Nonlinearity **12** (1999) 673–691. Printed in the UK PII: S0951-7715(99)95040-5

Excitation thresholds for nonlinear localized modes on lattices

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Received 1 June 1998
Recommended by V F Lazutkin

Abstract. We consider spatially localized and time periodic solutions to discrete extended Hamiltonian dynamical systems (coupled systems of infinitely many 'oscillators' which conserve total energy). These play a central role as carriers of energy in models of a variety of physical phenomena. Such phenomena include nonlinear waves in crystals, biological molecules and arrays of coupled optical waveguides. In this paper we study *excitation thresholds* for (nonlinearly dynamically stable) ground state localized modes, sometimes referred to as 'breathers', for networks of coupled nonlinear oscillators and wave equations of nonlinear Schrödinger (NLS) type. Excitation thresholds are rigorously characterized by variational methods. The excitation threshold is related to the optimal (best) constant in a class of discrete interpolation inequalities related to the Hamiltonian energy. We establish a precise connection among d, the dimensionality of the lattice, $2\sigma + 1$, the degree of the nonlinearity and the existence of an excitation threshold for discrete nonlinear Schrödinger systems (DNLS). We prove that if $\sigma \geqslant \frac{2}{d}$, then ground state standing waves exist if, and only if, the *total power* is larger than some strictly positive threshold, $\nu_{\rm thresh}(\sigma, d)$. This proves a conjecture of Flach *et al* (1997 Energy thresholds for discrete breathers in one-, two-, and three-dimensional lattices *Phys. Rev. Lett.* **78** 1207–10) in the context of DNLS. We also discuss upper and lower bounds for excitation thresholds for ground states of coupled systems of NLS equations, which arise in the modelling of pulse propagation in coupled arrays of optical fibres.

AMS classification scheme numbers: 35Q55, 35Q51, 78A40

1. Introduction

This paper concerns threshold behaviour of certain time-reversible, energy preserving nonlinear dynamical systems. Consider an infinite-dimensional Hamiltonian system (wave equation or network of discrete oscillators) defined on an infinite spatial domain. If the system is translation invariant (e.g., not having any localized potential wells), one expects that 'small-amplitude' or 'low-energy' solutions will disperse to zero; see, for example, [24]. If the system is nonlinear and having an *attractive* nonlinear potential, one can expect that sufficiently large 'amplitude' initial data will lead to an evolution consisting of a non-decaying 'bound state' component plus a dispersive component (*radiation*), which tends (weakly) to zero with increasing time. In this latter scenario, we think of permanent non-decaying structures as having been excited by the initial condition; a deep enough self-consistent potential well has been initialized in which one can sustain a permanent structure. Since the systems we are discussing are infinite dimensional, the sense in which one measures amplitude is crucial. In systems of physical interest, there is often a natural measure of a solution's size. Roughly speaking, if there is a critical size, $\nu_{\rm thresh} > 0$, such that there are permanent (non-decaying in time) states of size ν

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if and only if $\nu > \nu_{\text{thresh}}$, then we refer to ν_{thresh} as an *excitation threshold*. In this paper, we investigate the existence and nonexistence of excitation thresholds for a class of time-periodic and spatially localized standing wave states for two classes of dynamical systems. In certain models, these states have been called 'breathers'. See section 3 for a precise definition of and discussion concerning excitation thresholds. The dynamical systems we consider are: (1) the discrete nonlinear Schrödinger equation (1.1), and (2) a system of coupled nonlinear Schrödinger equations (1.7); see also (6.1).

Mathematical models which support discrete breathers are of interest in the study of vibrations in, for example, localized crystals and biological molecules [9, 14]. Recently, experimental observations of such discrete nonlinear localized modes have been made in coupled systems of optical waveguides [10]. With a view toward study of such structures in experiment it is of interest to understand under what circumstances a discrete breather is excited.

In [23] a formal variational argument is given suggesting the existence of such energy thresholds for the one-dimensional discrete nonlinear Schrödinger (DNLS) equation (also known as the discrete self-trapping equation [9]). For the related system of nearest neighbour coupled nonlinear Schrödinger equations, (1.7), such thresholds were rigorously demonstrated to exist [27].

In the recent paper of Flach *et al* [13], heuristic scaling arguments and numerical studies are presented which suggest that for a large class of Hamiltonian dynamical systems defined on one-, two- and three-dimensional lattices, there is a lower bound on the energy of a breather if the lattice dimension is greater than or equal to a certain critical value.

Theorem 3.1 resolves this conjecture for ground state breathers of the d-dimensional discrete nonlinear Schrödinger equation (DNLS):

$$\mathrm{i}\partial_t \psi_l = -\kappa(\delta^2\vec{\psi})_l - |\psi_l|^{2\sigma}\psi_l, \qquad \kappa > 0 \tag{1.1}$$

Here, $\vec{\psi} = \{\psi_l(t)\}$, $l \in \mathbb{Z}^d$, $t \in \mathbb{R}$, δ^2 denotes the d-dimensional discrete Laplacian on $\mathbb{Z}^d$ given by:

$$(\delta^2\vec{\psi})_l = \sum_{m\in N_l} \psi_m - 2d\psi_l, \tag{1.2}$$

where N_l denotes the set of $2d$ nearest neighbours of the point in $\mathbb{Z}^d$ with label l. The parameter κ can be interpreted as a discretization parameter, $\kappa \sim h^{-2}$, where h is the lattice spacing and $\psi_l = \psi(hl)$, $l \in \mathbb{Z}^d$. The parameter $\sigma > 0$ is a measure of the degree of nonlinearity.

Theorem 3.1 states that there is a ground state l^2-excitation threshold if and only if $\sigma \geqslant \frac{2}{d}$. For $\sigma < \frac{2}{d}$ breathers of arbitrarily small l^2 norm exist. See [11–13] for a study of the bifurcation of small amplitude states from the edge of the plane-wave spectrum. In contrast, the continuum limit nonlinear Schrödinger equation, (3.1), has an L^2 threshold only in the case of critical nonlinearity, $\sigma = \frac{2}{d}$. This is a manifestation of the role of discreteness, which breaks the dilation invariance of the continuum case; see the discussion and analysis of sections 3 and 4. Theorem 2.2 states that ground states are nonlinearly dynamically stable in an orbital sense; see also [19].

In section 5 we consider the limiting behaviour of ground states of *total power*: $\|\vec{\psi}\|_{l^2}^2 = \nu$, as ν tends to infinity. Such ground states are found to have large amplitude. As ν is increased they are increasingly concentrated about one lattice site. A phenomenon of this type has been observed for the systems (1.7), (6.1), and analytically studied in [27, 30]. The relation of this result to the numerical work of [4] and to the work on the *anti-integrable limit* [3, 21] is also discussed.

Studies of discrete breathers originated in the context of classical nonlinear wave equations.

An example is the one-dimensional Klein–Gordon equation:

$$\partial_t^2 u_n = D(u_{n+1} - 2u_n + u_{n-1}) - \Omega_0^2 u_n + u_n^3. \tag{1.3}$$

The techniques of this paper do not directly apply to give rigorous thresholds for discrete nonlinear Klein–Gordon equation localized states. However, our results concerning DNLS are related, through a multiple scale approximation, appropriate to the limit of large lattice spacing, h. Specifically, let $h = \kappa^{-\frac{1}{2}}\varepsilon^{-1}$, and therefore $D = \varepsilon^2\kappa$. Then, seeking a solution of the form:

$$u_n = \varepsilon\Psi_n + \varepsilon^2\Phi_n + \varepsilon^3\ \chi_n + \cdots \tag{1.4}$$

we find an approximate solution which is valid for times, t, of order ε^{-2} with

$$\Psi_n = \Psi_n(t, T) = \mathrm{e}^{-\mathrm{i}\Omega_o t}\psi_n(T) + \mathrm{e}^{\mathrm{i}\Omega_o t}\overline{\psi_n}(T), \qquad T = \tfrac{1}{2}\varepsilon^2 t \tag{1.5}$$

where $\psi_n(T)$ satisfies the discrete nonlinear Schrödinger equation (1.1). In particular, this yields, using the results of this paper on DNLS approximate solutions of the form:

$$u_n^{\varepsilon}(t;\omega) = 2\varepsilon\cos([\Omega_0 + \varepsilon^2\omega]t + \gamma)g_n + \mathcal{O}(\varepsilon^2) \tag{1.6}$$

where $\omega < 0$ and $\vec{g} = \vec{g}_\omega = \{g_n\}_{n\in\mathbb{Z}} \in l^2(\mathbb{Z})$.

