T H E U N I V E R S I T Y O F M I C H I G A N

COLLEGE OF ENGINEERING
Department of Engineering Mechanics
Department of Mechanical Engineering

Tire and Suspension Systems Research Group

First Yearly Report

THE MECHANICS OF PNEUMATIC TIRES:
I. PRELIMINARY INVESTIGATIONS AND EXPERIMENTS

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I. SURVEY OF GENERAL APPROACH

The purpose of this report is to outline the progress since last May of the Tire and Suspension Systems Research Group at The University of Michigan. At a meeting held in Ann Arbor in August, 1959, between representatives of the sponsoring companies and the Research Group, two areas were agreed upon in which the Research Group was to operate. These were (1) the determinations of the elastic constants of tire materials and (2) the use of these elastic constants in calculating the stress state and deformation in a tire resulting from a given set of loads, such as would be found by placing the tire under load on a flat surface. It was also agreed that the Research Group would compile a bibliography and abstracts of publications dealing with tires in a useful form.

Generally speaking, our goals are as follows:

1. Given the elastic properties of textile fibers and rubber compounds, or given the opportunity to measure these properties, one goal is to calculate completely the elastic properties at any point in a tire as a function of the known properties of textile and rubber and of the geometry of construction.

2. Given the elastic properties of the tire, as described, and given the shape to which the tire is molded and the loads acting on it, another goal is to calculate the state of stress and state of deformation at every point in the tire.

II. LITERATURE SURVEY

Some of this first year has been spent making a survey of the available literature dealing with the elastic and mechanical properties of tires. Realizing that we would have to limit our survey, we decided to investigate the following fields:

1. Wear of rubber and tires
2. Contact pressure between tires and roadway, steering forces, and car performance
3. Cord and rubber properties, both physical and mechanical
4. General tire design and the influence of specific variables on tire design
5. Calculation of stresses and deformations
6. Endurance and life testing of tires
7. Measurements of stress, deformations, and temperatures in tires

Copies of our reference files and translations are being made, and will continue to be made available to our sponsors.

Because new material is appearing regularly, we plan to continue adding references and translations to the files. To aid us in our survey of current literature, we have subscribed to many periodicals dealing with one or more of these topics. Many others are available through our own library system. By reviewing these often, we hope to keep abreast of developments in tire mechanics and to be able to evaluate them, use them, and to transmit them to our sponsors.

III. MEASUREMENTS ON TIRES

At this time we have planned and are beginning to execute several sets of measurements on tires or tire-like structures to obtain information which will be useful in our calculations, or useful in making assumptions to reduce the length of our calculations.

The first of these programs, now nearly completed, consists of obtaining the cross-sectional shape of an inner tube under different inflation pressures. This inner tube was bonded to the rim of a wheel with cement to simulate attached boundaries. From these measurements we can calculate the radii of curvature of the inner tube at any point, a quantity needed to start the stress calculations. It is our intention to use similar measurements at higher pressures as a physical check on the accuracy of some of our equations and of some of our digital computer techniques, since we can compare the calculated shapes from the computer with those actually measured on the tire. Later on we will extend this technique to a real tire with the tread removed. Figure 1 illustrates the equipment used here, in which a dial gauge moves on a circular cam.

We have not been able to develop simple methods of calculating the shape of the contact patch, even while the tire is stationary on a flat surface. Consequently a program is being begun in which we hope to be able to classify contact patches, both in shape and size, so as to use the contact patch as a known quantity from which to start our calculations, not as an unknown to be found. It is too early to say whether this attempt will be successful, but we are hopeful for it. Figure 2 shows the equipment used here, in which the tire is pressed against a ruled Plexiglas plate and then photographed for later reduction of data.

A very pressing need exists for good experimental data to aid in making realistic assumptions designed to simplify the present state of toroidal shell equations. We are anxious to make such simplifications, and so we have planned a program involving strain-gauge measurements at a number of points, in a number of directions, on a very slowly rolling loaded tire. Using the elastic properties of the tire carcass, we can reduce this to stress data, and these stress data will allow a re-examination of the shell equations of equilibrium in the light of experimental evidence. It is hoped to have this work completed by the summer; it is felt that the results will be extremely valuable.

