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Bianchi IX self-dual Einstein metrics and singular G_2 manifolds

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Abstract

We construct explicit cohomogeneity 2 metrics of G_2 holonomy, which are foliated by twistor spaces. The twistor spaces are S^2 bundles over four-dimensional Bianchi IX Einstein metrics with self-dual (or anti-self-dual) Weyl tensor. Generically the 4-metric is of triaxial Bianchi IX type, with $SU(2)$ isometry. We derive the first-order differential equations for the metric coefficients, and obtain the corresponding superpotential governing the equations of motion, in the general triaxial Bianchi IX case. In general our metrics have singularities, which are of orbifold or cosmic-string type. For the special case of biaxial Bianchi IX metrics, we give a complete analysis of their local and global properties, and the singularities. In the triaxial case, we find that a system of equations written down by Tod and Hitchin satisfies our first-order equations. The converse is not always true. A discussion is given of the possible implications of the singularity structure of these spaces for M-theory dynamics.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Concrete non-singular examples of seven-dimensional metrics with G_2 holonomy have been known only since about 1989. The original construction involved making an ansatz for metrics of cohomogeneity 1, where the six-dimensional principal orbits were $S^3 \times S^3$, or else the twistor spaces of S^4 or $\mathbb{CP}^2$ [1, 2]. The twistor space is a 2-sphere bundle over the S^4 or

$\mathbb{CP}^2$ base, with an $SU(2)$ or $SO(3)$ structure group associated with the chiral spin (or spin^c) bundle of the base. The local construction can be carried out for any base space M_4 equipped with an Einstein metric and for which the Weyl tensor is self-dual or anti-self-dual [1, 2][6]. By a theorem of Hitchin, the only non-singular such examples with positive Ricci tensor (which implies that M_4 is compact) occur when M_4 is S^4 or $\mathbb{CP}^2$ [3].

The current interest in G_2 manifolds in M-theory has been motivated in part by the role that they can play in compactifying to four dimensions, analogous to the compactification of ten-dimensional string theory on Calabi–Yau 6-manifolds. Unlike the latter, where non-singular Calabi–Yau manifolds can naturally give rise to chiral $\mathcal{N} = 1$ theories in four dimensions starting from the heterotic string in $D = 10$, non-singular G_2 compactifications of M-theory would necessarily give Abelian non-chiral $\mathcal{N} = 1$ theories in four dimensions. To get non-Abelian chiral theories from M-theory, one needs to consider compactifications on singular G_2 manifolds. One explicit realization of such an M-theory compactification has an interpretation as an S^1 lift of type IIA theory (compactified on an orientifold) with intersecting D6-branes and O6 orientifold planes [4]. Non-Abelian gauge fields arise at the locations of coincident branes, and chiral matter arises at the intersections of D6-branes. The S^1 lift of such configurations results in singular G_2 holonomy metrics in M-theory. Co-dimension four ADE-type singularities are associated with the location of the coincident D6-branes, and co-dimension seven singularities are associated with the location of the intersection of two D6-branes in type IIA theory [4–8].

Further analyses of co-dimension seven singularities of the G_2 holonomy spaces, leading to chiral matter, were given in [6–8] and the subsequent work [9–12]. It is expected that there exist wide classes of 7-manifolds with G_2 holonomy and the singularity structure that again would yield non-Abelian $\mathcal{N} = 1$ supersymmetric four-dimensional theories with chiral matter, and in particular the explicit construction of such metrics would provide a starting point for further studies of chiral M-theory dynamics.

Much research on finding new non-singular G_2 manifolds has been carried out in recent times (see, for example, [13–19] and references therein). In view of their potential phenomenological interest, it is appropriate also to investigate examples of singular G_2 manifolds. Typically, these singularities should be of co-dimension seven, and they should be of the relatively mild orbifold type [5, 6], where the curvature is bounded everywhere except for δ-function contributions.

One way to obtain singular G_2 holonomy spaces is by returning to the original G_2 construction in [1, 2], with principal orbits that are S^2 bundles over self-dual Einstein four-dimensional manifolds M_4 (forming the base of the twistor space), but with M_4 now chosen to be neither the S^4 nor the $\mathbb{CP}^2$ non-singular examples. Instead, one can choose M_4 to be a self-dual Einstein space with orbifold-type singularities. Some investigations of the G_2 metrics that result from such a construction have already been carried out [10]. In this paper we pursue the analysis further, by considering more general possibilities for the base space M_4. Since the procedure for obtaining the G_2 metric from a given self-dual Einstein base space M_4 is well established [1, 2], much of the paper will concentrate on the details of the self-dual Einstein metrics themselves.

There exists a large mathematical literature on self-dual Einstein metrics (sometimes called quaternionic Kähler). The focus of our study in this paper will be on self-dual Einstein metrics of the triaxial Bianchi IX type, where there is an $SU(2)$ isometry that acts transitively on three-dimensional orbits that are (locally) S^3. Quite a lot is known about this case

[6] Such metrics are generally referred to as 'self-dual Einstein', and unless the context makes it necessary in order to avoid confusion, we shall often use this term regardless of whether the Weyl tensor is actually self-dual or anti-self-dual.

[20, 22, 23], but we believe that our results go beyond what is in the existing literature, and that our viewpoint, derived as it is from the associated G_2 metric, is novel. In particular, we shall derive the general first-order equations for these metrics and analyse their local and global structures. For the special case of biaxial Bianchi IX metrics, we provide a complete analysis. In the triaxial case, we compare our analysis with that of Tod [21] and Hitchin [22, 23], and analyse some of the explicitly known solutions. Some implications for M-theory of these G_2 holonomy metrics are also discussed.

2. Asymptotically conical G_2 metrics

2.1. G_2 holonomy of R^3 bundles over self-dual Einstein 4-metrics

The metrics of G_2 holonomy that have twistor-space orbits take the form [1, 2]

$$\mathrm{d}s_7^2 = 4\left(1-\frac{1}{r^4}\right)^{-1}\mathrm{d}r^2 + r^2\left(1-\frac{1}{r^4}\right)(\mathrm{D}\mu_i)^2 + 2r^2\,\mathrm{d}s_4^2, \tag{1}$$

where $\mu_i\mu_i = 1$. The covariant exterior derivative is defined by $\mathrm{D}\mu_i \equiv \mathrm{d}\mu_i + \epsilon_{ijk}A^j\mu_k$, and the metric $\mathrm{d}s_4^2$ is required to be Einstein, with $R_{ab} = \Lambda g_{ab}$ (with Λ taken to be normalized to $\Lambda = 3$ in (1)). The Yang–Mills fields have the defining property that $\mathrm{D}J^i = 0$, where the quaternionic Kähler forms J^i on the base space M_4 have a definite duality, and satisfy

$$J^i_{ab}J^j_{bc} = -\delta_{ac}\delta^{ij} + \epsilon_{ijk}J^k_{ac}, \tag{2}$$

where the gauge-covariant exterior derivative is defined by

$$\mathrm{D}J^i_{ab} \equiv \nabla J^i_{ab} + \epsilon_{ijk}A^j J^k_{ab}, \tag{3}$$

with

$$\nabla J^i_{ab} \equiv \mathrm{d}J^i_{ab} + \omega_{ac}J^i_{cb} + \omega_{bc}J^i_{ac}. \tag{4}$$

The integrability condition $\mathrm{D}^2 J^i_{ab} = 0$ has, as a particular consequence,

$$F^i_{ab} = \tfrac{1}{2}J^i_{cd}R_{abcd}, \tag{5}$$

where $F^i \equiv \mathrm{d}A^i + \frac{1}{2}\epsilon_{ijk}A^j\wedge A^k$. We furthermore require that the Yang–Mills fields F^i be proportional to the quaternionic Kähler forms. We shall take J^i_{ab} to be self-dual, in which case we have the identity

$$J^i_{ab}J^i_{cd} = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bd} + \epsilon_{abcd}. \tag{6}$$

From this and equation (5), it can be seen that if the Weyl tensor

$$C_{abcd} \equiv R_{abcd} - \tfrac{1}{3}\Lambda(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) \tag{7}$$

of the Einstein metric $\mathrm{d}s_4^2$ is anti-self-dual, then we shall have

$$F^i = \tfrac{1}{3}\Lambda J^i. \tag{8}$$

We can change variables to a set of coordinates u_i on $\mathbb{R}^3$, which are unconstrained, by taking $u_i = \rho\mu_i$, and letting $\frac{1}{3}\Lambda\rho^2 = r^4 - 1$, leading to the expression

$$\mathrm{d}s_7^2 = \frac{(\mathrm{D}u_i)^2}{\sqrt{1+\frac{1}{3}\Lambda\rho^2}} + 2\sqrt{1+\frac{1}{3}\Lambda\rho^2}\,\mathrm{d}s_4^2, \tag{9}$$

where ρ means $\sqrt{u_iu_i}$, $\mathrm{D}u_i = \mathrm{d}u_i + \epsilon_{ijk}A^ju_k$, and we have rescaled so that $\mathrm{d}s_4^2$ has cosmological constant Λ.

The G_2 holonomy is easily established by noting that we may take the associative 3-form to be, reverting to $\Lambda = 3$ for convenience,

$$\Phi_{(3)} = \tfrac{1}{6}(1+\rho^2)^{-3/4}\epsilon_{ijk}\mathrm{D}u_i\mathrm{D}u_j\mathrm{D}u_k + 2(1+\rho^2)^{1/4}\mathrm{D}u_i \wedge J^i. \tag{10}$$

The dual of $\Phi_{(3)}$ in the metric (9) is therefore

$$*\Phi_{(3)} = 4(1+\rho^2)\Omega_{(4)} + \epsilon_{ijk}\mathrm{D}u_i \wedge \mathrm{D}u_j \wedge J^k, \tag{11}$$

where $\Omega_{(4)}$ is the volume form of $\mathrm{d}s_4^2$, which can also be written as $\Omega_{(4)} = \frac{1}{2}J^1 \wedge J^1 = \frac{1}{2}J^2 \wedge J^2 = \frac{1}{2}J^3 \wedge J^3$. From the identity $\mathrm{D}^2u_i = \epsilon_{ijk}F^j u_k$, one easily sees that $\Phi_{(3)}$ is closed and co-closed.

2.2. *Nearly Kähler geometry and G_2 holonomy*

If we go to the asymptotic region, where $\rho \longrightarrow \infty$, we get the metric on the cone over the twistor space of M_4,

$$\mathrm{d}s_7^2 = \frac{1}{\rho}(\mathrm{D}u_i)^2 + 2\rho\,\mathrm{d}s_4^2. \tag{12}$$

Defining $\rho = \frac{1}{4}r^2$, this becomes

$$\mathrm{d}s_7^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}s_6^2, \tag{13}$$

and so if $\mathrm{d}s_7^2$ has G_2 holonomy then

$$\mathrm{d}s_6^2 = \tfrac{1}{4}(\mathrm{D}\mu_i)^2 + \tfrac{1}{2}\,\mathrm{d}s_4^2 \tag{14}$$

is the nearly Kähler metric on the twistor space of M_4. The associative 3-form becomes

$$\Phi_{(3)} = \tfrac{1}{6}\rho^{-3/2}\epsilon_{ijk}\,\mathrm{D}u_i\,\mathrm{D}u_j\,\mathrm{D}u_k + 2\rho^{1/2}\,\mathrm{D}u_i \wedge J^i, \tag{15}$$

and its Hodge dual is

$$*\Phi_{(3)} = 4\rho^2\Omega_{(4)} + \epsilon_{ijk}\,\mathrm{D}u_i \wedge \mathrm{D}u_j \wedge J^k. \tag{16}$$

The conditions of closure and co-closure of $\Phi_{(3)}$ therefore imply that $\mathrm{d}s_6^2$ in (14) is nearly Kähler.

The definition of a nearly Kähler metric $\mathrm{d}s_6^2$ is that the cone over $\mathrm{d}s_6^2$, namely

$$\mathrm{d}\hat{s}_7^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}s_6^2 \tag{17}$$

has G_2 holonomy. A more suggestive, but equivalent, terminology for $\mathrm{d}s_6^2$ is therefore that it has *weak $SU(3)$ holonomy*; we discuss this briefly below.

If the cone metric $\mathrm{d}\hat{s}_7^2$ has G_2 holonomy, it follows that the associative 3-form $\Phi_{(3)}$, which may be written as

$$\Phi_{(3)} = r^2\,\mathrm{d}r \wedge J_{(2)} + r^3\rho_{(3)}, \tag{18}$$

must be closed and co-closed. This has the consequences

$$\mathrm{d}J_{(2)} = 3\rho_{(3)}, \qquad \mathrm{d}\tilde{\rho}_{(3)} + 2J_{(2)} \wedge J_{(2)} = 0, \tag{19}$$

where $\tilde{\rho}_{(3)} \equiv *_6\rho_{(3)}$. Immediate further consequences of these equations are $J_{(2)} \wedge \rho_{(3)} = 0$ and $\mathrm{d}\rho_{(3)} = 0$. The associativity relation

$$\Phi_{ABE}\Phi_{CDE} = \delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} + \tfrac{1}{6}\epsilon_{ABCDEFG}\Phi_{EFG} \tag{20}$$

for the 3-form $\Phi_{(3)}$ has the consequences that

$$J_{ab}J_{bc} = -\delta_{ac}, \qquad J_{ad}\rho_{abd} = \tilde{\rho}_{abc}. \tag{21}$$

This means that J defines an almost complex structure in ds_6^2, and that with respect to J, $\psi_{(3)} \equiv \rho + i\hat{\rho}$ is a holomorphic 3-form of type (3, 0).

