J. Phys. A: Math. Gen. 26 (1993) L941–L944. Printed in the UK

LETTER TO THE EDITOR

Hidden quantum group structure in Chern–Simons theory

A C T Wu

Randall Laboratory of Physic, University of Michigan, Ann Arbor, MI 48109, USA

Received 5 May 1993

Abstract. The unexpurgated K' matrix in the Chern–Simons theory of topological systems (such as the fractional Hall system, the chiral spin system and the anyon system) is viewed as a q-deformed Cartan matrix. The connection to the known generalized quantum groups is pointed out. An alternative interpretation in terms of quantum superalgebra in the graded Yang–Baxter basis also holds.

The $(2+1)$-dimensional Chern–Simons theory [1–8] has a number of interesting properties, for example, topological invariants [3], fractional statistics [4–7], link polynomials and knots [8], and connection to rational conformal field theory [8, 9]. Through the last two features, the connection with the Yang–Baxter equations and quantum groups [10, 11] is established.

Recently, Zee and his collaborators [12–14] have discussed the long-distance properties of two-dimensional topological fluids (such as the Hall fluid, the chiral spin fluid, and the anyon superfluid) in the Chern–Simons approach. The theory is characterized by a $m\times m$ K-matrix (see (4) below) which can be transformed into a K' matrix whose $(m-1)\times(m-1)$ block is the Cartan matrix for the Lie algebra $su(m)$. Thus a $SU(m)$ symmetry is claimed [12, 15] by ignoring the last row and the last column in the K' matrix.

In this letter we wish to point out that the unexpurgated K' matrix could be viewed as a q-deformed Cartan matrix which has been discussed in the generalized quantum groups [16]. This generalized quantum group structure arises in the non-standard braid group representations when the quantum group parameter q is changed into $-q^{-1}$ at certain strategic places in the Yang–Baxter R-matrix. In the conventional Yang–Baxter basis, the new algebra corresponds to a distorted $s\ell_q(m+1)$ with a special value of q (q being a root of unity). Alternatively, in the graded Yang–Baxter basis, the new algebra corresponds to the superalgebra $s\ell_q(m|1)$.

For the basic formalism of the K matrix in the Chern–Simons theory, we refer the reader to Zee [12]. The effective Lagrangian has the following form:

$$L=(1/4\pi)\varepsilon^{\mu\nu\lambda}\alpha_\mu K\partial_\nu\alpha_\lambda+\alpha^\mu j\mu \tag{1}$$

where α_μ is a gauge potential and j_μ is a reduced current (vortex current minus the electromagnetic current). K is the $m\times m$ matrix:

$$K=\begin{pmatrix} p+1 & p & \cdot & \cdot & p \\ p & p+1 & \cdot & \cdot & p \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ p & p & \cdot & \cdot & p+1 \end{pmatrix}. \tag{2}$$

Physically, the parameter p is a measure of the flux attached to each electron in the Hall effect; p enters in the fractional filling factor $\nu = m/(mp+1)$, for even p. In [12, 14], it is shown that the Fourier transform J_n of the J_0 current in the K-matrix Chern–Simons model satisfies the Kac–Moody algebra

$$[J^I_m, J^J_n] = m\delta_{m,-n}K^{IJ}. \tag{3}$$

Furthermore, the theory is invariant under a transformation on K, namely X^TKX with integer-valued matrix $X \,\varepsilon s\ell(m, Z)$ which would preserve the integer-valued topological vorticity. One finds [15, 14, 12] that

$$K' = X^TKX = \begin{pmatrix} 2 & -1 & 0 & \cdot & 0 \\ -1 & 2 & -1 & 0 & \cdot \\ 0 & -1 & 2 & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2 & -1 \\ 0 & \cdot & \cdot & -1 & p+1 \end{pmatrix} \tag{4}$$

by taking

$$X = \begin{pmatrix} 1 & 0 & \cdot & \cdot & \cdot \\ -1 & 1 & 0 & \cdot & \cdot \\ 0 & -1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & 1 \end{pmatrix}. \tag{5}$$

When the last row and the last column of the K' matrix are disregarded, one recognizes the $(m-1)\times(m-1)$ submatrix as the Cartan matrix for $su(m)$, thus a $SU(m)$ symmetry for the model [12, 14, 15].

Consider the unexpurgated $m \times m$ K' matrix given by (4). Equation (4) implies that the mth root vector (of the underlying algebra) has a norm $[(p+1)/2]^{1/2}$ instead of the usual 1. We can rescale this norm to be one, but at the cost of deforming its scalar product from $2\cos\theta = -1$ to $2\cos\theta = -[2/(p+1)]^{1/2}$. The rescaled K' matrix reads

$$K' = \begin{pmatrix} 2 & -1 & 0 & \cdot & \cdot & \cdot \\ -1 & 2 & -1 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 2 & -[2/(p+1)]^{1/2} \\ \cdot & \cdot & \cdot & \cdot & -[2/(p+1)]^{1/2} & 2 \end{pmatrix}. \tag{6}$$

A special class of the q-deformed Cartan matrix has been discussed in [16] in the context of non-standard braid group representations of the quantum group $s\ell_q(m)$. We here discuss the non-trivial case $p \neq 1$. (Physically relevant cases are when p is even.)