Finally, in section 6 we discuss and extend results on excitation thresholds for ground states of a class of coupled systems of nonlinear Schrödinger equations (CNLS), which arises in the modelling of pulse propagation through a coupled network of optical fibres [1, 2, 6, 20, 27, 30]:

$$\begin{aligned} &\mathrm{i}\partial_t\psi_l + \partial_x^2\psi_l + \kappa(\delta^2\vec{\psi})_l + 2|\psi_l|^2\psi_l = 0, \\ &\vec{\psi} = \{\psi_l(t, x)\}_{l\in\mathbb{Z}^d}, \qquad (t, x) \in \mathbb{R}^2. \end{aligned} \tag{1.7}$$

The cases of physical interest are $d = 1, 2$. Here, ψ_l denotes the slowly varying envelope of the highly oscillatory electric field in the fibre with position l in the lattice. We consider the case where l varies over $\mathbb{Z}^d$, with $\sum_l \|\psi_l\|^2_{L^2(\mathbb{R})} < \infty$. For the case $\sigma = d = 1$, we obtain numerical upper and lower bounds (6.23) for the excitation thresholds ν_c†. Other boundary conditions are discussed in [27, 30]. In particular, a result of the analysis is that there are no excitation thresholds in the case when the system is periodic in the discrete variable, l; ground states of arbitrary positive total power $\nu = \sum_l \|\psi_l\|_2^2$ exist.

In this paper, we use observations about the scaling structure of variational problems together with compactness methods in the calculus of variations; see e.g. [5, 22]. Thresholds for the excitation of breathers or nonlinear bound states are characterized in terms of the optimal (best) constant of discrete interpolation inequalities for elements of $l^2(\mathbb{Z}^d)$ in the case of (1.1) and for elements of $l^2(\mathbb{Z}^d; H^1(\mathbb{R}))$ in the case of (1.7). This is related to the approach taken in [28, 29] on a sharp criterion on initial conditions for global existence (no finite time blow-up) of solutions to the continuum nonlinear Schrödinger equation on $\mathbb{R}^d$, (3.1), with critical power nonlinearity. Results on excitation thresholds, stability, and other issues for the semi-discrete class of nonlinear Schrödinger equations were obtained by Yeary and this author [27, 30]. This paper is a detailed account with extensions of the work on excitation thresholds.

2. DNLS and a variational characterization of its ground state

By standard methods, one can check that for any $\vec{\psi}(t = 0) \in l^2(\mathbb{Z}^d)$, there is a unique global solution $\vec{\psi} \in C^1(\mathbb{R}; l^2(\mathbb{Z}^d))$ of DNLS, (1.1), and for which the following two quantities are

† An error in these bounds due to faulty algebra appeared in [27] and is corrected here.

independent of time:

$$\mathcal{H}_D[\vec{\psi}] = -\kappa(\delta^2\vec{\psi}, \vec{\psi}) - \frac{1}{\sigma+1}\sum_{l\in\mathbb{Z}^d} |\psi_l|^{2\sigma+2}, \tag{2.1}$$

$$\mathcal{N}_D[\vec{\psi}] = \sum_{l\in\mathbb{Z}^d} |\psi_l|^2. \tag{2.2}$$

The subscript, 'D', is used to indicate a quantity associated with the discrete equation (1.1). $\mathcal{H}_D$ is a Hamiltonian for (1.1), which can be written as:

$$\mathrm{i}\partial_t\vec{\psi} = \frac{\delta\mathcal{H}_D}{\delta\vec{\psi}^*}. \tag{2.3}$$

In various applications the invariant $\mathcal{N}$ has the interpretation of *total power* or of *particle number*. The term, $(-\delta^2\vec{\psi}, \vec{\psi})$, may be written out explicitly as:

$$(-\delta^2\vec{\psi}, \vec{\psi}) = \sum_{r=1}^{d}\sum_{l\in\mathbb{Z}^d} |\psi_l - \psi_{\tau_r l}|^2, \tag{2.4}$$

where τ_r denotes translation by one lattice unit in the rth coordinate direction.

Of particular interest are spatially localized and time-periodic solutions. We seek them in the form:

$$\begin{aligned} &\psi_l(t) = \mathrm{e}^{-\mathrm{i}\omega t} g_l, \qquad l \in \mathbb{Z}^d, \qquad t \in \mathbb{R}, \\ &\psi_l(t) \in l^2(\mathbb{Z}^d). \end{aligned} \tag{2.5}$$

where ω is real. A solution of this type is frequently called a nonlinear *bound state, standing wave* or *stationary state*. The term *discrete breather* is also used but is sometimes reserved for a localized state whose modulus oscillates.

Substitution of (2.5) into (1.1) yields the system of algebraic equations plus the 'boundary condition at infinity':

$$\omega g_l = -\kappa(\delta^2 g)_l - |g_l|^{2\sigma} g_l. \tag{2.6}$$

$$\vec{g} = \{g_l\}_{l\in\mathbb{Z}^d} \in l^2(\mathbb{Z}^d). \tag{2.7}$$

We construct a ground state by variational methods. To motivate our approach, we consider the quantum mechanical problem:

$$H\Psi = E\psi, \tag{2.8}$$

where $H = -\Delta + V(x)$ for a bound state, $\Psi \in L^2$ with $\|\Psi\|_2 = 1$. We assume $V(x)$ is a sufficiently smooth and rapidly decaying 'potential well'. Consider the constrained minimization problem:

$$\mathcal{I} = \inf\{(Hf, f) : \|f\|_2 = 1\}. \tag{2.9}$$

If $\mathcal{I} < 0$, then $E_g \equiv \mathcal{I}$ is the ground state (lowest) eigenvalue and there exists a ground state eigenstate $\Psi_g(x)$ such that

$$H\Psi_g = E_g\Psi_g, \qquad \|\Psi_g\|_2 = 1. \tag{2.10}$$

The time-periodic *breather* or standing wave, $\Psi_g(x)\mathrm{e}^{-\mathrm{i}E_g t}$, is a dynamically stable solution of the time-dependent Schrödinger equation

$$\mathrm{i}\partial_t\Psi = H\Psi. \tag{2.11}$$

We shall characterize the ground state of (1.1) using a nonlinear analogue of (2.9).

Definition. *Let*

$$\mathcal{I}_\nu = \inf\{\mathcal{H}_D[\vec{f}] : \mathcal{N}_D[\vec{f}] = \nu\}. \tag{2.12}$$

A minimizer of the variational problem (2.12) is called a ground state.

Clearly, $\mathcal{I}_\nu$ is bounded below: for,

$$\mathcal{H}_D[\vec{f}] \geqslant -\frac{1}{\sigma+1}\sum_l |f_l|^{2\sigma+2} \geqslant -\frac{1}{\sigma+1}\|\vec{f}\|_\infty^{2\sigma}\|\vec{f}\|_2^2 \geqslant -\frac{1}{\sigma+1}\nu^{\sigma+1}. \tag{2.13}$$

Theorem 2.1. *(a) If $-\infty < \mathcal{I}_\nu < 0$, then the minimum in (2.12) is attained.*

(b) Every minimizing sequence associated with the variational problem (2.12) is precompact modulo phase translations, i.e. for any minimizing sequence $\{\vec{g}^{(k)}\}$, there is a subsequence $\{\vec{g}^{(n_k)}\}$ and a sequence $\{\gamma_{n_k}\}$, and translations, $\tau(l^k)$ (where $\tau(l^k)\vec{g}^{(k)} = \{g_{j+l^k}^{(n_k)}\}_{j\in\mathbb{Z}^d}$), such that $\tau(l_k)\vec{g}^{(n_k)}\mathrm{e}^{\mathrm{i}\gamma_{n_k}}$ converges in $l^2(\mathbb{Z}^d)$ to a minimizer.

(c) If $\vec{g} = \{g_l\}_{l\in\mathbb{Z}^d}$ is a minimizer for the variational problem (2.12), then there exists $\omega = \omega(\nu) < 0$ such that the Euler–Lagrange equation:

$$\omega(\nu) g_l = -\kappa \left(\delta^2 g\right)_l - |g_l|^{2\sigma} g_l, \qquad l \in \mathbb{Z}^d \tag{2.14}$$

holds, together with the L^2 constraint:

$$\mathcal{N}_D[\vec{g}] = \sum_l |g_l|^2 = \nu. \tag{2.15}$$

This theorem can be proved by a standard application of concentration compactness ideas in the discrete context [22]; see [30]. An outline of the proof is presented in the appendix.

2.1. Dynamical stability

Before stating a precise result, we first introduce some terminology and notation.

Definitions.

(1) *Let $\mathcal{G}_\nu$ denote the set of all solutions of the minimization from (2.12), i.e. the set of ground states with $\mathcal{N} = \nu$.*

(2) *Given a particular ground state $\vec{g}$, we define its* orbit *to be the set:*

$$\mathcal{O}(\vec{g}) = \{\mathrm{e}^{\mathrm{i}\gamma}\vec{g} : \gamma \in [0, 2\pi)\}. \tag{2.16}$$

(3) *The* distance $\rho\left(\vec{\psi}, \mathcal{G}_\nu\right)$ *from $\psi \in l^2$ to the set of ground states, $\mathcal{G}_\nu$ is given by:*

$$\rho(\vec{\psi}, \mathcal{G}_\nu) \equiv \inf_{\vec{g}\in\mathcal{G}_\nu} \|\vec{\psi} - \vec{g}\|_{l^2(\mathbb{Z}^d)}. \tag{2.17}$$

Remark. We conjecture that the ground state with $\mathcal{N} = \nu$ is essentially unique, i.e. if $\vec{g}$ is any ground state with $\mathcal{N}[\vec{g}] = \nu$, then $\mathcal{G}_\nu = \mathcal{O}(\vec{g})$.