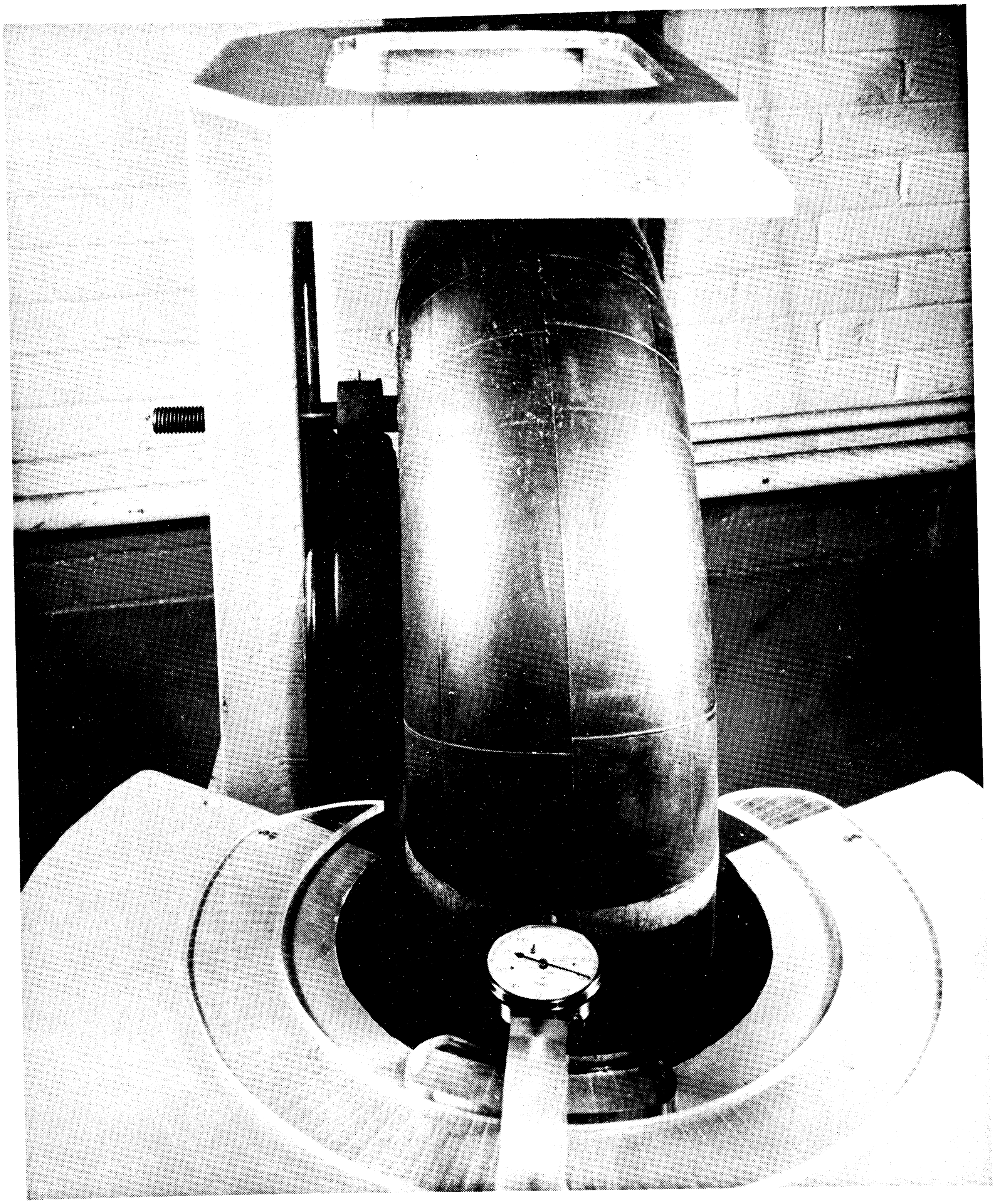

Fig. 1. Equipment to measure the shape of an inflated tire or inner tube.

