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Microscopic black hole entropy in theories with higher derivatives

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Abstract: We discuss higher derivative corrections to black hole entropy in theories that allow a near horizon $AdS_3 \times X$ geometry. In arbitrary theories with diffeomorphism invariance we show how to obtain the spacetime central charge in a simple way. Black hole entropy then follows from the euclidean partition function, and we show that this gives agreement with Wald's formula. In string theory there are certain diffeomorphism anomalies that we exploit. We thereby reproduce some recent computations of corrected entropy formulas, and extend them to the nonextremal, nonsupersymmetric context. Examples include black holes in M-theory on $K3 \times T^2$, whose entropy reproduces that of the perturbative heterotic string with both right and left movers excited and angular momentum included. Our anomaly based approach also sheds light on why exact results have been obtained in four dimensions while ignoring R^4 type corrections.

Keywords: Anomalies in Field and String Theories, Topological Strings, Black Holes in String Theory, Black Holes.

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1. Introduction

The famous area law of Bekenstein and Hawking relates the entropy of a black hole to the area of its event horizon as

$$S = \frac{1}{4G_D} A_{D-2}\,. \tag{1.1}$$

In string theory this law has been verified in examples where the entropy is interpreted statistically in terms of microstates and the area is that of a black hole with the same macroscopic charges as the statistical system. In such computations many details of the string spectrum are known, implying numerous corrections to the microscopic theory. Additionally, higher derivative terms in the action modify the classical geometry and also change the area law (1.1) into Wald's entropy formula [1]

$$S = -\frac{1}{8G_D}\int_{\rm hor} d^{D-2}x\sqrt{h}\,\frac{\delta \mathcal{L}_D}{\delta R_{\mu\nu\alpha\beta}}\epsilon^{\mu\nu}\epsilon^{\alpha\beta}\,, \tag{1.2}$$

which takes into account arbitrary derivative terms in the action.[1] Remarkably, agreement between microscopics and macroscopics is maintained also after all these corrections are taken into account, at least in some examples [2–4].

Recently it was pointed out that there are special cases of this agreement where the area of the black hole vanishes at leading order: $A_{D-2} = 0$ [5]. For example, the microstates of the heterotic string consist of the usual perturbative spectrum. The black hole with the same classical charges has vanishing area in the two-derivative approximation to the action, but after higher derivatives are taken into account the entropy found from (1.2) agrees with the microscopic result. This example is important because the microscopics is so simple, which should facilitate very detailed comparisons that can test the whole framework and its interpretation. In particular, this seems like an ideal setting for testing Mathur's conjecture [6] that all microstates can be realized as distinct geometries.

Ultimately one would like to understand which features of quantum gravity are responsible for these striking agreements between radically different representations of black hole physics. The purpose of this note is to emphasize the central role played by symmetries, particularly diffeomorphism invariance and its anomalies. Viewed in this light, some of the agreements between microscopic and macroscopic results seem less surprising.

The key assumption in our approach is the existence of a near horizon region that includes an AdS_3 factor, even after higher derivative terms have been included in the lagrangian. This assumption is suggested by the central role played by such near horizon geometries in the microscopics of black holes with finite area [7]. Additionally, in an appropriate duality frame, an AdS_3 factor appears in the corrected geometry in all examples where derivative corrections have successfully been taken into account, at least as far as we are aware. The power of the assumption is that it relates the lagrangian to the radius of the AdS_3 space and so, *via* generalized Brown-Henneaux [8] reasoning, to the central charges $c_{L,R}$ of the associated conformal field theory. As we will see, the saddlepoint approximation to the black hole entropy, including all higher derivative corrections, is then given by the Cardy formula

$$S = 2\pi \left[\sqrt{\frac{c_L h_L}{6}} + \sqrt{\frac{c_R h_R}{6}} \right], \tag{1.3}$$

where $h_{L,R}$ are the left and right moving momenta of the near horizon solution. Although the detailed form of the central charges $c_{L,R}$ depends critically on the spacetime lagrangian, it will turn out that the Cardy formula (1.3) agrees with Wald's formula (1.2) for general theories with diffeomorphism invariance. Thus, computation of the corrected black hole entropy reduces to finding the central charges. We will present a novel method for achieving this — *c-extremization* — which just involves solving algebraic equations. Given a higher derivative lagrangian it is then quite simple to compute the corrected entropy.

Recent work has shown that in favorable cases it is possible to reproduce microscopic degeneracies of states to all orders in an expansion in inverse powers of charges [9, 10]. This result emerges just as naturally in our approach. Knowledge of the central charge leads to

[1] Derivatives acting on the Riemann tensor are taken into account by the functional derivative: $\frac{\delta \mathcal{L}_D}{\delta R_{\mu\nu\alpha\beta}} \equiv \frac{\partial \mathcal{L}_D}{\partial R_{\mu\nu\alpha\beta}} - \nabla_\sigma \frac{\partial \mathcal{L}_D}{\partial \nabla_\sigma R_{\mu\nu\alpha\beta}} + \dots$.

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an expression for the black hole partition function which, when inverse Laplace transformed as in [10] yields the microscopic degeneracies including all power law corrections.

We stress that our considerations are independent of spacetime supersymmetry. This contrasts with the (much) more explicit approach of [2, 5] which relies on the full power of supergravity. In particular, the usual approach has so far been limited to four dimensions, where supergravity is best developed, while our results apply equally in five dimensions.

In string theory there is additional structure due to anomalies which affect diffeomorphism invariance. Some relevant aspects are discussed in [3, 4]. These anomalies ultimately arise from $M5$-branes on the compactification manifold but they can also be understood without reference to $M5$-branes, using standard AdS/CFT reasoning. In this way we recover formulae from [3, 4] using elementary methods.

A natural context for these considerations is M-theory on $AdS_3 \times S^2 \times X$ where X is some Calabi-Yau three-fold. A particularly striking example arises when $X = K3 \times T^2$, so that M-theory is dual to heterotic string theory on T^5. In this case we find $c_L = 12$ and $c_R = 24$ which are indeed the correct central charges for the heterotic string. The remarkable feature is that we are sensitive to *both* chiral sectors of the heterotic string, and that we thereby derive the entropy for the heterotic string with both sectors excited. This shows that agreement is possible even without supersymmetry.

The point we wish to emphasize is that the constraints of matching symmetries and anomalies are enough to explain the successful entropy comparisons, at least in the cases we have considered. One puzzle in existing work has been why exact results are obtained by keeping only R^2 corrections, and neglecting higher powers. Here we see that it is the R^2 terms which yield the relevant diffeomorphism anomalies, and they are uncorrected by additional higher derivative terms.

The conventional approach of [2, 5] involves near horizon geometries with an AdS_2 factor and uses results from topological string theory [11, 10]. These AdS_2 geometries are related to AdS_3 by compactification. The AdS_3 perspective is simpler because spacetime symmetries such as the Virasoro algebra become manifest. On the other hand, in our approach we have not yet exploited the effects that can be seen only after compactification. It would be interesting to analyze how these further constrain the black hole spectrum.