A consequence of part (*b*) of theorem 2.1 is the following [7, 26]:

Theorem 2.2. *Ground states of (1.1) are* orbitally Lyapunov stable *in the sense that: given any $\varepsilon > 0$, there is a $\delta > 0$ such that if the initial data $\vec{\psi}(t = 0) = \vec{\psi}_0$ satisfies*

$$\rho(\vec{\psi}_0, \mathcal{G}_\nu) < \delta, \tag{2.18}$$

then for all $t \neq 0$

$$\rho(\vec{\psi}(t), \mathcal{G}_\nu) < \varepsilon. \tag{2.19}$$

3. Excitation thresholds for DNLS

For a fixed lattice dimension, d, we consider the family of equations (1.1) parametrized by σ. Theorem 2.1 gives a criterion for the existence of a ground state.

Definition. *If for any $\nu > 0$ the variational problem (2.12) has a strictly negative infimum, $\mathcal{I}_\nu < 0$ then, by theorem 2.1, a ground state exists for any $\nu > 0$. In this case we say that there is* no excitation threshold*. However, if there is a strictly positive constant ν^D_{thresh} (which may depend on d and σ) such that $\mathcal{I}_\nu < 0$ if and only if $\nu > \nu^D_{\text{thresh}}$, then we call ν^D_{thresh} an* excitation threshold *or* L^2 excitation threshold *for a ground state.*

The main result concerning DNLS is the following:

Theorem 3.1. *(1) Let $0 < \sigma < \frac{2}{d}$. Then, $\mathcal{I}_\nu < 0$ for all $\nu > 0$. Therefore, the variational problem (2.12) has a solution for all $\nu > 0$ and there is no excitation threshold.*
(2) Let $\sigma \geqslant \frac{2}{d}$. Then, there exists a ground state excitation threshold, $\nu^D_{\text{thresh}} > 0$.

Remark on DNLS versus NLS. Here we contrast the discrete equation, DNLS, and its continuum limit. In particular, we comment on some consequences of the breaking of various symmetries in passing from NLS to DNLS.

(1) The continuum limit of (1.1) ($\kappa = h^{-2}$, $h =$ lattice spacing, and $h \to 0$) is the d-dimensional nonlinear Schrödinger equation:

$$\mathrm{i}\partial_t \phi = -\Delta\phi - |\phi|^{2\sigma}\phi. \tag{3.1}$$

For initial data $\phi(t = 0, x) \in H^1(\mathbb{R}^d)$, it has been shown that there exists a local solution which is continuous in time with values in $H^1(\mathbb{R}^d)$ and which satisfies the analogous conservation laws [16, 18]. If $\sigma < \frac{2}{d}$ solutions are always global in time, while for $\sigma \geqslant \frac{2}{d}$, finite energy initial data may give rise to a solution which leaves the space $H^1(\mathbb{R}^d)$ after a finite time [17, 25, 28]. In contrast, the evolution for (1.1) is globally defined in time.

(2) Solitary standing waves can be found by methods analogous to those used in section 2. An excitation threshold for standing waves, in terms of the natural L^2 invariant:

$$\mathcal{N}_{\text{NLS}}[\phi] = \int_{\mathbb{R}^d} |\phi(x)|^2\,\mathrm{d}x \tag{3.2}$$

can exist only in the case $\sigma = \frac{2}{d}$. This follows because under the scaling:

$$\phi(x, t) \mapsto \phi_\rho(x, t) \equiv \rho^{\frac{1}{\sigma}}\phi(\rho x, \rho^2 t), \tag{3.3}$$

we find

$$\mathcal{N}[\phi_\rho] = \rho^{\frac{2}{\sigma}-d}\mathcal{N}[\phi]. \tag{3.4}$$

Thus, given that a single standing wave exists, if $\sigma \neq \frac{2}{d}$, scaling can be used to find one of arbitrarily small total power, $\mathcal{N}_{\text{NLS}}$. In contrast, the dilation symmetry is broken in the discrete case.

(3) Let $\sigma = \frac{2}{d}$, and let R denote the ground state standing wave. That is, R is an H^1 solution of $\Delta R - R + R^{\frac{4}{d}+1} = 0$ of minimal power $\mathcal{N}_{\text{NLS}} \equiv \mathcal{N}_{\text{thresh}}$. In [28, 29] it was proved that if

$$\mathcal{N}[\phi_0] < \mathcal{N}_{\text{thresh}} \tag{3.5}$$

then the solution exists for all time and disperses to zero in the sense that $\|\phi(t)\|_{L^p} \to 0$, as $|t| \to \infty$, for $p > 2$ if $d = 1, 2$ and $2 < p < 2d/(d-2)$ if $d \geqslant 3$.

Conjecture. *If $\mathcal{N}[\vec{\psi}_0] < \nu^D_{\text{thresh}}$, then the solution of DNLS disperses to zero in the sense that for any $p \in (2, \infty]$:*

$$\|\vec{\psi}(t)\|_{l^p(\mathbb{Z}^d)} \to 0, \qquad \text{as} \quad |t| \to \infty. \tag{3.6}$$

Proofs of the assertions in (3) are given in [28, 29] and rely on the pseudo-conformal symmetry of the continuum limit NLS, a symmetry which is absent in DNLS.

Theorem 3.1 is a consequence of propositions 4.1 and 4.2 of section 4. We begin by investigating the conditions on σ, d, and ν under which $\mathcal{I}_\nu < 0$.

Proposition 3.1. $\mathcal{I}_\nu \geqslant 0$ *if and only if ν is such that the following inequality holds for all $\vec{u} \in l^2(\mathbb{Z}^d)$:*

$$\sum_{l\in\mathbb{Z}^d} |u_l|^{2\sigma+2} \leqslant (\sigma+1)\kappa\nu^{-\sigma}\left(\sum_{l\in\mathbb{Z}^d}|u_l|^2\right)^{\sigma}(-\delta^2\vec{u}, \vec{u}). \tag{3.7}$$

To prove proposition 3.1, we observe that $\mathcal{I}_\nu \geqslant 0$ if, and only if, for all $\vec{u} \in l^2(\mathbb{Z}^d)$, with $\|\vec{u}\|^2_{l^2} = \nu$

$$(\sigma+1)^{-1}\sum_{l\in\mathbb{Z}^d}|u_l|^{2\sigma+2} \leqslant \kappa(-\delta^2\vec{u}, \vec{u}). \tag{3.8}$$

Let $\vec{0} \neq \vec{v} \in l^2$ be arbitrary. Then, if $\vec{u}$ defined by:

$$\vec{u} \equiv \sqrt{\nu}\|\vec{v}\|^{-1}_{l^2}\vec{v} \tag{3.9}$$

satisfies the inequality (3.8), which after some algebra yields (3.7). Finally, if $\mathcal{I}_\nu \geqslant 0$ we have that $\mathcal{I}_\nu = 0$. This is seen by simply taking a sequence whose Nth element is a constant (depending on ν) on the set of sites satisfying $|l| \leqslant N$ and zero otherwise. Along such a sequence we have $\mathcal{N} = \nu$ and $\mathcal{H}$ tending to zero. Therefore, $\mathcal{I}_\nu = 0$.

Strategy of the proof of theorem 3.1. Clearly, if the inequality (3.7) holds for some ν_1 then it holds for all $\nu \leqslant \nu_1$. We shall prove in proposition 4.2(d) that a ground state does not exist for any $0 \leqslant \nu \leqslant \nu_1$. We are interested in characterizing ν_{thresh} defined by:

$$\nu^D_{\text{thresh}} \equiv \sup\{\nu : \text{inequality}(3.7)\text{ holds}\}. \tag{3.10}$$

In the following section we relate this threshold value to the optimal (best) constant in an interpolation estimate related to the Hamiltonian energy, $\mathcal{H}$. If a finite positive ν^D_{thresh} exists, then for any $\nu > \nu^D_{\text{thresh}}$ and element of $l^2(\mathbb{Z}^d)$, $\vec{u}_*$, can be found which violates the inequality (3.7). This choice of $\vec{u}_*$ shows that $\mathcal{I}_\nu < 0$, and by theorem 2.1 there is a ground state. If, however, for any choice of $\nu > 0$ one can construct an element of l^2 for which the inequality (3.7) is violated, theorems 3.3 and 2.1 imply that a ground state exists for any $\nu > 0$, i.e. there is no excitation threshold. The strategy used to prove theorem 3.1 is to show that if $0 < \sigma < \frac{2}{d}$, then there is no value of ν for which the inequality (3.7) holds for arbitrary $\vec{u} \in l^2$. However, if $\sigma \geqslant \frac{2}{d}$ we show it holds if and only if $\nu \leqslant \nu^D_{\text{thresh}}$, for some $\nu^D_{\text{thresh}} > 0$.

4. Best constants and excitation thresholds for DNLS

In this section we relate the problem of characterizing excitation thresholds to the problem of finding the optimal or best constant in discrete interpolation inequalities of Sobolev–Nirenberg–Gagliardo type.

The discussion concluding section 3 motivates the following question, answered in theorem 4.1 below.