The relation to weak $SU(3)$ holonomy can be made more explicit by considering the covariantly constant spinor $\hat{\eta}$ that exists in the G_2 metric (17). In the natural orthonormal basis $\hat{e}^0 = dr$, $\hat{e}^a = re^a$, one finds that the covariant exterior derivative $\hat{\nabla} = d + \frac{1}{4}\hat{\omega}_{AB}\Gamma^{AB}$ is given by $\hat{\nabla} = \nabla - \frac{1}{2}\Gamma_{0a}e^a + dr\frac{\partial}{\partial r}$. If ds_6^2 has weak $SU(3)$ holonomy then it admits Killing spinors $\eta_\pm$ satisfying $D_a^\pm \eta_\pm \equiv \left(\nabla_a \pm \frac{1}{2}\Gamma_{0a}\right)\eta_\pm = 0$, for which the integrability condition is $\left[D_a^\pm, D_b^\pm\right] = \frac{1}{4}C_{abcd}\Gamma^{cd}$. This admits solutions $\eta_\pm$ if $C_{abcd}\Gamma^{cd}$ generates the $SU(3)$ subgroup of the tangent-space group $SO(6) \sim SU(4)$. The two spinors are related by $\eta_\pm^* = \eta_\mp$ and $\Gamma_0\eta_\pm = -\eta_\mp$. The covariantly constant spinor in the G_2 metric $d\hat{s}_7^2$ is given by $\hat{\eta} = \eta_-$.

In terms of the Killing spinors $\eta_\pm$ in ds_6^2, the almost complex structure $J_{(2)}$ and the 3-form $\rho_{(3)}$ are given by

$$J_{ab} = i\eta_+^\dagger\Gamma_{ab}\eta_-, \qquad \rho_{abc} = i\eta_+^\dagger\Gamma_{0abc}\eta_-. \tag{22}$$

From the Killing spinor equations $D_a^\pm\eta_\pm = 0$ one can now easily derive the equations

$$\nabla_a J_{bc} = \rho_{abc}, \qquad \nabla_a\tilde{\rho}_{bcd} = -3J_{[ab}J_{cd]}. \tag{23}$$

These equations, which in particular imply (19), characterize nearly Kähler metrics. Note that by symmetrizing the first equation on a and b, we obtain the equation for a Yano Killing tensor, $\nabla_{(a}J_{b)c} = 0$ [24]. Thus the nearly Kähler 6-manifolds constructed in this paper provide new examples of supersymmetric quantum mechanical systems with hidden symmetries [25, 26]. In fact, because J_{ab} is an almost complex structure, the associated symmetric Staeckel Killing tensor is given by $J_{ab}J^b{}_c = -g_{ac}$, and hence is trivial in this case.

3. G_2 holonomy equations for Bianchi IX base

We now apply the formalism of section 2.1, with the four-dimensional base metric taken to be of the triaxial Bianchi IX form:

$$ds_4^2 = dt^2 + a_i^2\sigma_i^2. \tag{24}$$

The self-dual $SU(2)$ Yang–Mills connection is

$$A^i = -\omega_{0i} - \tfrac{1}{2}\epsilon_{ijk}\omega_{jk}, \tag{25}$$

where the spin connection of ds_4^2, in the vielbein basis $e^0 = dt$, $e^i = a_i\sigma_i$, is given by

$$\omega_{01} = \beta_1 e^1, \qquad \omega_{23} = \gamma_1 e^1, \tag{26}$$

and cyclically, with

$$\beta_1 \equiv -\frac{\dot{a}_1}{a_1}, \qquad \gamma_1 \equiv \frac{a_1^2 - a_2^2 - a_3^2}{2a_1a_2a_3}, \tag{27}$$

and cyclically. Since the Yang–Mills potentials are expressed in terms of the left-invariant 1-forms σ_i,

$$A^i = -a_i(\beta_i + \gamma_i)\sigma_i, \tag{28}$$

the field strengths are necessarily $SU(2)$ invariant, and are given by

$$F^i = -\Theta_{0i} - \epsilon_{ijk}\Theta_{jk}. \tag{29}$$

By imposing the closure and co-closure of $\Phi_{(3)}$ given by (10) (or equivalently, and more simply, (15)), we find that the first-order equations for a_i such that the 7-manifold has G_2

holonomy are then given by

$$\dot a_1 - \dot a_2\dot a_3 + \left(\frac{a_3^2-a_1^2-a_2^2}{2a_1a_2}\right)\dot a_2 + \left(\frac{a_2^2-a_1^2-a_3^2}{2a_1a_3}\right)\dot a_3 + \frac{a_2^4+a_3^4-3a_1^4+2\big(a_1^2a_2^2+a_1^2a_3^2-a_2^2a_3^2-\frac{2}{3}\Lambda a_1^2a_2^2a_3^2\big)}{4a_1^2a_2a_3} = 0, \tag{30}$$

together with the two equations obtained by cyclic permutation of the subscripts 1, 2 and 3. Note that we have restored the cosmological constant Λ, so that $\mathrm{d}s_4^2$ satisfies $R_{ab} = \Lambda g_{ab}$. It is straightforward to see that after using the first-order equations, (29) becomes

$$F^i = -\tfrac{1}{3}\Lambda\big(e^0\wedge e^i + \tfrac{1}{2}\epsilon_{ijk}e^j\wedge e^k\big) = \tfrac{1}{3}\Lambda J^i. \tag{31}$$

We saw in section 2 that the conditions for $\mathrm{d}s_7^2$ in (9) to have G_2 holonomy should be equivalent to the conditions for $\mathrm{d}s_4^2$ to have (anti)-self-dual Weyl tensor. In fact another way to derive the first-order equations (30) is as follows. We define the family of tensors

$$X_{abcd} \equiv R_{abcd} - \kappa(g_{ac}g_{bd} - g_{ad}g_{bc}), \tag{32}$$

where κ is an as-yet unspecified constant parameter. If we now require that X_{abcd} be anti-self-dual, we obtain the equation

$${}^*R_{abcd} + R_{abcd} - \kappa\epsilon_{abcd} - \kappa(g_{ac}g_{bd} - g_{ad}g_{bd}) = 0. \tag{33}$$

Contraction with g^{bd} gives $R_{ac} = 3\kappa g_{ac}$. It then follows that X_{abcd} is the Weyl tensor of an Einstein metric with scalar curvature 12κ, and moreover that the Einstein manifold has anti-self-dual Weyl tensor. We find that the equations

$$X_{0123} = -X_{2323}, \qquad X_{0231} = -X_{3131}, \qquad X_{0312} = -X_{1212} \tag{34}$$

give precisely (30), and that the remaining anti-self-duality equations for X_{abcd}, i.e.

$$X_{0101} = -X_{0123}, \qquad X_{0202} = -X_{0231}, \qquad X_{0303} = -X_{0312}, \tag{35}$$

give second-order equations that are nothing but the derivatives of (30).

One could, in principle, solve (30) for the $\dot a_i$ themselves, but this involves finding the roots of a quintic equation. It is, nevertheless, useful to present the first-order equations in a factorized form. Solving two of the equations (30) for $\dot a_2$ and $\dot a_3$, and substituting into the third, we get

$$\left(\dot a_1 - \frac{a_1^2-(a_2+a_3)^2}{2a_2a_3}\right)^2\left(\dot a_1 - \frac{a_1^2-(a_2-a_3)^2}{2a_2a_3}\right)^2\left(\dot a_1 - \frac{a_1^2-a_2^2-a_3^2}{2a_2a_3} - \frac{1}{3}\Lambda a_2a_3\right) - \frac{1}{9}\Lambda^2a_1^2a_2a_3\big(2a_2a_3\dot a_1+3a_2^2+a_3^2-a_1^2\big)\big(2a_2a_3\dot a_1+a_2^2+3a_3^2-a_1^2\big) = 0. \tag{36}$$

Of course the two equations following by cyclic permutation hold too, but it would be misleading to think of these three as the equations for the a_i, since one should not solve them independently. Rather, we can view (36) itself as the equation for $\dot a_1$, and then substitute this solution back into the cyclic set defined by (30) in order to obtain the equations for $\dot a_2$ and $\dot a_3$.

It is interesting to observe that in the limit when $\Lambda \longrightarrow 0$, then from (36) and (30) we can see that we get either the 'Atiyah–Hitchin' [27] first-order system[7]

$$\dot a_1 = \frac{a_1^2-(a_2+a_3)^2}{2a_2a_3}, \qquad \text{and cyclic,} \tag{37}$$

[7] Or an equivalent one with sign reversals of certain of the a_i functions.

or the 'BGPP' [28] system

$$\dot{a}_1 = \frac{a_1^2 - a_2^2 - a_3^2}{2a_2 a_3}, \qquad \text{and cyclic.} \tag{38}$$

Equations (37) admit the Atiyah–Hitchin [27] and self-dual Taub–NUT [29] metrics as particular solutions, whilst equations (38) admit the BGPP [28] and Eguchi–Hanson [30] metrics as solutions.

It is often more convenient to recast first-order equations such as (30) into a form where the metric functions $\alpha_i \equiv a_i^2$ themselves appear without square roots. This can be achieved by introducing a new radial variable ρ, defined by $\mathrm{d}t = a_1 a_2 a_3\, \mathrm{d}\rho$. We then find that (30) becomes

$$\begin{aligned} &2\frac{\mathrm{d}\alpha_1}{\mathrm{d}\rho} - \frac{1}{\alpha_2\alpha_3}\frac{\mathrm{d}\alpha_2}{\mathrm{d}\rho}\frac{\mathrm{d}\alpha_3}{\mathrm{d}\rho} + \frac{(\alpha_3 - \alpha_1 - \alpha_2)}{\alpha_2}\frac{\mathrm{d}\alpha_2}{\mathrm{d}\rho} + \frac{(\alpha_2 - \alpha_3 - \alpha_1)}{\alpha_3}\frac{\mathrm{d}\alpha_3}{\mathrm{d}\rho} \\ &\qquad + \alpha_2^2 + \alpha_3^2 - 3\alpha_1^2 + 2\alpha_1(\alpha_2 + \alpha_3) - 2\alpha_2\alpha_3 - \frac{4}{3}\Lambda\alpha_1\alpha_2\alpha_3 = 0, \end{aligned} \tag{39}$$

and cyclically. Note also that in terms of the β_i and γ_i coefficients defined in (26) and (27), the first-order equations (30) can be written as

$$(\beta_1 + \gamma_1)(\gamma_2 + \gamma_3) = (\beta_2 + \gamma_2)(\beta_3 + \gamma_3) + \tfrac{1}{3}\Lambda, \tag{40}$$

and cyclically.

If we consider the specialization where all three metric functions a_i are set equal, $a_i = a$, the first-order system (30) reduces to

$$\dot{a}^2 + \tfrac{1}{3}\Lambda a^2 = \tfrac{1}{4}. \tag{41}$$

This gives

$$\mathrm{d}s_4^2 = \mathrm{d}t^2 + \frac{3}{4\Lambda}\sin^2\left(\sqrt{\frac{1}{3}\Lambda t}\right)\sigma_i^2. \tag{42}$$

The metric extends to a complete non-singular metric on S^4 if $\Lambda > 0$, and to the hyperbolic space H^4 if $\Lambda < 0$.

The specialization to biaxial metrics, where two of the metric functions are set equal, is considerably more complicated. We shall study this in detail in the next section.

4. Biaxial anti-self-dual Bianchi IX metrics

In this section we shall specialize to the biaxial case, setting $a_2 = a_1$. The first-order equations (30) reduce to

$$\begin{aligned} \dot{a}_3 &= \dot{a}_1^2 + \frac{\dot{a}_1 a_3}{a_1} - 1 + \frac{3a_3^2}{4a_1^2} + \frac{1}{3}\Lambda a_1^2, \\ 0 &= (2a_1\dot{a}_1 + a_3)(2a_1\dot{a}_1 + a_3 - 2a_1)(2a_1\dot{a}_1 + a_3 + 2a_1) + \frac{1}{6}\Lambda a_1(2a_1\dot{a}_1 + 3a_3). \end{aligned} \tag{43}$$

It is easy to see that if we take the limit where Λ goes to zero, the cubic equation for $\dot{a}_1$ has roots giving

$$\dot{a}_1 = -\frac{a_3}{2a_1} + 1 \quad \text{or} \quad \dot{a}_1 = -\frac{a_3}{2a_1} - 1 \quad \text{or} \quad \dot{a}_1 = -\frac{a_3}{2a_1}. \tag{44}$$

The first two possibilities are associated with the first-order equations that yield the self-dual Ricci-flat Taub–NUT metrics, whilst the third yields the Eguchi–Hanson metric (which is also self-dual and Ricci-flat). In the self-dual Taub–NUT case, the $SO(3) \subset U(2)$ rotates the three hyper-Kähler forms as a triplet, while in the case of the Eguchi–Hanson metrics, they are singlets under $SO(3)$.