Another open question is to find a criterion that determines when a near horizon AdS_3 appears from a singular geometry after derivative corrections are taken into account. This would characterize any ultimate limitations of our approach.

The remainder of this paper is organized as follows. In section 2 we consider the higher derivative corrections in a rather general setting, assuming only that the lagrangian is formed from the metric in a diffeomorphism invariant way. In section 3 we apply these considerations to the case of M-theory on CY_3. In section 4, we discuss modifications due to gravitational anomalies and the appearance of the perturbative heterotic string spectrum. Finally, in section 5, we conclude with a discussion of how power law corrections to the entropy are taken into account using our approach.

2. Central charge and black hole entropy

In this section we first derive an expression for the central charge in terms of the lagrangian including higher derivative corrections. We then review the computation of BTZ black hole entropy from the central charge. Finally, we combine the results and write the entropy in a form that agrees with Wald's formula.

2.1 Computation of the central charge

We focus on brane configurations that have a near horizon geometry $\mathcal{M} = \mathrm{AdS}_3 \times S^p \times X$, where X is an arbitrary compact space. One familiar case is $p = 3$, which arises from the D1-D5 system in IIB string theory, where X is T^4 or $K3$. This system gives rise to black holes in $D = 5$. Another important example is $p = 2$ which corresponds to $D = 4$ black holes made from wrapping M2-branes and M5-branes on $X = CY_3$. We will come back to particular examples later, for now remaining in a general setting.

We take the near horizon limit, so that we have a theory of (not necessarily super) gravity on $\mathcal{M}$. In this section we will take the metric to have euclidean signature. For our purposes it is most convenient to perform a Kaluza-Klein reduction on X, to obtain a theory on $\mathrm{AdS}_3 \times S^p$. The action for this theory is

$$I = \frac{1}{16\pi G_{p+3}} \int d^{p+3}x \sqrt{g}\, \mathcal{L}_{p+3} + S_{\mathrm{bndy}} + S_{CS}\,. \tag{2.1}$$

At this stage, $\mathcal{L}_{p+3}$ is an arbitrary function of the gravitational and matter fields, which is diffeomorphism invariant up to total derivatives that are cancelled by S_{bndy}. In particular, it can include arbitrary higher derivative terms. The boundary terms indicated in (2.1) are needed to have a well-defined variational principle and also to define the boundary stress tensor [12, 13]; but we will not need their explicit form. S_{CS} denotes Chern-Simons terms built out of gauge fields; we isolate these for reasons that will become apparent as we proceed.

We will be assuming that this theory admits solutions of the form $\mathrm{AdS}_3 \times S^p$, over which $\mathcal{L}_{p+3}$ takes a constant value. This is indeed the case for the examples mentioned above. The radii of the two spaces are taken to be ℓ_{Ads} and ℓ_{S^p}. For a general action there is not necessarily a single preferred definition of the metric, and so the radii are defined with respect to some particular choice.

As originally shown by Brown and Henneaux [8], a theory of gravity on a space whose noncompact part is AdS_3 corresponds to a conformal field theory on the two dimensional boundary. The conformal field theory has left and right moving central charges, c_L and c_R, which are not necessarily equal. In this section we will consider the case in which they *are* equal. This is true for the D1-D5 system; for the M2-M5-brane case mentioned above it is only true for the leading part in an expansion in charges. We will come back to the case of unequal central charges later, for now just remarking that it leads to non-diffeomorphism invariant theories (gravitational anomalies), and so requires special care.

Our first task is to compute ℓ_{Ads} and ℓ_{S^p}. Suppose we consider a family of trial solutions with ℓ_{Ads} and ℓ_{S^p} left as free parameters. In particular, we write the metric as

$$ds^2 = \ell_{Ads}^2 \left(d\eta^2 + \sinh^2\eta\, d\Omega_2^2\right) + \ell_{S^p}^2 d\Omega_p^2\,. \tag{2.2}$$

The first two terms give AdS_3 in a convenient, but perhaps slightly unfamiliar, form. The actual values of the radii can then be obtained by demanding that the combination $\ell^3_{Ads}\ell^p_{S^p}\mathcal{L}_{p+3}$ be stationary under variation of ℓ_{Ads} and ℓ_{S^p}. Roughly speaking, this can be thought of as extremizing the bulk action. A little care is required to establish that this is the correct procedure. In particular, we should recall that when the equations of motion are satisfied the full action is stationary under variations which vanish at the boundary; but in our case variations of the radii lead to variations even at the boundary. Furthermore, we have the boundary terms in (2.1). A simple way to avoid these complications is to consider an analytic continuation so that our solutions take the form $S^3 \times S^p$. Then both complications are absent, and the total action is clearly proportional to $\ell^3_{Ads}\ell^p_{S^p}\mathcal{L}_{p+3}$. Hence this combination must be stationary. Our result follows after continuation back to $\mathrm{AdS}_3 \times S^p$. We note that in general $\mathcal{L}_{p+3}$ will be a complicated function of the radii, incorporating for example the contributions of the field strengths, whose values are fixed by their equations of motion.

This discussion makes it clear why we isolated the Chern-Simon terms. These are not necessarily constant over our solution. On the other hand, being topological they play no role in determining the radii, or the central charge, so we are free to neglect them at this stage.

With foresight, we define the central charge function

$$c(\ell_{Ads}, \ell_{S^p}) = \frac{3\Omega_2\Omega_p}{32\pi G_{p+3}}\ell^3_{Ads}\ell^p_{S^p}\mathcal{L}_{p+3}\,, \tag{2.3}$$

and so the actual values of the radii are determined by solving

$$\frac{\partial}{\partial \ell_{Ads}} c(\ell_{Ads}, \ell_{S^p})|_{\ell_{Ads}=\bar{\ell}_{Ads}} = \frac{\partial}{\partial \ell_{S^p}} c(\ell_{Ads}, \ell_{S^p})|_{\ell_{S^p}=\bar{\ell}_{S^p}} = 0\,. \tag{2.4}$$

We wish to establish that $c = c(\bar{\ell}_{Ads}, \bar{\ell}_{S^p})$ (equal on left and right!) is indeed the central charge, as defined by the conformal anomaly

$$T^i_i = -\frac{c}{12}{}^{(2)}R\,, \tag{2.5}$$

of the dual $D = 2$ CFT.

