]

$$\ddot{y}(t) = [-A^* + \dot{Q}(t)Q^{-1}(t) + Q(t)AQ^{-1}(t)]\dot{y}(t)$$
$$+ [\dot{Q}(t)Q\ \ (t)A^* + Q(t)AQ^{-1}A^* + Q(t)P^{-1}(t)]y(t) \qquad (4.30)$$

The general solution of this matrix differential equation contains two parameters (infinite tuplets)which may be determined through the use of the initial and terminal conditions imposed on $z(t)$. The optimal element u^* will then be given by $u^* = P^{-1}y$ and the solution of the problem is thus completely specified. To illustrate this last step, the following example is considered.

Example. Let

$$P = Q = I \qquad (4.31)$$

where I is the identity matrix. In this case Equations (4.27) and (4.30) reduces to the forms

$$\dot{u}^*(t) = -A^*u^*(t) + z^*(t), \quad t \in [t_o,t_1]$$

$$\ddot{u}^*(t) = [I + AA^*]u^*(t), \quad t \in [t_o,t_1]$$

respectively. Noticing that the matrix A is diagonal (see Equation (3.14)),

[1] It can be shown, in an analogous way, that for the general system defined by $\dot{z}(t) = A(t)z(t) + B(t)u(t)$, the corresponding equation is

$$\ddot{y}(t) = [-A^*(t) + \dot{Q}(t)Q^{-1}(t) + Q(t)A(t)Q^{-1}(t)]\dot{y}(t)$$
$$+ [\dot{Q}(t)Q^{-1}(t)A^*(t) + Q(t)A(t)Q^{-1}(t)A^*(t) + Q(t)B(t)P^{-1}(t)B^*(t)]y(t).$$

it follows that $A^* = A$ and the matrix $[I + AA^*]$ is also diagonal. Hence, u_n^*, the n^{th} component of u^*, satisfies the differential equation

$$\dot{u}_n^*(t) = (nk\pi/b)^2 u_n^*(t) + T_n^*(t), \quad t \in [t_o, t_1] \tag{4.32}$$

$$\ddot{u}_n^*(t) = [1 + (nk\pi/b)^4] u_n^*(t), \quad t \in [t_o, t_1] \tag{4.33}$$

where, as before, $T_n^*(t)$ is the n^{th} component of $z^*(t)$, $n = 1,2,\ldots$. The general solution of Equation (4.33) is given by

$$u_n^*(t) = C_n \exp\{-\gamma_n t\} + D_n \exp\{+\gamma_n t\} \tag{4.34}$$

where

$$\gamma_n = +\sqrt{1 + (nk\pi/b)^4}, \tag{4.35}$$

and C_n, D_n are scalars which may be computed as follows: Recalling that

$$z(t_1) = \int_{t_o}^{t_1} \Phi(t_1 - s) u(s) ds, \quad z(t_o) = 0$$

it follows that

$$\xi_n^1 = \int_{t_o}^{t_1} [\exp\{-(nk\pi/b)^2(t_1 - s)\}] [C_n \exp\{-\gamma_n s\} + D_n \exp\{+\gamma_n s\}] ds \tag{4.36}$$

where ξ_n^1, $n = 1,2,\ldots$, are the Fourier coefficients of the terminal state $x^1(\alpha)$ with respect to $\{\varphi_n = \sqrt{2/b}\sin(n\pi/b)\alpha\}$ (see Equation (4.11)). Also, by substituting Equation (4.34) in Equation (4.32) and noticing that

$$T_n(t_1) = \xi_n^1,$$

it follows that

$$\xi_n^1 = C_n[(nk\pi/b)^2 - \gamma_n]\exp\{-\gamma_n t_1\} + D_n[(nk\pi/b)^2 + \gamma_n]\exp\{+\gamma_n t_1\} \tag{4.37}$$

Equations (4.36) and (4.37), being linear independent equations in the unknowns C_n and D_n, can be easily solved, and thus the problem is completely solved.

It is interesting to notice that for sufficiently large n, Equation (4.35) reduces to the form

$$\gamma_n = (nk\pi/b)^2 ,$$

and thus the scalars C_n, D_n are given by

$$C_n = [t_1-t_o]^{-1} \exp\{(nk\pi/b)^2 t_1\} \xi_n^1 ,$$

$$D_n = [2(nk\pi/b)^2]^{-1} \exp\{-(nk\pi/b)^2 t_1\} \xi_n^1 .$$

Hence, it follows that for sufficiently large n, the function u_n is given by

$$u_n(t) = \Big[[t_1-t_o]^{-1}\exp\{(nk\pi/b)^2(t_1-t)\} + [2(nk\pi/b)^2]^{-1}\exp\{-(nk\pi/b)^2(t_1-t)\}\Big] \xi_n^1$$

for all $t \in [t_o,t_1]$.

4.1.4. A Necessary Condition for Optimality.

In this subsection a partial differential equation which should be satisfied by the optimal distributed input is derived for the case when $P = Q = I$.[1] This technique stems from the observation that the separation

[1] Note that the case when both P and Q are independent of t(i.e., $\bar{P}$ and $\bar{Q}$ are functions of α only) can be similarity treated.

of variables method associates with the partial differential equation an infinite system of ordinary differential equations. Therefore, if such an infinite system is given, it may be possible to find the partial differential equation which produces it.

It has been shown in subsection 4.1.3 that for the case when $P = Q = I$, the optimal control (vector-valued) function u^* satisfies the matrix first order ordinary differential equation

$$\dot{u}(t) = -A^* u(t) + z(t), \qquad t \in (t_0, t_1]. \qquad (4.38)$$

Noticing that the matrix equation

$$\dot{z}(t) = Az(t) + u(t)$$

results (see subsection 2.2.1) from the diffusion equation

$$\frac{\partial x(t,\alpha)}{\partial t} = + k^2 \frac{\partial^2 x(t,\alpha)}{\partial \alpha^2} + f(t,\alpha), \qquad (t,\alpha) \in \Delta \qquad (3.5)$$

it immediately follows that Equation (4.38) results from the backward diffusion equation

$$\frac{\partial f(t,\alpha)}{\partial t} = - k^2 \frac{\partial^2 f(t,\alpha)}{\partial \alpha^2} + x(t,\alpha) \qquad (t,\alpha) \in \Delta \qquad (4.39)$$

By differentiating Equation (4.39) with respect to t, and substituting Equation (3.5) in the resulting equation it follows that[1]

$$\frac{\partial^2 f(t,\alpha)}{\partial t^2} = -k^2 \frac{\partial^3 f(t,\alpha)}{\partial t \partial \alpha^2} + k^2 \frac{\partial^2 x(t,\alpha)}{\partial \alpha^2} + f(t,\alpha)$$

$$= -k^2 \frac{\partial^3 f(t,\alpha)}{\partial t \partial \alpha^2} + k^2 \frac{\partial^3 f(t,\alpha)}{\partial \alpha^2 \partial t} + k^4 \frac{\partial^4 f(t,\alpha)}{\partial \alpha^4} + f(t,\alpha)$$

---

[1] Here the differentiability of f with respect to t is assumed.

for all $(t,\alpha) \in \Delta$. Assuming that $\frac{\partial^3 f(t,\alpha)}{\partial t \partial \alpha^2} = \frac{\partial^3 f(t,\alpha)}{\partial \alpha^2 \partial t}$, it follow that the optimal control input function $f^*(t,\alpha)$ must satisfy the fourth order partial differential equation

$$k^4 \frac{\partial^4 f(t,\alpha)}{\partial \alpha^4} - \frac{\partial^2 f(t,\alpha)}{\partial t^2} + f(t,\alpha) = 0 , \qquad (t,\alpha) \in \Delta . \tag{4.40}$$

It can be easily verified that the function f^* given by

$$f^*(t,\alpha) = \sum_{n=1}^{\infty} u_n^*(t)\varphi_n(\alpha)$$

where $u_n^*(t)$ is given by Equation (3.34) , namely,

$$u_n^*(t) = C_n \exp\{-\gamma_n t\} + D_n \exp\{+\gamma_n t\} \tag{3.34}$$

does satisfy (formally) Equation (3.40), thus emphasizing the compatibility of both results.

This concludes the discussion of the generalized minimum energy problem stated at the opening of this section and attention is now directed to the study of another interesting optimization problem.

4.2. Second Generalized Problem - Controllers with Limited Energy

In many physical situations, it happens that the solution of the minimum energy problem requires a control function whose energy is beyond the capacity of the controller. In particular, if the terminal state z^1 is not an element of the range $R(F)$ of F , the controller needs infinite energy to bring the system to the required state. In such situations the question of how well the energy at hand may be utilized naturally arises and hence the motivation for the following optimization problem: For System I described by Equation (3.5), find the control element f of the Hilbert function space $\mathcal{F}$ (defined on page 40) which minimizes the functional

$$J(f) = \int_0^b [x^1(\alpha) - x_f(t_1,\alpha)]^2 d\alpha \qquad (4.41)$$

and satisfies the constraint

$$\int_{t_o}^{t_1} \int_0^b |f(t,\alpha)|^2 \, d\alpha dt \leq \kappa^2 \qquad (4.42)$$

where the function $x^1 \epsilon L_2(0,b)$ is a specified element (of the system state space), $x_f(t_1,\alpha)$ is the response of the system at the (specified) terminal time t_1 under the effect of the controlling input function f, and κ is a scalar.

Making use of the terminology of Section (3.2), this problem may be stated abstractly as follows: Find the element $u \epsilon U_\kappa = \{u{:}u \epsilon \bar{L}_2(t_o,t_1), ||u||^2 \leq \kappa^2\}$ which minimizes the functional[1]

$$J(u) = ||z^1 - F_1 u||^2_{\ell_2} .$$

Here, z^1 is the infinite tuplet of scalars $z^1 = (\xi_1^1,\ldots,\xi_n^1,\ldots)$ with ξ_n given by

$$\xi_n^1 = \langle x^1,\varphi_n \rangle_{L_2(0,b)} = \sqrt{2/b}\int_0^b x^1(\alpha)\sin(n\pi/b)\alpha d\alpha \qquad (4.11)$$

[1] In other words, U_κ is the sphere with radius κ in $\bar{L}_2(t_o,t_1)$.

and the transformation $F_1:\bar{L}_2(t_o,t_1) \to \ell_2$ is defined by Equation (3.17), namely

$$F_1u = \int_{t_o}^{t_1} \Phi(t_1-s)u(s)ds \qquad (3.17)$$

It is clear that this problem separates naturally into two cases depending on whether z^1 has a preimage under F_1 with norm less or equal to $|\kappa|$ or not. It is convenient to divide the discussion of the problem accordingly.