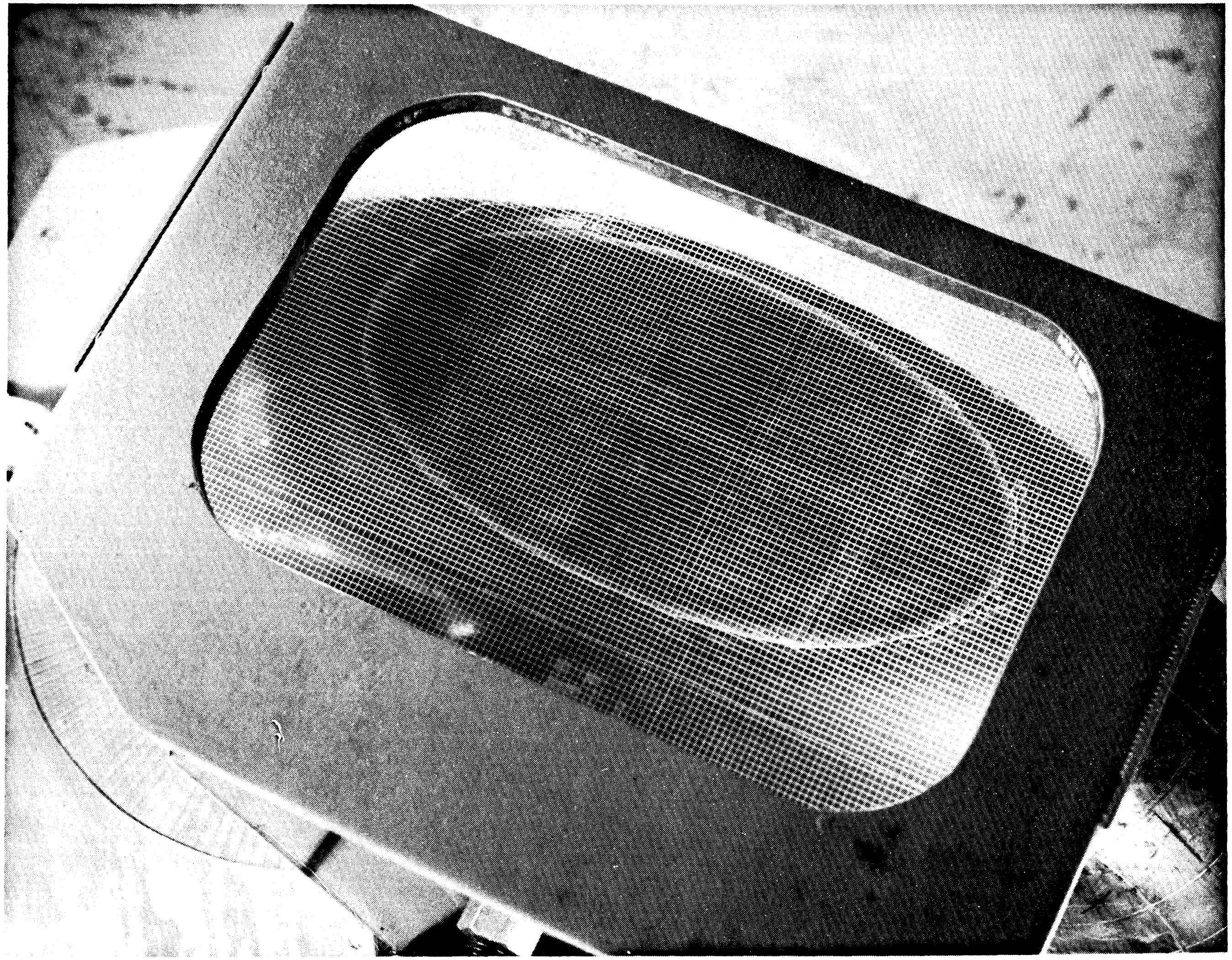

Fig. 2. Equipment used to measure the geometry of the contact patch between a tire or inner tube and a flat plate.

IV. PREDICTION OF PROPERTIES

A. THE CALCULATION OF TIRE CARCASS ELASTIC CONSTANTS USING TEXTILE AND RUBBER ELASTIC CONSTANTS

It should theoretically be possible to calculate exactly the elastic behavior under load of a typical element of tire carcass, made up of fibers and rubber, provided that one knows the elastic properties of the constituents and the geometry into which they are assembled. We have completed our first phase of analytical work in this area and are presently conducting some experimental work to check the accuracy of the theories we have developed so far. A recent paper by Hofferberth (translated, and sent under separate cover) dealt with this problem on a linear basis: the stress-strain curves of both rubber and textile, and hence of the composite body, were assumed linear. This assumption of linearity is not true for the textiles and rubbers commonly used, and so our efforts along this line have been directed toward devising a theory in which the stress-strain curves of arbitrary shape can be inserted. No serious difficulties are anticipated, but it should be realized that certain effects must be determined experimentally before the validity of the theory can be established. For example, we are presently uncertain about the importance of the torsion of the layer of insulating rubber between two cords at the point where they cross over one another in a two-ply structure, and so we are planning a set of experiments to evaluate this effect. An X-type specimen (see Fig. 3) will be used in these experiments.

Similarly, there is no clear evidence that a textile cord embedded in rubber cannot carry compression, and it is planned to test rubber cylinders (see Fig. 3) in compression and in tension very carefully to resolve this problem. Our present information from other sources concerning compression-carrying ability is somewhat contradictory. Its importance lies primarily in the areas of the bending and torsional rigidity of an element of tire carcass, for under these stress conditions it is possible, if a cord ceases to carry load in an abrupt manner, to get a discontinuity in the modulus of elasticity of the composite structure.

Under conditions in which the cord or cords do become ineffective in load-carrying, the rubber matrix used to embed the cords becomes of real importance. The stiffness of such a matrix has not, to our knowledge, been investigated in any way. Here again, some experiments have been planned and will be performed to indicate the magnitude of the effects.

In general substantial progress in this area has been made since our start in August, and we expect to issue a report on our calculations and experiments during the coming year.

Fig. 3. Typical specimens used in determining the elastic properties of carcass structures.

B. ELEMENTARY SHELL-STRUCTURE CALCULATIONS

We have been studying various aspects of the general problem of calculating the stress state in a tire since last September. There are several areas of extreme difficulty and we expect the problem to occupy us for quite a long time. Our method of approach has been to start in the simplest possible way and to add various degrees of complexity as we proceed. One advantage of this approach is that we will find out rather quickly whether some of our ideas are going to work.