To this end, put the $2D$ CFT on a sphere with metric

$$ds^2 = e^{2\omega} d\Omega_2^2\,, \tag{2.6}$$

and focus on the partition function, $Z = e^{-I}$, as a function of ω. Under constant shifts of ω we have

$$\delta I = \frac{1}{4\pi}\int d^2x\sqrt{g}\,T^{ij}\delta g_{ij} = \frac{\delta\omega}{2\pi}\int d^2x\sqrt{g}\;T^{ij}g_{ij} = -\frac{c}{24\pi}\delta\omega\int d^2x\sqrt{g}\;{}^{(2)}R = -\frac{c}{3}\delta\omega\,. \tag{2.7}$$

This is to be compared with the action (2.1) evaluated on (2.2):

$$I = \frac{\Omega_2\Omega_p}{16\pi G_{p+3}}\ell^3_{Ads}\ell^p_{S^p}\int d\eta\,\sinh^2\eta\,\mathcal{L}_{p+3} + S_{\mathrm{bndy}}\,. \tag{2.8}$$

To make sense of this we need to impose an upper cutoff on η. The integral gives

$$\int_0^{\eta_{\max}} d\eta\, \sinh^2\eta \mathcal{L}_{p+3} = \left(-\frac{1}{2}\eta_{\max} + \frac{1}{2}\cosh\eta_{\max}\sinh\eta_{\max}\right)\mathcal{L}_{p+3}\,. \tag{2.9}$$

Now, $S_{\rm bndy}$ is the integral of a expression defined locally on the AdS boundary. It is constructed out of the intrinsic and extrinsic curvature of the boundary. Such terms will never give a contribution linear in $\eta_{\max}$. Instead, they cancel the second term in (2.9), leaving the action

$$I = -\frac{\Omega_2\Omega_p}{32\pi G_{p+3}}\ell^3_{Ads}\ell^p_{S^p}\mathcal{L}_{p+3}\ \eta_{\max}\,. \tag{2.10}$$

Comparing (2.2) with (2.6) we have

$$\delta\omega = \delta\eta_{\max}\,, \tag{2.11}$$

which then yields

$$c = \frac{3\Omega_2\Omega_p}{32\pi G_{p+3}}\ell^3_{Ads}\ell^p_{S^p}\mathcal{L}_{p+3}\,, \tag{2.12}$$

as we wanted to show.

To summarize, we have shown that the central charge of an $\mathrm{AdS}_3 \times S^p \times X$ solution can be obtained simply by extremizing the central charge function (2.3) with respect to the AdS and sphere radii. For a given lagrangian this just means solving two algebraic equations. We will refer to our procedure as *c-extremization*.

We would like to emphasize a couple of important points. First, our result applies to an arbitrary higher derivative lagrangian including matter fields. The requirement is just that this lagrangian admits a solution with the assumed properties. Second, although we used some language familiar from the AdS/CFT correspondence, our result is completely independent of the validity of the AdS/CFT conjecture. Essentially, we have derived a result about how the gravitational action behaves under Weyl transformations of the AdS boundary.

2.2 Black hole entropy

Once the central charge is known, results for black hole entropy follow with little additional effort. We now review how this works in the general case, allowing independent values of the left and right moving central charges.

We consider black holes of the form $BTZ\times S^p\times X$. One way to compute the black hole entropy is by computing the action of the euclidean black hole. From there, one gets the free energy, and then thermodynamic quantities follow in the standard way. The euclidean BTZ black hole is a solid torus which can be continued to lorentzian signature in many different ways. Consider the cycles on the boundary of the torus which are noncontractible with respect to the boundary. There is clearly one such cycle which is contractible in the solid torus. If one calls the coordinate along the contractible cycle ϕ, and the other cycle coordinate t_E, then upon continuing $t_E \to -it$ one obtains the geometry of thermal

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AdS$_3$; that is, global AdS$_3$ with compact imaginary time. On the other hand, the opposite assignment of t_E and ϕ leads, upon continuation to lorentzian signature, to the BTZ black hole.[2]

From this point of view it becomes clear that the black hole partition function is just a rewriting of the thermal partition function. But the result for the thermal partition function follows directly from the central charges. Hence, so too does the black hole entropy.

Let us illustrate this in more detail; see [14]. An asymptotically AdS$_3$ solution carries energy H and angular momentum J. In the CFT on the boundary J is the momentum. We can also define the zero modes of the Virasoro generators as

$$\begin{aligned} h_L &= L_0 - \frac{c}{24} = \frac{H-J}{2}\,, \\ h_R &= \tilde{L}_0 - \frac{\tilde{c}}{24} = \frac{H+J}{2}\,. \end{aligned} \tag{2.13}$$

We can think of a bulk solution as a contribution to the partition function

$$\begin{aligned} Z(\beta,\mu) &= e^{-I} = \mathrm{Tr}\ e^{-\beta H - \mu J} \\ &= \mathrm{Tr}\ e^{2\pi i \tau h_L} e^{-2\pi i \overline{\tau} h_R}\,, \end{aligned} \tag{2.14}$$

where we defined

$$\tau = i\frac{\beta-\mu}{2\pi}\,, \qquad \overline{\tau} = -i\frac{\beta+\mu}{2\pi}\,. \tag{2.15}$$

When we go to euclidean signature μ becomes pure imaginary and $\overline{\tau}$ becomes the complex conjugate of τ. Also, it follows from (2.14) that τ is precisely the modular parameter of the euclidean boundary torus.

Now consider thermal AdS$_3$. In lorentzian signature thermal AdS$_3$ takes the same form as AdS$_3$ written in the usual global coordinates. On the other hand, we know that global AdS$_3$ corresponds to the NS-NS vacuum, and as such carries $L_0 = \tilde{L}_0 = 0$. Therefore, we conclude that the action of thermal AdS$_3$ is

$$I_{\text{thermal}}(\tau,\overline{\tau}) = \frac{i\pi}{12}(c\tau - \tilde{c}\overline{\tau})\,. \tag{2.16}$$

There are in fact additional contributions due to quantum fluctuations of massless fields. (2.16) just takes into account all the *local* contributions. The extra nonlocal contributions are, by definition, suppressed for large β, and will give subleading contributions to the entropy compared to the local piece.

We already noted that BTZ is obtained by interchanging t_E and ϕ. This is just a modular transformation of the boundary torus: $\tau \to -\frac{1}{\tau}$. Recall that a modular transformation is a diffeomorphism combined with a Weyl transformation. The action is invariant since if we take a flat metric on the torus then all potential anomalies vanish. We therefore conclude that

$$I_{\text{BTZ}}(\tau,\overline{\tau}) = -\frac{i\pi}{12}\left(\frac{c}{\tau} - \frac{\tilde{c}}{\overline{\tau}}\right)\,. \tag{2.17}$$

[2]Other choices lead to the so-called "SL$(2,\mathbb{Z})$" family of black holes.