Case I.

Assume here that the set $Y_\kappa = F_1^{-1}[z^1] \cap U_\kappa$ is nonempty where $F_1^{-1}[z^1]$ is the set of all preimages of z^1 under F_1. Obviously, $F_1^\dagger z^1$ is an element in Y_κ[1]. If $||F_1^\dagger z^1|| = |\kappa|$, the problem reduces exactly to that of subsection 3.2.1. Hence, the optimal control function is the unique element $F_1^\dagger z^1$ in M (the orthogonal complement of the null space $N(F_1)$ of F_1), with J taking the zero value.

If $||F_1^\dagger z^1|| < |\kappa|$, then $|F_1^\dagger z^1|$, as a solution loses its uniqueness property. Indeed, let v be an element in the set V_κ defined by

$$V_\kappa = \{v : v \in N(F_1),\ ||v||^2 \leq \kappa^2 - ||F_1^\dagger z^1||^2\} ,$$

then

$$\begin{aligned} J(F_1^\dagger z^1 + v) &= ||z^1 - F_1(F_1^\dagger z^1 + v)||^2 \\ &= ||z^1 - F_1F_1^\dagger z^1|| \\ &= 0 , \end{aligned}$$

---

[1] Note that $F_1^\dagger z^1$ is the element with minimum norm in the set $F_1^{-1}[z^1]$.

and

$$||(F_1^\dagger z^1 + v)||^2 = ||F_1^\dagger z^1||^2 + ||v||^2 \qquad (F^\dagger z^1 \perp v)$$

$$\leq \kappa^2$$

It thus follows that $(F_1^\dagger z^1 + v)$, $v \in V_\kappa$ is also a solution with J having the zero value. Evidently, $F^\dagger z^1$, besides being the solution with minimum norm, is the projection on M of all other solutions.

Case II.

The case where all preimages of z^1, if they exist, have norms exceeding $|\kappa|$ is more interesting. Here, even if $z^1 \in R(F_1)$, the element $F_1^\dagger z^1$ is not a solution $(F_1^\dagger z^1 \notin U_\kappa)$. Indeed, from Case 1, it follows that $F_1^\dagger z^1$ is a solution if and only if $||F_1^\dagger z^1|| \leq |\kappa|$.

Let W_κ be the set in ℓ_2 defined by[1]

$$W_\kappa = \{z(t_1) : z(t_1) = F_1 u, \; ||u||^2 \leq \kappa^2\} .$$

In Appendix F, it is shown that F_1, being compact, has the polar representation

$$F_1 = \sum_{n=1}^{\infty} e_n > \mu_n < g_n$$

where e_n is the usual basis of ℓ_2, and μ_n, g_n are defined by

$$\mu_n^2 = 2[nk\pi/b]^{-2} [1 - \exp\{-2(nk\pi/b)^2 (t_1 - t_o)\}] ,$$

$$g_n = \mu_n^{-1} [\exp\{-(nk\pi/b)^2 (t_1 - t)\}] e_n .$$

---

[1] In other words, W_κ is the image of U_κ under F_1.

Accordingly, the set W_κ may be written as[2]:

$$W_\kappa = \{z(t_1) : z(t_1) = \sum_{n=1}^{\infty} e_n > \mu_n\gamma_n, \sum_{n=1}^{\infty} |\gamma_n|^2 \leq \kappa^2\}$$

where

$$\gamma_n = < u, g_n > = \int_{t_o}^{t_1} [u_n(t)][g_n(t)]dt$$

Also, the performance index $J(u)$ takes the form

$$J(u) = ||z^1 - F_1 u||^2_{\ell_2} = || \sum_{n=1}^{\infty} (\xi_n^1 - \mu_n\gamma_n)e_n||^2 = \sum_{n=1}^{\infty} (\xi_n^1 - \mu_n\gamma_n)^2$$

where ξ_n^1 is given by Equation (4.11).

From above, it is obvious that the optimization problem under consideration reduces to the following ordinary constraint minimization problem: From the tuplets $\gamma = (\gamma_1, \ldots, \gamma_n, \ldots)$ satisfying the condition $[\gamma, \gamma] \leq \kappa^2$, determine the element which minimizes the functional

$$J(\gamma) = [(z^1 - \Lambda\gamma), (z^1 - \Lambda\gamma)]$$

where [,] represents the usual inner product in ℓ_2, and Λ is the infinite diagonal matrix given by

$$\Lambda = \text{diag}[\mu_1, \ldots, \mu_n, \ldots] . \qquad (4.43)$$

The solution of this problem can be easily obtained through the application of the following theorem whose proof is given in [30, page 210].

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[2] Note that $\{g_n\}$ is an orthonormal basis for $\overline{L}_2(t_o, t_1)$ and thus

$$||u||^2 = || \sum_{n=1}^{\infty} \gamma_n g_n||^2 = \sum_{n=1}^{\infty} |\gamma_n|^2$$

Theorem.

Let J, h_i, $i=1,\ldots,r$ be given differentiable functionals defined on a normed linear space E . Let $y^* \in E$ be a local extremum for the functional J subject to the constraint.

$$h_i(y^*) = 0 , \qquad i=1,\ldots,r$$

then, it is necessary that there exist scalars $(\lambda_1,\ldots,\lambda_r)$ for which

$$J^*(y) = J(y) + \sum_{i=1}^{r} \lambda_i h_i(y) , \qquad y \in E$$

has an unconstrained extremum at y^* .[1]

It can be readily shown[2] that to solve the above problem it is sufficient to consider only the tuplets $[\gamma,\gamma] = \kappa^2$. Hence, according to the above theorem, the functional

$$J^*(\gamma) = [(z^1-\Lambda\gamma),(z^1-\Lambda\gamma)] + \lambda\,([\gamma,\gamma] - \kappa^2)$$

$$= \sum_{n=1}^{\infty} (\xi_n^1 - \mu_n\gamma_n)^2 + \lambda(\sum_{n=1}^{\infty} \gamma_n^2 - \kappa^2) \tag{4.44}$$

has an unconstrained extremum at γ^* , or in other words, the relations

$$\frac{\partial J^*}{\partial \gamma_n} = 0 \tag{4.45}$$

hold at $\gamma_n = \gamma_n^*$, $n = 1,2,\ldots$. Substitution of Equation (4.44) in

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[1] Note that this theorem is a generalization of that dealing with functions of finite number of variables (see [7, Section 76].

[2] For instance, the slack variables "technique (see [44, page 108]) may be used to reduce the inequality constraint to an equivalent equality form.

Equation (4.45) gives

$$-2\mu_n(\xi_n^1 - \mu_n\gamma_n^*) + 2\lambda\ \gamma_n^* = 0\ , \qquad n=1,2,\ldots$$

or equivalently,

$$\gamma_n^* = \mu_n \xi_n^1\ [\mu_n^2 + \lambda]^{-1}\ . \tag{4.46}$$

It thus follows that

$$[\gamma^*,\gamma^*] = \sum_{n=1}^{\infty}\ [\mu_n\ \xi_n^1]^2[\mu_n^2 + \lambda]^{-2}$$

$$= \sum_{n=1}^{\infty} \frac{2(nk\pi/b)^2[1-\exp\{-2(nk\pi/b)^2(t_1-t_o)\}|\xi_n^1|^2}{[2(1-\exp\{-2(nk\pi/b)^2(t_1-t_o)\}) + (nk\pi/b)^2\lambda]}$$

This infinite series is monotically decreasing with λ and such that

(1) $[\gamma^*,\gamma^*]\ (\lambda) \to 0$ as $\lambda \to \infty$, and

(2) $[\lambda^*,\lambda^*]\ (\lambda) \to \infty$ as $\lambda \to \lambda_o$ from the right

where

$$\lambda_o = -\ 2\ [k\pi/b]^2[1 - \exp\{ - 2(k\pi/b)^2(t_1-t_o)\}]\ .$$

It thus follows that a λ can always be found such that this infinite series converges to κ^2 . Substituting this value of λ in Equation (4.46), $\lambda_n, n = 1,2,\ldots,$ are computed and thus the optimal control function is given by

$$f^*(t,\alpha) = \sqrt{2/b}\ \sum_{n=1}^{\infty}\ \gamma_n^*[g_n(t)]\sin(n\pi/b)\alpha.\ t_o \leq t \leq t_1,\ 0 < \alpha < b.$$

The conclusion that f^* is indeed the optimal control element is a direct consequence of the fact that the set W_κ is a convex set in a Hilbert space and hence z^1 has <u>at most</u> one closest point in the set W_κ (see [40, section 4.2.]).

CHAPTER 5

APPROXIMATION TECHNIQUES

5.1. Introduction

It was shown in Chapter 3 that explicit results for the minimum energy control problem depend upon the computation of the inverse of the self-adjoint matrix operator $F_iF_i^*(i=1,2,s)$. Only $F_1F_1^*$ was fortunate enough to be diagonal and thus its inverse $(F_1F_1^*)^{-1}$ was easily determined. Since there is no technique available to perform such an inversion operation in closed form for the nondiagonal case, a suitable approximation technique must be used to determine the (possibly pointwise) inverse of the above operators. In other words, one must try to find the (possibly approximate) solution of the equation

$$F_iF_i^*y = z^1 , \qquad i=2,3, \qquad y,z^1 \in \ell_2 \tag{5.1}$$

where z^1 is the given terminal state.

In Chapter 4, similar difficulties arose. The (integral) operator equation

$$(P + \hat{F}_1^*Q\hat{F}_1)u = v \tag{4.14}$$

for some $v \in M$ was to be solved to determine the explicit solution of the first generalized problem for System I. Again such an equation is not solvable in a closed form and one has to resort to an approximation technique there by defining a sequence that converges to the required solution.