During the course of our work, Mr. Claus has developed a rather general method of handling the problem of a tire which is inflated inside a ring, such as shown in Fig. 4. This solution is quite different from that developed by Prof. Rivlin and by Dr. Hofferberth for a network of cords. While it does not represent a physically real case, it does have many features of interest; some aspects of it are presented in Section V. We plan to describe this work in detail in a future technical report.

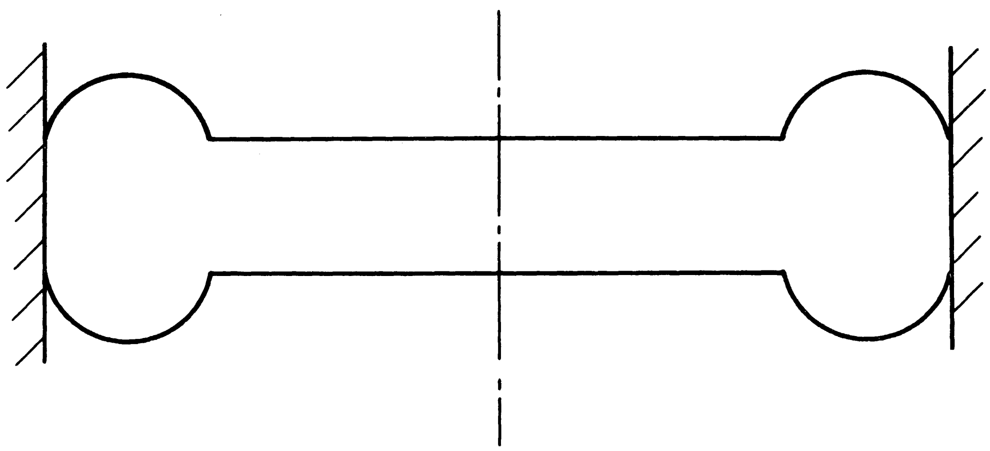

Fig. 4. A tire inflated inside a drum or ring.

Our primary interest has been, and is now, in the area of a tire subjected to an unsymmetrical load. Nothing has yet been published about this, although the recent paper of Ames and Lauterbach begins to touch on the subject.

To arrive at a simplified model which we can handle mathematically, we have made the following assumptions about a tire:

(a) Its primary load-carrying structure is the carcass. Hence the tread will be neglected for the time being.
(b) Bending stiffness of the carcass is negligible.
(c) Transverse shearing forces are negligible.

With these assumptions, it is possible to write a set of equations which are perfectly valid and perfectly general for any piece of carcass properly oriented. These are force-equilibrium equations and as such are independent of the materials of which the tire is made, being dependent only on the tire shape and loading. A general solution is not available for these equations. Even for a specified geometry and set of loads, it is difficult to arrive at a particular solution, and we are at present investigating several means by which we may simplify these equations and still retain their most important features.

One method of simplification is to specify the loads on a tire due to contact with the ground by means of a known pressure distributed over a known area. In view of our assumptions of negligible bending stiffness in the tire, we may easily prove that, barring buckling, the pressure distribution over the contact patch is uniform. We now have studies underway to measure the contact-patch areas experimentally in a given tire, and we believe that we have several ways for classifying these areas in terms of the variables upon which they depend.

This and other methods of simplifying the equilibrium equations will be tried on the simplest tire-like structure available, an inner tube. We plan to calculate the stresses and deformations in an inner tube while under pressure and loaded against a flat plate, and to compare these quantities with what we measure experimentally on an actual inner tube under the conditions for which the calculations are to be made. We hope that these results will become available soon.

If we find that the shapes of loaded inner tubes can be easily predicted, then it will be easy to proceed to the anisotropic properties of tires, since the methods we are using allow the insertion of various physical properties at different points.

Our calculations are being performed on the University's IBM 704 computer due to their extreme length.

C. IMPLICATIONS OF PRESENT WORK

We feel quite strongly that success in calculating the state of stress and the resulting deformation in a tire carcass will be beneficial. First, it will provide a rational basis for evaluating the strength of various tires and the need for given proportions of textiles. Secondly, it may permit examination of the carcass motion in the contact patch and will allow, perhaps, some conclusions concerning the wear properties of tires due to their scrubbing action on the road. Third, in many instances it will provide the tire designer with a means of calculation, enabling him to obtain more precise information on tire characteristics. Finally, it will provide a much more rational basis for the calculation of automobile behavior, which ultimately rests on a detailed knowledge of the tire behavior.