When does there exist a constant $C > 0$ such that for all $\vec{u} = \{u_l\} \in l^2(\mathbb{Z}^d)$:

$$\sum_{l\in\mathbb{Z}^d} |u_l|^{2\sigma+2} \leqslant C \left(\sum_{l\in\mathbb{Z}^d} |u_l|^2\right)^\sigma (-\delta^2\vec{u}, \vec{u})? \tag{4.1}$$

If (4.1) holds for some $C > 0$ and C_* is the infimum over all such constants, then ν^D_{thresh} defined by

$$(\sigma + 1)\kappa(\nu^D_{\text{thresh}})^{-\sigma} \equiv C_* \tag{4.2}$$

is a ground state excitation threshold. Therefore, we seek to characterize the optimal constant, C_*. If there is a strictly positive and finite C_*, then

$$\frac{1}{C_*} = \mathcal{J}^{\sigma,d} \equiv \inf \frac{(\sum_{l\in\mathbb{Z}^d} |u_l|^2)^\sigma(-\delta^2\vec{u}, \vec{u})}{\sum_{l\in\mathbb{Z}^d} |u_l|^{2\sigma+2}} \tag{4.3}$$

and we have:

$$\nu^D_{\text{thresh}} = ((\sigma + 1)\kappa\mathcal{J}^{\sigma,d})^{\frac{1}{\sigma}}. \tag{4.4}$$

Remark. If $\mathcal{J}^{\sigma,d} > 0$, then by proposition 4.2 below, there exists a strictly positive lower bound on the energy, $\mathcal{N}$, of a ground state.

Note that (4.4) is consistent with the simple observation that for the case of uncoupled lattice sites, $\kappa = 0$, there is no excitation threshold. For example, in this case the solution

$$\begin{aligned} \psi_0(t) &= \nu^{\frac{1}{2}} e^{i|\nu|^\sigma t} \\ \psi_l(t) &= 0, \qquad l \neq 0, \end{aligned} \tag{4.5}$$

is a $l^2(\mathbb{Z}^d)$ solution of (1.1) with $\mathcal{N} = \nu$. This limit is also called the *anti-integrable limit* [3, 21]. In section 5 we shall relate the anti-integrable limit to the large amplitude limit of our variationally constructed ground states.

Proposition 4.1. *If $\sigma < \frac{2}{d}$, then $\mathcal{J}^{\sigma,d} = 0$. Therefore, for $\sigma < \frac{2}{d}$, and there is no ground state excitation threshold ($\nu^D_{\text{thresh}} = 0$). In other words, ground states of arbitrary energy, $\mathcal{N}$, exist.*

Proof of proposition 4.1. Consider the one parameter family of trial functions, $\vec{u}(\alpha)$ defined by:

$$u_l(\alpha) = e^{-\alpha|l|}, \tag{4.6}$$

where $l = (l_1, \ldots, l_d) \in \mathbb{Z}^d$, $|l| = |l_1| + \ldots + |l_d|$ and $\alpha > 0$. Evaluation of the terms of the quotient in (4.3) yields, for $\alpha \downarrow 0$:

$$\sum_{l\in\mathbb{Z}^d} |u_l|^2 \sim \alpha^{-d}, \qquad (-\delta^2\vec{u}, \vec{u}) \sim \alpha^{2-d}, \qquad \sum_{l\in\mathbb{Z}^d} |u_l|^{2\sigma+2} \sim \alpha^{-d}. \tag{4.7}$$

Therefore, the quotient in (4.3) is of order $\alpha^{2-d\sigma}$, which tends to zero as α tends to zero if $\sigma < \frac{2}{d}$. This proves the proposition 4.1.

□

Proposition 4.2. *Let $\sigma \geqslant \frac{2}{d}$. Then,*

(a) $\mathcal{J}^{\sigma,d} > 0$.
(b) If $\|\vec{\psi}\|^2_{l^2} = \nu$, then

$$\mathcal{H}_D[\vec{\psi}] \geqslant \kappa(-\delta^2\vec{\psi}, \vec{\psi}) \left[1 - \left(\frac{\nu}{\nu^D_{\text{thresh}}}\right)^\sigma\right], \tag{4.8}$$

where $\nu^D_{\text{thresh}} > 0$ is given by (4.4). For $\sigma \geqslant \frac{2}{d}$, $\nu^D_{\text{thresh}}(\sigma, d)$ is an excitation threshold, i.e.

(c) if $\nu > \nu^D_{\mathrm{thresh}}(\sigma, d)$ *then* $\mathcal{I}_\nu < 0$ *and a ground state exists, and*
(d) if $\nu < \nu^D_{\mathrm{thresh}}(\sigma, d)$*, then* $I_\nu = 0$ *and there is no ground state minimizer of (2.12).*

Proof of proposition 4.2. To prove part (a), it suffices to show that the inequality (4.1) holds for *some* positive constant, C. Then part (c) follows from the discussion at the end of section 3. We proceed as follows. For functions $f \in H^1(\mathbb{R}^n)$, one has the Sobolev–Nirenberg–Gagliardo inequality [15]:

$$\|f\|^{2\sigma+2}_{2\sigma+2} \leqslant C\|\nabla f\|^{\sigma n}_2 \; \|f\|^{2+\sigma(2-n)}_2, \tag{4.9}$$

where σ is restricted to satisfy:

$$\begin{aligned} &0 < \sigma < \infty, \qquad n = 1, 2 \\ &0 < \sigma < 2(n-2)^{-1}, \qquad n \geqslant 3. \end{aligned} \tag{4.10}$$

The proof of (4.9) can be followed closely to yield, under the same restrictions on σ, the following estimate in the discrete case for $\vec{u} \in l^2(\mathbb{Z}^d)$:

$$\sum_{l\in\mathbb{Z}^d} |u_l|^{2\sigma+2} \leqslant C\left(\sum_{l\in\mathbb{Z}^d} |u_l|^2\right)^{1+\frac{\sigma}{2}(d-2)} (-\delta^2\vec{u}, \vec{u})^{\frac{\sigma d}{2}}. \tag{4.11}$$

To give the idea, we present the proof of (4.11) in the case $d = 2$. We write $u_l = u_{ab}$, $(a, b) \in \mathbb{Z}^2$. Without loss of generality we can take $u_{ab} \geqslant 0$. Note that

$$u^{\sigma+1}_{ab} = \sum_{\alpha=-\infty}^{a} (u^{\sigma+1}_{\alpha b} - \, u^{\sigma+1}_{\alpha-1,b}). \tag{4.12}$$

By the fundamental theorem of calculus,

$$\begin{aligned} u^{\sigma+1}_{\alpha b} - u^{\sigma+1}_{\alpha-1,b} &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}[s u_{\alpha b} + (1-s)u_{\alpha-1,b}]^{\sigma+1}\,\mathrm{d}s \\ &= (\sigma+1)\int_0^1 [s u_{\alpha b} + (1-s)u_{\alpha-1,b}]^{\sigma}\,\mathrm{d}s (u_{\alpha b} - u_{\alpha-1,b}). \end{aligned}$$

Therefore, (using the convention that sums without specified upper and lower limits are understood to be taken over all $\mathbb{Z}$)

$$|u^{\sigma+1}_{\alpha b} - u^{\sigma+1}_{\alpha-1,b}| \leqslant |\sigma+1| \sum_\alpha \max(|u_{\alpha b}|^\sigma, |u_{\alpha-1,b}|^\sigma)|u_{\alpha b} - u_{\alpha-1,b}|. \tag{4.13}$$

It follows by summing over α and applying the Cauchy–Schwarz inequality, that

$$\sum_\alpha |u^{\sigma+1}_{\alpha b} - u^{\sigma+1}_{\alpha-1,b}| \leqslant 2^{\frac{1}{2}}|\sigma+1|\left(\sum_\alpha |u_{\alpha b}|^{2\sigma}\right)^{\frac{1}{2}}\left(\sum_\alpha |u_{\alpha b} - u_{\alpha-1,b}|^2\right)^{\frac{1}{2}}. \tag{4.14}$$

The analogous computation can be performed by summing on the second index, to get:

$$\sum_\beta |u^{\sigma+1}_{a\beta} - u^{\sigma+1}_{a,\beta-1}| \leqslant 2^{\frac{1}{2}}|\sigma+1|\left(\sum_\beta |u_{a\beta}|^{2\sigma}\right)^{\frac{1}{2}}\left(\sum_\beta |u_{a\beta} - u_{a,\beta-1}|^2\right)^{\frac{1}{2}}. \tag{4.15}$$

The product of the last two estimates yields:

$$\begin{aligned} |u_{ab}|^{2\sigma+2} \leqslant 2|\sigma+1|^2 &\left(\sum_\alpha |u_{\alpha b}|^{2\sigma}\right)^{\frac{1}{2}}\left(\sum_\alpha |u_{\alpha b} - u_{\alpha-1,b}|^2\right)^{\frac{1}{2}}\left(\sum_\beta |u_{a\beta}|^{2\sigma}\right)^{\frac{1}{2}} \\ &\times\left(\sum_\beta |u_{a\beta} - u_{a,\beta-1}|^2\right)^{\frac{1}{2}}. \end{aligned}$$

Summing on a and applying the Cauchy–Schwarz inequality gives:

$$\sum_a |u_{ab}|^{2\sigma+2} \leqslant 2|\sigma+1|^2 \left(\sum_\alpha |u_{\alpha b}|^{2\sigma}\right)^{\frac{1}{2}} \left(\sum_\alpha |u_{\alpha b} - u_{\alpha-1,b}|^2\right)^{\frac{1}{2}} \left(\sum_{a,\beta} |u_{a\beta}|^{2\sigma}\right)^{\frac{1}{2}}$$
$$\times \left(\sum_{a,\beta} |u_{a\beta} - u_{a,\beta-1}|^2\right)^{\frac{1}{2}}.$$

Finally, summing this result on b and applying the Cauchy–Schwarz inequality gives (4.11) for the case $d = 2$ and arbitrary $\sigma > 0$.