We go to the non-standard braid group representation [16] of $s\ell_q(m+1)$ by making one deformation $q \to -1/q$ in the last entry in the $(m+1)^2 \times (m+1)^2$ R-matrix. The net result is the following generalized algebra:

(*a*) $\quad (X^{\pm}_m)^2 = 0 \quad$ for the last mth element. (7*a*) (*b*) Corresponding to the regular Cartan matrix element $a_{ij}=3\delta_{ij}-1$, $|i-j|\leqslant 1$, for i, $j=1,\ldots,m-1$, ($a_{ij}=0$, $|i-j|>1$), we have the standard quantum algebra $s\ell_q(m)$:

$$K_iX_j^{\pm}K_i^{-1}=q^{\pm a_{ij}/2}X_j^{\pm} \qquad i,j=1,\ldots,m-1. \tag{7b}$$

(c) Corresponding to the entry $a_{m-1,m}$, we obtain

$$K_jX_i^{\pm}K_j^{-1}=(-q)^{\pm 1/2}X_i^{\pm} \tag{7c}$$

$$=q^{\pm 1/2\, a}m-1,X_i^{\pm} \qquad i,j=m-1,m. \tag{7d}$$

(*d*) Inserting the value from (6)

$$a_{m-1,m}=-[2/(p+1)]^{1/2} \tag{8}$$

we see that (7*c*) and (7*d*) are compatible for q being a root of unity:

$$q=\exp(-\mathrm{i}\pi/\{1+[2/(p+1)]^{1/2}). \tag{9}$$

This shows that the unexpurgated K' matrix of (4) can be interpreted as a q-deformed Cartan matrix which can be accommodated in the non-standard braid group representation $s\ell_q(m)$ with special value of q given by (9). Alternatively, in the graded Yang–Baxter basis, the non-standard braid group representations can be reinterpreted as quantum superalgebra [16, 17]. Thus for the present case of (6), we would get the quantum supersymmetry $SL_q(m|1)$. Such supersymmetry is perhaps not a great surprise for the anyon systems. A concrete realization of generalized quantum group structure in two-dimensional quantum fluids would be of interest and the details remain to be worked out.

The author thanks A Zee for an illuminating lecture and for a copy of his Kyoto Lectures.

References

[1] Schwarz A 1978 *Lett. Math. Phys.* 2 247
Schonfeld J 1981 *Nucl. Phys.* B **185** 157
[2] Jackiw R and Templeton S 1981 *Phys. Rev.* D **23** 2291
Deser S, Jackiw R and Templeton S 1983 *Phys. Rev. Lett.* **48** 975; 1984 *Ann. Phys., NY* **140** 372.
[3] Zuckerman G *Proceedings of 1986 San Diego Summer Workshop* ed S T Yau
[4] Polyakov A M 1988 *Mod. Phys. Lett.* A 3 325
[5] Fröhlich J 1987 *Cargese Lecture, in Nonperturbative Quantum Field Theory* eds G't Hooft et al (New York: Plenum Press)
[6] Hagen C R 1984 *Ann. Phys., NY* **157** 342
[7] Arovas D, Schrieffer R, Wilczek F and Zee A 1985 *Nucl. Phys.* B **251** 117
[8] Witten E 1989 *Commun. Math. Phys.* **121** 351
[9] Kohno T (ed) 1990 *New Development in the Theory of Knots* (Singapore: World Scientific) and references therein
[10] Yang C N and Ge M L (ed) 1989 *Braid Group, Knot Theory and Statistical Mechanics* (Singapore: World Scientific)
[11] Jimbo M (ed) 1990 *Yang–Baxter Equation in Integrable Systems* (Singapore: World Scientific)
[12] Zee A *Proceedings of 1991 Kyoto Conference on Low Dimensional Field Theories and Condensed Matter Physics*
[13] Wen X G and Zee A 1992 *Phys. Rev.* B **44** 274; 1990 *Nucl. Phys.* B **15** 135
[14] Fröhlich J and Zee A 1992 *Nucl. Phys.* B **364** 517

[15] Blok B and Wen X G 1990 *Phys. Rev.* B **42** 8133
Read N 1990 *Phys. Rev. Lett.* **65** 1502
[16] Ge M L and Wu A C T 1992 *J. Phys. A: Math. Gen.* **25** L807
[17] Kulish P P and Skylanin E K 1980 *Zapiski Nauch Sem.* **LOMI 95** 129 (English translation); 1982 *J. Math.* **19** 1596
Bazhanov and Shadrikov 1987 *Theor. Math. Phys.* **73** 1302
Schnidke W B, Vokos S P and Zumino B 1990 *Z. Phys.* C **48** 249
Liao L and Song X C 1991 *Mod. Phys. Lett.* A **6** 959