From (2.14) it follows that

$$\begin{aligned} h_L &= \frac{-1}{2\pi i}\frac{\partial I}{\partial \tau} = -\frac{c}{24\tau^2}\,, \\ h_R &= \frac{1}{2\pi i}\frac{\partial I}{\partial \overline{\tau}} = -\frac{\tilde{c}}{24\overline{\tau}^2}\,. \end{aligned} \tag{2.18}$$

From the thermodynamic relation $I = \beta H + \mu J - S$ we compute the entropy S to be

$$S_{BTZ} = 2\pi\left(\sqrt{\frac{c}{6}h_L} + \sqrt{\frac{\tilde{c}}{6}h_R}\right). \tag{2.19}$$

Three facts about this computation are worth stressing. First, the result (2.19) holds for an arbitrary theory of gravity admitting a BTZ black hole (times an arbitrary compactification space). Second, the result is valid entirely independent of the AdS/CFT correspondence. One can just think of it as a result for computing the action of the euclidean black hole. Finally, (2.19) gives the entropy in terms of the black hole mass and angular momentum, and with the central charges appearing as "undetermined parameters". This shows that once we can compute the central charges, the black hole entropy follows directly from (2.19). But we have seen in the last section how the central charge — in the case of $\tilde{c} = c$ — follows from a simple extremization principle. Altogether, we have arrived at an efficient method of computing black hole entropy.

2.3 Equivalence with Wald's approach

The Wald formula (1.2) gives the black hole entropy in an arbitrary diffeomorphism invariant theory [1]. In his approach, one integrates a certain expression over the black hole horizon. The power of this result is its complete generality. However, for black holes with a near horizon $\mathrm{AdS}_3 \times S^p \times X$ structure, the method we have described above is actually much simpler to implement. In particular, one is not required to locate the horizon at all: c-extremization gives the entropy directly. In any case, it is worthwhile to check that our result agrees with Wald's formula, as we do now.

The essential ideas for demonstrating this equivalence appear in the paper [15] where it is shown that Wald's approach leads to a black hole entropy in the form (2.19). We will follow a slightly different procedure from [15].

We first want to write the central charge in a form suitable for comparison with the Wald formula. It is convenient to work directly in the theory compactified all the way to $D = 3$. By assumption, all matter fields take constant values, so that we can write the action purely in terms of the metric. In $D = 3$ the Riemann tensor can be expressed in terms of the Ricci tensor; so the general action will be a function of the Ricci tensor and its covariant derivatives[3]

$$S = \frac{1}{16\pi G_3}\int d^3x\sqrt{-g}\;\mathcal{L}_3(g^{\mu\nu}, R_{\mu\nu}) + S_{\mathrm{bndy}}\,. \tag{2.20}$$

[3] Actually, one can also include a Chern-Simons term, $S \sim \int \mathrm{Tr}\;\omega \wedge R$, but for now we exclude such a term. It would lead to $\tilde{c} \neq c$ and associated subtleties, which we postpone till a later section.

Schematically we have

$$\mathcal{L}_3(g^{\mu\nu}, R_{\mu\nu}) \sim \sum_n a_n (g^{\mu\nu})^n (R_{\mu\nu})^n \,, \tag{2.21}$$

where the a_n include covariant derivatives and contractions are not written out explicitly. The central charge function (2.3) is

$$c(\ell_{Ads}) = \frac{3\Omega_2}{32\pi G_3} \ell_{Ads}^3 \mathcal{L}_3 \,. \tag{2.22}$$

If we write introduce rescaled variables through [9]

$$g_{\mu\nu} = \ell_{Ads}^2 \hat{g}_{\mu\nu} \,, \quad g^{\mu\nu} = \frac{1}{\ell_{Ads}^2} \hat{g}^{\mu\nu} \,, \qquad R_{\mu\nu} = 2\hat{g}_{\mu\nu} = \hat{R}_{\mu\nu} \,, \tag{2.23}$$

then ℓ_{Ads} satisfies

$$3\mathcal{L}_3 + 2\ell_{Ads}^2 \frac{\partial \mathcal{L}_3}{\partial \ell_{Ads}^2} = 0 \,. \tag{2.24}$$

Furthermore, in the rescaled variables (2.23) the action reads

$$S = \frac{1}{16\pi G_3} \int d^3x \sqrt{-\hat{g}}\, \ell_{Ads}^3 \mathcal{L}_3 (\frac{\hat{g}^{\mu\nu}}{\ell_{Ads}^2}, \hat{R}_{\mu\nu}) \,, \tag{2.25}$$

so the derivative in (2.24) can be evaluated as[4]

$$\ell_{Ads}^2 \frac{\partial \mathcal{L}_3}{\partial \ell_{Ads}^2} = -\hat{R}_{\mu\nu} \frac{\partial \mathcal{L}_3}{\partial \hat{R}_{\mu\nu}} = -\frac{2}{\ell_{Ads}^2} g_{\mu\nu} \frac{\partial \mathcal{L}_3}{\partial R_{\mu\nu}} \quad . \tag{2.26}$$

Simplifying (2.22) using (2.24) and (2.26) we find

$$c = \frac{\ell_{Ads}}{2G_3}\, g_{\mu\nu} \frac{\partial \mathcal{L}_3}{\partial R_{\mu\nu}} \,. \tag{2.27}$$

This formula generalizes the usual Brown-Henneaux central charge

$$c_0 = \frac{3\ell_{Ads}}{2G_3} \,, \tag{2.28}$$

by taking higher derivative corrections into account. The net result amounts to a rescaling of the AdS_3 radius $\ell_{Ads} \to \ell_{\text{eff}} = \Omega \ell_{Ads}$ where

$$\Omega = \frac{1}{3} g_{\mu\nu} \frac{\partial \mathcal{L}_3}{\partial R_{\mu\nu}} = \frac{2G_3}{3\ell_{Ads}} c \,. \tag{2.29}$$

We are now ready to make the connection with Wald's approach, since the latter involves an integration over the horizon of the derivative of the lagrangian with respect to the curvature. Presently, the black hole entropy takes the form (2.19) with the central charge (2.27). The BTZ black hole, as usually written, is expressed in terms of the parameters M_3 and J_3 which, for a 2-derivative action are identified with the mass and angular momentum of the black hole. However, in the presence of higher derivatives the relation

[4]We use the fact that all covariant derivatives vanish on the background.

is rescaled by the conformal factor (2.29) and we have instead

$$h_{L,R} = \Omega\, \frac{M_3 \mp J_3}{2}\,. \tag{2.30}$$

We now find the entropy (2.19)

$$\begin{aligned} S &= \frac{\pi}{12G_3} g^{\mu\nu} \frac{\partial \mathcal{L}_3}{\partial R_{\mu\nu}} \left[\sqrt{8G_3 \ell_{Ads}(M_3+J_3)} + \sqrt{8G_3 \ell_{Ads}(M_3-J_3)} \right] \\ &= \frac{A_{\rm BTZ}}{4G_3}\, \Omega\,, \end{aligned} \tag{2.31}$$

where $A_{\rm BTZ}$ is the standard expression for the area of the BTZ black hole, i.e. a specific function of M_3, J_3, ℓ_{Ads} and G_3; and Ω is the rescaling factor (2.29) that encodes the correction due to higher derivative terms. It is now straightforward to show that Wald's formula

$$S = -\frac{1}{8G_3} \int_{\rm hor} d\phi \sqrt{g_{\phi\phi}} \frac{\partial \mathcal{L}_3}{\partial R_{\mu\alpha}} g^{\nu\beta} \epsilon_{\mu\nu} \epsilon_{\alpha\beta}\,, \tag{2.32}$$

agrees precisely with (2.31), and so with Cardy's formula (2.19).