V. GENERAL SHELL EQUATIONS FOR FUTURE SOLUTIONS

This section discusses different methods for determining the stresses in a loaded tire. Although it is possible to include every effect that arises in the mathematical derivation, the resulting equations would make even a numerical analysis extremely difficult, if not impossible, at the present time. The features that are mainly responsible for rendering the problem difficult mathematically are the lack of symmetry and the fact that the final geometry of the tire after loading differs from the original geometry before loading by a finite amount. The lack of symmetry changes the system of ordinary differential equations into one involving partial differential equations, while the effect of finite geometry change is to make a nonlinear problem instead of a linear one. Neglecting or taking bending into account does not alter these conclusions, although it is true that including bending would result in considerably more complicated equations. However, when a few simplifying assumptions are made, solutions can be carried out in some cases. The assumptions selected should not only permit solution of the problem, but also should reduce the problem to one which exhibits at least some of the characteristics of the real situation. Although it is true that no exact quantitative results can be expected, this method of attack has the advantage that it introduces the difficulties one by one.

CASE 1. EFFECTS OF BENDING NEGLECTED—SYMMETRICAL SHELL

a. Geometry not dependent on load provided increments are taken small.— In this case, three equations (the equations of equilibrium) are available to determine the three unknown membrane stresses: the two normal stresses N_θ, N_ϕ, and the shear stress $N_{\theta\phi}$:

$$
(\mathrm{I})\quad \left\{
\begin{aligned}
&\frac{\partial}{\partial\phi}(N_\phi\, r_o) + \frac{\partial N_\theta}{\partial\theta} r_1 = N_\theta\, r_1 \cos\phi - p_y\, r_1 r_o && (1)\\
&\frac{\partial}{\partial\phi}(N_{\theta\phi}\, r_o) + \frac{\partial N_\theta}{\partial\theta} = - N_{\theta\phi}\, r_1 \cos\phi + p_x\, r_o r_1 && (2)\\
&\frac{N_\phi}{r_1} + \frac{N_\theta}{r_2} = p && (3)
\end{aligned}
\right.
$$

It may be observed that the above equations do not contain any elastic constant and the stress field is therefore independent of the material. However, one has to interpret this latter statement carefully. Equations (1), (2), and (3) determine the stress field, assuming r_1, r_2, r_o to be constant and known, i.e., final geometry exactly equal to the original geometry, which is equivalent to assuming a completely rigid structure. This, in turn, corresponds to infinite elastic constants. The above assumption of final geometry equal to original geometry is sufficiently accurate for a tire provided the load is small enough. As a matter of fact, the magnitude of the applied load can always be taken in such a way as to make the difference between the original and final geometry small enough. A possible method of solution would then consist of applying a small internal pressure Δp, and solving system (I) for this load, i.e., find the stress field N_θ, N_ϕ, $N_{\theta\phi}$, for this load. The stress-strain relations then give the strains corresponding to the known stresses. The anisotropy of the material does not introduce a conceptual difficulty at this point, provided, of course, that all material constants describing the anisotropy are known. From these strains, the displacements may be evaluated and the new geometry, such as radii of curvature, may be found. Again, a new load Δp is applied and the whole cycle is repeated. This is done as many times as is necessary to build up the pressure at which the stress field, strain field, and displacement field are to be found

Finding the displacement from the strains and the strain from the stresses presents no special difficulties. However, methods of solution of system (I) must be found. In the search for methods of attack, the equation governing the characteristic directions of system (I) was set up:

$$\frac{\partial^2 N_\phi}{\partial \theta^2} + \frac{r_o^2}{r_1 r_2} \frac{\partial^2 N_\phi}{\partial \phi^2} = 0$$

It can be seen that, if r_1 and r_2 have different signs, system (I) is hyperbolic; if r_1 and r_2 have the same sign, (I) is elliptic. This means that the equilibrium equations are hyperbolic in part A of the tire and they are elliptic in part B (see Fig. 5).

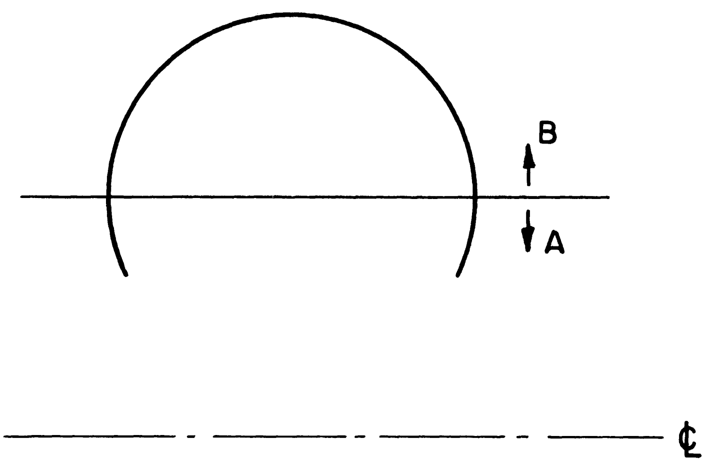

Fig. 5. A represents the hyperbolic and B the elliptic regions of a toroidal-like membrane.