For future reference, we note that equations (43) imply that the Weyl tensor of ds_4^2 satisfies the relation

$$Y^2 = X^3, \tag{45}$$

where

$$X \equiv \tfrac{1}{24} C_{abcd} C^{abcd}, \qquad Y \equiv \tfrac{1}{48} C_{abcd} C^{cdef} C_{ef}{}^{ab}. \tag{46}$$

4.1. Self-dual Taub–NUT–de Sitter metrics

The general biaxial Bianchi IX Einstein metrics have long been known; these are the Taub–NUT–de Sitter solutions. Their local form can straightforwardly be derived by directly solving the Einstein equations in a suitable coordinate gauge. Writing (24) as

$$ds_4^2 = \frac{dr^2}{a_3^2} + a_1^2\big(\sigma_1^2 + \sigma_2^2\big) + a_3^2 \sigma_3^2, \tag{47}$$

the Ricci tensor is given (in the natural orthonormal frame) by

$$\begin{aligned}
R_{00} &= -a_3 a_3'' - a_3'^2 - \frac{2\big(a_1' a_3' a_3 + a_1'' a_3^2\big)}{a_1},\\
R_{11} &= R_{22} = -\frac{a_1'' a_3^2}{a_1} - \frac{a_1'^2 a_3^2}{a_1^2} - \frac{2a_1' a_3' a_3}{a_1} - \frac{a_3^2}{2a_1^4} + \frac{1}{a_1^2},\\
R_{33} &= -a_3 a_3'' - a_3'^2 - \frac{2a_1' a_3' a_3}{a_1} + \frac{a_3^2}{2a_1^4}.
\end{aligned} \tag{48}$$

From this we see that $R_{00} - R_{33} = -a_3^2\big(a_1^{-4} + 4a_1'' a_1^{-1}\big)/2$, and since this must vanish by the Einstein condition, it is easy to solve for a_1, and hence, using the remaining Einstein equations, for a_3. Any Einstein solution to (48) is by definition a Taub–NUT–de Sitter metric. Apart from special limiting cases, the general solution has three parameters that we can think of as the mass m, the NUT charge n and the cosmological constant Λ. This general metric is given by[8]

$$ds_4^2 = \frac{r^2 - n^2}{\Delta}\, dr^2 + \frac{4n^2\Delta}{r^2 - n^2}\sigma_3^2 + (r^2 - n^2)\big(\sigma_1^2 + \sigma_2^2\big), \tag{49}$$

where

$$\Delta \equiv r^2 - 2mr + n^2 + \Lambda\big(n^4 + 2n^2 r^2 - \tfrac{1}{3} r^4\big). \tag{50}$$

The metric (49) has a self-dual or anti-self-dual Weyl tensor if [35]

$$m = \pm n\big(1 + \tfrac{4}{3}\Lambda n^2\big), \tag{51}$$

in which case we find

$$\Delta = (r \mp n)^2\big(1 - \tfrac{1}{3}\Lambda (r \mp n)(r \pm n)\big). \tag{52}$$

Making the specific choice of the upper sign, we obtain the self-dual Taub–NUT–de Sitter metric

$$ds_4^2 = \frac{dr^2}{F} + 4n^2 F \sigma_3^2 + (r^2 - n^2)\big(\sigma_1^2 + \sigma_2^2\big), \tag{53}$$

[8] The metric (49), parametrized by m, n and Λ, covers an open dense set in the modulus space of solutions of (48). However, for special choices of the relation between the parameters, it may be necessary to change the radial coordinate r because (49) degenerates unless a limit is taken.

where

$$F = \left(\frac{r-n}{r+n}\right)\left(1 - \frac{1}{3}\Lambda(r-n)(r+3n)\right),$$
$$= \frac{\Lambda}{3}\left(\frac{r-n}{r+n}\right)(r_+ - r)(r - r_-), \qquad r_\pm \equiv -n \pm \sqrt{4n^2 + \frac{3}{\Lambda}}. \tag{54}$$

The Weyl tensor is given by

$$C_{0101} = C_{2323} = -C_{0123} = -\frac{n\left(1 + \frac{1}{3}\Lambda n^2\right)}{(r+n)^3},$$
$$C_{0303} = C_{1212} = -C_{0312} = \frac{2n\left(1 + \frac{1}{3}\Lambda n^2\right)}{(r+n)^3}. \tag{55}$$

It can easily be verified that this metric satisfies the first-order equations (30). Note that because it is biaxial, and thus satisfies our reduced first-order system (43), it follows that the Weyl tensor of the self-dual Taub–NUT–de Sitter metrics obeys relation (45). It is evident that if we send Λ to zero in (53), we obtain the self-dual Taub–NUT metric first written down as a Euclidean-signature metric in [29]:

$$ds_4^2 = \left(\frac{r+n}{r-n}\right)dr^2 + 4n^2\left(\frac{r-n}{r+n}\right)\sigma_3^2 + (r^2 - n^2)\left(\sigma_1^2 + \sigma_2^2\right). \tag{56}$$

We saw, however, that the first-order equations (43) have three branches, and in the limit where Λ goes to zero two of these should lead to the self-dual Taub–NUT metric, whilst the third should lead instead to the Eguchi–Hanson metric. As noted above, the metric form (49) with parameters m, n and Λ, and radial coordinate r, does not necessarily cover all regions of the modulus space, and in the present case the existence of three branches suggests that there should exist a different parameterization of biaxial self-dual Einstein metrics whose limiting form when Λ goes to zero is the Eguchi–Hanson metric.

The required metrics cannot be those found in [35], which are referred to as the Eguchi–Hanson–de Sitter metrics,

$$ds_4^2 = \frac{dr^2}{F} + \frac{1}{4}r^2 F\sigma_3^2 + \frac{1}{4}r^2\left(\sigma_1^2 + \sigma_2^2\right), \tag{57}$$

where $F = 1 - \ell^4 r^{-4} - \frac{1}{6}\Lambda r^2$, because these metrics have neither self-dual nor anti-self-dual Weyl tensor, when Λ and ℓ are both non-zero, and thus they do not satisfy (30). They are in fact Einstein–Kähler, and the Weyl tensor has a definite duality only if $\ell = 0$ (giving the Fubini–Study metric on $\mathbb{CP}^2$ if $\Lambda > 0$, and the Bergmann metric on the open ball in $\mathbb{C}^2$ if $\Lambda < 0$), or if $\Lambda = 0$, in which case the Weyl tensor has the opposite duality and the metric is Eguchi–Hanson[9]. In order to find the 'missing' metrics, which we shall distinguish from (57) by giving them the name 'self-dual Eguchi–Hanson–de Sitter', it is helpful to study the first-order equations (43) in greater detail. This forms the topic of the next subsection.

4.2. Biaxial first-order equations and self-dual Eguchi–Hanson–de Sitter

To proceed with studying the biaxial first-order equations (43), we define

$$u \equiv \dot{a}_1 + \frac{a_3}{2a_1}. \tag{58}$$

[9] We shall discuss Bianchi IX Einstein–Kähler metrics briefly in appendix A.

The cubic equation for $\dot{a}_1$ now becomes

$$3u^3 + \left(\Lambda a_1^2 - 3\right)u - \Lambda a_1 a_3 = 0. \tag{59}$$

One approach is to follow Cardano's procedure for solving the cubic equation, but other than establishing the principle that there will be two roots whose $\Lambda \longrightarrow 0$ limit yields the self-dual Taub–NUT first-order equation $u = \pm 1$, with the third yielding the Eguchi–Hanson first-order equation $u = 0$ (see (44)), the direct solution of the cubic equation is not very enlightening.

A more profitable route is to view (59) as an equation expressing a_3 in terms of u,

$$a_3 = -\frac{u\left(u^2 + \lambda a_1^2 - 1\right)}{\lambda a_1}. \tag{60}$$

Note that we are defining

$$\lambda \equiv \tfrac{1}{3}\Lambda \tag{61}$$

for convenience. In view of (60), we can now choose to regard (a_1, u) as our two metric functions, rather than (a_1, a_3). From the first-order equations (43) we can now deduce that a_1 and u satisfy the first-order equations

$$\dot{u} = -\lambda a_1, \qquad \dot{a}_1 = \frac{u\left(u^2 + 3\lambda a_1^2 - 1\right)}{2\lambda a_1^2}. \tag{62}$$

In order to find the solution that gives rise to Eguchi–Hanson in the $\lambda \equiv \frac{1}{3}\Lambda \longrightarrow 0$ limit, it is useful to make a redefinition that casts equations (62) and (60) into a form where this limit can be taken smoothly, and such that u tends to zero in the limit. This is easily done, by letting $u = \lambda w$. The first-order equations (62) become

$$\dot{w} = -a_1, \qquad \dot{a}_1 = \frac{w\left(\lambda^2 w^2 + 3\lambda a_1^2 - 1\right)}{2a_1^2}, \tag{63}$$

and (60) gives

$$a_3 = -\frac{w\left(\lambda^2 w + 3\lambda a_1^2 - 1\right)}{a_1}. \tag{64}$$

It follows that the solution to (63) for general non-vanishing λ will give the required self-dual Eguchi–Hanson–de Sitter metrics. By defining a new radial variable x such that $\mathrm{d}x = -a_1\,\mathrm{d}t$, the equation for w can be solved to give $w = x$, and hence the solution for a_1 can be found. After a further simple coordinate redefinition, the solution can be expressed as

$$\mathrm{d}s_4^2 = \frac{\mathrm{d}\rho^2}{UV} + \frac{1}{4}\rho^2 \frac{V}{1 - 2\mu\ell^2}\left(\sigma_1^2 + \sigma_2^2\right) + \frac{1}{4}\rho^2 \left(\frac{1 - \mu\ell^2}{1 - 2\mu\ell^2}\right)^2 UV\sigma_3^2, \tag{65}$$

where

$$U \equiv 1 - \frac{\ell^4}{\rho^4}, \qquad V \equiv 1 - \mu(\ell^2 + \rho^2). \tag{66}$$

The metric is Einstein, with cosmological constant $\Lambda = 12\mu$, and its Weyl tensor is anti-self-dual. In fact, we find that the tangent-frame components of the Weyl tensor are given by

$$\begin{aligned} C_{0101} &= C_{2323} = -C_{0123} = -\frac{2\ell^4(1 - \mu\ell^2)}{\rho^6}, \\ C_{0303} &= C_{1212} = -C_{0312} = \frac{4\ell^4(1 - \mu\ell^2)}{\rho^6}. \end{aligned} \tag{67}$$

Since (65) is Einstein and of biaxial Bianchi IX type, it must be contained within the general Taub–NUT–de Sitter class of solutions (49). Furthermore, since its Weyl tensor is anti-self-dual, it can be expected to lie within the subclass of (49) that satisfy (51). After simple algebra we find that there is indeed a transformation that maps (49) with the anti-self-dual specialization given by (51) into (65), namely

$$r^2 = -\frac{(12 - \Lambda\ell^2 - 2\Lambda\rho^2)^2}{32\Lambda(6 - \Lambda\ell^2)}, \qquad n^2 = -\frac{(12 - \Lambda\ell^2)^2}{32\Lambda(6 - \Lambda\ell^2)}. \tag{68}$$

Substituting these redefinitions into (49) with (52), we recover (63). It should be noted that when $\Lambda\ell^2 < 6$, the self-dual Eguchi–Hanson–de Sitter metric corresponds to a section of the self-dual Taub–NUT–de Sitter metric in which the NUT parameter and radial coordinate are imaginary. Thus from the point of view of the real geometry, the self-dual Taub–NUT–de Sitter and self-dual Eguchi–Hanson–de Sitter metrics should be viewed as inequivalent.

In order to clarify the relations between the self-dual Taub–NUT–de Sitter and self-dual Eguchi–Hanson–de Sitter metrics, and more generally to investigate the full solution space of the self-dual biaxial metrics, it is useful to study the phase-plane for the first-order system (62). Before doing so, we shall close this subsection by showing where two well-known self-dual Einstein metrics that are contained within the biaxial Bianchi IX class fit in, namely S^4 and $\mathbb{CP}^2$.

Setting $\ell = 0$ in (65) gives S^4, as can be seen by changing to the radial coordinate t defined by $\sqrt{\mu}\rho = \sin\frac{1}{2}t$. This gives

$$\mathrm{d}s_4^2 = \frac{3}{\Lambda}\left(\mathrm{d}t^2 + \frac{1}{4}\sin^2 t\sigma_i^2\right). \tag{69}$$

From (68), this corresponds to $n^2 = -3/(4\Lambda)$ in the self-dual Taub–NUT–de Sitter parametrization.

Another special case of (65), which arises when $\Lambda\ell^2 = 12$, also gives rise to S^4. This is a singular limit, for which we must first rescale the Euler angle ψ that appears in $\sigma_3 = \mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi$ according to $\psi = \left(1 - \frac{1}{12}\Lambda\ell^2\right)^{-1}\tau$. Substituting into (65), and then sending $\ell^2 \longrightarrow 12/\Lambda$, we obtain

$$\mathrm{d}s_4^2 = \frac{3}{\Lambda}\left[\mathrm{d}\chi^2 + \sin^2\chi\left(\sigma_1^2 + \sigma_2^2\right) + \cos^2\chi\,\mathrm{d}\tau^2\right], \tag{70}$$

where we have also set $\Lambda\rho^2 = 12\sin\chi$. We can recognize (70) as the metric on S^4, written as a foliation by $S^2 \times S^1$ surfaces. The fact that (65) describes S^4 both for $\ell^2 = 0$ and $\ell^2 = 12/\Lambda$ is not unexpected in view of the expressions (67), since the Weyl tensor can be seen to vanish for these two values of ℓ^2. Note that from (68) the value of the NUT parameter in the self-dual Taub–NUT–de Sitter parametrization corresponding to the S^4 limit with $\Lambda\ell^2 = 12$ is $n = 0$.

A further special case of (65) is when $\Lambda\ell^2 = 6$. This gives $\mathbb{CP}^2$. One must first define a new radial coordinate, for example by setting $\rho^2 = (2\mu)^{-1}[1 + (1 - 2\mu\ell^2)\cos 2\chi]$, before taking the limit. We then obtain the metric

$$\mathrm{d}s_4^2 = \frac{6}{\Lambda}\left[\mathrm{d}\chi^2 + \frac{1}{4}\sin^2\chi\left(\sigma_1^2 + \sigma_2^2\right) + \frac{1}{4}\sin^2\chi\cos^2\chi\,\sigma_3^2\right], \tag{71}$$

which can be recognized as the Fubini–Study metric on $\mathbb{CP}^2$ [35]. From (68), it corresponds, in the self-dual Taub–NUT–de Sitter parametrization, to sending the NUT parameter n to infinity. Note that with the conventions of this paper, the Weyl tensor is anti-self-dual, as is the (covariantly constant) Kähler form $J = e^0 \wedge e^3 - e^1 \wedge e^2$. Of course none of the self-dual quaternionic Kähler forms J^i is covariantly constant, since the right-handed $SU(2)$ part of the spin connection is non-vanishing.