3. Example: M-theory on CY_3

To illustrate our approach, we now consider the example of M-theory compactified on a Calabi-Yau 3-fold $X = CY_3$, yielding a supergravity theory in $D = 5$. This is a rich example that includes includes black holes in both four and five noncompact dimensions and also BPS black ring solutions.

3.1 Two-derivative action

We will follow the conventions in [16], to which we refer for more details. In particular, in this section we set $G_5 = \frac{\pi}{4}$, which is convenient since it leads to integrally quantized charges q^I. The hypermultiplets are assumed to be consistently set to constant values. Then the $D = 5$ action for the metric and vectormultiplets is given by (as in (2.1) with $p = 2$)

$$\begin{aligned} \mathcal{L}_5 &= -R + \frac{1}{2} G_{IJ} \partial_\mu X^I \partial^\mu X^J + \frac{1}{4} G_{IJ} F^I_{\mu\nu} F^{J\mu\nu} + (\text{fermions}) \ + \ (\text{higher derivs}) \\ S_{CS} &= \frac{1}{96\pi^2} \int d^5x\, C_{IJK} \epsilon_{\mu_1 \ldots \mu_5} F^{I\mu_1\mu_2} F^{J\mu_3\mu_4} A^{K\mu_5}\,. \end{aligned} \tag{3.1}$$

At first we neglect the higher derivative terms.

We consider $\mathrm{AdS}_3 \times S^2$ vacua of this theory supported by magnetic flux. The magnetic charges are given by

$$q^I = -\frac{1}{2\pi} \int_{S^2} F^I\,, \tag{3.2}$$

where

$$F^I = -\frac{q^I}{2\ell^2_{S^2}} \epsilon_{S^2} \tag{3.3}$$

is interpreted microscopically as q^I M5-branes wrapped on the Ith 4-cycle of X. The scalars X^I are taken to have constant values, fixed by the attractor mechanism to be

$$X^I = \frac{q^I}{(\frac{1}{6}C_{IJK}q^I q^J q^K)^{1/3}} \,. \tag{3.4}$$

The central charge function (2.3) becomes

$$c(\ell_{Ads}, \ell_{S^2}) = -6\ell_{Ads}^3 \ell_{S^2}^2 \left(-\frac{6}{\ell_{Ads}^2} + \frac{2}{\ell_{S^2}^2} - \frac{G_{IJ}q^I q^J}{4\ell_{S^2}^4} \right) . \tag{3.5}$$

Extremizing, we find

$$\ell_{Ads} = 2\ell_{S^2} = \frac{1}{3}\sqrt{6 G_{IJ}q^I q^J}\,, \quad c = 4\sqrt{\frac{2}{3}}(G_{IJ}q^I q^J)^{3/2} \,. \tag{3.6}$$

Special geometry relations (reviewed in [16]) give

$$G_{IJ}q^I q^J = \frac{3}{2}\left(\frac{1}{6}C_{IJK}q^I q^J q^K\right)^{2/3} \,, \tag{3.7}$$

which then yields

$$\ell_{Ads} = 2\ell_{S^2} = \left(\frac{1}{6}C_{IJK}q^I q^J q^K\right)^{1/3} \,, , \qquad c = C_{IJK}q^I q^J q^K \,. \tag{3.8}$$

In the appendix we review how these relations appear in the explicit solutions.

3.2 Higher derivative corrections

Our approach makes it simple to include the effects of higher derivatives. As an example we consider adding to the action the term

$$\Delta\mathcal{L}_5 = A\left(R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} - 4R^{\mu\nu}R_{\mu\nu} + R^2\right) , \tag{3.9}$$

for some constant A. If we were in $D = 4$ this term would be the Euler invariant. It is one particular higher derivative term present in M-theory on CY_3. Since other terms are present as well, we don't expect (3.9) to capture the complete microscopic correction to the central charge or the black hole entropy. Later, we will do better, but this example is a good illustration.

Evaluated on $\mathrm{AdS}_3 \times S^2$ we have

$$\Delta\mathcal{L}_5 = -\frac{24A}{\ell_{Ads}^2 \ell_{S^2}^2} \,, \tag{3.10}$$

and so the central charge function is now

$$c(\ell_{Ads}, \ell_{S^2}) = -6\ell_{Ads}^3 \ell_{S^2}^2 \left(-\frac{6}{\ell_{Ads}^2} + \frac{2}{\ell_{S^2}^2} - \frac{G_{IJ}q^I q^J}{4\ell_{S^2}^4} - \frac{24A}{\ell_{Ads}^2 \ell_{S^2}^2} \right) . \tag{3.11}$$

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Extremizing and using (3.7), we find that both the radii and the central charge are corrected:

$$\begin{aligned}\ell_{Ads} &= \left(\frac{1}{6}C_{IJK}q^Iq^Jq^K\right)^{1/3} + \frac{4A}{\left(\frac{1}{6}C_{IJK}q^Iq^Jq^K\right)^{1/3}} + O(A^2)\\ \ell_{S^2} &= \frac{1}{2}\left(\frac{1}{6}C_{IJK}q^Iq^Jq^K\right)^{1/3} + \frac{A}{\left(\frac{1}{6}C_{IJK}q^Iq^Jq^K\right)^{1/3}} + O(A^2)\\ c &= 7C_{IJK}q^Iq^Jq^K + 144A\left(\frac{1}{6}C_{IJK}q^Iq^Jq^K\right)^{1/3} + O(A^2)\,. \end{aligned} \tag{3.12}$$

3.3 Black hole entropy

The corrected central charge (3.12) gives the black hole entropy according to (2.19)

$$S = 2\pi\sqrt{\frac{c}{6}h_L} + 2\pi\sqrt{\frac{c}{6}h_R}\,. \tag{3.13}$$

This formula could refer to either asymptotically AdS or asymptotically flat black holes, with slightly different interpretations. In the AdS case, the Virasoro generators are related to the mass and angular momentum of the black hole as in (2.13). The entropy formula (3.13) then holds for an arbitrary (i.e. nonextremal, nonsupersymmetric) BTZ black hole in this theory.