Part of the tire is governed by a hyperbolic system of equations, which is most unusual since equilibrium equations are invariably elliptic. However, all this has followed from the assumption of a completely rigid structure. The peculiarity of an equilibrium equation of the hyperbolic type is that the crucial problem in this type of equations is the initial value problem. Furthermore, each point in the hyperbolic region has a so-called domain of dependence outside of which the edge tractions do not influence the state of stress at the point considered. A numerical solution valid in part A of system (I) can be found provided the surface tractions are prescribed on the boundary. From this, it is possible to determine the stress state on the line separating the regions A and B (parabolic line). Solving (I) in region B then becomes a problem in elliptic equations, the values of the unknown functions being specified on the boundary. It therefore seems possible, starting from known normal and shear stresses at the rim, to find the corresponding stress field through the whole tire. The rest of the cycle can then be completed, as pointed out earlier.

b. Geometry dependent on load.—It is clear that in system (I) r_1, r_2 have to be the values corresponding to the final geometry if one wants to find an exact solution. This is just another way of saying that the system is in equilibrium in its final position. The quantities r_1, r_2 appear therefore as unknowns and (I) is no longer sufficient to determine the stress field. The hyperbolic nature of the equations then disappears and the whole problem becomes essentially elliptic as is expected from an equilibrium problem. This is a mathematically consistent problem, but is an extremely difficult one in nonlinear partial differential equations, an analytic solution of which is probably out of the question. Even numerical solutions represent great labor.

The three equilibrium equations (I) can be converted into three equations in the three unknown displacements u, v, w, since the stresses N_θ, N_ϕ, and $N_{\theta\phi}$ are expressible in terms of the strains, and the strains can be expressed in terms of the displacements. All this involves a great deal of labor but it could be done. Furthermore, r_1, r_2, r_o are known functions of u, v, w so (I) reduces to a system of three equations in the three unknowns u, v, w. This system is so complicated that it probably could not be solved on a general basis. This method has been outlined very briefly; but the symmetrically equivalent case will be handled in detail. It is hoped that this will throw some light on the general procedure.

In everything we have done so far, the complications that arise from the contact patch have been carefully avoided. The resultant pressure over the contact area is equal to zero and this must be considered. The boundary of the contact area is unknown, and this introduces an additional difficulty. Since we are dealing with a nonlinear problem, the principle of superposition does not hold and it seems that methods utilizing Green's function are not applicable; it may be observed that obtaining a Green's function would in itself be an extremely hard problem. The only way of handling the contact patch so far devised is by assuming an approximate shape and letting the external load be equal to zero within its boundary. The final geometry obtained in this way has to have a flat portion equal to the originally assumed contact patch. The extent to which this is true will then be a measure of the correctness of the shape of the assumed boundary.

It will be useful to outline in some detail a method that could be used to solve the symmetrical case with a contact patch. Consider the following problem: a tire with no bending stiffness is inflated inside a cylindrical surface, the centerline of the tire coinciding with the centerline of the cylinder. The radius of the cylinder is small enough so that the inflated tire is in contact with the surface. The problem is to find the final geometry and final stress field in the tire.

Consider a curve Γ in the x-y plane of Fig. 6. This curve can be translated to any position Γ' by associating a displacement $\overline{PP'}$ with components u and v with any point P on AB. Note that u and v are functions of a parameter such as arc length s along Γ locating P on Γ. If $x(s)$, $y(s)$ are the coordinates of P, and $X(s)$, $Y(s)$ those of P', the parametric equations of Γ' are:

$$\begin{aligned} X(s) &= x(s) + u(s) \\ Y(s) &= y(s) + v(s) \end{aligned}$$

(note that $x' + y' = 1$, primes denoting derivatives with respect to s)

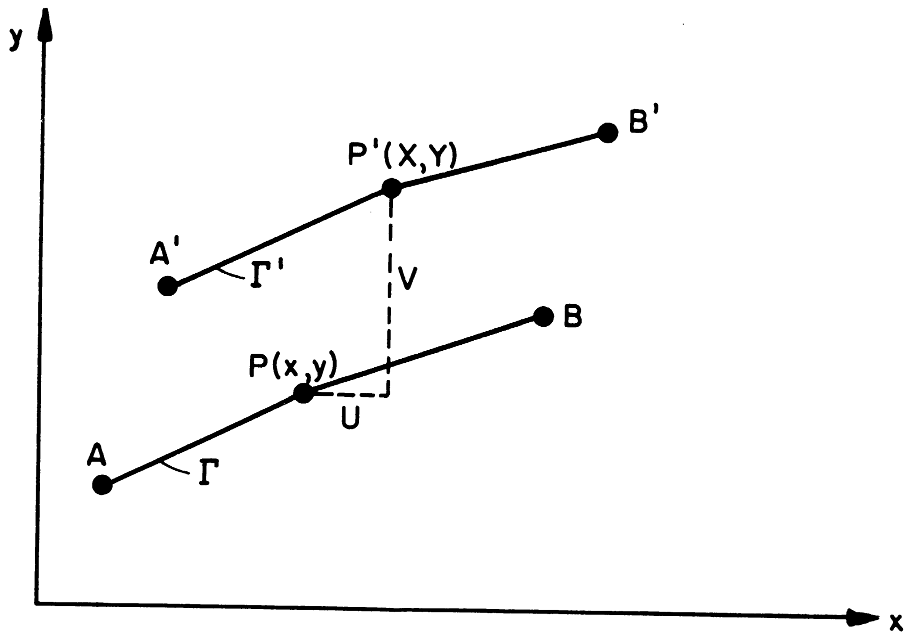

Fig. 6. The geometry of deformation in general.