4.3. Phase-plane analysis for the biaxial system

As we have seen above, finding a uniform parametrization of the space of solutions, even in the biaxial case, is non-trivial. The approach taken in this section will be to classify all the possible orbits in the phase space of the first-order equations (62). We shall find that not all solutions can be parametrized by giving real and finite values of ℓ or n.

We begin by making the definition $v = 3a_1$, choosing the scale size $\lambda = 3$ for convenience, and sending $t \longrightarrow -t$ for inconvenience. The first-order equations (62) become

$$\dot{u} = v, \qquad \dot{v} = \frac{9u(1-u^2-v^2)}{2v^2}, \tag{72}$$

and so the solutions can be represented as flows in the (u, v) plane. We can divide the two equations to get

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{9u(1-u^2-v^2)}{2v^3}. \tag{73}$$

In general, equation (73) can be integrated to give the flows for any biaxial self-dual solution. The constant of integration is related to the NUT parameter n, or, equivalently, the scale parameter ℓ in the self-dual Eguchi–Hanson–de Sitter formulation. In terms of n, the integral of (73) is given by

$$\left(u^2 + \tfrac{1}{3}v^2 - 1\right)^2 = 12n^2\left(u^2 + \tfrac{2}{3}v^2 - 1\right). \tag{74}$$

Since this is symmetrical under reflections in the u and v axes, it suffices to consider flows within the positive quadrant.

It follows from (73) that flow lines inside the unit circle have positive gradient, whilst those outside the unit circle have negative gradient. The v axis corresponds to $a_3 = 0$, signifying an endpoint of the metric at which the three-dimensional orbits degenerate to an S^2 bolt. The u axis, on the other hand, corresponds to $a_1 = 0$, and the metric will be singular here unless it happens that $u = \pm 1$ or $u = 0$, in which case the orbits degenerate to a point, implying a NUT endpoint in the metric. By a theorem of Hitchin, the only complete and non-singular metrics with positive Λ are S^4 and $\mathbb{CP}^2$.

It is straightforward to establish that the $\mathbb{CP}^2$ solution (71) corresponds to the ellipse $u^2 + \frac{2}{3}v^2 = 1$. The flow starts on the v axis at $v^2 = \frac{3}{2}$ at a bolt, and runs along the ellipse to a NUT on the u axis at $u = 1$. Since we have chosen the normalization $\mu = \frac{1}{12}\Lambda = \frac{1}{4}\lambda = \frac{3}{4}$ in this subsection, it follows that this occurs for $\ell^2 = \frac{2}{3}$.

The S^4 solution (69) with $\ell^2 = 0$ corresponds to the ellipse $\left(u - \frac{1}{2}\right)^2 + \frac{1}{3}v^2 = \frac{1}{4}$. This runs from the NUT at $u = 0$, $v = 0$ to the NUT at $u = 1$, $v = 0$. The other S^4 solution (70), with $\ell^2 = 12/\Lambda = 4/3$, corresponds to the ellipse $u^2 + \frac{1}{3}v^2 = 1$. Although this appears to be singular since, from (60), we have $a_3 = 0$, we saw that to obtain (70) it was necessary to rescale the ψ coordinate and this has the effect of compensating for the vanishing of a_3.

The phase-plane plot, with the various ellipses and unit circle mentioned above displayed, is given in figure 1.

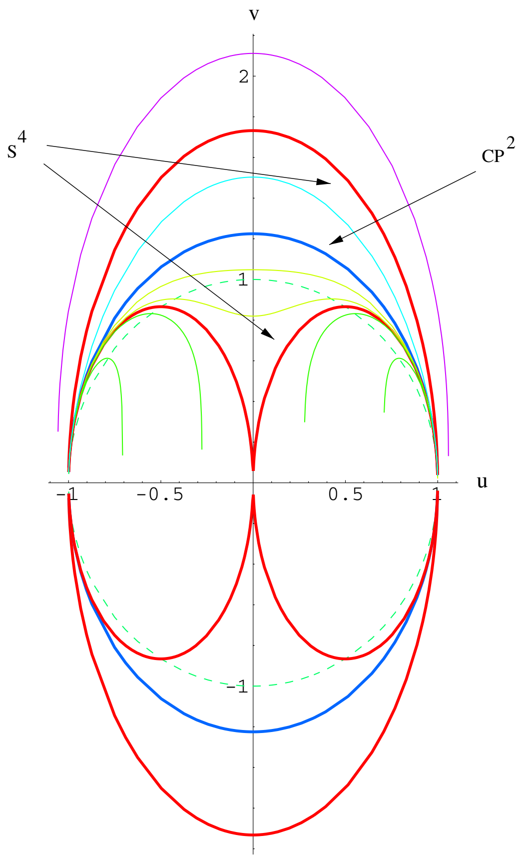

Figure 1. The phase plane for the first-order system of equation (72). The solid blue ellipse corresponds to the $\mathbb{CP}^2$ flow, and the three solid red ellipses to S^4 flows. The dashed green circle is $u^2 + v^2 = 1$; all solutions that cross this do so horizontally. To label distinct metrics it is sufficient to consider flows lying within the positive quadrant. We label qualitatively similar flows by A, B, C and D, which indicate the regions they occupy and their initial points. Thus the regions A, B and C indicate starting points for solutions on the v axis. Region A ranges from $v = 0$ to the intersection of the $\mathbb{CP}^2$ ellipse with the v axis. Region B ranges from this intersection to the intersection of the outer ellipse (the S^4 solution (70)) with the v axis. Region C ranges from this point to $v = +\infty$. Region D denotes (singular) starting points on the u axis for solutions in the range $0 < u < 1$. For clarity we have plotted complete ellipses for the S^4 and $\mathbb{CP}^2$ special cases, but only flows in the upper half-plane for the other representative examples.

(This figure is in colour only in the electronic version)

From (74), we see that solutions starting from a bolt on the v axis are specified at $u = 0$ by

$$\begin{aligned}
&\text{region A}: &&-\infty < n^2 < -\tfrac{1}{12}: &&\tfrac{3}{2} > v^2 > 0,\\
&\text{region B}: &&-\infty < n < 0: &&\tfrac{3}{2} < v^2 < 3,\\
&\text{region C}: &&0 < n < \infty: &&3 < v^2 < \infty.
\end{aligned}$$

We also have solutions starting from the singular curvature singularity along the u axis, specified at $v = 0$ by

$$\text{region D:}\quad -\tfrac{1}{12} < n^2 < 0: \qquad 0 < u < 1. \tag{75}$$

In terms of the parameter ℓ^2 of the self-dual Eguchi–Hanson–de Sitter formulation of the metrics, we see from (68) that region A, where n is imaginary, is covered by real values of ℓ, and so the self-dual Eguchi–Hanson–de Sitter form of the metrics is better adapted to describing this region of the phase plane. On the other hand, in region D, where n is again imaginary, ℓ is complex, and so neither the self-dual Taub–NUT–de Sitter nor the self-dual Eguchi–Hanson–de Sitter formulation is well adapted to describing this region of the phase plane. It is straightforward to find an adapted parametrization where the analogue of the NUT parameter, and the radial coordinate, is real in region D, but since the metrics there have power-law curvature singularities there is not much value in writing them down.

It is instructive to express the Weyl tensor for the biaxial self-dual metrics in terms of u and v. We find that it is given by

$$C_{0101} = -C_{0123} = C_{2323} = -2C_{0303} = 2C_{0312} = -2C_{1212} = f, \tag{76}$$

where

$$f = -\tfrac{81}{2} v^{-6}\big(u^2 + \tfrac{1}{3}v^2 - 1\big)\Big(\big(u - \tfrac{1}{2}\big)^2 + \tfrac{1}{3}v^2 - \tfrac{1}{4}\Big)\Big(\big(u + \tfrac{1}{2}\big)^2 + \tfrac{1}{3}v^2 - \tfrac{1}{4}\Big). \tag{77}$$

As expected, this vanishes only on the S^4 ellipses, and it diverges everywhere on the u axis except at the points $u = 0, \pm 1$, provided they are approached along the S^4 flows.

4.4. Global structure of the biaxial solutions

As we have already remarked, a theorem of Hitchin implies that when the cosmological constant Λ is positive, only the S^4 and $\mathbb{CP}^2$ self-dual Einstein metrics can be non-singular. In particular, therefore, this means that the self-dual Taub–NUT–de Sitter and self-dual Eguchi–Hanson–de Sitter metrics will be singular except for the special values of n or ℓ^2 for which they reduce to S^4 or $\mathbb{CP}^2$.

We shall analyse the self-dual Taub–NUT–de Sitter metrics first, described by (53) and (54). The coordinate r is taken to lie in the interval $n \leqslant r \leqslant r_+$. For convenience, we shall again set $\Lambda = 9$ here. Near $r = n$, letting $r - n = \rho^2$, the metric becomes

$$ds_4^2 \sim 8n\big[d\rho^2 + \tfrac{1}{4}\rho^2\big(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\big)\big], \tag{78}$$

which describes a NUT. The metric smoothly approaches the origin of $\mathbb{R}^4$, provided that the Euler angle ψ appearing in $\sigma_3 = d\psi + \cos\theta\, d\phi$ has its canonical period 4π.

Near $r = r_+$, by letting $r_+ - r = \rho^2$ we see that the metric becomes

$$ds_4^2 \sim \frac{2}{\big(\sqrt{3}\sqrt{12n^2+1} - 6n\big)}\big(d\rho^2 + 4n^2\big(\sqrt{3}\sqrt{12n^2+1} - 6n\big)^2\rho^2\sigma_3^2\big) + \big(r_+^2 - n^2\big)\big(\sigma_1^2 + \sigma_2^2\big). \tag{79}$$

This approaches $\mathbb{R}^2 \times S^2$ locally, but in general there will be a conical singularity. If ψ has period $\Delta\psi = 4\pi/N$, then regularity at $r = r_+$ is achieved if

$$N = 4n\big(-6n + \sqrt{3}\sqrt{12n^2+1}\big). \tag{80}$$

Regularity at $r = n$ required $N = 1$. This is compatible with (80) if $n = \infty$, which is the limit where the self-dual Taub–NUT–de Sitter metric becomes $\mathbb{CP}^2$ [35]. (Another case where the singularity can be avoided is by taking a limit where $n^2 \longrightarrow -3/(4\Lambda) = -\frac{1}{12}$, in which case one must first rescale coordinates in the metric. This case is S^4.) For all other values of n, there will be a deficit angle at the origin, and hence a conical singularity.

The G_2 metrics (9) obtained by taking ds_4^2 to be self-dual Taub–NUT–de Sitter were discussed recently in [10]. They have cohomogeneity 2, since there are two 'radial' coordinates ρ and t. The conical singularities in the Taub–NUT–de Sitter metrics imply, of course, that the corresponding G_2 metrics will have conical singularities too.

A further class of geometries within the biaxial Bianchi IX class is obtained by considering instead the self-dual Eguchi–Hanson–de Sitter form of the metrics, given by (65). If $2\mu\ell^2 < 1$, meaning that $\Lambda\ell^2 < 6$, the radial coordinate ρ can be chosen to lie in the interval $\ell \leqslant \rho \leqslant \rho_0$, where $\rho_0^2 = 1/\mu - \ell^2$. Near $\rho = \ell$, setting $\rho = \ell + x^2$, we find

$$ds_4^2 \sim \frac{\ell}{1-2\mu\ell^2}\big[dx^2 + (1-\mu\ell^2)^2 x^2\sigma_3^2\big] + \frac{1}{4}\ell^2\big(\sigma_1^2+\sigma_2^2\big), \tag{81}$$

whilst near $\rho = \rho_0$, we have, setting $\rho = \rho_0 - x^2$,

$$ds_4^2 \sim \frac{1-\mu\ell^2}{1-2\mu\ell^2}\left(dx^2 + \frac{1}{4}x^2\sigma_i^2\right). \tag{82}$$

Thus regularity at the NUT at $\rho = \rho_0$ requires that ψ have period 4π, which implies that there is a conical singularity on the bolt at $\rho = \ell$.

4.5. Phase plane and global structure for negative Λ

The phase-plane analysis of section 4.3 can be repeated for the case where the cosmological constant Λ is taken to be negative. Starting from (62) and (60), and fixing the scale by choosing $\lambda \equiv \frac{1}{3}\Lambda = -3$, we now have

$$\begin{aligned} &\dot{u} = v, \qquad \dot{v} = \frac{9u(1-u^2+v^2)}{2v^2}, \\ &\frac{dv}{du} = \frac{9u(1-u^2+v^2)}{2v^3}, \qquad a_3 = -\frac{u}{v}\big(1-u^2+\tfrac{1}{3}v^2\big). \end{aligned} \tag{83}$$

The flow can be integrated, giving

$$\big(u^2 - \tfrac{1}{3}v^2 - 1\big)^2 + 12n^2\big(u^2 - \tfrac{2}{3}v^2 - 1\big) = 0. \tag{84}$$

As in the case when $\Lambda > 0$, this is symmetric under reflections in the u and v axes. The hyperbola $u^2 - \frac{2}{3}v^2 = 1$, which arises when $n = \infty$, corresponds to the Bergmann metric on the open ball in $\mathbb{C}^2$ (i.e. the Fubini–Study metric with negative Λ, which is the coset $SU(2,1)/U(2)$). The hyperbolic 4-space H^4 arises if $n = 0$, giving the hyperbola $u^2 - \frac{1}{3}v^2 = 1$. It also arises if $n^2 = \frac{1}{12}$, giving the hyperbolae $\big(u \pm \frac{1}{2}\big)^2 - \frac{1}{3}v^2 = \frac{1}{4}$.