To complete the proof of proposition 4.2, we write estimate (4.11) as:

$$\sum_{l\in\mathbb{Z}^d} |u_l|^{2\sigma+2} \leqslant C \left(\sum_{l\in\mathbb{Z}^d} |u_l|^2\right)^\sigma (-\delta^2\vec{u}, \vec{u}) \left(\frac{(-\delta^2\vec{u}, \vec{u})}{\sum_{l\in\mathbb{Z}^d} |u_l|^2}\right)^{\frac{\sigma d}{2}-1} \tag{4.16}$$

The last factor in (4.16) is bounded by a constant for $\sigma \geqslant \frac{2}{d}$; the discrete Laplacian is a bounded operator. Therefore, if in addition to (4.10), we have $\sigma \geqslant \frac{2}{d}$, then the estimate (4.1) holds.

Finally, we want to show that for $d \geqslant 3$ we can relax the constraint $0 < 2 - \sigma(d-2)$. Suppose $d \geqslant 3$ and take σ in the range for which we know the estimate (4.1) to hold. This estimate is equivalent to:

$$\sum_{l\in\mathbb{Z}^d} |u_l|^{2\sigma+2} \leqslant C(-\delta^2\vec{u}, \vec{u}) \tag{4.17}$$

subject where $\vec{u}$ satisfies the constraint

$$\sum_{l\in\mathbb{Z}^d} |u_l|^2 = 1. \tag{4.18}$$

The constraint (4.18) implies that for all $l \in \mathbb{Z}^d$, $|u_l| \leqslant 1$ and therefore if σ_1, is *any* number satisfying $\sigma_1 > \sigma \geqslant \frac{2}{d}$, then the estimate (4.17) holds with σ replaced by σ_1. This implies the following result which completes the proof of part (a) of proposition 4.2:

Theorem 4.1. *For $\sigma \geqslant \frac{2}{d}$, the interpolation inequality (4.1) holds.*

Remark. Note that there is no upper restriction for $n \geqslant 3$ on σ as in the continuum case (4.9). Through (4.16), the boundedness of the discrete Laplacian, $-\delta^2$, on $l^2(\mathbb{Z}^d)$ plays a key role.

Part (b) of proposition 4.2 follows from the definition of $\mathcal{H}_D$ and the inequality (3.7) with optimal choice $\nu = \nu^D_{\text{thresh}}$ given by (4.2).

Finally, we prove part (d) of proposition 4.2. Suppose $\nu < \nu^D_{\text{thresh}}$. Then, by part ($b$), $\mathcal{I}_\nu \geqslant 0$. On the other hand, as at the end of the proof of proposition 3.1, we have that $\mathcal{I}_\nu \leqslant 0$. It follows that $I_\nu = 0$ for any $\nu < \nu^D_{\text{thresh}}$. If the minimum is attained at a state $\vec{\psi}$, then

$$\kappa(-\delta^2\vec{\psi}, \vec{\psi}) = \frac{1}{\sigma+1} \sum_l |\psi_l|^{2\sigma+2}$$
$$\sum_l |\psi_l|^2 = \nu.$$

Since $\sigma \geqslant \frac{2}{d}$, ν^D_{thresh} defined by (4.4) is strictly positive and by (3.7), with the optimal choice $\nu = \nu^D_{\text{thresh}}$, we have

$$\kappa(-\delta^2\vec{\psi}, \vec{\psi}) \leqslant \kappa \left(\frac{\nu}{\nu^D_{\text{thresh}}}\right)^\sigma (-\delta^2\vec{\psi}\ \vec{\psi}) < \kappa(-\delta^2\vec{\psi}, \vec{\psi}), \tag{4.19}$$

a contradiction.

□

Theorem 3.1 now follows from propositions 4.1 and 4.2.

5. Large amplitude and the anti-integrable limit

5.1. Large ν limit of ground states

As discussed in [21], breather solutions of DNLS can also be constructed perturbatively in the limit of zero coupling, $\kappa \equiv 0$, also called the *anti-integrable limit*; see also [3]. In [21], as an explanation for the numerical studies in [4], it is conjectured that the large amplitude *anti-integrable limit breathers* play an important role in the dynamics of DNLS. We now give evidence of this, by showing the connection between the nonlinearly stable ground state breathers constructed by variational methods and the large amplitude *anti-integrable breathers*. We also prove that as ν increases, ground state breathers of 'total power' ν grow in amplitude and become increasingly concentrated about one lattice site. This property of ground states and their nonlinear stability (theorem 2.2) elucidate the numerical simulations in [4].

We begin by considering a scaled version of the variational problem (2.12), for the DNLS ground state:

$$\mathcal{I}_\nu = \inf \left\{ -\kappa(\delta^2 \vec{f}, \vec{f}) - \frac{1}{\sigma+1} \sum_l |f_l|^{2\sigma+2} : \sum_l |f_l|^2 = \nu \right\}. \tag{5.1}$$

In anticipation of our taking $\nu \uparrow \infty$, we set

$$f_l = \nu^{\frac{1}{2}} F_l, \qquad \sum_l |F_l|^2 = 1. \tag{5.2}$$

and introduce the parameter

$$\alpha = \alpha(\nu, \kappa; \sigma) \equiv \frac{\kappa}{\nu^\sigma}, \tag{5.3}$$

which tends to zero as ν tends to infinity. The variational problem, (5.1), is then equivalent to:

$$\mathcal{K}(\alpha) = \inf \left\{ \alpha(-\delta^2 \vec{F}, \vec{F}) - \frac{1}{\sigma+1} \sum_l |F_l|^{2\sigma+2} : \sum_l |F_l|^2 = 1 \right\}. \tag{5.4}$$

By theorem 3.1, if $\nu \geqslant \nu^D_{\text{thresh}} \geqslant 0$, then there is a ground state breather:

$$\vec{G} = \vec{G}(\alpha) = \{G_l(\alpha)\}_{l\in\mathbb{Z}^d}, \tag{5.5}$$

satisfying the Euler–Lagrange equation:

$$-\alpha(\delta^2 G)_l - |G_l|^{2\sigma} G_l = \lambda_\alpha G_l, \tag{5.6}$$

where λ_α is a Lagrange multiplier. By (5.2), this gives rise to a ground state family of solutions of (5.1):

$$\vec{\psi}_g(t) = \nu^{\frac{1}{2}} \vec{G}(\alpha) \mathrm{e}^{-\mathrm{i}\lambda_\alpha \nu^\sigma t} \mathrm{e}^{\mathrm{i}\gamma}, \qquad \gamma \in [0, 2\pi). \tag{5.7}$$

By theorem 2.2, the ground state family is nonlinearly orbitally stable, and is therefore expected to participate in the dynamics.

What is the structure of ground states for large ν? We next show that as $\nu \to \infty$, ground states become concentrated on the lattice about a single site.

To see this, we first observe that by the methods of the appendix (see also [8, 27]):

(a) As α tends to zero ($\nu \uparrow \infty$) through a sequence, $\{\vec{G}(\alpha)\}$, is a minimizing sequence for the limit variational problem:

$$\mathcal{K}_\infty = \inf \left\{ -\frac{1}{\sigma+1} \sum_l |G_l|^{2\sigma+2} : \sum_l |G_l|^2 = 1 \right\}. \tag{5.8}$$

(b) A subsequence can be extracted, which (modulo phase adjustments) converges to a minimizer, $\vec{G}^\infty$. $\vec{G}^\infty$, which satisfies the Euler–Lagrange equation associated with the limit problem (5.8):

$$\begin{aligned} &-|G_l^\infty|^{2\sigma} G_l^\infty = \lambda_0 G_l^\infty, \\ &\sum_l |G_l^\infty|^2 = 1. \end{aligned} \tag{5.9}$$

Thus, for each $l \in \mathbb{Z}^d$, $G_l^\infty \in \{0\} \cup \{(-\lambda)^{\frac{1}{2\sigma}} \mathrm{e}^{\mathrm{i}\gamma} : \gamma \in [0, 2\pi)\}$. Since $\|\vec{G}^\infty\|_{l^2(\mathbb{Z}^d)} = 1$, G_l^∞ can be nonzero only at a finite number of sites, $N \geqslant 1$. Therefore,

$$\begin{aligned} &\|\vec{G}^\infty\|_{l^2(\mathbb{Z}^d)} = 1 \qquad \text{implies} \qquad -\lambda = N^{-\sigma} \\ &-\frac{1}{\sigma+1} \sum_l |G_l^\infty|^{2\sigma+2} = -\frac{N^{-\sigma}}{\sigma+1}. \end{aligned}$$

The minimum is therefore attained for $N = 1$ and we have: $\vec{G}_l^\infty = \pm\delta_{l,l_0}$ for some $l_0 \in \mathbb{Z}^d$, and $\lambda_0 = -1$.