The radii of curvature r_1 and r_2 of the surface of revolution generated by rotation of Γ' about the x-axis are then:

$$r_1 = \frac{(X'^2+Y'^2)^{3/2}}{\begin{vmatrix} X' & Y' \\ X'' & Y'' \end{vmatrix}} = \frac{1+2(x'u'+y'v')+u'^2+v'^2}{\begin{vmatrix} x'+u' & y'+v' \\ x''+u'' & y''+v'' \end{vmatrix}} = r_1\,(x'',y'',x',y',u'',v'',u',v') \tag{4}$$

$$r_2 = y\sqrt{1 + (y'/x')^2} = y\sqrt{1 + [(y'+v')/(x'+u')]^2} = r_2\,(y,x',y',u',v') \tag{5}$$

An element ds on Γ corresponds to an element dS on Γ' and we will define the strains:

(1) meridianal $\epsilon_1 = (dS-ds)/ds$

(2) along a parallel circle $\epsilon_2 = v/y$

but

$$dS^2 = dX^2 + dY^2 = [(x'+u')^2 + (y'+v')^2]ds^2 = [1 + 2(x'u'+y'v') + u'^2 + v'^2]ds^2$$

so

$$\epsilon_1 = \sqrt{1 + 2\,(x'u'+y'v') + u'^2+v'^2} - 1 = \epsilon_1\,(x',y',u',v') \tag{6}$$

$$\epsilon_2 = \epsilon_2\,(v,y) \tag{7}$$

The stress-strain relations can be represented as

$$N_1 = N_1\,(\epsilon_1,\ \epsilon_2)$$

$$N_2 = N_2\,(\epsilon_1,\ \epsilon_2)$$

N_1 and N_2 are known functions of ϵ_1 and ϵ_2 provided it has been decided what sort of anisotropy will be assumed.

Since ϵ_1 and ϵ_2 are known functions of x', y', u', v', v, and y, respectively, we can write:

$$\left.\begin{aligned} N_1 &= f_1\,(y,\ x',\ y',\ v,\ u',\ v') \\ N_2 &= f_2\,(y,\ x',\ y',\ v,\ u',\ v') \end{aligned}\right\} \tag{8}$$

The equilibrium equations (1) and (3) [noting that (2) becomes trivial in the symmetric case] can be written as

$$\frac{d(N_1 y)}{ds} = N_2 \cos\phi\,\frac{dS}{ds}$$

$$\frac{N_1}{r_1} + \frac{N_2}{r_2} = -p$$

Since $N_2 \cos\phi\,(dS/ds)$ is a known function of y, v, x', y', u', v', say $g(y,v,x',y',u',v')$, we have

$$N_1'y + N_1y' = g(y,v,x',y',u',v')$$

so that the equilibrium equations become:

$$\frac{\partial f_1}{\partial y} y' + \frac{\partial f_1}{\partial x'} x'' + \frac{\partial f_1}{\partial y'} + \frac{\partial f_1}{\partial v} v' + \frac{\partial f_1}{\partial u'} u'' + \frac{\partial f}{\partial v'} v'' + f_1 \frac{y'}{y} = \frac{g}{y} \tag{9}$$

$$\frac{N_1}{r_1} + \frac{N_2}{r_2} = -p \tag{10}$$

x and y are given functions of s; they represent the parametric form of the original meridian curve of the tire, before loading. Note also that all derivatives of x and y with respect to s are known at any point of the curve. The unknown functions are u(s) and v(s) which are the components of the displacement vector at point x(s), y(s). Assume that at point A of Fig. 7 u, v, u', v' are known. The following quantities at that point are then also known:

(1) r_2 from (5)
(2) N_1 and N_2 from (8)
(3) r_1 from (10) knowing N_1, N_2, r_2

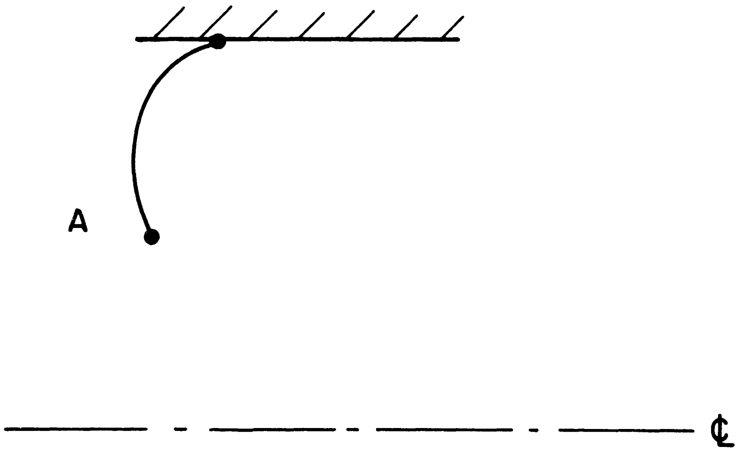

Fig. 7. Starting point A for numerical integration.