Equations (5.1) and (4.14) may be considered as specific forms of the following general problem: Let A be a linear self-adjoing positive definite[1] bounded[2] operator defined on the (dense) set D(A) of the Hilbert space H . Find the element $x^* \in D(A)$ which satisfies the equation

$$Ax = y \tag{5.2}$$

for given $y \in H$.

The positive-definiteness property of the operators under consideration is obvious from the following equalities

$$< F_i F_i^* x, x > = < F_i^* x, F_i^* x > = ||F_i^* x||^2$$

and

$$\begin{aligned} < (P + \hat{F}_i^* Q \hat{F}_i) y, y > &= < Py, y > + < \hat{F}_i^* Q \hat{F}_i y, y > \\ &= < Py, y > + < QF_i y, F_i y > \\ &> 0 \end{aligned}$$

the last inequality follows from the fact that both P and Q are positive definite symmetric matrices (see Section 4.1). These operators inherit boundedness from the boundedness of F_i .

[1] An operator A defined on a set contained in a Hilbert space H is said to be positive definite if the inequality
$$< Ax, x > > 0 , \qquad x \neq 0$$
holds for all x in the domain of A .

[2] The operator A is said to be bounded if there exists a scalar M such that $||Ax|| < M||x||$ for all $x \in D(A)$.

In Section 5.2, the application of the steepest descent method for solving Equation (5.2) is explored. Other techniques which may be used when A satisfies more restrictive conditions are discussed in Section 5.3 and 5.4. In these sections it is assumed that $y \in R(A)$, the range of A and thus a solution for Equation (5.2) always exists. Finally, Section 5.5 briefly discusses in a general way the various techniques used in approximating the mathematical models of distributed-parameter systems.

5.2. <u>The Steepest Descent Method</u>

The method of steepest descent in Hilbert spaces was originated by Kantrovich in 1948. Many practical problems such as the solution of algebraic, differential, integral and other types of minimization problems can be solved by this approximation technique. Recently, this method has been applied by a number of authors to solve problems in the optimal control area (see [5, 20, 29]).

The gist of this method is first to convert the problem under consideration to that of seeking a minimum and then to devise an iterative scheme for obtaining this minimum. More specifically, a functional $F(x)$ is constructed such that if there exists in $D(A)$, the domain of A , a function which gives a minimal value to $F(x)$, then this function is the solution of Equation (5.2). It can be shown (see [31, page 318]) that such a functional is given by the following quadractic form:

$$F(x) = \langle Ax,x \rangle - 2 \langle x,y \rangle . \tag{5.3}$$

The iterative scheme for obtaining the minimum of $F(x)$ is as follows: Take an arbitrary function x_o. If it happens that $Ax_o = y$, then the problem is solved. If $Ax_o \neq y$, let any point x in the neighborhood of x_o be denoted by

$$x = x_o + \epsilon z , \qquad z \in H$$

where ϵ is a scalar. In order to seek the minimum of Equation (5.3), both ϵ and z are chosen such that (1) $F(x_o + \epsilon z)$ is smaller than $F(x_o)$, and (2) the change between $F(x_o)$ and $F(x_o + \epsilon z)$ is maximum.[1] Following this line of thought, it can be shown (see [25]) that the first approximation for x_1 is determined by the formula

$$x_1 = x_o - \frac{\langle Lx_o, Lx_o \rangle}{\langle ALx_o, Lx_o \rangle} Lx_o .$$

where

$$L(x) \triangleq Ax - y .$$

The succeeding approximations are determined inductively by the analogous formula,

$$x_{n+1} = x_n - \frac{\langle Lx_n, Lx_n \rangle}{\langle ALx_n, Lx_n \rangle} Lx_n \qquad (n=0,1,2,\ldots) . \qquad (5.4)$$

The convergence of such a sequence to the solution of Equation (5.2) is guaranteed by the following theorem, the proof of which is given in [25].

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[1] This second condition is imposed to increase the rate of convergence of this iterative scheme.

<u>Theorem</u>. If a self-adjoint bounded operator A satisfies that

$$0 \leq \langle Ax,x \rangle \leq M||s||^2 , \tag{5.5}$$

where M is a scalar, and if Equation (5.2) has a solution x^* , then the successive approximations x_n determined by Equation (5.4) give a minimizing sequence, in the sense that

$$\lim_{n\to\infty} F(x_n) = F(x^*)$$

and

$$\lim_{n\to\infty} \langle A(x_n-x^*),(x_n-x^*) \rangle = 0$$

<u>Example</u>. Consider the infinite system of linear algebraic equations given by

$$\sum_{k=1}^{\infty} a_{ik}\xi_k = b_i \qquad (i=1,2,\ldots) \tag{5.6}$$

with the b_i satisfying the condition

$$\sum_{i=1}^{\infty} b_i^2 < \infty .$$

In the ℓ_2 space Equation (5.6) may be written as

$$Ax = y \tag{5.6}$$

where A is the matrix operator $[a_{ik}]$, $y = (b_1,b_2,\ldots)$, and $x = (\xi_1, \xi_2,\ldots)$. Assuming that the conditions of the above theorem are satisfied let the required solution be denoted by $x^* = (\xi_1^*,\xi_2^*,\ldots)$. Take an arbitrary element $x_o = (\xi_1^o, \xi_2^o, \dots)$ in ℓ_2 as the initial approximation to x^*. According to Equation (5.4), the second approximation, for example, must be taken as

$$x_1 = x_o + \epsilon_1 z_1 ,$$

where

$$z_1 = (\xi_1^{(1)}, \xi_2^{(1)}, \dots),$$

with

$$\xi_i^{(1)} = \left\{ \sum_{k=1}^{\infty} a_{ik} \xi_k^o - b_i \right\},$$

and

$$\epsilon_1 = -\left[\sum_{i=1}^{\infty} (\xi_i^{(1)})^2 \right] \left[\sum_{i,k} a_{ik} \xi_i^{(1)} \xi_k^{(1)} \right]^{-1} .$$

It may happen that the matrix A is such that the convergence of the series obtained during the determination of the successive approximations is slow. To improve such a situation, the equations of the infinite may be multiplied by suitable factors $\mu_i, i=1,2,\dots$.[1] Also, the choice of the initial value x_o is another factor which affects the rate of convergence.

It is worth noticing that the above theorem does not give any information about the speed of convergence of the approximating sequence

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[1] Note that the determination of $\{\mu_i\}$ may be, in itself, a difficult problem.

thus obtained. However, if the more restrictive condition that A be positive-bounded-below is imposed, i.e., the inequality

$$m||x||^2 \leq < Ax,x > \leq M||x||^2 \qquad (5.7)$$

with $M > m > 0$ holds for all $||x|| \in D(A)$, it can be shown (see [25])that the speed of convergence of the sequence (5.4) is given by the inequality

$$||x_n - x^*|| \leq [||Lx_o||/m][(M - m)/(M + m)]^n \, . \qquad (5.8)$$

To increase this rate of convergence, the above steepest descent method can be modified by combining several steps of the iteration into one (see [1]). Evidently, such a modification has a similar effect on the speed of convergence in the general case where A satisfies the inequality (5.5).

5.3. Ritz Method

It was mentioned in the previous section that the problem of finding the solution of the equation $Ax = y$ can be reduced to that of finding the minimum of the functional

$$F(x) = < Ax,x > - 2 < x,y > \qquad (5.3)$$

In Section 5.2 the steepest descent method was used to construct a minimizing sequence for $F(x)$ when A is a self-adjoint positive definite linear operator. The Ritz method discussed here (see [33]) is an alternative technique for obtaining such a minimizing sequence if A satisfies the following two conditions:

(1) A is self-adjoint, i.e., $A = A^*$

(2) A is positive-bounded-below, i.e., there exists a positive m such that

$$< Ax,x > \geq m||x|| , \qquad x \in D(A)$$

where $D(A)$, the domain of A , is assumed to be dense in a separable Hilbert space H .

The minimizing sequence of the functional $F(x)$ given by Equation (5.3) is constructed as follows: An auxiliary Hilbert space H_A with the scalar product

$$< u,v >_A = < Au,v > , \qquad u,v \in H_A \tag{5.9}$$

where $< , >$ denotes the scalar product of the original Hilbert space H , is defined to be the completion of $D(A)$ in the norm $||u||_A = < Au,u >^{1/2}$. A sequence $\{\varphi_n\}$ is chosen in $D(A)$ such that

(1) $\{\varphi_n\}$ is complete in H_A

(2) For any n , the functions $\varphi_n,\ldots,\varphi_n$ are linearly independent.

Let H_n be the finite-dimensional subspace generated by the elements $\varphi_1,\ldots,\varphi_n$. The approximate solution x_n is considered as the element $x \in H_n$ which minimizes the functional $F(x)$ in this subspace. Since all the elements $x \in H_n$ have the form

$$x = \sum_{i=1}^{n} a_i\varphi_i$$

the determination of x_n reduces to the algebraic problem of finding the n unknown coefficients a_i, $i=1,\ldots,n$. Following this line of thought, it is easy to see that the a_i are determined by the relation

$$\frac{\partial}{\partial a_j} F\left(\sum_{i=1}^{n} a_i\varphi_i\right) = 0 , \qquad j=1,\ldots,n . \tag{5.10}$$

Equation (5.10) can be reduced (see [33], Section 14) to the following system of n linear algebraic equations in the n unknowns a_i ,

$$\sum_{k=1}^{n} < A\varphi_i,\varphi_k > a_k = < y,\varphi_i > , \qquad i=1,\ldots,n . \tag{5.11}$$

Solving Equation (5.11) for a_i, $i=1,2,\ldots,n$, the approximate solution $x_n = \sum_{i=1}^{n} a_i\varphi_i$ becomes well-defined. It can be shown that the sequence $\{x_n\}$ thus obtained converges to the exact solution of Equation (5.2) if A satisfies the above-mentioned conditions (i.e., A is self-adjoint positive-bounded-below). Since the operator $(P + F_1^*QF_1)$ of Equation (4.14) satisfies these conditions, it is obvious that Ritz method can be applied to obtain the approximate solution of this equation.