The Weyl tensor is given by (76), where f is now given by

$$f = \tfrac{81}{2}v^{-6}\big(u^2 - \tfrac{1}{3}v^2 - 1\big)\big(\big(u-\tfrac{1}{2}\big)^2 - \tfrac{1}{3}v^2 - \tfrac{1}{4}\big)\big(\big(u+\tfrac{1}{2}\big)^2 - \tfrac{1}{3}v^2 - \tfrac{1}{4}\big). \tag{85}$$

The Weyl tensor therefore vanishes on the H^4 hyperbolae, and has a power-law divergence at all points on the u axis except if one approaches $u = 0, \pm 1$ along the H^4 flows.

Writing the metric in the self-dual Eguchi–Hanson–de Sitter form (65), where now $\mu \equiv \frac{1}{12}\Lambda$ is taken to be negative, say $-\mu \equiv \nu > 0$, we see that the radial variable can be taken in the range $\rho \geqslant \ell$. Near $\rho = \ell$ we set $\rho = \ell + x^2$, giving

$$ds_4^2 \sim \frac{\ell}{1+2\nu\ell^2}\big[dx^2 + (1+\nu\ell^2)^2 x^2\sigma_3^2\big] + \frac{1}{4}\ell^2\big(\sigma_1^2+\sigma_2^2\big). \tag{86}$$

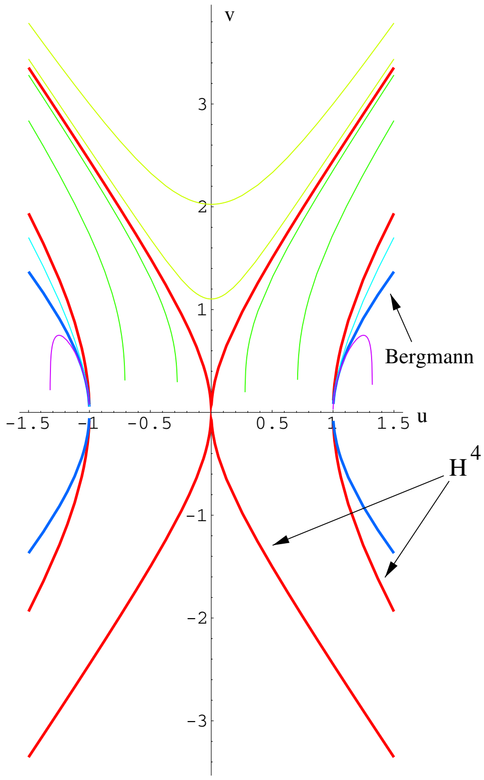

Figure 2. The phase plane for the first-order system of equation (83). The solid blue hyperbola corresponds to the Bergmann flow, and the three solid red hyperbolae to H^4 flows. The discussion of other flows is analogous to that for $\Lambda > 0$; some representative examples are depicted.

(This figure is in colour only in the electronic version)

Thus we have a regular S^2 bolt, provided that the period $\Delta\psi$ of ψ is chosen to be

$$\Delta\psi = \frac{2\pi}{1+\nu\ell^2}. \tag{87}$$

Provided that ℓ is such that this period is $4N$, for N an integer, we shall have a regular metric, with S^3/Z_N orbits.

Now consider instead writing the metric in the self-dual Taub–NUT–de Sitter form (53). Taking $\Lambda = -9$ for simplicity, the roots $r_\pm$ are given by $r_\pm = -n \pm \frac{1}{\sqrt{3}}\sqrt{12n^2 - 1}$. Assuming $n^2 > \frac{1}{12}$, this means that the roots $r_\pm$ are both less than n (assumed positive), and so we can take $r \geqslant n$. Near $r = n$ we set $r = n + x^2$, finding

$$ds_4^2 \sim 8n\left(dx^2 + \tfrac{1}{4}x^2\sigma_i^2\right). \tag{88}$$

Thus $r = n$ is a regular NUT, provided ψ has period 4π.

The regular solutions with a bolt, which we described in the self-dual Eguchi–Hanson–de Sitter form (65) above, can also be expressed in the self-dual Taub–NUT–de Sitter form. They correspond to running the radial coordinate r from $r = r_-$ to $r = -\infty$ (note that $r_- < -n$, so the curvature singularity at $r = -n$ is avoided).

All the other solutions represented in figure 2 have flows that intersect the u axis at points other than $u = 0$ or ± 1, and thus they have power-law curvature singularities.

4.6. Superpotential for the biaxial system

Although the $D = 4$ self-dual Einstein spaces do not themselves have special holonomy, the existence of the first-order system implies that it might be possible to derive it from a superpotential. To obtain such a superpotential, we first note that the Hamiltonian of the cohomogeneity one Einstein space is given by $H = T + U$, where

$$\begin{aligned} T &= \frac{2a_1'a_2'}{a_1a_2} + \frac{2a_1'a_3'}{a_1a_3} + \frac{2a_2'a_3'}{a_2a_3}, \\ U &= \tfrac{1}{2}\left(a_1^4 + a_2^4 + a_3^4 - 2a_1^2a_2^2 - 2a_1^2a_3^2 - 2a_2^2a_3^2 + 12\lambda a_1^2a_2^2a_3^2\right), \end{aligned} \tag{89}$$

and a prime denotes a derivative with respect to η defined by $\mathrm{d}t = a_1a_2a_3\,\mathrm{d}\eta$.

Here, we shall consider the biaxial system with $a_2 = a_1$, and use the u and a_1 variables defined in section 4.2. We can write $T = \frac{1}{2}g_{ij}(\mathrm{d}\alpha^i/\mathrm{d}\eta)(\mathrm{d}\alpha^j/\mathrm{d}\eta)$, with $\alpha_i = (a_1, u)$, which implies that g_{ij} is given by

$$\begin{aligned} g_{ij} &= \begin{pmatrix} \frac{4x}{a_1^2 y} & \frac{4z}{a_1uy} \\ \frac{4z}{a_1uy} & 0 \end{pmatrix}, \qquad && x \equiv 1 + 3\lambda a_1^2 - u^2 \\ y &\equiv -1 + \lambda a_1^2 + u^2, && z \equiv -1 + \lambda a_1^2 + 3u^2. \end{aligned} \tag{90}$$

We find that the potential U can then be expressed as $U = -\frac{1}{2}g^{ij}(\partial W/\partial\alpha_i)(\partial W/\partial\alpha_j)$, with the superpotential W given by

$$W = -\frac{u^2(u^2-1)^2}{\lambda^2a_1^2} - \frac{u^2(5u^2-4)}{\lambda} - a_1^2(3u^2+2) + \lambda a_1^4. \tag{91}$$

It is straightforward to derive the first-order equations from this superpotential.

5. Triaxial anti-self-dual Bianchi IX metrics

In this section we discuss the full triaxial system of equations, which are considerably more complicated than the biaxial case.

5.1. Phase-plane and superpotential for triaxial system

We begin with an outline of a phase-plane analysis for the triaxial system, using methods similar to those that we used for the biaxial case.

Starting from the first-order equation for $\dot{a}_1$ obtained in (36), it is natural to define the auxiliary variable w, by

$$w \equiv \dot{a}_1 - \frac{a_1^2 - a_2^2 - a_3^2}{2a_2a_3}. \tag{92}$$

In terms of w, equation (36) becomes

$$(w^2-1)^2(w - \lambda a_2a_3) - \lambda^2a_1^2(a_2w + a_3)(a_3w + a_2) = 0. \tag{93}$$

The remaining first-order equations in (30), namely those for $\dot{a}_2$ and $\dot{a}_3$, then become

$$\begin{aligned} \dot{a}_2 &= \frac{a_2^2 - a_1^2 - a_3^2}{2a_1a_3} - \frac{\lambda a_1(a_2w + a_3)}{w^2-1}, \\ \dot{a}_3 &= \frac{a_3^2 - a_1^2 - a_2^2}{2a_1a_2} - \frac{\lambda a_1(a_3w + a_2)}{w^2-1}. \end{aligned} \tag{94}$$

We can now try following the strategy of treating (w, a_2, a_3) as the independent variables, instead of (a_1, a_2, a_3). This is similar to the strategy used in the biaxial case, although not exactly parallel. Differentiating (93), using (94) and (92), and then using (93) itself to substitute for a_1^2, we get

$$(\dot{w}+\lambda a_1)(w^2-1)^2\big[a_2a_3(3w^4+6w^2-1\big)-4\big(a_2^2+a_3^2\big)w^3 \\ -\lambda a_2a_3\big(\big(a_2^2+a_3^2\big)(1+3w^2)+2a_2a_3w(w^2+3)\big)\big]=0. \tag{95}$$

Unless the algebraic expression contained in square brackets vanishes, we therefore have the first-order equation

$$\dot{w}=-\lambda a_1. \tag{96}$$

We should think of a_1 as being solved for here, using (93). Since this would involve the use of square roots, it seems preferable to introduce a new radial variable ρ, defined by $\mathrm{d}\rho=-a_1\,\mathrm{d}t$. We then have

$$w'=\lambda. \tag{97}$$

The remaining first-order equations (94) will also involve a_1 only through a_1^2, and so we shall have the system

$$\begin{aligned} w' &= \lambda, \\ a_2' &= -\frac{a_2^2-a_1^2-a_3^2}{2a_1^2a_3}+\frac{\lambda(a_2w+a_3)}{w^2-1}, \\ a_3' &= -\frac{a_3^2-a_1^2-a_2^2}{2a_1^2a_2}+\frac{\lambda(a_3w+a_2)}{w^2-1}, \end{aligned} \tag{98}$$

where from (93), a_1^2 is given by

$$a_1^2=\frac{(w^2-1)^2(w-\lambda a_2a_3)}{\lambda^2(a_2w+a_3)(a_3w+a_2)}. \tag{99}$$

Analogously to the biaxial case, we see from (93) that when $\lambda=0$ we have $w=0$ corresponding to the 'BGPP' first-order equations, and $w=\pm1$ corresponding to the 'Atiyah–Hitchin' first-order equations.

The problem of solving the general triaxial first-order equations can be reduced to a second-order equation in a single variable. Defining $y\equiv a_2/a_3$ and $z\equiv a_2a_3$, we find, after normalizing so that $\lambda=1$, that y satisfies the equation

$$y''-\frac{3y^2-2\rho y+1}{y(y^2-1)}y'^2-\frac{2\rho y^2-3\rho^2y-y+2\rho}{\rho(\rho^2-1)y}y'+\frac{2(y^2-1)}{\rho(\rho^2-1)^2}=0, \tag{100}$$

and that z is given by

$$z=\frac{\rho(\rho^2-1)y[(\rho^2-1)y'+1-y^2]}{(\rho^2-1)^2yy'+\rho(y^4-1)+2y^2(y^2-1)}. \tag{101}$$

We find that it is possible to derive the triaxial first-order system from a superpotential also. We use $\alpha_i=(u,a_2,a_3)$ as variables, as discussed above. The kinetic energy T given in (89) can be straightforwardly rewritten in terms of derivatives of (u,a_2,a_3), and hence we can read off the components of the sigma-model metric g_{ij} in $T=\frac{1}{2}g_{ij}(\mathrm{d}\alpha^i/\mathrm{d}\eta)(\mathrm{d}\alpha^j/\mathrm{d}\eta)$, where as before $\mathrm{d}t=a_1a_2a_3\,\mathrm{d}\eta$. Since the expression for g_{ij} is quite complicated, we shall not present it here. Then, we find after some calculation that the potential U given in (89) can be written in terms of a superpotential W as $U=-\frac{1}{2}g^{ij}(\partial W/\partial\alpha^i)(\partial W/\partial\alpha^j)$, with

$$W=-a_2^2-a_3^2-2\lambda^{-1}w^2+\frac{a_2a_3(3w^2-1)}{w}+\frac{\big(\lambda^2a_2^2a_3^2-w^2\big)(w^2-1)^2}{\lambda^2w(a_2w+a_3)(a_3w+a_2)}. \tag{102}$$

It is easily verified that if one sets $a_2=a_1$, then w reduces to the function u of the biaxial system, and, after using (60) to replace a_3 by u, then $W+\lambda^{-1}$ becomes the biaxial superpotential given in (91), where W denotes the triaxial superpotential (102) after the biaxial specializations.

5.2. The Tod–Hitchin first-order system

In this section we shall follow Tod [21] and Hitchin [22, 23], who use a different approach to study the general triaxial system (30). The metric is written as

$$\mathrm{d}s_4^2 = F\left(\frac{\mathrm{d}x^2}{x(1-x)} + \frac{\sigma_1^2}{\Omega_1^2} + \frac{(1-x)\sigma_2^2}{\Omega_2^2} + \frac{x\sigma_3^2}{\Omega_3^2}\right). \tag{103}$$

Tod [21] shows that $\mathrm{d}s_4^2$ is Einstein with anti-self-dual Weyl tensor if the functions Ω_i satisfy

$$\Omega_1' = -\frac{\Omega_2\Omega_3}{x(1-x)}, \qquad \Omega_2' = -\frac{\Omega_3\Omega_1}{x}, \qquad \Omega_3' = -\frac{\Omega_1\Omega_2}{1-x}, \tag{104}$$

where a prime denotes a derivative with respect to x, and F is given by

$$F = -\frac{8x\,\Omega_1^2\Omega_2^2\Omega_3^2 + 2\Omega_1\Omega_2\Omega_3\left[x\left(\Omega_1^2+\Omega_2^2\right) - \left(1-4\Omega_3^2\right)\left(\Omega_2^2-(1-x)\Omega_1^2\right)\right]}{4\left(x\,\Omega_1\Omega_2 + 2\Omega_3\left(\Omega_2^2-(1-x)\Omega_1^2\right)\right)^2}. \tag{105}$$

(We have normalized the Einstein constant so that $R_{ab} = 3g_{ab}$.)