Therefore, as $\nu \to \infty$, a subsequence of ground states converges, to a limiting state:

$$G_l(\alpha) \to G_l^\infty \equiv \delta_{l,l_0}, \qquad \text{in} \quad l^2(\mathbb{Z}^d) \tag{5.10}$$

for some $l_0 \in \mathbb{Z}^d$. Therefore the large ν (α small) limit of ground states behaves as a large amplitude *one-site breather*:

$$\psi_l(t) \sim \pm\nu^{\frac{1}{2}} \delta_{l,l_0} \mathrm{e}^{-\mathrm{i}\nu^\sigma t}, \qquad \text{for} \quad \nu \quad \text{large}. \tag{5.11}$$

5.2. *Connection with the anti-integrable limit*

For small α, equation (5.6) is the anti-integrable limit studied in [21]. The approach taken in [3, 21] is to first observe that for $\alpha = 0$ each lattice site evolves independently and that (5.6) has solutions $\Psi_l(t)$, where for each $l \in \mathbb{Z}^d$, Ψ_l satisfies the equation:

$$\mathrm{i}\partial_t \Psi(t) = -|\Psi(t)|^{2\sigma} \Psi(t). \tag{5.12}$$

The solutions of (5.12) are:

$$\Psi(t) = \omega \mathrm{e}^{\mathrm{i}|\omega|^{2\sigma} t} \mathrm{e}^{\mathrm{i}\gamma}, \tag{5.13}$$

with $\omega, \gamma \in \mathbb{R}$. Fix a solution which is supported at lattice sites $q \in I \subset \mathbb{Z}^d$, where I is finite or infinite and such that at each site the evolution is an oscillation of the form

$$\Psi_q(t) = \omega_q \mathrm{e}^{\mathrm{i}|\omega_q|^{2\sigma} t} \mathrm{e}^{\mathrm{i}\gamma_q}, \qquad q \in I \tag{5.14}$$

and such that the frequencies ω_q are all commensurate. The implicit function theorem implies that these solutions have a continuation for α sufficiently small in the space of time-periodic solutions. These range in spatial complexity from those that are small perturbations of the simplest $\alpha = 0$ breather, consisting of a solution of the form (4.5), to those which are small perturbations of an $\alpha = 0$ 'spatially chaotic' configuration of oscillators.

Consider the continuation from the anti-integrable limit of one-site breathers. These are solutions of the form:

$$\Psi_l(t, \alpha, \mu) = A_l(\alpha, \mu) \mathrm{e}^{-\mathrm{i}\mu t}, \qquad l \in \mathbb{Z}^d \tag{5.15}$$

where

$$\begin{aligned} &\mu A_l = \alpha(\delta^2 A)_l - |A_l|^{2\sigma} A_l, \\ &A_l(\alpha = 0, \mu) = (-\mu)^{\frac{1}{2\sigma}} \delta_{l,l_0}, \qquad \text{for some} \quad l_0 \in \mathbb{Z}^d. \end{aligned} \tag{5.16}$$

We wish to relate the two-parameter family $\vec{A}(\alpha, \mu)$ to the family of scaled ground states $\vec{G}(\alpha)$, for small α. Note that $\vec{A}(0, -1)$ is such that $\|\vec{A}(0, -1)\|_{l^2(\mathbb{Z}^d)} = 1$. It is easy to check, by the implicit function theorem that a locally unique solution $(\alpha, \mu(\alpha))$ defined in a neighbourhood of $\alpha = 0$ exists such that

$$\begin{aligned} &\|\vec{A}(\alpha, \mu(\alpha))\|_{l^2(\mathbb{Z}^d)} = 1, \\ &\mu(0) = -1. \end{aligned}$$

Therefore, by our variational arguments and local uniqueness:

$$\vec{G}(\alpha) = \vec{A}(\alpha, \mu(\alpha)). \tag{5.17}$$

6. Thresholds for coupled systems of nonlinear Schrödinger equations

In this section we discuss results for systems of coupled on nonlinear Schrödinger equations (CNLS) (1.7):

$$\begin{aligned} &\mathrm{i}\partial_t \psi_l + \partial_x^2 \psi_l + \kappa(\delta^2 \vec{\psi})_l + (\sigma + 1)|\psi_l|^{2\sigma} \psi_l = 0, \\ &\vec{\psi} = \{\psi_l(t, x)\}_{l \in \mathbb{Z}^d}, \qquad d = 1, 2, \qquad (t, x) \in \mathbb{R}^2. \end{aligned} \tag{6.1}$$

CNLS has been introduced as a model governing the propagation of light pulses in a coupled $d = 1$ or $d = 2$ dimensional array of optical fibres. We consider the case where the discrete variable varies over $\mathbb{Z}^d$, and such that $\psi_l(t, x)$ decays as l and x tend to infinity. Other boundary conditions (e.g. periodic) are considered in [6, 30]. We follow a similar outline for the CNLS as that followed in our analysis of DNLS. Certain details are omitted and for them we refer to [27, 30].

Given initial data $\vec{\psi}_0(x)$ for CNLS satisfying

$$\sum_{l \in \mathbb{Z}^d} \|\psi_{0l}\|_{H^1}^2 < \infty, \tag{6.2}$$

there is a unique solution $t \mapsto \vec{\psi}(t, x)$ which is continuous in t with values in $l^2(\mathbb{Z}^d) \times H^1(\mathbb{R})$. The following two functionals, evaluated on solutions, are independent in time:

$$\mathcal{H}[\vec{\psi}] = \int_{\mathbb{R}} (-\delta^2 \vec{\psi}(x), \vec{\psi}(x)) + \sum_{l \in \mathbb{Z}^d} \int_{\mathbb{R}} |\partial_x \psi_l(x)|^2 \, \mathrm{d}x - \int_{\mathbb{R}} |\psi_(x)|^{2\sigma+2} \, \mathrm{d}x$$

$$\mathcal{N}[\vec{\psi}] = \sum_l \int_{\mathbb{R}} |\psi_l|^2 \, \mathrm{d}x.$$

$\mathcal{H}$ is a Hamiltonian energy of the CNLS in the sense that CNLS can be expressed as:

$$\mathrm{i}\partial_t \vec{\psi} = \frac{\delta \mathcal{H}}{\delta \vec{\psi}^*}. \tag{6.3}$$

The functional $\mathcal{N}$ corresponds to the *total input power* in the system.

Of interest are nonlinear bound states of CNLS. These are solutions of the form:

$$\vec{\psi} = \mathrm{e}^{\mathrm{i}\lambda^2 t} \vec{g}(x; \lambda), \tag{6.4}$$

for which the invariants $\mathcal{H}$ and $\mathcal{N}$ are finite. The components of $\vec{\psi}$ satisfy the coupled system of equations:

$$-\lambda^2 \psi_l + \partial_x^2 \psi_l + \kappa(\delta^2 \vec{\psi})_l + (\sigma + 1)|\psi_l|^{2\sigma} \psi = 0, \qquad l \in \mathbb{Z}^d. \tag{6.5}$$

In analogy with the discrete case, we seek to characterize the ground state of the system by variational methods.

Definition. Let

$$\mathcal{J}_\nu = \inf\{\mathcal{H}[\vec{f}] : \mathcal{H}[\vec{f}] = \nu\}. \tag{6.6}$$

Because of the similarity of the arguments to those in the previous sections and the more detailed treatment in [27, 30] we provide a summary.

(1) If $0 < \sigma < 2$, then $\mathcal{J}_\nu > -\infty$ for any $\nu > 0$.
(2) In analogy with theorem 2.1, we can show the following theorem.

Theorem 6.1. *The infimum in (6.6) is attained if and only if $\mathcal{J}_\nu < 0$. Moreover, any minimizing sequence has a subsequence which converges strongly in $l^2\left(\mathbb{Z}^d; H^1(\mathbb{R})\right)$ modulo translations in space and phase. Furthermore, any minimizer satisfies the equation (6.5).*

In view of this result, we study the question: for which σ, d and ν do we have $\mathcal{J}_\nu < 0$?

(3) **Proposition 6.1.** *$\mathcal{J}_\nu = 0$ if, and only if, for any $\vec{\psi} \in l^2\left(\mathbb{Z}^d; H^1(\mathbb{R})\right)$ we have the estimate:*

$$\|\vec{\psi}\|_{2\sigma+2}^{2\sigma+2} \leqslant \nu^{-\sigma} a_\sigma^{\frac{\sigma}{2}-1} \|\vec{\psi}\|_2^{2\sigma} \langle -\delta^2\vec{\psi}, \vec{\psi}\rangle^{1-\frac{\sigma}{2}} \|\partial_x\vec{\psi}\|_2^\sigma, \tag{6.7}$$

where $a_\sigma = (\frac{\sigma}{2})^{\frac{\sigma}{2-\sigma}} - (\frac{\sigma}{2})^{\frac{2}{2-\sigma}}$.

Here, $\|\vec{f}\|_p^p = (\sum_i \|f_i\|_{L^p(\mathbb{R})}^p)^{\frac{1}{p}}$.