In the asymptotically flat case (3.13) still holds, but additional work is required to relate h_L and h_R to the asymptotic charges of the black hole. A good example is the case of M5-branes and M2-branes in M-theory on $CY_3 \times S^1$. This was the case considered in [3]. As before, we consider the M5-branes (with charges q^I) to wrap 4-cycles in CY_3, and in addition we take the M2-branes (with charges Q_I) to wrap 2-cycles. The asymptotically flat solution, in the case of $CY_3 = T^6$ compactification, was given in [17]. After taking the near horizon limit (which was all that was needed for the analysis in [3]) we find that AdS_3 becomes a extremal rotating BTZ black hole, with

$$h_R = Q_0 + \frac{1}{2}C^{IJ}Q_IQ_J\,. \tag{3.14}$$

Here Q_0 is momentum running around the asymptotic S^1, i.e. the Kaluza-Klein electric charge, and C^{IJ} is related to the intersection matrix of the compactification manifold. The extra term in h_R is due to the nonzero M2-brane charges. More discussion of this effect can be found in [3, 18, 19]. With this identification (3.13) gives the entropy in terms of the charges measured at asymptotic infinity. As we have already stressed, once higher derivatives are included (3.13) will still hold but the central charges will be corrected.

4. Anomalies, central charges, and entropy

Up until this point we have restricted attention to cases with $\tilde{c} = c$, and focussed on computing the central charge and black hole entropy from the conformal anomaly. The approach is quite powerful, but for certain cases one can do even better. Indeed, one potential disadvantage is that to compute the conformal anomaly one needs to know all the terms in the action which are nonzero in the given background. But all such terms are not necessarily known when one is considering higher derivative theories. Furthermore, when $\tilde{c} \neq c$, this approach is clearly insufficient to determine both central charges.

It is wise to take advantage of any other anomalies in the problem, as well as the relations among them following from symmetries. For the M5-brane example considered in the previous section gravitational anomalies are especially powerful. As we will review, there are two anomalies — the tangent and normal bundle anomalies — which follow from knowledge of a single term in the action, and which suffice to determine the corrections to both central charges [4]. So from this point of view the corrected entropy formula for the M5-brane emerges rather easily.

In the absence of gravitational anomalies the on-shell bulk supergravity action is a diffeomorphism invariant function of the boundary geometry. By the AdS/CFT correspondence it is supposed to yield the partition function of the CFT on the boundary. In the presence of gravitational anomalies, one is still led to conjecture the correspondence, but with each side suffering a loss of diffeomorphism invariance. This manifests itself in the non-conservation of the boundary stress tensor.

4.1 Some higher derivative terms

Several higher derivative terms in the effective action of M-theory are known (some relevant references are [20 − 22]). Those involving R^4 terms take the schematic forms

$$\begin{aligned} &t_8 t_8 RRRR\,, \\ &\epsilon_{11}\cdot\epsilon_{11} RRRR\,, \\ &\epsilon_{11} C_3 \left[\mathrm{Tr}\, R^4 - \frac{1}{4}(\mathrm{Tr}\, R^2)^2\right]. \end{aligned} \tag{4.1}$$

For the precise definitions of these, and their coefficients in the action, see, e.g. [21]. Of most interest to us is the term given in the third line since this term yields corrections to central charges and black hole entropy. The coefficient of this term is determined by requiring that its anomalous variation under diffeomorphisms cancel anomalous terms on the M5-brane worldvolume. We will review this in the dimensionally reduced context below.

Dimensional reduction of these terms on CY_3 leads to various higher derivative terms in $D = 5$ [23], as well as shifts in the coefficients of some two-derivative terms. One of the terms that appear this way is the dimensionally continued Euler invariant (3.9). Here we focus on

$$S_{\mathrm{anom}} = \frac{c_2 \cdot P_0}{48} \int_{M_5} A \wedge p_1\,, \tag{4.2}$$

which arises from reduction of the third term in (4.1). In (4.2) p_1 is the first Pontryagin class

$$p_1 = -\tfrac{1}{2}\left(\frac{1}{2\pi}\right)^2 \mathrm{Tr}\, R\wedge R\,. \tag{4.3}$$

We take the M5-brane to wrap the cycle $P_0 = P_0^I \sigma_I$, where $\{\sigma_I\}$ form a basis for $H_4(X, Z)$. The choice of 4-cycle then determines a particular linear combination of gauge fields in five dimensions, which was denoted by A in (4.2). Finally, c_2 is the second Chern class of X, which has coefficients c_{2I} in its expansion with respect to chosen basis for $H^4(X, Z)$.

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After reduction on X, the wrapped M5-branes correspond to a string in five dimensions, on which lives a chiral CFT. As explained in [4], the term (4.2) cancels the gravitational anomalies of the CFT.[5]

4.2 Anomalies

Anomaly cancellation occurs via the inflow mechanism, as we now recall. First of all, since A is ill-defined in the presence of a magnetic charge, (4.2) should really be written after performing an integration by parts and discarding the boundary term. So the actual term of interest is

$$S = \tfrac{1}{2}\left(\frac{1}{2\pi}\right)^2 \frac{c_2 \cdot P_0}{48} \int_{M_5} F \wedge \omega_3 \,, \tag{4.4}$$

where ω_3 is the Lorentz Chern-Simons 3-form:

$$\omega_3 = \mathrm{Tr}(\omega d\omega + \frac{2}{3}\omega^3) \,, \tag{4.5}$$

with ω being the spin connection. Now under a local Lorentz transformation parameterized by Θ,

$$\delta\omega = d\Theta + [\omega, \Theta] \,, \tag{4.6}$$

the action changes as

$$\delta S_{\mathrm{bulk}} = \tfrac{1}{2}\left(\frac{1}{2\pi}\right)^2 \frac{c_2 \cdot P_0}{48} \int_{M_5} F \wedge \mathrm{Tr}(d\Theta \wedge d\omega) \,. \tag{4.7}$$

At this point we encounter two distinct interpretations. The approach of [4] was to consider the magnetic string essentially as a pointlike defect placed in an ambient space. The presence of the magnetic string corresponds to dF having delta function support at the location of the string. In this approach, one integrates (4.7) by parts, and then uses the delta function to perform the integral over the directions transverse to the string. What remains is an integral over the string worldvolume, which cancels a term coming from the variation of the path integral over the string degrees of freedom.

The interpretation in our case is somewhat different. We are dealing with a smooth supergravity solution with geometry $\mathrm{AdS}_3 \times S^2 \times X$ and $dF = 0$. The branes have been replaced by flux. Instead of cancelling the anomaly at the brane location, we get a contribution at the AdS boundary. It is clear that this contribution yields the anomalous variation of the CFT on the boundary. This mechanism is well known in AdS/CFT, going back to the treatment of the R-symmetry anomaly of $\mathcal{N} = 4$ super-Yang-Mills in [24]. In particular, (4.7) gives the boundary term

$$\delta S_{\mathrm{bulk}} = \frac{1}{2}\left(\frac{1}{2\pi}\right)^2 \frac{c_2 \cdot P_0}{48} \int_{\partial M_5} F \wedge \mathrm{Tr}(\Theta d\omega) \,. \tag{4.8}$$

[5]More precisely, it cancels the part of the anomaly linear in M5-brane charge. There are also cubic terms which we'll discuss momentarily.