This first-order system can be reduced to the problem of solving the Painlevé VI equation [21]. One introduces a function $y(x)$, in terms of which the Ω_i^2 are written as

$$\begin{aligned}
\Omega_1^2 &= \frac{(y-x)^2y(y-1)}{x(1-x)}\left(z-\frac{1}{2(y-1)}\right)\left(z-\frac{1}{2y}\right),\\
\Omega_2^2 &= \frac{y^2(y-1)(y-x)}{x}\left(z-\frac{1}{2(y-x)}\right)\left(z-\frac{1}{2(y-1)}\right),\\
\Omega_3^2 &= \frac{(y-1)^2y(y-x)}{(1-x)}\left(z-\frac{1}{2y}\right)\left(z-\frac{1}{2(y-x)}\right),
\end{aligned} \tag{106}$$

where

$$z = \frac{x-2xy+y^2-2x(1-x)y'}{4y(y-1)(y-x)}. \tag{107}$$

(Note that $\Omega_1^2 - \Omega_2^2 - \Omega_3^2$, which is conserved, must take the value $-\frac{1}{4}$ in order that $\mathrm{d}s_4^2$ be Einstein.) The claim then is that the first-order equations are satisfied if y satisfies the Painlevé VI equation

$$\begin{aligned}
y'' = \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\right)y'^2 - \left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\right)y'\\
+ \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\alpha+\beta\frac{x}{y^2}+\gamma\frac{x-1}{(y-1)^2}+\delta\frac{x(x-1)}{(y-x)^2}\right),
\end{aligned} \tag{108}$$

with $(\alpha,\beta,\gamma,\delta) = \left(\frac{1}{8},-\frac{1}{8},\frac{1}{8},\frac{3}{8}\right)$. Note that expression (105) for F is actually quite simple, expressed in terms of y:

$$\begin{aligned}
F &= \frac{y(1-y)(y-x)z}{2x(1-x)}\\
&= \frac{x-2xy+y^2-2x(1-x)y'}{8x(1-x)}.
\end{aligned} \tag{109}$$

It is a straightforward, although somewhat involved, exercise to show that if the first-order equations (104) are satisfied, then the metric functions a_i indeed satisfy our first-order equations (30). Note, however, that the converse is not true; not every solution of the general first-order equations (30) for anti-self-dual Einstein metrics gives a solution of (104). For example, the uniaxial solutions certainly do not satisfy equations (104); setting the a_i^2 equal implies that $\Omega_2^2 = (1-x)\Omega_1^2$ and $\Omega_3^2 = x\Omega_1^2$, and one can easily see that substituting into (104) leads to a contradiction. Likewise, one can show that setting any two of the metric functions equal leads to a degeneration in (104). This can be understood from the fact that the radial coordinate used in [21, 22] becomes a constant if any two of the metric functions are set equal.

The first-order equations (104) were obtained in [21–23] by first solving the conditions for metrics with anti-self-dual Weyl tensor and vanishing Ricci scalar, and then performing a conformal rescaling of the metric to arrive at one that was Einstein. We have shown that every solution of the Tod–Hitchin system provides a solution of our system of first-order equations. Our equations are valid not only for the triaxial case but also for the biaxial and uniaxial cases, and yield all possible Bianchi IX self-dual Einstein metrics. The method of Tod and Hitchin breaks down in the biaxial and uniaxial cases. The arguments from twistor theory presented in [23] show that the Tod–Hitchin method gives the general triaxial metric, but the explicit correspondence to our first-order equations remains unclear.

5.3. *Explicit examples*

Hitchin gives explicit solutions to (108) characterized by an integer k, with $k = 3, 4, 6, 8$ [22, 23]. The case $k = 3$ corresponds to the round metric on S^4, written in triaxial form [32], whilst $k = 4$ corresponds to the Fubini–Study metric on $\mathbb{CP}^2$, again written in triaxial form [38][10]. For $k \geqslant 5$ the metrics will necessarily have orbifold-type singularities.

In general it is easiest to give these solutions $y(x)$ by introducing a 'parametric variable' r, with y and x both expressed in terms of r. Thus one has

$$\begin{aligned}
k=3:&\quad y=\frac{r^2(2r^2+5r+2)}{(2r+1)(r^2+r+1)}, && x=\frac{r^3(r+2)}{2r+1},\\
k=4:&\quad y=r, && x=r^2,\\
k=6:&\quad y=\frac{r(r^2+r+1)}{(2r+1)}, && x=\frac{r^3(r+2)}{(2r+1)},\\
k=8:&\quad \frac{4r(3r^2-2r+1)}{(r+1)(1-r)^3(r^2+2r+3)}, && x=\left(\frac{2r}{1-r^2}\right)^4.
\end{aligned} \tag{110}$$

It is straightforward to verify that these expressions all satisfy the Painlevé equation (108).

For $k = 3$, after normalizing so that $R_{ab} = 3g_{ab}$, the metric (103) becomes [22]

$$h^2=\frac{3}{(1+r+r^2)^2},\qquad a_1^2=(1+2r)^2h^2,\qquad a_2^2=(1-r^2)^2h^2,\qquad a_3^2=r^2(2+r)^2h^2. \tag{111}$$

Note that the radial variable r being used here is precisely the parametric variable in (110). Defining a new radial variable t by $r = -\frac{1}{2} + \frac{\sqrt{3}}{2}\tan\left(\frac{1}{2}\sqrt{3}t\right)$, the $k = 3$ metric becomes

$$\mathrm{d}s_4^2=\mathrm{d}t^2+4\sin^2 t\,\sigma_1^2+4\sin^2\left(t-\tfrac{2}{3}\pi\right)\sigma_2^2+4\sin^2\left(t+\tfrac{2}{3}\pi\right)\sigma_3^2, \tag{112}$$

which can be recognized as the triaxial form of the Einstein metric on S^4, discussed in [32].

[10] The triaxial form of the Fubini–Study metric is derived in appendix B.

For $k = 4$, and normalizing for convenience so that $R_{ab} = 6g_{ab}$, the metric in [22] has

$$h^2 = \frac{1}{4r(1+r)^2}, \qquad a_1^2 = \frac{1}{1+r}, \qquad a_2^2 = \frac{(1-r)^2}{(1+r)^2}, \qquad a_3^2 = \frac{r}{1+r}. \tag{113}$$

Defining a new radial variable by $r = \tan^2 t$, the $k = 4$ metric becomes

$$\mathrm{d}s_4^2 = \mathrm{d}t^2 + \cos^2 t\sigma_1^2 + \cos^2 2t\sigma_2^2 + \sin^2 t\sigma_3^2, \tag{114}$$

which can be recognized as the triaxial $\mathbb{CP}^2$ metric [38], discussed in appendix B.

For $k = 6$, the metric functions are given by

$$\begin{aligned} h^2 &= \frac{3(1+r+r^2)}{r(r+2)^2(2r+1)^2}, & a_1^2 &= \frac{3(1+r+r^2)}{(r+2)(2r+1)^2}, \\ a_2^2 &= \frac{3(r^2-1)^2}{(1+r+r^2)(r+2)(2r+1)}, & a_3^2 &= \frac{3r(1+r+r^2)}{(r+2)^2(2r+1)}. \end{aligned} \tag{115}$$

The radial coordinate runs from $r = 1$ to $r = \infty$, and we have normalized the metric so that $R_{ab} = 3g_{ab}$.

For $k = 8$, after rederiving the metric using the construction given in [22], we find that the metric functions are given by

$$\begin{aligned} h^2 &= \frac{4(1+r)(3-2r+r^2)(1-2r+3r^2)(1+2r+3r^2)}{(1-r)r(1+r^2)(1+2r-r^2)^2(3+2r+r^2)^2}, \\ a_1^2 &= \frac{4(1-r)(1+r)^3(3-2r+r^2)(1-2r+3r^2)}{(1+2r-r^2)(3+2r+r^2)^2(1+2r+3r^2)}, \\ a_2^2 &= \frac{4(1+r^2)(3-2r+r^2)(1-2r-r^2)^2(1+2r+3r^2)}{(1+2r-r^2)^2(3+2r+r^2)^2(1-2r+3r^2)}, \\ a_3^2 &= \frac{16r(1-2r+3r^2)(1+2r+3r^2)}{(1+2r-r^2)(3-2r+r^2)(3+2r+r^2)^2}, \end{aligned} \tag{116}$$

where we have again chosen the normalization so that $R_{ab} = 3g_{ab}$. (This corrects a typographical error in [22], where there is an extra factor $(1 + r)^2$ in the coefficient of σ_3^2 that should not be there.) The radial coordinate lies in the interval $\sqrt{2} - 1 < r < 1$.

The $k = 3$ and $k = 4$ Tod–Hitchin metrics are S^4 and $\mathbb{CP}^2$ respectively, albeit in their less common triaxial forms. The existence of more than one Bianchi IX form is a consequence of the homogeneity of these metrics. The isometry algebra contains more than one $SU(2)$ subalgebra, and the orbits are different. The full set of homogeneous Einstein 4-manifolds is known, and from that list we deduce that this can only happen for self-dual Einstein metrics in the case of S^4 and $\mathbb{CP}^2$. Thus for higher values of k, the Tod–Hitchin metrics and the biaxial Bianchi IX self-dual Einstein metrics form disjoint classes. An explicit demonstration of this for the $k = 6$ and $k = 8$ metrics can be given by computing the quantity X^3/Y^2, where X and Y are the quadratic and cubic Weyl tensor invariants defined in (46). We showed that any biaxial self-dual Einstein metric must satisfy $X^3/Y^2 = 1$ (see (45)), and an elementary calculation shows that whilst this is true for the $k = 3$ and $k = 4$ metrics, it does not hold for the $k = 6$ and $k = 8$ metrics.

5.4. Global structure of the metrics

The global structure of the Tod–Hitchin metrics is described in detail in [22, 23]. Here, we summarize the conclusions, presenting them in a way that is perhaps more readily accessible to physicists.

The key to understanding the global structure is to understand the nature of the degenerate orbits where metric coefficients vanish. An important feature of the metrics, for all k including $k = 3$ and $k = 4$, is that at one end of the radial coordinate range the coefficient of σ_1 vanishes, while at the other end it is the coefficient of σ_2 that vanishes instead. This 'slumping' is reminiscent of the metric behaviour in the Atiyah–Hitchin metric, where the coefficient of one of the σ_i vanishes at short distance, while the coefficient of another of them stabilizes in the asymptotic region. In fact, as shown in [22], the Atiyah–Hitchin metric itself arises as the $k \longrightarrow \infty$ limit of the Tod–Hitchin metrics.

Because of the slumping, it is useful to introduce two different Euler-angle parametrizations of the left-invariant 1-forms, one adapted to the region where σ_1 collapses, and the other adapted to the region where σ_2 collapses. The procedure was described in [33], and elaborated somewhat in [34]. Here we shall present a brief summary of the description in [34], with labelling adapted to our present conventions.

Let us introduce Euler angles (θ, ϕ, ψ) and $(\tilde\theta, \tilde\phi, \tilde\psi)$, such that

$$\sigma_1 = \mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi, \qquad \sigma_2 + \mathrm{i}\sigma_3 = \mathrm{e}^{\mathrm{i}\psi}(\mathrm{d}\theta + \mathrm{i}\sin\theta\,\mathrm{d}\phi), \tag{117}$$

$$\sigma_2 = \mathrm{d}\tilde\psi + \cos\tilde\theta\,\mathrm{d}\tilde\phi, \qquad \sigma_3 + \mathrm{i}\sigma_1 = \mathrm{e}^{\mathrm{i}\tilde\psi}(\mathrm{d}\tilde\theta + \mathrm{i}\sin\tilde\theta\,\mathrm{d}\tilde\phi). \tag{118}$$

We begin by taking ψ and $\tilde\psi$ both to have period 2π, so that the orbits are $\mathbb{RP}^3$. Clearly one could, in principle, solve for the transformation that relates the tilded and untilded coordinates, but we shall not need this.

