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ADDENDUM

Proof of crossing formula for 2D percolation

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Abstract. The author's recently conjectured expression for Cardy's crossing formula in 2D percolation is rewritten in terms of theta and elliptic functions, and verified explicitly. Exact results for aspect ratio r equal to integral powers of two are also given.

In [1], the author conjectured that Cardy's result [2] (see also [3]) for the crossing probability in percolation

$$\pi_v(r) = c\,\eta^{1/3}{}_2F_1(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{4}{3}; \eta) \tag{1}$$

where $\eta = (1-k)^2/(1+k)^2$, $r = 2K(k^2)/K(1-k^2)$ and $c = 3\Gamma(\frac{2}{3})/\Gamma(\frac{1}{3})^2$, can be written directly in terms of r as

$$\pi_v'(r) = -\frac{2^{4/3}\pi c}{3}\left[\sum_{n=-\infty}^{\infty}(-1)^n \mathrm{e}^{-3\pi r(n+1/6)^2}\right]^4 \tag{2}$$

where $\pi_v(r)$ is the probability density of crossing a rectangular system of height r and of unit width in the vertical direction, and $\pi_v'(r)$ (the derivative with respect to r) gives the probability density that the maximum height of clusters grown from the bottom of an infinitely high rectangular system is equal to r (assuming free boundaries on the sides in both cases). The form of (2) was conjectured from a series development and verified to high order, but not proven explicitly. In this addendum, I provide that proof, and also give alternative expressions for (2).

Those alternative expressions are

$$\pi_v'(r) = -\frac{2^{4/3}\pi c}{3}\mathrm{e}^{-\pi r/3}\prod_{n=1}^{\infty}(1-\mathrm{e}^{-2\pi r n})^4 \tag{3a}$$

$$= -\frac{\pi c}{3}\left(\vartheta_1'\right)^{4/3} \tag{3b}$$

$$= -\frac{4c}{3\pi}\left[\eta(1-\eta)\right]^{1/3}K^2(\eta)\,. \tag{3c}$$

The first result is implied by 24.2.1 of [4] or (13a,b) of [1], and the second puts this product in terms of the Jacobi theta function $\vartheta_1' = 2q^{1/4}\prod_{n=1}^{\infty}(1-q^{2n})^3 = 2q^{1/4}\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n^2+n}$ where $q = \mathrm{e}^{-\pi r}$. The third expression follows by applying $\vartheta_1' = \vartheta_2\vartheta_3\vartheta_4$ and formulae for ϑ_i in terms of the elliptic integral K. Note that in [1] it was shown that r and η are directly related according to $r = K(1-\eta)/K(\eta)$. Expanding (3*a*) or (3*b*) in powers of q yields the series expansion of $\pi_v'(r)$ given in [1].

Equation (3*c*) above leads directly to an explicit proof of (2). The derivative of Cardy's result (1) is given by [1]

$$\pi_v'(r) = -\frac{c}{3[\eta(1-\eta)]^{2/3}} \frac{K^2(\eta)}{\dot{K}(1-\eta)K(\eta) + K(1-\eta)\dot{K}(\eta)} \tag{4}$$

where the dot represents differentiation with respect to the argument (a prime is used to indicate the complementary argument $K'(\eta) = K(\eta_1) = K(1-\eta)$). This result is equivalent to (3*c*) if the relation

$$\dot{K}(1-\eta)K(\eta) + K(1-\eta)\dot{K}(\eta) = \frac{\pi}{4\eta(1-\eta)} \tag{5}$$

is valid. But this identity follows directly from $\dot{K} = (E - \eta_1 K)/2\eta\eta_1$ and Legendre's relation $EK' + E'K - KK' = \pi/2$ [5], and thus, the equivalence of (1) and (2) follows.

In [1] it was shown that Landen's transformation can be used to find how η scales with r: $\eta(2r) = \{(1 - [1-\eta(r)]^{1/2})/(1 + [1-\eta(r)]^{1/2})\}^2$. Applying this same transformation to (3*c*) yields

$$\pi_v'(2r) = \pi_v'(r)\left(\frac{\eta^2(r)}{256(1-\eta(r))}\right)^{1/6} \tag{6}$$

which also implies $[\pi_v'(2r)]^6 = [\pi_v'(4r)]^2[\pi_v'(r)]^4 + 16[\pi_v'(4r)]^4[\pi_v'(r)]^2$. These yield $\pi_v'(2)/\pi_v'(1) = 2^{-3/2}$, $\pi_v'(4)/\pi_v'(2) = 2^{-7/4}(2^{1/2} - 1)$, etc. Furthermore, $\pi_v'(1) = \Gamma(\frac{1}{4})^4/(\Gamma(\frac{1}{3})^3 2^{5/3} 3^{1/2}\pi) \approx 0.520\,246\,1715$, so closed expressions for $\pi_v'(r)$ for all r equal to powers of two follow. (Note $\pi_v'(1/r) = r^2\pi_v'(r)$.) Finally, a plot of $\pi_v'(r)$ shows that its maximum is at $r \approx 0.523\,5217$ (where $(\mathrm{d}/\mathrm{d}q)\vartheta_1' = 0$) with value $\pi_v' \approx -0.737\,3222$.

Corrections to [1] are as follows: $k(4) = 2^{5/4}/(2^{1/2}+1) \approx 0.985\,171\,431$ and $\eta(4) = [(2^{1/4}-1)/(2^{1/4}+1)]^4$ on p 1253. Also, the series in (16) can be found to all orders directly by using $\eta = \vartheta_2^4/\vartheta_3^4$.

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