Proposition 6.1 is proved by a simple scaling argument. For any $\vec{\psi}$, such that $\|\vec{\psi}\|_2^2 = \nu$, we define the scaling $\vec{\psi}^r(x) = r^{\frac{1}{2}}\vec{\psi}(rx)$, which preserves the L^2 norm. Evaluation of the Hamiltonian on $\vec{\psi}^r$ and minimization over $r > 0$ gives:

$$\mathcal{H}[\vec{\psi}^r] \geqslant \mathcal{H}[\vec{\psi}^{r_{min}}] = \langle -\delta^2\vec{\psi}, \vec{\psi}\rangle - a_\sigma \|\vec{\psi}\|_{2\sigma+2}^{\frac{4(1+\sigma)}{2-\sigma}} \|\partial_x\vec{\psi}\|_2^{-\frac{2\sigma}{2-\sigma}}. \tag{6.8}$$

We can pass to an expression for arbitrary $\vec{\psi}$ by replacing $\vec{\psi}$ by $\nu^{\frac{1}{2}}\|\vec{\psi}\|_2^{-1}\vec{\psi}$ in (6.8). This gives for *any* $\vec{\psi} \in l^2\left(\mathbb{Z}^d; H^1(\mathbb{R})\right)$:

$$\mathcal{H}[\vec{\psi}^{r_{min}}] = \left(\frac{\nu}{\|\vec{\psi}\|_2^2}\right)\langle -\delta^2\vec{\psi}, \vec{\psi}\rangle - a_\sigma \left(\frac{\nu}{\|\vec{\psi}\|_2^2}\right)^{\frac{2+\sigma}{2-\sigma}} \|\vec{\psi}\|_{2\sigma+2}^{\frac{4(1+\sigma)}{2-\sigma}} \|\partial_x\vec{\psi}\|_2^{-\frac{2\sigma}{2-\sigma}}. \tag{6.9}$$

It follows that $\mathcal{J}_\nu$ can be realized as the infinimum of the expression in (6.9) over all $\vec{\psi} \in l^2(\mathbb{Z}^d; H^1(\mathbb{R}))$. Thus $\mathcal{J}_\nu \geqslant 0$ if and only if (6.7) holds for all $\vec{\psi} \in l^2(\mathbb{Z}^d; H^1(\mathbb{R}))$. As in the discrete case, it is simple to construct a sequence along which the L^2 constraint is satisfied and the Hamiltonian tends to zero.

(4) Suppose an estimate of the type (6.7) holds. In particular, we let C_* denote the smallest constant for which this estimate holds. That is,

$$\|\vec{\psi}\|_{2\sigma+2}^{2\sigma+2} \leqslant C_* \|\vec{\psi}\|_2^{2\sigma} \langle -\delta^2\vec{\psi}, \vec{\psi}\rangle^{1-\frac{\sigma}{2}} \|\partial_x\vec{\psi}\|_2^\sigma, \tag{6.10}$$

where

$$C_*^{-1} = \mathcal{K}^{\sigma,d} \equiv \inf \frac{\|\vec{\psi}\|_2^{2\sigma} \langle -\delta^2\vec{\psi}, \vec{\psi}\rangle^{1-\frac{\sigma}{2}} \|\partial_x\vec{\psi}\|_2^\sigma}{\|\vec{\psi}\|_{2\sigma+2}^{2\sigma+2}}. \tag{6.11}$$

There are two possibilities. First, if $\mathcal{K}^{\sigma,d} = C_*^{-1} = 0$, then for any $\nu > 0$, there is a choice of $\vec{\psi}$ which makes the Hamiltonian negative. In this case, by assertion (2), a ground state of any prescribed L^2 norm exists; there is no L^2-excitation threshold. The second possibility is that $0 < C_*^{-1} = \mathcal{K}^{\sigma,d} < \infty$. In this case, we have that $J_\nu \geqslant 0$ if and only if $C_* \leqslant \nu^{-\sigma} a_\sigma^{\frac{\sigma}{2}-1}$. Therefore, we can define the *threshold power*, $\nu_c = \nu_c(\sigma, d)$ by:

$$\nu_c = a_\sigma^{\frac{1}{2}-\frac{1}{\sigma}} C_*^{-\frac{1}{\sigma}} = a_\sigma^{\frac{1}{2}-\frac{1}{\sigma}} (\mathcal{K}^{\sigma,d})^{\frac{1}{\sigma}}. \tag{6.12}$$

By use of the estimate (6.7) with the optimal choice $\nu = \nu_c$, we obtain the sharp lower bound for the Hamiltonian, in analogy with the discrete case (cf (4.8)):

$$\mathcal{H}[\vec{\psi}] \geqslant \langle -\delta^2 \vec{\psi}, \vec{\psi} \rangle \left(1 - \left(\frac{\nu}{\nu_c}\right)^\sigma\right) \tag{6.13}$$

for any $\vec{\psi}$ with $\|\vec{\psi}\|_2^2 = \nu$.

(5) The question of when an L^2 threshold exists is reduced to the determination of the range of values of σ and d for which $\mathcal{K}^{\sigma,d} > 0$. Formula (6.12) then gives an expression for the threshold. To determine when $\mathcal{K}^{\sigma,d}$ is strictly positive amounts to determining when one can prove an inequality of type (6.10) for some (not necessarily optimal) choice of C_*. This is addressed in [27, 30]. Ranges of σ, d for which this inequality fails to hold are determined by scaling arguments, while a proof of such inequalities for certain σ, d can be obtained following the strategy used in the fully discrete case, where we mimic the proof of continuum interpolation estimates (e.g. see the proof of proposition 4.2) or alternatively by applying the continuum interpolation estimates to functions of $d + 1$ variables and where the functions are taken to be piecewise linear in the variable corresponding to the d discrete variables; see [27, 30]. The results obtained are that $\mathcal{K}^{\sigma,d} > 0$ for all $\sigma \in [1, 2)$ and for all $\sigma \in (\frac{2}{d+1}, \frac{2}{d-1})$. In summary we have the following theorem.

Theorem 6.2. *Let $\sigma \in [1, 2)$ or $\sigma \in (\frac{2}{d+1}, \frac{2}{d-1})$. Then, there exists an L^2 excitation threshold given by ν_c in (6.12).*

6.1. Estimates on ν_c for $\sigma = 1$, and $d = 1$

We now consider the case $\sigma = 1$ and $d = 1$, an infinite one-dimensional array:

$$\mathrm{i}\partial_t \psi_n + \partial_x^2 \psi_n + \kappa(\psi_{n-1} - 2\psi_n + \psi_{n+1}) + 2|\psi_n|^2 \psi_n = 0, \qquad n \in \mathbb{Z}. \tag{6.14}$$

We show how to get upper and lower estimates for the threshold power. A sketch was given in [27], where an error appears in the displayed upper and lower bounds (due to an error in algebra).

By the above discussion, we know that there is an L^2 excitation threshold. That is, there is a constant $\nu_c > 0$ such that there are no ground states of L^2 norm less than $\nu_c^{\frac{1}{2}}$ and there are ground states of L^2 norm $\nu^{\frac{1}{2}}$ for any $\nu \geqslant \nu_c$. By (6.12) (using that we must replace δ^2 by $\kappa\delta^2$) we have

$$\nu_c(1, 1; \kappa) = 2\kappa^{\frac{1}{2}} \mathcal{K}^{1,1} = 2\kappa^{\frac{1}{2}} \inf \frac{\|\vec{\psi}\|_2^2 \langle -\delta^2 \vec{\psi}, \vec{\psi} \rangle^{\frac{1}{2}} \|\partial_x \vec{\psi}\|_2}{\|\vec{\psi}\|_4^4}. \tag{6.15}$$

Upper estimate on $\nu_c(1, 1; \kappa)$: An upper estimate is obtained by evaluation of the functional in (6.15) on any $\vec{\psi} \neq 0$. In particular, if we use as a trial function the exact *one-soliton* supported on one site of the lattice, $\psi_j(x) = \mathrm{sech}(x)\delta_{j0}$, we obtain the upper bound

$$\nu_c(1, 1; \kappa) \leqslant \kappa^{\frac{1}{2}} 2\sqrt{6} \sim \kappa^{\frac{1}{2}} 4.89 \ldots . \tag{6.16}$$

Lower estimate on $\nu_c(1, 1; \kappa)$: To obtain a lower bound we follow the strategy in [27]. First note that for arbitrary functions $\psi \in H^1(\mathbb{R}^2)$,

$$\|\psi\|_4^4 \leqslant C_{\mathrm{SNG}}(\|\partial_x\psi\|_2^2 + \|\partial_y\psi\|_2^2)\|\psi\|_2^2. \tag{6.17}$$

By scaling in y, $\psi(x, y) \mapsto \psi(x, ry)$, we have from (6.17) the estimate:

$$\|\psi\|_4^4 \leqslant 2C_{\mathrm{SNG}}\|\partial_x\psi\|_2\|\partial_y\psi\|_2\|\psi\|_2^3. \tag{6.18}$$

In [28] the best constant in (6.17) is calculated and was found to be:

$$C_{\mathrm{SNG}} = (\pi \times 1.862\,25\ldots)^{-1} \tag{6.19}$$

By (6.15), to obtain a lower bound for ν_c it sufficies to obtain an lower bound for $\mathcal{K}^{1,1}$ or equivalently an upper bound for C_*.