We now consider the operation, which we shall denote by I_1, which implements the identification $\psi \approx \psi + \pi$. It is easily seen that in terms of the tilded coordinates, this corresponds to $\tilde\theta \longrightarrow \pi - \tilde\theta$, $\tilde\phi \longrightarrow \tilde\phi + \pi$, $\tilde\psi \longrightarrow -\tilde\psi$. Likewise we define $\tilde I_2$ which implements $\tilde\psi \approx \tilde\psi + \pi$. Since the tilded basis is related to the untilded by a cyclic permutation of $(\sigma_1, \sigma_2, \sigma_3)$, we can see that in our notation we shall have $I_i = \tilde I_i$, and so we can deduce that the effect of the I_i on the untilded coordinates is

$$\begin{array}{llll} I_1: & \theta \longrightarrow \theta, & \phi \longrightarrow \phi, & \psi \longrightarrow \psi + \pi, \\ I_2: & \theta \longrightarrow \pi - \theta, & \phi \longrightarrow \phi + \pi, & \psi \longrightarrow -\psi, \\ I_3: & \theta \longrightarrow \pi - \theta, & \phi \longrightarrow \phi + \pi, & \psi \longrightarrow \pi - \psi, \end{array} \tag{119}$$

while on the tilded coordinates we have

$$\begin{array}{llll} I_1: & \tilde\theta \longrightarrow \pi - \tilde\theta, & \tilde\phi \longrightarrow \tilde\phi + \pi, & \tilde\psi \longrightarrow \pi - \tilde\psi, \\ I_2: & \tilde\theta \longrightarrow \tilde\theta, & \tilde\phi \longrightarrow \tilde\phi, & \tilde\psi \longrightarrow \tilde\psi + \pi, \\ I_3: & \tilde\theta \longrightarrow \pi - \tilde\theta, & \tilde\phi \longrightarrow \tilde\phi + \pi, & \tilde\psi \longrightarrow -\tilde\psi, \end{array} \tag{120}$$

Consider first the case $k = 3$, which gives the triaxial metric (112) on S^4. Near $t = 0$ we have

$$\mathrm{d}s_4^2 \sim \mathrm{d}t^2 + 4t^2\sigma_1^2 + \sigma_2^2 + \sigma_3^2. \tag{121}$$

From expression (117) we see that regularity at $t = 0$ requires that ψ have period π, and so from (119) we should impose the identification I_1. Near the other endpoint $t = \frac{2}{3}\pi$, we set $t = \frac{2}{3}\pi - \tau$, and so the metric takes the form

$$\mathrm{d}s_4^2 \sim \mathrm{d}\tau^2 + 4\tau^2\sigma_2^2 + \sigma_1^2 + \sigma_3^2. \tag{122}$$

From (118) we see that regularity requires that $\tilde\psi$ have period π, and so from (120) we should in addition impose the identification I_2. Thus the principal orbits are $SO(3)/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. We also see that the two-dimensional bolt described by $\sigma_2^2 + \sigma_3^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$ at $t = 0$, and the two-dimensional bolt described by $\sigma_1^2 + \sigma_3^2 = \mathrm{d}\tilde\theta^2 + \sin^2\tilde\theta\,\mathrm{d}\tilde\phi^2$ at $t = \frac{2}{3}\pi$ each has the topology of $\mathbb{RP}^2$, since there is an antipodal identification in the former implied by I_2 in (119), and in the latter implied by I_1 in (120). The metric therefore extends smoothly on the Veronese surfaces $\mathbb{RP}^2$ at each endpoint [22].

The case $k = 4$ gives the triaxial $\mathbb{CP}^2$ metric (114). We can take the two endpoints to be at $t = \frac{1}{2}\pi$ and $t = \frac{1}{4}\pi$. Near $t = \frac{1}{2}\pi$, after setting $t = \frac{1}{2}\pi - \tau$ the metric takes the form

$$\mathrm{d}s_4^2 \sim \mathrm{d}\tau^2 + \tau^2\sigma_1^2 + \sigma_2^2 + \sigma_3^2. \tag{123}$$

Regularity therefore requires that we *not* impose the identification I_1. On the other hand, at the other endpoint $t = \frac{1}{4}\pi$, after defining $t = \frac{1}{4}\pi - \tau$ we have

$$\mathrm{d}s_4^2 \sim \mathrm{d}\tau^2 + 4\tau^2\sigma_2^2 + \sigma_1^2 + \sigma_3^2. \tag{124}$$

Regularity therefore requires that we impose the identification I_2. This means that the principal orbits are $SO(3)/\mathbb{Z}_2$, and that the metric extends smoothly onto $\mathbb{RP}^2$ at $t = \frac{1}{2}\pi$, and onto S^2 at $t = \frac{1}{4}\pi$ [22]. This reflects the fact that $\mathbb{CP}^2$ can be described as the double covering of S^4 branched over $\mathbb{RP}^2$.

For $k = 6$, we see by letting $r = 1 + 3\rho$ that near $r = 1$ the metric (115) takes the form

$$\mathrm{d}s_4^2 \sim \mathrm{d}\rho^2 + 4\rho^2\sigma_2^2 + \tfrac{1}{3}\left(\sigma_1^2 + \sigma_3^2\right), \tag{125}$$

whilst letting $r = 1/\rho^2$ the metric near $r = \infty$ has the form

$$\mathrm{d}s_4^2 \sim \mathrm{d}\rho^2 + \tfrac{1}{4}\rho^2\sigma_1^2 + \tfrac{3}{2}\left(\sigma_2^2 + \sigma_3^2\right). \tag{126}$$

Thus if we impose the identification I_2 the metric extends smoothly over $\mathbb{RP}^2$ at $r = 1$, and extends over $\mathbb{RP}^2$ with an orbifold singularity having angle $\frac{1}{2}\pi$ at $r = \infty$ [22].

For $k = 8$, after letting $r = \sqrt{2} - 1 + (\sqrt{2 - \sqrt{2}})\rho$, the metric near $r = \sqrt{2} - 1$ can be seen to have the form

$$\mathrm{d}s_4^2 \sim \mathrm{d}\rho^2 + 4\rho^2\sigma_2^2 + (3 - 2\sqrt{2})\left(\sigma_1^2 + \sigma_3^2\right). \tag{127}$$

Letting $r = 1 - \frac{3}{8}\rho^2$, the metric near $r = 1$ takes the form

$$\mathrm{d}s_4^2 \sim \mathrm{d}\rho^2 + \tfrac{1}{9}\rho^2\sigma_1^2 + \tfrac{4}{3}\left(\sigma_2^2 + \sigma_3^2\right). \tag{128}$$

Thus by imposing the identification I_2 the metric extends smoothly over $\mathbb{RP}^2$ at $r = \sqrt{2} - 1$, and extends over $\mathbb{RP}^2$ with an orbifold singularity having angle $\frac{1}{3}\pi$ at $r = 1$ [22].

In [22] it is shown that all the metrics obtained from solving the Painlevé equation are positive definite with x lying in the interval $1 < x < \infty$, for all values of the constant k parametrizing the solutions described in [22]. Near $x = 1$, the metric takes the form

$$\mathrm{d}s_4^2 \sim \frac{1}{16}\cos^2\frac{\pi}{k}\left(\mathrm{d}r^2 + 4r^2\sigma_2^2\right) + \sigma_1^2 + \sigma_3^2. \tag{129}$$

This shows that the metric extends over the degenerate orbit at $r = 0$, with $\sigma_1^2 + \sigma_3^2$ describing $\mathbb{RP}^2$ [22]. As $x \longrightarrow \infty$ the metric assumes the form

$$\mathrm{d}s_4^2 \sim \mathrm{d}\rho^2 + \frac{4\rho^2}{(k-2)^2}\sigma_1^2 + 2^{8/k-2}\left(\sigma_2^2 + \sigma_3^2\right), \tag{130}$$

where $x = \rho^{-k}$, which shows that there is an orbifold singularity with angle $2\pi/(k-2)$ around $\mathbb{RP}^2$ [22]. These results are consistent with the explicit calculations for the $k = 6$ and $k = 8$ cases above.

6. Singularity structure and M-theory

In this paper, we have extended the analysis of G_2 holonomy spaces to those whose principal orbits are twistor spaces, constructed as S^2 bundles over four-dimensional self-dual Einstein metrics of the general Bianchi IX type. We obtained the general first-order differential equations for these triaxial Bianchi IX metrics, and we showed how they can be derived from a superpotential. In special cases, the self-dual Einstein metrics reduce to S^4, $\mathbb{CP}^2$ and the (biaxial) Taub–NUT–de Sitter metrics,

We focused on the analysis of the local and global structures of the self-dual Einstein Bianchi IX metrics. For the biaxial specialization, where the local form of the general solution is well known, we gave a complete analysis of the solutions by studying the flows in the phase plane of the first-order equations. Even in this biaxial case the analysis is quite subtle, since there is no single local expression for the metric that directly covers all the possible regions of flows in the phase plane. Some regions are well described by the standard expression for the self-dual Taub–NUT–de Sitter metrics, but our analysis reveals that in another region there are flows that are more appropriately described by a different local form of the solution, which we refer to as the self-dual Eguchi–Hanson–de Sitter metrics. These metrics, which as far as we are aware have not been presented explicitly before, describe flows in a region of the phase plane that can be viewed as generalizations of the Eguchi–Hanson metric in which the cosmological constant is non-zero. Unlike the usual Eguchi–Hanson–de Sitter metrics [31], which are Kähler but neither self-dual nor anti-self-dual, the new metrics have a self-dual Weyl tensor even when the cosmological constant is non-zero. In the self-dual Taub–NUT–de Sitter form, the two parameters of biaxial solutions can be thought of as the NUT parameter and the cosmological constant. In the self-dual Eguchi–Hanson–de Sitter form, the two parameters can be thought of as the Eguchi–Hanson scale size and the cosmological constant.

We discussed the global structure for the biaxial self-dual metrics, both for positive and negative cosmological constant. For the positive cosmological constant the metrics are compact, in general with singularities. The radial coordinate ranges over an interval that terminates at endpoints where the $SU(2)$ principal orbits degenerate; to a point (a NUT) at one end, and to a two-dimensional surface (a bolt) that is (locally) S^2 at the other. For generic choices of the NUT parameter (or, in the alternative local description, the Eguchi–Hanson scale size), the metrics cannot be smoothly extended on the NUT and bolt endpoints simultaneously. This is because the periodicity requirements needed for regularity at one end are in general incommensurate with the periodicity requirements at the other end. Only for very special values of the NUT parameter is the metric regular at both endpoints. In general, however, one encounters singularities at either endpoint of the four-dimensional radial coordinate.

In the generic case, a specific choice of the period for the azimuthal angle ψ allows the singularity at the S^2 bolt to be removed, but then the NUT has a co-dimension four orbifold singularity. Alternatively, choosing the periodicity appropriate for regularity at the NUT, there will be a co-dimension two singularity on the S^2 bolt. The associated seven-dimensional G_2 holonomy space therefore has singularities of the same co-dimensions. The co-dimension four NUT singularities may admit an M-theory interpretation associated with the appearance of non-Abelian gauge symmetries [10] and the circle reduction of M-theory on these G_2 holonomy spaces may have a type IIA interpretation in terms of a location of coincident D6-branes [10]. On the other hand, the co-dimension two singularities at the bolts do not seem to have a straightforward interpretation in M-theory dynamics. Since neither type of singularity is of co-dimension seven, these spaces do not seem to shed light on the appearance of chiral matter.

The triaxial self-dual Einstein Bianchi IX metrics described by the Tod–Hitchin system are defined on compact spaces with bolts at each endpoint. For the solutions discussed in section 5.3, with $k \geqslant 6$, one endpoint has an $\mathbb{RP}^2$ bolt, while the other endpoint is an $\mathbb{RP}^2$ bolt with a $\mathbb{Z}_{k-2}$ conical co-dimension two singularity. The corresponding G_2 holonomy spaces again have co-dimension two singularities, and so M-theory on these spaces does not have a straightforward interpretation; in particular, their relevance for obtaining non-Abelian gauge group enhancement or the appearance of chiral matter is not clear.

Despite the fact that the role of the singularities in our metrics in M-theory is unclear, one thing is certain: the singularities do not affect the amount of supersymmetry. Because the Killing spinor is a singlet, it is invariant under all elements of the isometry group. In particular, it is invariant under the action of the binary dihedral group generated by I_1, I_2 and I_3, and in the biaxial case it is invariant under arbitrary shifts of the coordinate ψ. Since it was these symmetries that entered into the discussion of singularities, it is clear that no matter what identifications we choose to make, it will not affect the existence of the Killing spinor. This should be contrasted with the co-dimension two and co-dimension four singularities discussed in [43]. In that case, the Killing spinors are not singlets, and identifications may or may not leave them invariant. The singularities for which the identifications are incompatible with the existence of Killing spinors are believed to be unstable, due to closed-string tachyons, whilst those that are compatible with the Killing spinors are believed to be stable. In our case, it is clear that there is no room for a closed-string tachyon instability, or its M-theoretic analogue. In other words, “Don’t Panic, it’s G_2!”

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Appendix A. Bianchi IX Einstein–Kähler metrics

The purpose of this appendix is to clarify the distinction between the anti-self-dual Einstein metrics considered in this paper and Bianchi IX Einstein–Kähler metrics. These two classes do not overlap except when the metrics are Ricci-flat, or else the Fubini–Study metric on $\mathbb{CP}^2$ (or the Bergmann metric on the open ball in $\mathbb{C}^2$ if $\Lambda < 0$). In the case that the metrics are biaxial, the general Einstein–Kähler solutions, together with their Kähler potential, were obtained in [31], where they were called the Eguchi–Hanson–de Sitter metrics (see equation (57)). A subsequent discussion was given in [36].

The triaxial case has been considered by Dancer and Strachan in [37], where a first-order system was obtained. This generalizes that for hyper-Kähler metrics with triholomorphic $SU(2)$ action, written down and solved in [28]. The general solution of the Dancer–Strachan system is not known, but particular cases, such as triaxial forms of the Fubini–Study metric on $\mathbb{CP}^2$ and the product metric on $\mathbb{CP}^1 \times \mathbb{CP}^1$, are known, and turn out to be remarkably simple.