The application of Ritz method in practice involves great difficulties in finding a complete system $\{\varphi_n\}$ in the space H_A . However, a sufficient condition for $\{\varphi_n\}$ to be complete in H_A is that the system $\{A\varphi_n\}$ be complete in H (see [1, Section 1]). In this case, Equation (5.11) may be replaced by

$$\sum_{k=1}^{n} < A\varphi_i,A\varphi_k > a_k = < y,A\varphi_i > , \qquad i=1,\ldots,n , \tag{5.12}$$

and an iteration process for the approximate solution is given by

$$x_1 = (\langle y, A\bar{\varphi}_1 \rangle / ||A\bar{\varphi}_1||^2)\bar{\varphi}_1 ,$$

$$x_{n+1} = x_n + (\langle y, A\bar{\varphi}_{n+1} \rangle / ||A\bar{\varphi}_{n+1}||^2)\bar{\varphi}_{n+1} , \tag{5.13}$$

where

$$\bar{\varphi}_1 = \varphi_1 ,$$

$$\bar{\varphi}_{n+1} = \varphi_{n+1} - \sum_{i=1}^{n} (\langle A\varphi_{n+1}, A\bar{\varphi}_i \rangle / ||A\bar{\varphi}_i||^2)\bar{\varphi}_i .$$

The extimate of error in this approximate solution is given by the recurrent formula

$$\delta_1 = ||y||^2 - [\langle y, A\bar{\varphi}_1 \rangle]^2 / ||A\bar{\varphi}_1||^2 ,$$

$$\delta_{n+1} = \delta_n - [\langle y, A\bar{\varphi}_{n+1} \rangle]^2 / ||A\bar{\varphi}_{n+1}||^2$$

where $\delta_n = ||y - Ax_n||$. According to this formula, it is not necessary to find all the approximate solutions in consecutive order. Instead, the estimates for the corresponding approximate solution is computed until an acceptable one is obtained. Only then, the corresponding approximate solution is computed.

5.4. The Bubnov-Galerkin Method

The Bubnov-Galerkin method (see [33]) can be considered as a generalization of the Ritz method for an equation of the form $Ax = y$ where A is a linear positive definite operator. Although the basic ideas of the two methods are radically different, they yield identical results when A is self-adjoint positive-bounded-below.

This method can be summarized as follows: Let A be an arbitrary linear operator defined on a dense set $D(A)$ contained in a separable Hilbert space H where it is required to solve the equation

$$Ax - y = 0 \tag{5.2}$$

To solve this problem, a set of elements $\{\varphi_n\}$, $\varphi_n \in D(A)$, which are linearly independent and complete are chosen. It follows, by definition of completeness, that the only element which is orthogonal to all φ_i is the null element. Therefore, the solution of Equation (5.2) can be thought of as the $x \in D(A)$ which forces the element $(Ax - y)$ to be orthogonal to all φ_i. The approximate solution is constructed by introducing the subspace H_n generated by the element $\varphi_1, \varphi_2, \ldots, \varphi_n$ and seeking the element $x \in H_n$ for which

$$(Ax-y) \perp \varphi_i \qquad i=1,\ldots,n .$$

where $\perp$ denotes the orthogonality condition. Since any element in H_n can be expressed as

$$x = \sum_{j=1}^{n} a_j \varphi_j , \qquad x \in H_n$$

this orthogonality condition can be written in terms of scalar products as

$$< A(\sum_{j=1}^{n} a_j \varphi_j) - y, \varphi_i > = 0 , \qquad i=1,\ldots,n ,$$

or equivalently,

$$< A \sum_{j=1}^{n} a_j\varphi_j, \varphi_i > = < y,\varphi_i > , \qquad i=1,\ldots,n$$

which is the same condition obtained by Ritz method (see Equation (5.11)).

Of great importance is the question of the convergence of the Bulnov-Galerkin method. This question has a long history since the method was proposed in 1915. In 1948, Mikhlin [34] gave a sufficient criterion for the convergence of the method when the operator A has the form

$$A = A_o + K \tag{5.14}$$

where A_o is a self-adjoint positive-bounded-below operator in the Hilbert space H and such that $D(A_o) \subset D(K)$. As in Ritz method the space H_o is introduced where the scalar product is given by

$$< u,v >_{H_o} = < A_o u,v > ,$$

and the sequence $\{\varphi_n\}$, $\varphi_n \in D(A_o)$ is chosen to be complete in the space H_o. Using these notations, the following theorem was proved.

Theorem: The approximate solutions of the equation

$$Ax = (A_o + K)x = y \tag{5.15}$$

constructed by the Bubnov-Galerkin method converge in the space H_o to the exact solution of this equation (i.e., $< A(x^* - x_n),(x^* - x_n) > \to 0$, where x^* is the exact solution) if the following conditions are satisfied:

(1) Equation (5.15) has not more than one solution in H_o,

(2) The operator $T = A_o^{-1}K$ is completely continuous in H_o.

It is obvious that the operator $(P + F^{*}QF)$ of the Equation (3.14) satisfies the above conditions. In fact, it is a self-adjoint positive-bounded-below operator and thus the approximate solutions constructed by this method converge (in the mean) to the exact solution.

5.5. Approximate Mathematical Models

Until very recently (see Section 1.2), the common approach to the solution of control problems of distributed parameter systems was to approximate the given mathematical model by a discretized one and then applying the existing theory for the lumped parameter systems to the approximate model. In this section, several forms of such approximation are discussed. In fact, all these forms stem from one basic idea; to replace the original partial differential equation (whose solution at any time t may be considered as being an element in an infinite-dimensional Hilbert space) by an equation whose solution at any time t is an element in a finite dimensional Hilbert space.

Spatial Discretization. This form of approximation is used for the analog computer solution of partial differential equations. Here, the discretized mathematical model consists of a finite-dimensional system of continuous-time ordinary differential equations. Such form of approximation is quite natural since the derivation of dynamic equations, for many distributed systems, usually, starts with this discrete form. As an example of this type, the minimum energy control problem is solved in Appendix G for the approximate model of System III given by Equation (3.7).

Time Discretization. Here the discretized model consists of a finite-dimensional system of spatially-continuous ordinary differential equations. This form of approximation may be used in discrete-time distributed parameter control systems where the spatial distribution of the physical variables are sampled in time.[1]

Space and Time Discretization. This is the form used in the digital computer solution of partial differential equations. Here, the discretized model consists of a finite-dimensional system of difference equations. Recently, this method has received extensive attemtion as a result of the rapidly increasing use of digital computers in the optimum control area. (see [46]).

Generally, there are numerous discretization schemes for any of the above three forms of approximation. Before choosing a specific scheme, one should make sure that the chosen scheme is consistent in the sense that the corresponding approximate equations approaches the original continuous equation (in some definite sense) as the spatial and/or time increments $\to 0$. Another fundamental associated problem is that of the convergence of the approximate solution to the exact one as the increments $\to 0$. These problems are considered out of scope of this work and are not given any further consideration.

Dyadic Approximation. For a large class of linear distributed parameter systems, the functional relation between the state of the

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[1] This type of approximation is not applicable to the diffusion equation (see [46]).

system $x(t,\alpha)$ and the input function $f(t,\alpha)$ takes the form[1]

$$x(t,\alpha) = \int_{t_0}^{t} \int_{\Omega} G(t,\alpha;t',\alpha')f(t'\alpha')d\alpha'dt \tag{5.16}$$

for all $t \in \tau = [t_0,t_1]$ and $\alpha \in \Omega$, where x and f are elements in the function Hilbert space $L_2(\tau \times \Omega)$ and $G(t,\alpha;t',\alpha')$ satisfies the conditions

$$G(t,\alpha;t',\alpha') = G(t',\alpha';t,\alpha)$$

and

$$\int_{\tau} \int_{\Omega} \int_{\tau'} \int_{\Omega'} G\,(t,\alpha;t',\alpha')d\alpha'dt'd\alpha dt < \infty$$

Under these conditions, it can be shown (see [28, Vol. 2, page 120]) that the operator A defined on $L_2(\tau \times \Omega)$ by

$$x = Af$$

where x and f are related by Equation (5.16), is compact (completely continuous) self-adjoint operator and thus has the dyadic representation (see Appendix F)

$$A = \sum_{n=1}^{\infty} \varphi_n > \lambda_n < \varphi_n; \qquad \lambda_n \to 0 ,$$

in the sense that

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[1] Here the boundary input forces and the initial state of the system are assumed to be zero to clarify the discussion; the general case is treated analogously.

$$\lim_{N\to\infty} \| A - \sum_{n=1}^{N} \varphi_n > \lambda_n < \varphi_n \| = 0 .$$

Here $\{\varphi_n\}$ is an orthonormal set of eigenvectors of A corresponding to the eigenvalues $\{\lambda_n\}$ and the ordering is taken according to decreasing λ. It thus follows that the operator A defined by Equation (5.17) can be approximated, to any degree of accuracy, by the operator A_N defined by

$$A_N = \sum_{n=1}^{N} \varphi_n > \lambda_n < \varphi_n$$

Physically, this type of approximation amounts to considering the system as being of a low-pass nature in the sense that the high frequency effects are neglected.

CHAPTER 6

CONCLUSIONS AND SUGGESTIONS FOR FUTURE RESEARCH

6.1. Concluding Summary

In this thesis, a solution technique is proposed for a class of optimal control problems in distributed-parameter systems. The method in question utilizes the fact that the (scalar-valued) multivariable Hilbert function spaces which arise naturally in describing the system response and input spaces are congruent to appropriate (vector-valued) univariable Hilbert function spaces. By using a suitable unitary mapping the original optimal control problem is transformed to an equivalent one in these congruent spaces. Such a transformation is obvious when the equations representing the dynamic behavior of the system are solvable by the well-known method of separation of variables, or more generally, when the Green's function of the system has the separable form $G(t,\alpha) = \sum_{n=1}^{\infty} T_n(t)\varphi_n(\alpha)$.

The principle advantage of this approach is that it provides a systematic technique to get explicit expressions for the solution of the optimization problem; a result of considerable importance to the practicing engineer.