It is then possible to evaluate u" and v" at point A, using the following two equations in which the only unknowns are u" and v":

$$r_1 \left(x''_A, y''_A, x'_A, y'_A, u''_A, v''_A, u'_A, v'_A\right) = (r_1)_A$$

$$\left(\frac{\partial f_1}{\partial y}\right)_A y'_A + \left(\frac{\partial f_1}{\partial x'}\right)_A x''_A + \left(\frac{\partial f_1}{\partial y}\right)_A y''_A + \left(\frac{\partial f_1}{\partial v}\right)_A v'_A + \left(\frac{\partial f_1}{\partial u'}\right)_A u''_A$$

$$+ \left(\frac{\partial f_1}{\partial v'}\right)_A v''_A + \left(f_1 \frac{y'}{y}\right)_A = \frac{g_A}{y_A}$$

The final curve can now be constructed using a standard numerical integration technique. The Euler method will be utilized here. The application of the more refined and much more accurate Runge-Kutta scheme would not introduce any conceptual difficulties, but the labor would increase considerably. The general idea is that from known values of u, v, u', v' at, for example, point A, we can compute the values of u, v, u', v' at neighboring point B. Let $s = 0$ locate point A, and $s = \Delta s$ locate point B. The functions u(s) and v(s) can be expanded in a MacLaurin series to evaluate their values at point B:

$$u_B = u_A + \Delta s\, u'_A + \frac{\Delta s^2}{2!} u''_A + \cdots$$

$$v_B = v_A + \Delta s\, v'_A + \frac{\Delta s^2}{2!} v''_A + \cdots$$

Also

$$u'_B = u'_A + \Delta s\, u''_A + \frac{\Delta s^2}{2!} u'''_A + \cdots$$

$$v'_B = v'_A + \Delta s\, v''_A + \frac{\Delta s^2}{2!} v'''_A + \cdots$$

If Δs is chosen small enough, the following approximation can be made:

$$\begin{aligned} u_B &\cong u_A + \Delta s\, u'_A + \frac{\Delta s^2}{2} v''_A \\ v_B &\cong v_A + \Delta s\, v'_A + \frac{\Delta s^2}{2} v''_A \end{aligned} \qquad (12)$$

$$\begin{aligned} u'_B &= u'_A + \Delta s\, u''_A \\ v'_B &= v'_A + \Delta s\, v''_A \end{aligned} \qquad (13)$$

It was shown that, knowing u_A, u'_A, v'_A, v_A the values of u''_A could be found. From (12) and (13) then, u_B, v_B, u'_B, v'_B can be found. It is therefore clear that a step-by-step construction of the final curve of the tire shape can be carried out. The beginning of the contact patch is then indicated by a zero value of the slope, namely, by a zero of (y'+v'). From that point on, the resultant pressure becomes an unknown but it is necessary that (y'+v') remains equal to zero; this should be enough to determine the pressure variation over the contact patch and the total load applied on the tire by the cylinder.

CASE 2. EFFECTS OF BENDING NEGLECTED—UNSYMMETRICAL SHELL

A few words may be said about extending the method described above, which applies only to the symmetric case, to the general nonsymmetric case. It is clear how to go about extending this procedure up to the point where the differential equations are set up. The original surface can be represented as a function of independent parameters α and β; the curves α = constant, β = constant are curves on the surface and α,β can be selected so that the two families of curves are orthogonal. A displacement field $u(\alpha,\beta)$, $v(\alpha,\beta)$, $w(\alpha,\beta)$ transforms the original surface into its final geometry when the load is applied. One can then carry out operations analogous to those described in the presentation of the symmetric case. This leads, in principle, to a consistent set of equations in u, v, w, which are, of course, partial differential equations, the independent variables being α and β. It can well be imagined how complicated this set would be.

Energy methods, which in many cases are powerful tools for finding approximate equilibrium configurations, seem to be of limited use in the nonsymmetric case. If we consider, for example, the Ritz method, then we know that a function has to be minimized. This function is the total potential energy of the system including strain energy, energy in compressed air, etc., and can be expressed as a function of the displacements u, v, w. In selecting the coordinate functions, linear combinations of which will give the approximate equilibrium geometry, it is known that nongeometrically admissible functions are not permitted. However, although the displacements are known to be zero at the rim, they are not known in the contact patch and the selection of geometrically admissible functions is therefore highly complicated.

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