Writing the Bianchi IX metrics in the form (24), with $e^0 = \mathrm{d}t$ and $e^i = a_i\sigma_i$, a basis for anti-self-dual 2-forms is $\Omega_i = e^0 \wedge e^i - \frac{1}{2}\epsilon_{ijk}\epsilon^j \wedge e^k$, and so an ansatz for the $SU(2)$-invariant anti-self-dual Kähler form is

$$\Omega = \alpha_i\Omega_i, \tag{A.1}$$

where the coefficients α_i depend only on t, and $\alpha_i^2 = 1$. The metric will be Kähler if Ω is covariantly constant, which leads to the first-order equations

$$\dot{\alpha}_1 = (\beta_3 + \gamma_3)\alpha_2 - (\beta_2 + \gamma_2)\alpha_3, \qquad \text{and cyclic}, \tag{A.2}$$

where β_i and γ_i are defined in (27). From these, and the Einstein equations, one can show that $\alpha_1 = \alpha_2 = 0$ and $\alpha_3 = 1$ (or cyclic permutations) [37], and hence that the metric coefficients satisfy the first-order equations

$$\begin{aligned} \dot{a}_1 &= -\frac{a_1^2 - a_2^2 - a_3^2}{2a_2a_3}, \\ \dot{a}_2 &= -\frac{a_2^2 - a_1^2 - a_3^2}{2a_1a_3}, \\ \dot{a}_3 &= -\frac{a_3^2 - a_1^2 - a_2^2 + 2\Lambda a_1^2a_2^2}{2a_1a_2}. \end{aligned} \tag{A.3}$$

Rewriting in terms of the radial variable η, defined by $\mathrm{d}t = a_1a_2a_3\mathrm{d}\eta$, it is easily seen that the first-order equations can be derived from a superpotential. In the notation of section 4.6, the potential U in (89) can be written as $U = -\frac{1}{2}g^{ij}(\partial W/\partial\alpha^i)(\partial W/\partial\alpha^j)$, where we now define $\alpha^i = (\log a_1, \log a_2, \log a_3)$, and hence $g_{ij} = 2 - 2\delta_{ij}$. We find that the superpotential is then given by

$$W = -\left(a_1^2 + a_2^2 + a_3^2\right) + \Lambda a_1^2a_2^2. \tag{A.4}$$

Two particular triaxial solutions of the first-order Einstein–Kähler system (A.3) are the Fubini–Study metric on $\mathbb{CP}^2$, which can be written (setting $\Lambda = 6$ for convenience) as [38]

$$\mathrm{d}s_4^2 = \mathrm{d}t^2 + \sin^2 t\sigma_1^2 + \cos^2 t\sigma_2^2 + \cos^2 2t\sigma_3^2, \tag{A.5}$$

and the product metric on $S^2 \times S^2$, which can be written (setting $\Lambda = 2$ for convenience) as [39]

$$\mathrm{d}s_4^2 = \mathrm{d}t^2 + \sin^2 t\sigma_1^2 + \sigma_2^2 + \cos^2 t\sigma_3^2. \tag{A.6}$$

In view of the somewhat unfamiliar forms of these metrics, we shall give a brief description of them below.

Appendix B. Iwai's construction, Dragt coordinates and the Guichardet connection

In this appendix, we shall derive the triaxial forms of the Einstein metrics on $\mathbb{CP}^2$ and $S^2 \times S^2$. The method used differs slightly from those in [38, 39], but it has the merit of giving a unified description of the two cases. The basic idea is to express the metric in flat Euclidean 6-space in an appropriate coordinate system, adapted to an $SO(3)$ action. We shall here follow the paper of Iwai [40], who was interested in the three-body problem in molecular physics. It turns out that we can use his results not only to obtain Bianchi IX metrics but we can also use Scherk–Schwarz reduction to obtain some insight into global monopoles of the sort recently studied by Hartnoll [41].

We think of $\mathbb{E}^6$ as $\mathbb{E}^3 \oplus \mathbb{E}^3 \ni (\mathbf{x}, \mathbf{y})$ and consider the diagonal action[11] of $SO(3)$. Projection from the principal orbits is Iwai's generalization of the standard Hopf map used in the Taub–NUT metric. The standard Hopf map $\pi : \mathbb{E}^4 \equiv (\mathbb{C} \oplus \mathbb{C}) \ni (z^1, z^2) \to \mathbb{R}^3 \equiv \mathbb{R} \oplus \mathbb{C}$ onto the orbits of the diagonal action of $U(1)$ is given by

$$(z^1, z^2) \to (|z^1|^2 - |z^2|^2, 2z^1\bar{z}^2). \tag{B.1}$$

Introducing polar coordinates on $\mathbb{R}^3$, and an angle ψ along the Hopf fibres, we may write the flat metric on $\mathbb{E}^4$ as a special case of the multi-centre metrics, which have an interpretation in terms of Kaluza–Klein monopoles and D6-branes. Iwai's procedure is rather similar and may have a corresponding generalization.

In the case of flat six dimensions, Iwai's map is $\pi : \mathbb{E}^6 \to \mathbb{R}^3_+ \ni (w^1, w^2, w^3)$, given by

$$(\mathbf{x}, \mathbf{y}) \to (\mathbf{x}^2 - \mathbf{y}^2, 2\mathbf{x}\cdot\mathbf{y}, 2|\mathbf{x}\times\mathbf{y}|) = (w^1, w^2, w^3), \tag{B.2}$$

with $w^3 \geqslant 0$. The orbit space $\mathbb{R}^3_+$ may be given coordinates (ρ, ψ, χ), called Dragt coordinates, such that

$$(w^1, w^2, w^3) = (\rho^2 \cos\psi \cos\chi, \rho^2 \sin\psi \cos\chi, \rho^2 \sin\chi), \tag{B.3}$$

with $0 \leqslant \rho < \infty, 0 \leqslant \psi < 2\pi, 0 \leqslant \chi < \frac{\pi}{2}$. Note the range of χ. One checks that

$$\mathbf{x}^2 + \mathbf{y}^2 = \rho^2 = \sqrt{(w^1)^2 + (w^2)^2 + (w^3)^2}. \tag{B.4}$$

To fix the $SO(3)$ freedom we introduce an orthonormal moving frame $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ related to a fixed orthonormal frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ by a rotation with standard Euler angles and left-invariant 1-forms $(\sigma_1, \sigma_2, \sigma_3)$ say. Now if

$$\mathbf{x} = \rho\cos\frac{\psi}{2}\cos\frac{\chi}{2}\mathbf{u}_1 - \rho\sin\frac{\psi}{2}\sin\frac{\chi}{2}\mathbf{u}_2, \tag{B.5}$$

and

$$\mathbf{y} = \rho\sin\frac{\psi}{2}\cos\frac{\chi}{2}\mathbf{u}_1 + \rho\cos\frac{\psi}{2}\sin\frac{\chi}{2}\mathbf{u}_2, \tag{B.6}$$

Iwai finds that the flat metric on $\mathbb{E}^6$ is given by

$$\begin{aligned} \mathrm{d}s^2 = {} & \mathrm{d}\rho^2 + \frac{1}{4}\rho^2(\mathrm{d}\chi^2 + \cos^2\chi\,\mathrm{d}\psi^2) + \rho^2\sin^2\frac{\chi}{2}\sigma_1^2 \\ & + \rho^2\cos^2\frac{\chi}{2}\sigma_2^2 + \rho^2\left(\sigma_3 - \frac{1}{2}\sin\chi\,\mathrm{d}\psi\right)^2. \end{aligned} \tag{B.7}$$

If we set $\psi = \frac{1}{2}\pi$ and $\rho^2 = 2$, the vectors $\mathbf{x}$ and $\mathbf{y}$ have unit magnitudes, and thus parametrize points on $S^2 \times S^2$, embedded in $\mathbb{R}^3 \times \mathbb{R}^3$. The result is the metric (A.6) on $S^2 \times S^2$, obtained in [39].

If instead we set $\rho = 1$ we obtain the unit S^5. The angle ψ is a coordinate along the Hopf fibres. Projecting orthogonally to the Hopf fibres, we obtain the triaxial form (A.5) of the Fubini–Study metric on $\mathbb{CP}^2$ obtained in [38].

We note *en passant* that we could consider the seven-dimensional flat metric on $\mathbb{E}^{6,1}$ as a trivial solution of supergravity, and perform a Scherk–Schwarz reduction on the orbits of $SO(3)$. We get in four dimensions a global monopole coupled to an $SO(3)$ gauge field $A^i, i = 1, 2, 3$, with the Higgs field in the symmetric tensor (i.e. the **5**) representation of

[11] Note that the triaxial form of the standard round metric on S^4 can also be obtained from the flat metric on $\mathbb{E}^6$, but now the action of $SO(3)$ is different. In this case, one identifies $\mathbb{E}^6$ with the space of real symmetric 3×3 matrices on which $SO(3)$ acts by conjugation [32].

$SO(3)$. The gauge connection coincides with the Guichardet connection, and in the present case the only non-zero gauge field is

$$A^3 = -\tfrac{1}{2}\sin\chi\,\mathrm{d}\psi. \tag{B.8}$$

Ignoring the Weyl rescaling, the interpretation is as follows. One should think of ψ as an azimuthal angle, i.e. a longitude, while χ is to be thought of as a latitude. Because $\chi \in [0, \frac{\pi}{2})$, there is a deficit solid angle, and hence a conical singularity at the origin. Moreover, the metric is not asymptotically flat. We have an embedding of an Abelian monopole into the non-Abelian gauge group $SO(3)$. This monopole may be thought of as sitting at the centre of a global monopole supported by a Higgs field.

Appendix C. Killing spinors

Since the seven-dimensional metric constructed from the anti-self-dual Einstein 4-metric according to (12) has G_2 holonomy, it follows that it admits a covariantly constant spinor. It is instructive to look at how this is related to spinors in the four-dimensional base space. To do this, we begin by calculating the Lorentz-covariant exterior derivative on spinors in seven dimensions in terms of quantities in the four-dimensional base metric. We adopt a notation where quantities in seven dimensions carry hats, and so we write (12) as $\mathrm{d}\hat{s}_7^2 = \rho^{-1}(\mathrm{D}u_i)^2 + 2\rho\,\mathrm{d}s_4^2$, for which we choose the natural vielbein basis $\hat{e}^i = \rho^{-1/2}\mathrm{D}u_i$, $\hat{e}^a = \sqrt{2\rho}\,e^a$. The spinor-covariant exterior derivative is given by $\hat{\nabla} \equiv \mathrm{d} + \frac{1}{4}\hat{\omega}_{AB}\hat{\Gamma}^{AB}$, and after some calculation we find that this is given by

$$\begin{aligned}\hat{\nabla} = \mathrm{d} + \tfrac{1}{4}\omega_{ab}\hat{\Gamma}^{ab} - \tfrac{1}{4}\epsilon_{ijk}A^k\hat{\Gamma}^{ij} + \tfrac{1}{16}\rho^{-3/2}u_j\big(\epsilon_{ijk}J^k_{ab}\hat{\Gamma}^{ab} - 4\hat{\Gamma}^{ij}\big)\hat{e}^i \\ + \tfrac{1}{8}\rho^{-3/2}u_i\big(\epsilon_{ijk}J^j_{ab}\hat{\Gamma}^{kb} - 2\hat{\Gamma}^{ia}\big)\hat{e}^a.\end{aligned} \tag{C.1}$$

The covariantly constant spinor $\hat{\eta}$ in the seven-dimensional G_2 metric satisfies $\hat{\nabla}\hat{\eta} = 0$. It can be seen from (C.1) that this spinor is annihilated by the terms involving the $\mathbb{R}^3$ coordinates u_i, and that it is independent of u_i. In fact in this basis we find that $\hat{\eta}$ is the spinor that is determined, up to overall u_i-independent scale, by the conditions

$$\hat{\Gamma}_{ij}\hat{\eta} = \tfrac{1}{4}\epsilon_{ijk}J^k_{ab}\hat{\Gamma}_{ab}\hat{\eta}. \tag{C.2}$$

It then follows from (C.1) that $\hat{\eta}$ satisfies

$$\big(\mathrm{d} + \tfrac{1}{4}\omega_{ab}\hat{\Gamma}^{ab} - \tfrac{1}{4}\epsilon_{ijk}A^k\hat{\Gamma}^{ij}\big)\hat{\eta} = 0. \tag{C.3}$$

Decomposing spinors into the tensor product of spinors in the four-dimensional base and the $\mathbb{R}^3$ fibres, we choose Dirac matrices $\hat{\Gamma}_a = \Gamma_a \otimes \mathbb{1}$ and $\hat{\Gamma}_i = \Gamma_5 \otimes \tau_i$. The Pauli matrices τ_i can be viewed as the generators of an internal $SU(2)$ isospin, and so (C.2) and (C.3) can be written as

$$J^i_{ab}\Gamma^{ab}\eta^\alpha = 4\mathrm{i}(\tau_i)^\alpha{}_\beta\eta^\beta, \qquad \nabla\eta^\alpha - \frac{\mathrm{i}}{4}A^i(\tau_i)^\alpha{}_\beta\eta^\beta = 0. \tag{C.4}$$

The second equation is the condition for the 4-component spinor η^α with its isospin doublet index α to be gauge covariantly constant with respect to the $SU(2)$ Yang–Mills covariant derivative.

Using (C.2) we can rewrite (C.3) as the four-dimensional equation

$$\mathrm{d}\eta^\alpha + \tfrac{1}{4}\big(\omega_{ab} - \tfrac{1}{2}A^i J_{ab}\big)\Gamma^{ab}\eta^\alpha = 0. \tag{C.5}$$

With the Yang–Mills connection taken to be the self-dual part of the four-dimensional spin connection as in (25), we therefore find that (C.5) is nothing but

$$\mathrm{d}\eta^\alpha + \tfrac{1}{4}\omega^-_{ab}\Gamma^{ab}\eta^\alpha = 0, \tag{C.6}$$

where $\omega^-_{ab} \equiv \frac{1}{2}\left(\omega_{ab} - \frac{1}{2}\epsilon_{abcd}\omega^{cd}\right)$ is the anti-self-dual part of the spin connection. In fact it follows from the conditions (C.2) satisfied by $\hat{\eta}$ that $\hat{\Gamma}_{ab}\hat{\eta}$ is self-dual in the four-dimensional base space, and hence (C.6) reduces simply to $d\eta^\alpha = 0$.

It is interesting to note that in the special case of $\mathbb{CP}^2$, which does not admit an ordinary spin structure, η^α is a generalized spinor (in the terminology of [42]) that is charged with respect to the Yang–Mills connection A^i. In this case the connection is actually $SO(3)$-valued, as opposed to $SU(2)$-valued, and it is this that serves to compensate for the minus sign that ordinary spinors would acquire upon parallel propagation around a family of curves spanning the bolt in $\mathbb{CP}^2$ [42].

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