In Chapter 3, this technique is illustrated by solving the minimum energy control problem for systems described by the diffusion equation. Here, the fact that the theorems of Section 2.2. do not depend on the rank of the considered transformation is used. However, since the ranges are dense rather than closed (or onto), these theorems are applied in the sense of subsection (2.2.2.). The complicated case where the input control function is composed of all the three possible kinds of control (i.e., distributed, spatially discreted, and boundary forces) was solved. The solution is based on two fundamental ideas: the superposition principle of linear systems and the notion of cartesian products of Hilbert spaces. The applicability of the proposed technique to systems with separable Green's function was illustrated by solving the minimum energy control problem of Section 3.3. The chapter was concluded by considering the synthesis problem of feedback loops and some general expressions were derived (see Equations (3.67) and (3.69)) which include, as a special case, the results obtained by Wang using the technique of dynamic programming (see [54, Equation 5.48]).

In Chapter 4, two generalized minimum energy problems were considered. The performance index of the first problem is given by Equation (4.1), namely,

$$J(f) = \frac{1}{2}\int_{t_o}^{t_1}\int_{o}^{b} \bar{P}(t,\alpha)\ |\ f(t,\alpha)\ |^2\ d\alpha dt + \frac{1}{2}\int_{to}^{t_1}\int_{o}^{b} \bar{Q}(t,\alpha)\ |x_f(t,\alpha)\ |^2\ d\alpha dt.$$

Through the use of the notion of the graph of a transformation, the optimal control input f^* for this problem was computed by two different methods. Also, a necessary condition of optimality was derived in subsection 4.1.4. The second problem deals with controllers with limited energy which minimize the performance index

$$J(f) = \int_{o}^{b} [x^1(\alpha) - x_f(t_1,\alpha)]^2\ d\alpha\ , \qquad (4.41)$$

and satisfy the constraint

$$\int_{t_o}^{t_1}\int_{o}^{b} |f(t,\alpha)|^2\ d\alpha dt \leq \kappa^2\ . \qquad (4.42)$$

This problem was solved through the application of a theorem generalizing the Lagrange multiplier rule to functions with infinite number of variables.

In the course of obtaining explicit expressions for the solution of the above mentioned problems, the computation of the inverse of certain linear (matrix or integral) operators is required. In Chapter 5, several applicable approximation techniques were discussed showing that an approximate solution can be computed to any desired degree of accuracy.

6.2. Suggestions for Future Research

In this thesis, the developed technique was successfully applied to the solution of the minimum energy control problem of linear distributed parameter systems with fixed domain, i.e., the spatial domain boundaries remian time-invariant with respect to a given coordinate system. Other types of systems to which the applicability of the proposed technique is worth investigation are the following:

(1) Variable-Domain Systems: In these systems, the domain boundaries vary with time. The boundary motion may either be a specified function of time or depend on certain variables defined over the entire domain or subsets of the domain. An example of such a system is the process of heating a solid substance beyond its melting point. Also, re-entry vehicles with ablative surfaces (see Example 2 of Section 1.1.) is another example of a system where the domain boundary motion depends on certain system variables evaluated at the boundary.

(2) Composite Systems: Here the distributed parameter system is coupled to a lumped parameter system. The spatial domain of the distributed

parameter system may be either fixed or variable. A familiar example to the electrical engineers is that of a transmission line with a load at the receiving end. Another important example is a transport system consisting of a fluid-actuated, free, rigid carrier enclosed in a cylinder (see [55]).

(3) Banach function spaces: In this thesis, the study is restricted to the case where the function spaces involved are Hilbert spaces. However, the theorems of Section 2.2. have been generalized for the Banach spaces case (see [37], [39]). The application of these theorems to distributed parameter systems in such spaces deserves future study.

(4) Aside from the optimal control problem, the proposed technique seems to be applicable to the study of several aspects of the general control problem associated with linear distributed parameter systems such as sensitivity, controllability and observability.

APPENDIX A

STURM-LIOUVILLE PROBLEMS

It was shown in Section 2.1 that the separation of variables method for the solution of partial differential equations leads to a boundary value problem (see Equations (2.6) and (2.7)). An important class of boundary value problems arising in mathematical physics is that of Sturm-Liouville. To display the important aspects and properties of solutions of such problems, an ordinary second order differential equation with boundary conditions at two points is considered.

Let $\mathcal{L}$ be a second order differential operator defined by

$$(\mathcal{L}\, z)(s) = \frac{d}{ds}\left[p(s)\frac{dz(s)}{ds}\right] + q(s)z(s), \qquad s \in [a,b] \tag{1}$$

where p, $\frac{dp}{ds}$ and q are real valued continuous functions on $[a,b]$, and $p(s) > 0$ there. A boundary value problem of the Sturm-Liouville type is to find non-trivial solutions to the equations

$$(\mathcal{L}\, z)(s) = \lambda\, r(s)z(s)\ , \qquad s \in [a,b] \tag{2}$$

satisfying the boundary conditions

$$A_1 z(a) + A_2 \left.\frac{dz}{ds}\right|_{s=a} = 0\ , \tag{3}$$

$$B_1 z(b) + B_2 \left.\frac{dz}{ds}\right|_{s=b} = 0\ . \tag{4}$$

Here, $r(s)$ is a real valued continuous function on $[a,b]$, λ is a parameter independent of s and A_1, A_2, B_1, B_2 are constants such that

$$|A_1| + |A_2| \neq 0 , \quad |B_1| + |B_2| \neq 0 .$$

It can be shown (see [42, Section 12.1]) that the existence of non-trivial solutions of a Sturm-Liouville problem depends upon the value of the parameter λ in the differential equation of the problem. The values of the parameter λ for which there exist non-trivial solutions of the problem are called the eigenvalues (or characteristic values) of the problem. The corresponding non-trivial solutions themselves are called the eigenvectors (or the eigenfunctions) of the problem. The following theorem summarizes the properties of solutions to the above Sturm-Liouville problem.

Theorem: Consider the Sturm-Liouville problem given by Equations (1) to (3) above.

(1) There exists an infinite number of eigenvalues λ_n of the given problem. These eigenvalues can be arranged in a monotomic increasing sequence

$$\lambda_1 < \lambda_2 < \lambda_3 < \dots$$

such that $\lambda_n \to + \infty$. All these eigenvalues will be non-negative if the function $q(s)$ of Equation (1) is non-positive on the interval $[a,b]$.

(2) Corresponding to each eigenvalue λ_n there exists a one-parameter family of eigenfunctions φ_n . Each of these eigenfunctions is defined on $[a,b]$ and any two eigenfunctions corresponding to the same eigenvalue are nonzero constant multiples of each other.

(3) Each eigenfunction φ_n corresponding to the eigenvalue λ_n (n=1,2,...) has exactly (n-1) zeros in the open interval $a < s < b$.

(4) Any two eigenfunctions φ_m , φ_n corresponding to two different eigenvalues λ_m , λ_n are orthogonal with respect to the weight function r on the interval $a \leq s \leq b$, i.e.

$$\int_a^b \varphi_m(s)\varphi_n(s)r(s)ds = 0 \qquad \text{for all } m \neq n .$$

(5) Any set $\{\varphi_n\}$ of eigenfunctions[1] corresponding to the set $\{\lambda_n\}$ of eigenvalues can be normalized to form a complete orthonormal basis $\{e_n\}$ for the $L_2(a,b)$. In the present setting, the inner product is defined by

$$< x,y >_r = \int_0^b x(s)y(s)r(s)ds .$$

The Sturm-Liouville problem can also be studied through considering the inverse of the operator $\mathcal{L}$ and thus applying the results of the integral operators theory. It can be shown (see [21, section 6.4]) that if the boundary conditions satisfy consistency relations then a Green's function $G(t,s)$ exists such that whenever the relation

$$z(t) = \lambda \int_a^b G(t,s)r(s)z(s)ds \qquad t \in [a,b] \qquad (5)$$

holds, z also satisfies Equations (2) to (4) . The reader is referred to the above reference for the discussion of the properties of this

[1] It is obvious that in the set $\{\varphi_n\}$ no two elements are allowed to correspond to the same eigenvalue.

Green's function. It suffices here to mention that $G(t,s)$ is uniformly continuous in both variables and that $G(t,s) = G(s,t)$. Therefore, the linear operator $y \to z$ of Equation (5) is self-adjoint and compact on the space $L_2(a,b)$; a result which has quite gratifying consequences.

APPENDIX B

PROOF THAT $R(F_i)$, i=1,2,3, IS DENSE IN ℓ_2

Consider first the (bounded) linear transformation F_1 defined by Equation (3.17). It is shown in subsection 3.2.1 that $F_1F_1^*$ is given by Equation (3.21), namely,

$$F_1F_1^* = \mathrm{diag}[\gamma_1,\ldots,\gamma_n,\ldots] \tag{3.21}$$

where

$$\gamma_n = [2(nk\pi/b)^2]^{-1}[1-\exp\{-2(nk\pi/b)^2(t_1-t_o)\}] \;.$$

Let $\{e_n\}$ be the usual basis of the Hilbert space ℓ_2 . By direct substitution in Equation (3.21), it is clear that $F_1F_1^*e_n = \gamma_n e_n$, n = 1,2,..., and thus $R(F_1F_1^*)$ is dense in ℓ_2 . Since $R(F_1)\supseteq R(F_1F_1^*)$, it immediately follows that $R(F_1)$ is also dense in ℓ_2 .

Such a direct proof is not applicable to F_2 and F_3 due to the fact that both $F_2F_2^*$ and $F_3F_3^*$ are nondiagonal (see Equations (3.27) and (3.35)). However, the dense property of $R(F_2)$ and $R(F_3)$ can be proved by considering the transformations $\overline{F}_2$: $[L_2(t_o,t_1)]^m \rightarrow L_2(o,b)$, and $\overline{F}_3$: $[L_2(t_o,t_1)]^2 \rightarrow L_2(o,b)$ defined by

$$F_i = U\overline{F}_i \;, \qquad i = 2,3$$

where U is the Fourier transformation from $L_2(o,b)$ onto ℓ_2 . Since U is an isometry, it follows that $R(F_i)$ is dense in ℓ_2 if and only if $R(\overline{F}_i)$ is dense in $L_2(o,b)$, or equivalently, if and only if the orthogonal complement of the closure of $R(\overline{F}_i)$, $[R(\overline{F})]$ is vacuous.