We consider pure $\mathrm{AdS}_3 \times S^2$ with the components of F given by (3.3). Integrating (4.8) over the S^2 we obtain

$$\delta S_{\text{bulk}} = -\tfrac{1}{2}\, \frac{c_2 \cdot q}{48}\, \frac{1}{2\pi} \int_{\partial \mathrm{AdS}_3} \mathrm{Tr}(\Theta d\omega)\,. \tag{4.9}$$

Importantly the matrices Θ and $d\omega$ are still by 5×5; they include indices along the AdS_3 boundary and also in the radial and S^2 directions. Accordingly, we can study two kinds of anomalies, associated with diffeomorphisms that map the boundary to itself (tangent bundle anomaly) and with diffeomorphisms acting on the vectors normal to the boundary (normal bundle anomaly). From the point of view of the D=2 CFT, these are gravitational and $SU(2)_R$ symmetry anomalies.

In the CFT the gravitational anomaly is obtained via descent from $I_4 = 2\pi\, \frac{1}{24}(c-\tilde{c})p_1$, yielding

$$\delta S_{\text{CFT}} = \frac{c-\tilde{c}}{48}\, \frac{1}{2\pi} \int_{\partial \mathrm{AdS}_3} \mathrm{Tr}(\Theta d\omega)\,. \tag{4.10}$$

Equating this with (4.9) we find[6]

$$\tilde{c} - c = \tfrac{1}{2} c_2 \cdot q\,. \tag{4.11}$$

The computation of the normal bundle anomaly is similar. In this case the corresponding CFT anomaly is in the $SU(2)_R$ symmetry which, in our conventions, acts on the leftmovers so that the normal bundle anomaly contributes

$$c_{\text{lin}} = \tfrac{1}{2} c_2 \cdot q\,, \tag{4.12}$$

to c. The form of (4.11) and (4.12) are the same because these contributions arise from the same anomaly (4.9), decomposed into tangent and normal bundle part, and interpreted appropriately. These expressions capture the linear contributions to the central charges exactly. However, there are also $\mathcal{O}(q^3)$ contributions (see (3.8)) coming from the two-derivative part of the action, and so altogether we have

$$c = C_{IJK} q^I q^J q^K + \frac{1}{2} c_2 \cdot q\,, \qquad \tilde{c} = C_{IJK} q^I q^J q^K + c_2 \cdot q\,. \tag{4.13}$$

These are the results found in [3, 4].

The $C_{IJK} q^I q^J q^K$ contributions are actually quite subtle in the context of anomaly cancellation for M5-branes viewed as pointlike defects [25, 26]. The $\mathcal{O}(q^3)$ contribution to the normal bundle anomaly requires a subtle modification of the M-theory Chern-Simons term. By contrast, in the context of the smooth supergravity backgrounds considered here, this contribution is simple to understand because it comes from the leading two-derivative part of the action. We have phrased this in terms of computing the conformal anomaly, but we could have equally well computed the normal bundle anomaly directly. In our problem supersymmetry related these anomalies to one another, so a computation of either suffices.

[6] There are two (cancelling) sign changes relative to the anomaly inflow in [4]: the boundary at infinity has normal opposite to that of a defect in bulk; and also we are *equating* the two anomalies, as in AdS/CFT, rather than *cancelling* them, as in the anomaly inflow.

4.3 Application: heterotic strings

An important special case of our computations is M-theory on $K3 \times T^2$. Consider an $M5$-brane wrapped around the $K3$ and transverse to the T^2. In this case we have $C_{IJK}q^I \times q^J q^K = 0$, and $c_2 \cdot q = 24$ because the Euler number of $K3$ is 24. Therefore, (4.13) gives $c = 12$ and $\tilde{c} = 24$. These are the correct assignments for the heterotic string which, indeed, is a dual representation of an M5-brane on $K3 \times T^2$. Thus we find the central charges of *both* sides of the heterotic strings; so we are sensitive to all excitations, rather than just the BPS states. In particular, from the Cardy formula (1.3) we get the entropy of nonsupersymmetric small black holes in agreement with the non-BPS entropy of the heterotic string. Although the formulae (4.13) have been known for some time, this agreement apparently has not been noticed before.

The recovery of both the central charges of the heterotic string sounds like an extremely powerful and surprising result when put, as above, in terms of the near horizon geometry, corrected by higher derivative terms in the action. However, from another point of view the agreement is almost trivial: a heterotic string propagating in a curved background suffers gravitational anomalies, because $c \neq \tilde{c}$, and these must be cancelled by bulk terms, *via* the inflow mechanism. This works, of course; indeed, it would be one way to derive the anomalous coupling S_{anom}, including the coefficient. Related to this, heterotic string theory in $AdS_3 \times N$ has linear corrections that precisely reproduce the ones seen here [27]. From either point of view, we should hardly be surprised when these couplings give back the heterotic string, when interpreted in terms of the near horizon geometry and its boundary at infinity. On the other hand, the fact that the agreement is essentially automatic does not make it any less valid, nor any less interesting.

4.4 Application: inclusion of angular momentum

Consider the BPS states of a heterotic string wrapped on an S^1 in T^5, with fixed winding number, rightmoving momentum, and angular momentum in a given 2-plane. The microscopic entropy is known to be [28]

$$S = 4\pi\sqrt{N_w N_p - J}\,. \tag{4.14}$$

Geometrically, the states correspond to rotating helical strings. The maximal angular momentum, $J = N_w N_p$ is attained when all the momentum is placed in oscillators of the lowest mode number, with polarizations in the angular momentum 2-plane. The profile of the helix is then a circle. If we decrease J from its maximal value while holding $N_{w,p}$ fixed, then there are additional microstates in which the string wiggles away from its circular shape, either in the noncompact or internal dimensions. These additional states give rise to the entropy (4.14). As we will now argue, there is also a black object with the same charges and whose entropy agrees with (4.14).

Using heterotic/IIA duality, and lifting to M-theory, the configuration above describes a rotating helical M5-brane wrapped on $K3 \times T^2$. The supergravity solution will have near horizon limit $\mathrm{AdS}_3 \times S^2 \times K^3 \times T^2$. The rightmoving central charge is $\tilde{c} = 24N_w$, since the M5-brane wraps $K3$ N_w times. The level number h_R appearing in the near horizon region differs from the total rightmoving momentum N_p measured at infinity. Although we have not checked this explicitly in the present context, in other very similar cases (see, e.g. [29]) one finds that h_R is obtained by subtracting from N_p the momentum used up by the gyration:

$$h_R = N_p - N_{\rm gyro}\,. \tag{4.15}$$

The mechanical gyration of the string carries momentum and angular momentum related by $J_{\rm gyro} = \frac{\lambda}{2\pi} P_{\rm gyro}$, where λ is the wavelength of the gyration. Since our brane is wrapped N_w times around a circle of radius R, the largest possible wavelength (which yields the highest entropy) is $\lambda = 2\pi R N_w$, and so

$$h_R = N_p - \frac{J}{N_w}\,. \tag{4.16}$$

The near horizon geometry will thus be a BTZ black hole with entropy given by the Cardy formula as

$$S = 2\pi\sqrt{\frac{c}{6}h_R} = 4\pi\sqrt{N_w N_p - J}\,, \tag{4.17}$$

in agreement with (4.14). The black object could be thought of as a "small" black ring.