We next relate C_* to C_{SNG}. This can be done by considering (6.17) for the restricted class of functions, $\psi(x, y)$, which are smooth in x and piecewise linear in y with jumps in $\partial_y\psi(x, y)$ at the integers. In particular, let

$$\psi(x, y) = (1-\theta)\psi_n(x) + \theta\psi_{n+1}(x),$$
$$y = n + \theta, \qquad 0 \leqslant \theta \leqslant 1.$$

Direct calculation gives:

$$\begin{aligned}
&\tfrac{2}{5}\sum_n \int |\psi_n(x)|^4\,\mathrm{d}x \leqslant \int |\psi(x, y)|^4\,\mathrm{d}x\,\mathrm{d}y \\
&\sum_n \int |\psi_n(x)|^2\,\mathrm{d}x \geqslant \int |\psi(x, y)|^2\,\mathrm{d}x\,\mathrm{d}y \\
&\sum_n \int |\partial_x\psi_n(x)|^2\,\mathrm{d}x \geqslant \int |\partial_x\psi(x, y)|^2\,\mathrm{d}x\,\mathrm{d}y \\
&\sum_n \int |\psi_{n+1}(x) - \psi_n(x)|^2\,\mathrm{d}x = \int |\partial_y\psi(x, y)|^2\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned} \tag{6.20}$$

This, together with (6.18) yields:

$$\|\vec{\psi}\|_4^4 \leqslant 5C_{\mathrm{SNG}}\|\vec{\psi}\|_2^2\langle\vec{\psi}, \vec{\psi}\rangle^{\frac{1}{2}}\|\vec{\psi}\|_2. \tag{6.21}$$

with a non-optimal constant, $\tilde{C} = 5C_{\mathrm{SNG}}$ which is an *upper* bound for C_*. Thus,

$$\begin{aligned}
\nu_c(1, 1, ; \kappa) &\geqslant 2\kappa^{\frac{1}{2}}C_*^{-1} \geqslant 2\kappa^{\frac{1}{2}}(5C_{\mathrm{SNG}})^{-1} \\
&= \kappa^{\frac{1}{2}}\frac{2}{5}\pi \times 1.862\,25\ldots \geqslant \kappa^{\frac{1}{2}}2.3402\ldots.
\end{aligned} \tag{6.22}$$

Combining (6.16) and (6.22) we obtain:

$$\kappa^{\frac{1}{2}}2.34\ldots \leqslant \nu_c(1, 1, ; \kappa) \leqslant \kappa^{\frac{1}{2}}4.89\ldots. \tag{6.23}$$

A careful numerical simulation [30] indicates $\kappa^{-\frac{1}{2}}\nu_c(1, 1; \kappa) \sim 4.08$.

Acknowledgments

The author wishes to thank Peter D Miller for very helpful comments. This work was supported in part by a grant from the US National Science Foundation.

Appendix. Concentration compactness methods for DNLS

Theorem 2.1 can be proved using the concentration compactness principle; see, for example, [22]. Since arguments follow, quite closely, those for the continuum case (see [22, 30] for a detailed implementation), we present here an outline of the ideas.

Let $\vec{u}^{(k)} = \{u_l^{(k)}\}$, denote a sequence in $l^2(\mathbb{Z}^d)$, and such that

$$\sum_l |u_l^{(k)}|^2 = \nu. \tag{7.1}$$

Let $B_t(m)$ denote $\{l \in \mathbb{Z}^d : |l-m| < t\}$, and the norm $|l-m| = \max_{1\leqslant i\leqslant d} |l_i - m_i|$.

Theorem 7.1. *(Concentration compactness principle.)*

There exists a subsequence $\vec{u}^{(n_k)}$ satisfying one of the following three scenarios:

(1) *Compactness (*the 'mass' of the sequence concentrates*). There exists $m_k \in \mathbb{Z}^d$ such that for every $\varepsilon > 0$, there exists a real positive number R_ε (independent of k), such that*

$$\sum_{l\in B_{R_\varepsilon}(m_k)} |u_l^{(n_k)}|^2 \geqslant \nu - \varepsilon. \tag{7.2}$$

(2) *Vanishing (*the sequence spreads its mass over larger and larger sets and tends to zero*). For all $R < \infty$,*

$$\lim_{k\to\infty} \sup_{m\in\mathbb{Z}^d} \sum_{l\in B_R(m)} |u_l^{(n_k)}|^2 = 0. \tag{7.3}$$

(3) *Dichotomy (*the sequence concentrates its mass in at least two regions which become increasingly distant*). There exists $\alpha \in (0, \nu)$ such that , for all $\varepsilon > 0$, there exist $k_0 \geqslant 1$ and disjointly supported sequences $\vec{a}^{(k)}, \vec{b}^{(k)}$ in $l^2(\mathbb{Z}^d)$ satisfying for all $k \geqslant k_0$:*

$$\|\vec{u}^{(n_k)} - (\vec{a}^{(k)} + \vec{b}^{(k)})\|_{l^2(\mathbb{Z}^d)} \leqslant \varepsilon$$
$$|\|\vec{a}^{(k)}\|^2_{l^2(\mathbb{Z}^d)} - \alpha| \leqslant \varepsilon$$
$$|\|\vec{b}^{(k)}\|^2_{l^2(\mathbb{Z}^d)} - (\nu - \alpha)| \leqslant \varepsilon$$
$$\text{distance}(\text{supp}(a^{(k)}, \text{supp}(b^{(k)}) \to \infty$$

as $k \to \infty$.

To prove this result, we introduce the sequence of *concentration functions*:

$$Q^{(k)}(t) = \sup_{m\in\mathbb{Z}^d} \sum_{l\in B_t(m)} |u_l^{(k)}|^2. \tag{7.4}$$

By following the arguments in [22] it can be shown that:

(1) Along a subsequence $n_k \to \infty$, $Q^{n_k}(t)$ converges to a nondecreasing and nonnegative function, $Q(t)$ with limit:

$$\lim_{t\to\infty} Q(t) = \alpha \in (0, \nu). \tag{7.5}$$

(2) The cases $\alpha = 0$, $\alpha = \nu$ and $0 < \alpha < \nu$ correspond, respectively, to the above scenarios: *vanishing, compactness* and *dichotomy*.

To prove theorem 2.1 we must rule out the vanishing and dichotomy scenarios.

Vanishing is ruled out as follows. Let $\vec{u}_k$ denote a minimizing sequence. Then, $\mathcal{H}[\vec{u}_k] = \mathcal{I}_\nu + \epsilon_k$, where $\epsilon_k \downarrow 0$ as $k \to \infty$. From the definition of $\mathcal{H}$ and the hypotheses $\mathcal{I}_\nu < 0$ we have

$$\begin{aligned}\mathcal{I}_\nu + \epsilon_k &= \mathcal{H}[\vec{u}^{(k)}]\\ &= \langle -\delta^2\vec{u}^{(k)}, \vec{u}^{(k)}\rangle - (\sigma+1)^{-1} \sum_{l\in\mathbb{Z}^d} |u_l^{(k)}|^{2\sigma+2}\\ &\geqslant -(\sigma+1)^{-1} \sum_{l\in\mathbb{Z}^d} |u_l^{(k)}|^{2\sigma+2},\end{aligned}$$

and therefore

$$\frac{(\sigma+1)}{2}|\mathcal{I}_\nu| \leqslant \nu \|\vec{u}^{(k)}\|_{l^\infty}. \tag{7.6}$$

Since vanishing implies $\|\vec{u}_k\|_{l^\infty} \to 0$ as $k \to \infty$, the lower bound (7.6) precludes vanishing.

Dichotomy is ruled out, as in the continuum case [22], using the strict subadditivity of the functional $\mathcal{I}_\nu$, i.e. if $0 < \alpha < \nu$, then

$$\mathcal{I}_\nu < \mathcal{I}_\alpha + \mathcal{I}_{\nu-\alpha}. \tag{7.7}$$

The idea is as follows. If dichotomy occurs (see (3) above) then as $k \to \infty$ and $\varepsilon \to 0$ we have

$$\mathcal{I}_\nu = \mathcal{H}[\vec{u}^{(k)}] + \mathrm{o}(1) = \mathcal{H}[\vec{a}^{(k)}] + \mathcal{H}[\vec{b}^{(k)}] + \mathrm{o}(1), \tag{7.8}$$

where we have used that $\vec{a}^{(k)}$ and $\vec{b}^{(k)}$ have disjoint supports. Furthermore,

$$\mathcal{H}[\vec{a}^{(k)}] \geqslant \mathcal{I}_{\alpha+\mathrm{o}(\varepsilon)} \qquad \text{and} \qquad \mathcal{H}[\vec{b}^{(k)}] \geqslant \mathcal{I}_{\nu-\alpha+\mathrm{o}(\varepsilon)}, \tag{7.9}$$

by definition of $\mathcal{I}_\theta$ Therefore, taking $k \to \infty$ and $\varepsilon \to 0$ we get

$$\mathcal{I}_\nu \geqslant \mathcal{I}_\alpha + \mathcal{I}_{\nu-\alpha}. \tag{7.10}$$

This contradicts (7.7).

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