As an illustration of the method of proof, System III (see Equation (3.7)) with $x^o(\alpha) = h_1(t) = 0$ is considered. In this case, it can be shown (see [18, page 230]) that the state of the system, $x(t,\alpha)$, may be written as

$$x(t,\alpha) = \int_{t_o}^{t} h_2(\tau)G(\alpha,t-\tau)d\tau$$

where

$$G(\alpha,t-\tau) = \sum_{n=1}^{\infty} 2\pi(-1)^{n+1} n \exp\{-(nk\pi/b)^2(t-\tau)\} \sin(n\pi/b)\alpha$$

for all $t > t_o$. Let $\bar{F}_3' : L_2(t_o,t_1) \to L_2(o,b)$ be defined by

$$(\bar{F}_3' h_2)(\alpha) = x(t_1,\alpha) = \int_{t_o}^{t_1} h_2(\tau)G(\alpha,t_1-\tau)d\tau .$$

Suppose that $R(\bar{F}_3')$ is not dense in $L_2(o,b)$. Let v be an element in the orthogonal complement of the closure of $R(\bar{F}_3')$, i.e.,

$$v \in [R(\bar{F}_3')] .$$

It thus follows that

$$< \bar{F}_3' h_2 , v >_{L_2(o,b)} = 0 , \quad \text{for all } \bar{F}_3' h_2 ,$$

or equivalently,

$$\int_o^b v(\alpha)d\alpha \int_{t_o}^{t_1} h_2(\tau)G(\alpha,t_1-\tau)d\tau = 0 , \quad \text{for all } h_2 \in L_2(t_o,t_1) .$$

By interchanging the order of integration, the above condition may be written as

$$\int_{t_o}^{t_1} h_2(\tau)\, d\tau \int_0^b v(\alpha)G(\alpha,t_1-\tau)d\alpha=0 , \quad \text{for all } h_2 \in L_2 (t_o,t_1) ,$$

and thus

$$\int_0^b v(\alpha)G(\alpha,t_1-\tau)d\alpha = 0 , \qquad \text{for all } \tau \in (t_o,t_1) .$$

It can be shown (see [18, Section 8.3]) that G is continuous for all $t > t_o$ and $0 \leq \alpha \leq b$ if it is defined to vanish on the line $t > t_o$ and $\alpha = b$. Therefore,

$$\int_0^b v(\alpha)G(\alpha,t_1-\tau)d\alpha = 2\pi \sum_{n=1}^{\infty} (-1)^{n+1} n\exp\{-(nk\pi/b)^2(t_1-\tau) \int_0^b v(\alpha)\sin(n\pi/b)\alpha d\alpha$$

$$= 2\pi \sum_{n=1}^{\infty} (-1)^{n+1} n\exp\{-(nk\pi/b)^2(t_1-\tau)\}\eta_n$$

where $\qquad = 0 , \qquad \tau \in (t_o,t_1)$.

$$\eta_n = \int_0^b v(\alpha)\sin(n\pi/b)\alpha d\alpha , \quad n = 1,2,\ldots .$$

The above identity implies that $\eta_n = 0$, $n = 1,2,\ldots$. Indeed, let $z = \exp -(k\pi/b)^2(t_1-\tau)$ and consider the function

$$\varphi(z) = \sum_{n=1}^{\infty} (-1)^{n+1}\, n\, \eta_n z^{n^2} .$$

It is clear that (z) is analytic function in the disc $|z|$ 1 According to the above identity, $(z) \equiv 0$ on the interval $(\exp\{-(k\pi/b)^2 (t_1-t)\} , 1)$ (0.1) . It thus follows (see [26 , Theorem B, page 522]) that

$$\varphi(z) = 0 \text{ in } |z| < 1 .$$

Therefore, every coefficient of the infinite series $\varphi(z)$ is identically zero (see [26 , page 355]), i.e.,

$$\eta_n = 0 \quad , \qquad n = 1,2,\ldots$$

which implies that $v = 0$ and thus $R(\overline{F}_3')$ is dense in $L_2(o,b)$.[1]

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[1] The author is indebted to Professor H. W. Hedstrom for the method of this proof.

APPENDIX C

EXISTENCE OF SOLUTION TO THE DIFFUSION EQUATION FOR EVERY $f \in \mathcal{F}$

Consider the inhomogeneous diffusion equation

$$\frac{\partial x(t,\alpha)}{\partial t} = k^2 \frac{\partial^2 x(t,\alpha)}{\partial \alpha^2} + f(t,\alpha) , \qquad t_o \leq t \leq t_1 , \ o < \alpha < b$$

with the auxiliary conditions

$$x(t_o,\alpha) = x(t,o) = x(t,b) = 0$$

In reference [19, Section 5.2] a collection, W_M^{Ω}, of function spaces is introduced. Each space in this collection consists of entire functions on a suitable domain. It is shown (see [19, Section 7.11]) that a (generalized) solution exists to the above equation for every $f \in \Phi^*$ where Φ^* is the adjoint of a member Φ of W_M^{Ω}.

To show that $\mathcal{F} \subset \Phi^*$, it suffices to show that $\Phi \subset \mathcal{F}$. Since every $y \in \Phi$ is entire, it is both bounded and measurable on the domain $[t_o,t_1] \times [o,b]$ and hence

$$\int_{t_o}^{t_1} \int_{o}^{b} |y(t,\alpha)|^2 dt d\alpha < \infty$$

holds which implies that every $y \in \Phi$ is an element in $\mathcal{F}$. Noticing also that the $\mathcal{F}$-norm of any $y \in \Phi$ is small whenever its Φ-norm is small, it follows that $\Phi \subset \mathcal{F}$ and thus $\mathcal{F}^* = \mathcal{F} \subset \Phi^*$.

APPENDIX D

PROOF OF EQUATION (3.55)

The system to be considered is that described by Equations (3.48) to (3.52) which, for convenience, are repeated here:

$$\frac{\partial x(t,\alpha)}{\partial t} = k^2 \frac{\partial^2 x(t,\alpha)}{\partial \alpha^2} , \qquad 0 < t \leq t_1 , \; 0 < \alpha < b \tag{3.48}$$

$$x(0,\alpha) = 0 , \qquad 0 \leq \alpha \leq b , \tag{3.49}$$

$$\frac{\partial x(t,0)}{\partial \alpha} - \gamma^2 x(t,0) = -\gamma^2 v(t) , \quad 0 \leq t \leq t_1 \tag{3.50}$$

$$\frac{\partial x(t,b)}{\partial \alpha} = 0 \qquad 0 \leq t \leq t_1 \tag{3.51}$$

$$\frac{dv(t)}{dt} + \sigma^2 v(t) = q(t) . \tag{3.52}$$

The Laplace Transform with respect to t of the above system of equations is given by

$$\frac{d^2 X(s,\alpha)}{d\alpha^2} = \frac{s}{k^2} X(s,\alpha) , \tag{1}$$

$$\frac{dX(s,0)}{d\alpha} - \gamma^2 X(s,0) = -\gamma^2 V(s) , \tag{2}$$

$$\frac{dX(s,b)}{d\alpha} = 0 , \tag{3}$$

$$[s + \sigma^2] \, V(s) = Q(s) \qquad (\text{assuming } v(0) = 0) \tag{4}$$

where

$$X(s,\alpha) = \mathcal{L}_t\{x(t,\alpha)\}$$
$$V(s) = \mathcal{L}\{v(t)\}$$
$$Q(s) = \mathcal{L}\{q(t)\}$$

The general solution of Equation (1) is given by

$$X(s,\alpha) = A_1(s)\sinh(\sqrt{s}/k)\alpha + A_2(s)\cosh(\sqrt{s}/k)\alpha \ . \qquad (5)$$

Using the boundary conditions (2) and (3), it is easy to show that

$$A_1(s) = [-\gamma^2 V(s)\sinh(\sqrt{s}/k)b][(\sqrt{s}/k)\sinh(\sqrt{s}/k)b + \gamma^2\cosh(\sqrt{s}/k)b]^{-1} \ ,$$

and

$$A_2(s) = [\gamma^2 V(s)\cosh(\sqrt{s}/k)b][(\sqrt{s}/k)\sinh(\sqrt{s}/k)b + \gamma^2\cosh(\sqrt{s}/k)b]^{-1}$$

Defining $H(s)$ as

$$H(s) = \gamma^2[(\sqrt{s}/k)\sinh(\sqrt{s}/k)b + \gamma^2\cosh(\sqrt{s}/k)b]^{-1} \ ,$$

it follows by substitution in Equation (5) that

$$X(s,\alpha) = H(s)[\cosh\{(\sqrt{s}/k)b\}\cosh(\sqrt{s}/k)\alpha - \sinh\{(\sqrt{s}/k)b\}\sinh(\sqrt{s}/k)\alpha]V(s)$$

$$= H(s)[\cosh(\sqrt{s}/k)(1 - \alpha/b)b]V(s)$$

$$= [\cosh(b/k)(1 - \alpha/b)\sqrt{s}][s + \sigma^2]^{-1} H(s)Q(s) \ .$$

Putting

$$g(t,\alpha) = \mathcal{L}^{-1}\{X(s,\alpha)/Q(s)\} \ ,$$

it follows by the convolution theorem that

$$x(t,\alpha) = \int_0^t g(t - \tau,\alpha)u(t)dt , \quad 0 < t \leq t_1, \ a \leq \alpha \leq b$$