With the replacements $N_w \to N_5$ and $N_p \to N_1$, (4.14) also gives the ground state entropy of the D1-D5 system on K3. Indeed there is a duality chain that relates the two systems. The M-theory configuration can be interpreted as IIA on $K3 \times S^1$ with NS5-branes wrapped on the compact space and carrying momentum on the S^1. A T-duality on the S^1 followed by S-duality then yields the D1-D5 system. The ground state entropy of the D1-D5 system has recently been obtained in a different approach by Iizuka and Shigemori [30].

5. Discussion: corrections to all orders in $1/Q$ and beyond

The black hole entropy discussed in this paper has been presented in all cases in terms of the Cardy formula which is essentially semi-classical. It is interesting to think about how further corrections might be included. In particular, recent work has shown that is possible to reproduce the BPS entropy of the heterotic string to all orders in an expansion in inverse powers of the charges [10]. Let us now show how our approach is naturally extended to include this agreement.

In evaluating the black hole partition function in section 2.3 we specified the black hole temperature β and chemical potential μ, which are conjugate to the mass and angular momentum of the black hole. We now note that we could also specify the values of any conserved charges or, alternatively, the boundary values of the corresponding gauge potentials.

In the case of M-theory on $K3 \times T^2$ we have gauge fields A^I that couple to charges Q_I that correspond to wrapped M2-branes. We thus need the euclidean action of black holes carrying these charges (as well as the M5-brane charge q^I). According to (3.14) this just gives a shift in h_R which, from (2.14), changes the action to

$$I_{BH}(\overline{\tau}, Q_I) = \frac{i\pi}{12}\frac{\tilde{c}}{\overline{\tau}} + 2\pi i\overline{\tau}\;\frac{1}{2}C^{IJ}Q_I Q_J\,. \tag{5.1}$$

To focus on BPS states we set the left moving temperature to zero: $\frac{1}{\tau} = 0$. Semi-classically, the potentials are related to the charges as

$$\phi^I = \frac{1}{\pi}\frac{\partial I_{BH}}{\partial Q_I}\,, \tag{5.2}$$

so

$$I_{BH}(\phi^0, \phi^I) = \frac{\pi\tilde{c}}{6\phi^0} - \frac{\pi}{2}\frac{C_{IJ}\phi^I\phi^J}{\phi^0}\,, \tag{5.3}$$

where we renamed the right moving temperature

$$\phi^0 = \frac{2}{i}\overline{\tau}\,. \tag{5.4}$$

The potentials ϕ^0, ϕ^I defined in (5.2) and (5.4) were designed to agree with the conventions in the topological string literature [11, 10] which amounts to the equality

$$e^{-\pi(Q_0\phi^0 + q_I\phi^I)} = e^{2\pi i\overline{\tau}(Q_0 + Q_I A_t^I)}\,. \tag{5.5}$$

Now, the expression (5.3) for the action is precisely the same as (the negative of) the free energy $\mathcal{F}_{\rm pert}$ appearing in [10, (2.6)], and at this point we can simply follow their analysis. In particular, the degeneracy of states $\Omega(Q_0, Q_I)$ with the specified charge is given by the relation between the canonical and microcanonical ensembles

$$\Omega(Q_0, Q_I) = \int d\phi\, e^{-I_{BH}(\phi^0,\phi^I) + \pi(Q_0\phi^0 + Q_I\phi^I)}\,. \tag{5.6}$$

Carrying out the integral yields a Bessel function which correctly accounts for the number of heterotic string states to all orders in inverse powers of charges. We refer the reader to [10] for the details (and also to [9] for an alternative approach). The point we wish to emphasize here is that the power law corrections to the black hole entropy are semi-classical in nature, and so they can be captured by our approach.

Ultimately, several other corrections must be included in order to account completely for the microscopic degeneracies including exponentially suppressed terms. For example, there are contributions from world-sheet and brane instantons and also, more dramatically, from semi-classical geometries distinct from the one contributing to the leading term. These corrections remain to be understood, both in the 4D topological string approach, and in the approach considered here.

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A. Asymptotically flat M5-brane solution

For convenience, we give here the asymptotically flat solution representing M5-branes wrapped on 4-cycles of CY_3. We follow the conventions of [16]. The CY_3 has harmonic $(1,1)$ forms J_I and Kahler moduli X^I. The metric and 3-form are

$$\begin{aligned} ds^2 &= ds_5^2 + ds_{CY_3}^2 \\ \mathcal{A} &= A^I \wedge J_I \end{aligned} \tag{A.1}$$

with

$$\begin{aligned} ds_5^2 &= \left(\frac{1}{6}C_{IJK}H^I H^J H^K\right)^{-1/3}\left(-dt^2 + dx_4^2\right) + \left(\frac{1}{6}C_{IJK}H^I H^J H^K\right)^{2/3}\left(dr^2 + r^2 d\Omega_2^2\right) \\ A^I &= \frac{1}{6}q^I(1+\cos\theta)d\phi \\ X^I &= \frac{H^I}{(\frac{1}{6}C_{IJK}H^I H^J H^K)^{1/3}} \\ H^I &= \overline{X}^I + \frac{q^I}{2r}\,. \end{aligned} \tag{A.2}$$

To examine the near horizon geometry we write

$$r = \frac{\frac{1}{6}C_{IJK}q^I q^J q^K}{2z^2}\,. \tag{A.3}$$

For $z \to \infty$ we then find the following $\mathrm{AdS}_3 \times S^2 \times CY_3$ geometry

$$\begin{aligned} ds_5^2 &= \ell_{Ads}^2 \frac{-dt^2 + dx_4^2 + dz^2}{z^2} + \ell_{S^2}^2 d\Omega_2^2 \\ X^I &= \frac{q^I}{(\frac{1}{6}C_{IJK}q^I q^J q^K)^{1/3}} \end{aligned} \tag{A.4}$$

with

$$\ell_{Ads} = 2\ell_{S^2} = \left(\frac{1}{6}C_{IJK}q^I q^J q^K\right)^{1/3}\,. \tag{A.5}$$

The Brown-Henneaux computation of the central charge applied to this case gives

$$c = C_{IJK}q^I q^J q^K\,. \tag{A.6}$$

(A.5) and (A.6) are in perfect agreement with (3.8).

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