It is clear that $G(s,\alpha) = \mathcal{L}\{g(t,\alpha)\}$ has simple poles at

$$s_0 = - \sigma^2 ,$$

$$s_i = - [(k/b)\beta_i]^2 , \qquad i=1,2,\ldots$$

where β_i are the (real) roots of the equation

$$\beta \tan \beta = b\gamma^2 .$$

Therefore,

$$g(t,\alpha) = \mathcal{L}^{-1}\{G(s,\alpha)\} = \mathcal{L}^{-1}\{N(s,\alpha)/D(s,\alpha)\}$$

$$= \sum_{i=0}^{\infty} [N(s_i,\alpha)/D'(s_i)]e^{s_i t} , \qquad (6)$$

where

$$N(s,\alpha) = \cosh(b/k)(1 - \alpha/b) \sqrt{s} , \qquad (7)$$

$$D(s) = [s + \sigma^2][(\sqrt{s}/k\gamma^2)\sinh(\sqrt{s}/k)b + \cosh(\sqrt{s}/k)b],$$

and $D'(s)$ is the derivative of $D(s)$ with respect to s , i.e.,

$$D'(s) = [(\sqrt{s}/\gamma^2 k)\sinh(b/k) \sqrt{s} + \cosh(b/k) \sqrt{s}]$$

$$+[s+\sigma^2][\{1/(2k \sqrt{s})\}\{\gamma^{-2}+b\}\sinh(b/k) \sqrt{s} + (b/2k^2\gamma^2)\cosh(b/k) \sqrt{s}] ,$$

It thus follows that

$$D'(s_o) = -(\sigma/\gamma^2 k)\sin(b/k)\sigma + \cos(b/k)\sigma \ , \qquad (8)$$

and

$$D'(s_i) = (b^2/2k^2)[\sigma^2-(k^2/b^2)\beta_i{}^2][b/\gamma^2 + (1+b\gamma^2)/\beta_i{}^2]\cos\beta_i . \qquad (9)$$

for $i=1,2,\ldots$. Substituting Equations (7), (8) and (9) in Equation (6) gives

$$g(t,\alpha) = \frac{\cos[(b/k)(1-\alpha/b)\sigma]}{\cos(b/k)\sigma - \sigma/k\gamma^2} \exp\{-\sigma^2 t\}$$

$$+ 2(k/b)^2 \sum_{i=1}^{\infty} \frac{\exp\{-(k\beta_i/b)t\} \ [\cos(1-\alpha/b)\beta_i]}{[\sigma^2 -(k^2/b^2)\beta_i{}^2][(b/\gamma^2)+(1+b\gamma^2)/\beta_i{}^2]\cos\beta_i}$$

which is the required result.

APPENDIX E

EVALUATION OF THE INTEGRAL

$$\int_0^b \cos^2 \{\beta_n(1-\alpha/b)\} \, d\alpha \, .$$

Let

$$S_n(\alpha) = \cos \{\beta_n(1-\alpha/b)\} = \cos \{\lambda_n(b-\alpha)\}$$

where

$$\lambda_n = (\beta_n/b) \, .$$

Differentiating $S_n(\alpha)$ twice with respect to α gives

$$S_n''(\alpha) = -\lambda_n^2 \, S_n(\alpha)$$

which implies that

$$\lambda_n^2 \, S_n^2 = - \, S_n \, S_n'' \tag{1}$$

Integrating Equation (1) from 0 to b gives

$$\lambda_n^2 \int_0^b S_n^2(\alpha) \, d\alpha = - \, [S_n(\alpha)S_n'(\alpha)]_0^b + \int_0^b \left|S_n'(\alpha)\right|^2 d\alpha \tag{2}$$

Noticing that

$$\lambda_n^2 |S_n(\alpha)|^2 + |S_n'(\alpha)|^2 = \lambda_n^2 \, ,$$

it follows, by integrating between o and b , that

$$\lambda_n^2 \int_0^b |S_n(\alpha)|^2 d\alpha + \int_0^b |S'(\alpha)|^2 d\alpha = \lambda_n^2 b \, ,$$

or equivalently,

$$\int_0^b |S_n'(\alpha)|^2 d\alpha = \lambda_n^2 b - \lambda_n^2 \int_0^b |S_n(\alpha)|^2 d\alpha . \qquad (3)$$

Substitution of Equation (3) in Equation (2) gives

$$2\lambda_n^2 \int_0^b |S_n(\alpha)|^2 d\alpha = -[S_n(\alpha)S_n'(\alpha)]_0^b + \lambda_n^2 b$$

Therefore,

$$\int_0^b |S_n(\alpha)|^2 d\alpha = (1/2\lambda_n^2)[\lambda_n^2 b + \lambda_n \sin \lambda_n b \cos \lambda_n b]$$

$$= (b/2)[1 + (\sin 2\beta_n)/2\beta_n]$$

which is the required result.

APPENDIX F

DYADIC REPRESENTATION OF LINEAR TRANSFORMATIONS

A useful form of representing linear operators is that using the dyadic notation. Here the bracket $>$ is attached to vectors while the bracket $<$ is attached to functionals. Thus, $< f,u >$ reads: the functional $< f$ acting on the vector $u >$.

Let X and Y be normed linear spaces, and let the sets E and F be defined as

$$E = \{e_1 >, \ldots, e_n >\} \subset Y ,$$
$$F = \{< f_1, \ldots, < f_n\} \subset X^*$$

where X^* is the conjugate space of X. The transformation A written as

$$A = \sum_{i=1}^{n} e_i > f_i$$

and defined by

$$Ax = \sum_{i=1}^{n} e_i >< f_i , x > , \qquad x \in X$$

is called an n^{th} order dyad. It is clear that A is linear and bounded if the functionals $< f_1, \ldots, < f_n$ are bounded, and the range of A is the linear manifold spanned by $e_1 >, \ldots, e_n >$. If the sets $\{e_i\}_1^n$ and $\{f_i\}_1^n$ are both linearly independent, the transformation A above is called P-normal. The following theorem (which will be generalized to the infinite dimensional cases) proves to be useful in studying linear physical systems.

Theorem . (see [37, page 29]) Let T denote a linear transformation $T:H \rightarrow R^n$ where H is a separable Hilbert space and R^n is the n-dimensional Euclidean space. Then there exist orthonormal sets $\{e_i\}_1^n \subset R^n$ and $G = \{g_i\}_1^n \subset H$ and non-negative real scalars $\{\mu_i\}_1^n$ such that

$$T = \sum_{i=1}^{n} e_i > \mu_i < g_i \tag{1}$$

Proof. Assume that an expansion for T of the prescribed form exists. A simple computation shows that the transformation conjugate to T is given by

$$T^* = \sum_{j=1}^{n} g_j > \mu_j < e_j \tag{2}$$

Using the orthonormality of the set G, it follows directly from Equations (1) and (2) that

$$TT^* = \sum_{i=1}^{n} e_i > \mu_i^2 < e_i \tag{3}$$

It is clear from Equation (3) that the scalars $\{\mu_i^2\}_1^n$ must constitute the spectrum of TT^* and the elements e_i are eigenvectors of TT^*. It is thus clear how the vectors e_i, g_i and the scalars μ_i should be chosen. Indeed, since TT^* is a non-negative definite self-adjoint operator on R^n, it has non-negative eigenvalues $\lambda_1, \ldots, \lambda_n$ (each counted according to its multiplicity) with corresponding eigenvectors $e_1, e_2, \ldots, e_n$. If the λ's are subscripted such that the nonzero ones occur first, say $\lambda_j > 0$ for $j=1,2,\ldots,k$ and $\lambda_j = 0$ for $j=k+1,\ldots,n$, then the sequence $\{g_i\}_1^n$ is defined as follows

$$g_i = (1/\lambda_i)^{1/2} T^* e_i , \qquad 1 \le i \le k , \tag{4}$$

$g_{k+\ell}$ is any unit vector orthogonal to $g_1, \ldots, g_k, \ldots, g_{k+\ell-1}$; $1 \le \ell \le (n-k)$. It is easy to check that the sequence $\{g_i\}_1^n$ defined by Equation (4) above is an orthonormal set of vectors and that Equation (1) holds for $\mu_i = (\lambda_i)^{1/2}$.

This canonical form for T is unique when the spectrum of TT^* is distinct. When multiple values occur, a non-uniqueness exists in the choice of basis within the eigenspace. It is also noteworthy that the computation of the canonical decomposition takes place in the simplest function space involved, namely R^n . Indeed, TT^* is a matrix and the elements $\{e_i\}_1^n$ are n-tuplets. The Hilbert space vectors $\{g_i\}_1^n$ are computed in terms of the e_i as indicated in the preceding proof.

The following theorem generalizes the above canonical decomposition to the class of linear bounded compact[1] transformations which are often met in the study of distributed parameter systems.

Theorem II. If $T:H_1 \to H_2$ is a bounded linear transformation from one Hilbert space into another, and if T is compact, then there exist positive scalars μ_n and a pair of orthonormal sequences $\{e_n\}, \{g_n\}$ such that

$$T = \sum_1^\infty e_n > \mu_n < g_n \tag{5}$$

---

[1] A transformation $A:H_1 \to H_2$ from one Hilbert space to another is called compact (or completely continuous) if it maps every bounded set of H_1 into a compact set in H_2 .

in the sense that

$$\lim_{N\to\infty} ||T - \sum_{1}^{N} e_n > \mu_n < g_n|| = 0 .$$

Proof. It can be shown (see [41, page 286]) that T may be written as

$$T = US$$

where $S = (T^*T)^{1/2}$ is a positive self-adjoint operator on H_1, and U is a bounded linear transformation from H_1 into H_2 characterized by

$$Uv = Tu \quad \text{if} \quad v = Su ,$$
$$Uv = 0 \quad \text{if} \quad v \perp R(S)$$

where $\perp$ denotes the orthonogonality condition and $R(S)$ is the range of S. Moreover, it follows from the compactness property of T that S is also compact. By the spectral theorem,

$$Su = \sum \lambda_n < u,e_n > e_n$$

where the λ_n are the nonzero eigenvalues of S counted according to multiplicity and $\{e_n\}$ is a corresponding sequence of eigenvectors. Since $Se_n = \lambda_n e_n$ and $\lambda_n > 0$, it follows that the e_n belong to $R(S)$ and therefore $g_n = Ue_n$ is an orthonormal sequence of vector:

$$< g_n,g_m > = < Ue_n,Ue_m >$$

$$= < T(\lambda_n^{-1} e_n),\ T(\lambda_m^{-1} e_m) >$$

$$= \lambda_n^{-1} \lambda_m^{-1} < e_n,\ T^* T e_m >$$

$$= \lambda_n^{-1} \lambda_m^{-1} < e_n,\ S^2 e_m >$$

$$= \lambda_n^{-1} \lambda_m^{-1} < e_n,\ \lambda_m^2 e_m > = \delta_{mn} .$$

It thus follows that

$$Tu = USu = U \sum \lambda_n < u, e_n > e_n = \sum \lambda_n < u, e_n > g_n$$

which is the desired decomposition.

Example. Consider the linear bounded transformation F_1 defined by Equation (3.17). $F_1 F_1^*$ is given by Equation (3.21), namely

$$F_1 F_1^* = \mathrm{diag}[\gamma_1, \ldots, \gamma_n, \ldots] \tag{3.21}$$

where

$$\gamma_n = [2(nk\pi/b)^2]^{-1} [1 - \exp\{-2(nk\pi/b)^2 (t_1 - t_o)\}] .$$

It is clear that

$$\sum_{n=1}^{\infty} |\gamma_n|^2 < [2(k\pi/b)^2]^{-2} \sum_{n=1}^{\infty} (1/n)^4 < \infty ,$$

since $\sum_{n=1}^{\infty} (1/n)^p$ converges for $p > 1$. It thus follows (see [2, page 58]) that $F_1 F_1^*$ is compact and hence F_1 is itself compact.

Since F_1 satisfies all the conditions of Theorem II above, it follows that it has the polar decomposition

$$F_1 = \sum_{n=1}^{\infty} e_n > \mu_n < g_n.$$

Here, the μ_n^2 are the eigenvalues of the matrix operator $F_1F_1^*$,i.e.,

$$\mu_n = \gamma_n^{1/2} = [2(nk\pi/b)^2]^{-1/2}[1-\exp\{-2(nk\pi/b)^2(t_1-t_o)]^{1/2} .$$

The sequence $\{e_n\}$ is the usual orthonormal basis of ℓ_2 , and the g_n , according to Equation (4), are given by

$$g_n(t) = (1/\mu_n)^{1/2}F_1^* e_n$$

$$= (1/\mu_n)^{1/2}\exp\{-(nk\pi/b)^2(t_1-t)\}e_n , \quad t \in (t_o,t_1) .$$

It is worth noticing that $F_1^\dagger$ is readily given by

$$F_1^\dagger = \Sigma\, g_n > \mu_n^{-1} < e_n$$

which exhibits the unboundedness of $F_1^\dagger$ resulting from the fact that $\mu_n^{-1} \to \infty$ (see subsection 2.2.2).

APPENDIX G

SOLUTION OF THE MINIMUM ENERGY CONTROL PROBLEM OF A SPATIALLY-DISCRETED APPROXIMATE MODEL OF SYSTEM III

The mathematical model of System III is given by Equation (3.7), namely,

$$\begin{aligned}
\frac{\partial x(t,\alpha)}{\partial t} &= k^2 \frac{\partial^2 x}{\partial \alpha^2} && t_o < t \leq t_1 \; ; \; 0 < \alpha < b \\
x(t_o,\alpha) &= x^o(\alpha) && 0 \leq \alpha \leq b \qquad (3.7) \\
x(t,o) &= h_1(t) && t_o < t < t_1 \\
x(t,b) &= h_2(t) && t_o \leq t \leq t_1
\end{aligned}$$

Using a Taylor series expansion, it can be shown that

$$\frac{\partial^2 x(t,\alpha)}{\partial \alpha^2} = \lim_{\Delta\alpha \to 0} \frac{x(t,\alpha + \Delta\alpha) + x(t,\alpha - \Delta\alpha) - 2x(t,\alpha)}{(\Delta\alpha)^2}$$

It thus follows that as $\Delta\alpha \to 0$, Equation $(3.7)_1$ can be written as

$$\frac{\partial x(t,\alpha)}{\partial t} = \frac{k^2}{(\Delta\alpha)^2} [x(t,\alpha + \Delta\alpha) + x(t,\alpha - \Delta\alpha) - 2x(t,\alpha)]$$

Taking $\alpha_o = 0$, and $\Delta\alpha = b/4$, the spatial-discreted points α_n are expressed as

$$\alpha_n = \alpha_o + n\Delta\alpha = nb/4 \; , \qquad n=0,1,\ldots,4 \; ,$$

and the approximate mathematical model of System III is thus given by

$$\frac{dX_n(t)}{dt} = (k/\Delta\alpha)^2[X_{n+1}(t) + X_{n-1}(t) - 2X_n(t)] \qquad (1)$$

where

$$X_n(t) = x(t,\alpha_n) , \qquad n=1,2,3 ,$$

together with the initial conditions

$$X_n(t_o) = x^o(n\Delta\alpha) ; \qquad n=1,2,3 \tag{2}$$

and the boundary conditions

$$X_o(t) = h_1(t) , \qquad t_o \leq t \leq t_1 \tag{3}$$

$$X_4(t) = h_2(t) , \qquad t_o < t < t_1 . \tag{4}$$

Equations (1) and (2) has the matrix form

$$\dot{X}(t) = AX(t) + Bu(t) ; \quad X(t_o) = X^o , \quad t_o < t < t_1 \tag{5}$$

where

$$X(t) = \text{col}(X_1(t),X_2(t),X_3(t)) ,$$

$$X^o = \text{col}(X_1(t_o),X_2(t_o),X_3(t_o)) ,$$

$$u(t) = \text{col}(h_1(t),h_2(t)) ,$$

A is the 3 x 3 matrix given by

$$A = M \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} , \qquad M = (k\Delta\alpha)^2$$

and B is the 3 x 2 matrix given by

$$B = M \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Solution of Equation (5) may be written as

$$X(t) = \Phi(t-t_o)X^o + \int_{t_o}^{t} \Phi(t-s)Bu(s)ds \, .$$

where $\Phi(t)$, the transition matrix of the system, is a 3 x 3 matrix with its elements φ_{ij} given by

$$\varphi_{11} = (1/4)\exp\{\lambda_1 t\} + (1/4)\exp\{\lambda_2 t\} + (1/2)\exp\{\lambda_3 t\} ,$$

$$\varphi_{12} = -(\sqrt{2}/4) \exp\{\lambda_1 t\} + (\sqrt{2}/4) \exp\{\lambda_2 t\} ,$$

$$\varphi_{13} = (1/4)\exp\{\lambda_1 t\} + (1/4)\exp\{\lambda_2 t\} - (1/2)\exp\{\lambda_3 t\} ,$$

$$\varphi_{21} = \varphi_{12} = \varphi_{23} = \varphi_{32} ,$$

$$\varphi_{22} = (1/2)\exp\{\lambda_1 t\} + (1/2)\exp\{\lambda_2 t\} ,$$

$$\varphi_{33} = \varphi_{11} .$$

Here, λ_i , $i=1,2,3$, are the eigenvalues of the matrix A . By simple computation, it can be shown that

$$\lambda_1 = -[2 + \sqrt{2}] M ,$$

$$\lambda_2 = -[2 - \sqrt{2}] M ,$$

$$\lambda_3 = -2 M .$$

The optimum control problem discussed here is to find the element $u^*(t) \in [L_2(t_o,t_1)]^2$ which brings the system to a specified state[1]

[1] Notice that in this approximate problem, the terminal state is specified only at the discrete spatial points α_n while in the exact one the terminal state is specified at all $\alpha \in [0,b]$.

$X^1 = \text{col}(X_1(t_1), X_2(t_1), X_3(t_1))$ at a specified time t_1, while minimizing the functional

$$J = \frac{1}{2}\int_{t_o}^{t_1} \{h_1^2(t) + h_2^2(t)\}dt \ .$$

It can be shown (see [29]) that the optimal control input u^* is given by

$$u^*(t) = B^*(t)\Phi^*(t_o - t)\mathcal{F}^{-1}(t_1)\beta \qquad (6)$$

where

$$\mathcal{F}(t_1) = \int_{t_o}^{t_1} \Phi(t_o - s)B(s)\Phi^*(t_o - s)ds \ , \qquad (7)$$

and

$$\beta = \Phi(t_o - t_1)X^1 - X^o \qquad (8)$$

The computation of the matrix $\mathcal{F}(t_1)$ is straight forward and leads to the result that

$$\mathcal{F}(t_1) = M^2[f_{ij}] \ , \qquad i=1,2,3, \qquad j = 1,2,3$$

where

$$f_{11} = [16\lambda_1]^{-1}[1-\exp\{-2\lambda_1(t_1-t_o)\}] + [16\lambda_2]^{-1}[1-\exp\{-2\lambda_2(t_1-t_o)\}]$$

$$+ [4\lambda_3]^{-1}[1-\exp\{-2\lambda_3(t_1-t_o)\}] + [4(\lambda_1+\lambda_2]^{-1}[1-\exp\{-(\lambda_1+\lambda_2)(t_1-t_o)\}]$$

$$f_{12} = [-\sqrt{2}/16\lambda_1][1-\exp\{-2\lambda_1(t_1-t_o)\}] + [\sqrt{2}/16\lambda_2][1-\exp\{-2\lambda_2(t_1-t_o)\}]$$

$$f_{13} = [16\lambda_1]^{-1}[1-\exp\{-2\lambda_1(t_1-t_o)\}] + [16\lambda_2]^{-1}[1-\exp\{-2\lambda_2(t_1-t_o)\}]$$

$$- [4\lambda_3]^{-1}[1-\exp\{-2\lambda_3(t_1-t_o)\}] + [4(\lambda_1+\lambda_2)]^{-1}[1-\exp\{-(\lambda_1+\lambda_2)(t_1-t_o)\}]$$

$$f_{21} = f_{12} = f_{23} = f_{32} \ ,$$

$$f_{22} = [8\lambda_1]^{-1}[1-\exp\{-2\lambda_1(t_1-t_o)\}] + [8\lambda_2]^{-1}[1-\exp\{-2\lambda_2(t_1-t_o)\}]$$

$$-[2(\lambda_1+\lambda_2)]^{-1}[1-\exp\{-(\lambda_1+\lambda_2)(t_1-t_o)\}] \ ,$$

$$f_{31} = f_{13} \ ,$$

$$f_{33} = f_{11} \ .$$

By inverting $\mathcal{F}(t_1)$ and noticing that the adjoint of a matrix is its transpose, the optimal control element u^* can be computed from Equation (6) above.

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