THE UNIVERSITY OF MICHIGAN
COLLEGE OF LITERATURE, SCIENCE, AND THE ARTS
Computer and Communication Sciences Department

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A Conjecture on S^*-semigroups of Automata

Dennis Paul Geller

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Introduction

In [2] Hedetniemi and Fleck define the S^*-semigroup of an automaton. It was conjectured that if A and A' are any two strong machines with the same number of states then $S^*(A)$ is isomorphically embedded in $S^*(A')$, and vice versa. In this note we prove a stronger result which settles the conjecture except in the case where exactly one of the machines is autonomous.

S^*-semigroups

Let $A = (S,I)$ be an automaton with states S and input set I; if $s \in S$, $i \in I$ then we write si for the successor of s under input i. If $s,t \in S$ and x is a string of symbols from $I(x \in I^*)$ such that $sx = t$ then we say that (s,x,t) is a <u>triple</u> of A; if $x = i \in I$ then (s,x,t) is an <u>elementary triple</u>. Let U and V be finite sets of triples of A; we define the product $U \circ V$ by

$$U \circ V = \{(s,x,t) | \exists (s,y,r) \in U, (r,z,t) \in V \text{ such that } x=yz\}.$$

Under the operation $\circ$ the finite sets of triples form a semigroup; we call this semigroup $S^*(A)$.

Let A be a strong automaton with n states and at least two inputs, η and $\bar{\eta}$. Since A is strong, for any states s and t of A there is a string w_{st} of length at least one such that $sw_{st}=t$.

Let A' be any automaton with at most n states, and let ϕ be a 1-1 map from the states of A onto the states of A'; thus, unless $|A'|=|A|$ the domain of ϕ will be a proper subset $\bar{S}$ of the states of A. If the input set of A' is $I' = \{i_1', i_2', \ldots, i_a' | a = |I'|\}$ we define a map $h: \bar{S} \times I' \to I^*$ in the following manner. Let $s \in \bar{S}$ and $i_j' \in I'$ be such that $(\phi(s), i_j', t')$ is a triple of A'; choose $t \in \bar{S}$ such that $\phi(t) = t'$ and define

$$h(s,i_j') = h_s(i_j') = \eta^j \bar{\eta} w_{qt},$$

where $q = s(\eta^{j}\bar{\eta})$. Clearly h_s is 1-1 for each $s \varepsilon S$; also for each $s \varepsilon S$, $i_j' \varepsilon I'$, the relation $\phi(sh_s(i_j')) = \phi(s)i_j'$ holds. (In fact, the pair (ϕ,h) defines a generalization of the classical automata-theoretic notion of realization; this is dealt with in detail in [1]). We can also extend h to domain $\bar{S} \times (I')^*$ inductively by $h_s(i_j'x') = h_s(i_j')h_t(x')$, where $x' \varepsilon (I')^*$ and $t = sh_s(i_j') \varepsilon \bar{S}$.

<u>Lemma</u>. The map $h: \bar{S} \times (I')^* \to I^*$ is 1-1 for each $s \varepsilon \bar{S}$.

<u>Proof</u>. Let $j_1 \ldots j_m$ and k_1 and $k_1 \ldots_{m'}$ be two strings from $(I')^*$ and let $h_s(j_1 \ldots j_m) = h_s(k_1 \ldots k_{m'}) = w$. Then, by definition, there are states $r,t \varepsilon \bar{S}$ such that $w = h_s(j_1)h_r(j_2 \ldots j_m) = h_s(k_1)h_t(k_2 \ldots k_{m'})$. But there is a unique positive integer ℓ such that the prefix of w having length $\ell + 1$ is the string $\eta^{\ell}\bar{\eta}$. This uniquely determines $j_1 = k_1 = i_\ell'$, so that r=t= si_ℓ'. Then $h_r(j_2 \ldots j_m) = h_r(k_2 \ldots k_{m'})$, and we can repeat the above process until we arrive at m=m' and $j_\rho = k_\rho$, $\rho = 1, 2, \ldots, m$.

Note that the lemma would not simply follow if h_s was 1-1 on symbols for each $s \varepsilon \bar{S}$. Using the notation of the lemma, suppose $h_s(j_1) = a$, $h_s(k_1) = ab$, $h_r(j_2) = bc$, $h_t(k_2) = c$. Then $h_s(j_1j_2) = h_s(k_1k_2) = abc$, but $j_1j_2 \neq k_1k_2$.

<u>Theorem</u>: Let A be a strong automaton with n states and at least two inputs, and let A' be an automaton with $n' \leq n$ states. Then $\mathcal{S}^*(A')$ is isomorphic to a subsemigroup of $\mathcal{S}^*(A)$.

Proof: We use the maps ϕ and h above to define the isomorphism.

Let $b' = \{(s', x', t')\}$ be a singleton in $\mathcal{S}^*(A')$ and set

$$g(b') = \{(s,h_s(x'),t) | \phi(s) = s'\}.$$

Note that this implies that $\phi(t) = t'$. Also, since h_s is 1-1 for each $s\varepsilon\bar{S}$, $g(b_1') = g(b_2')$ if and only if $b_1' = b_2'$; i.e., g is 1-1. Let $b_1' = \{(s_1',x_1',r')\}$ and $b_2' = \{(r',x_2',t_2')\}$. Let $b_1 = \{(s_1,x_1,r)\} = g(b_1')$ $b_2 = \{(r,x_2,t_2)\} = g(b_2')$. Then $b_1 \circ b_2 = \{(s_1,h_{s_1}(x_1'x_2'), t_2)\}$ is a singleton of $\mathcal{S}^*(A)$ and, as $\phi(s_1) = s_1'$, $\phi(t_2) = t_2'$, $b_1 \circ b_2 = g(b_1' \circ b_2')$. Thus $g(b_1') \circ g(b_2') = g(b_1' \circ b_2')$. On the other hand, if $b_1' = \{(s_1',x_1',t_1')\}$, $b_2' = \{(s_2',x_2',t_2')\}$, $g(b_1') = \{(s_1,x_1,t_1)\}$, $g(b_2') = \{(s_2,x_2,t_2)\}$ and $t_1' \neq s_2'$ then $t_1 \neq s_2$, so that $b_1' \circ b_2' = \phi$ and $g(b_1') \circ g(b_2') = \phi$.

Now let V' be any element of $\mathcal{S}^*(A')$; V' is a finite set of triples of A'. Extend g to g* by $g^*(V') = \{g(b') | b\varepsilon V'\}$. Let $\mathcal{S}^*_{A'}(A)$ be $\{g^*(V') | V' \varepsilon \mathcal{S}^*(A')\}$.

Now, if $V_1', V_2' \varepsilon \mathcal{S}^*(A')$,

$$\begin{aligned} g^*(V_1') \circ g^*(V_2') &= [\cup\{g(b_i') | b_i' \varepsilon V_1'\}] \circ [\cup \{g(d_j') | d_j' \varepsilon V_2'\}] \\ &= \bigcup_{i,j} \{g(b_i') \circ g(d_j') | b_i' \varepsilon V_1', d_j' \varepsilon V_2'\} \\ &= \bigcup_{i,j} \{g(b_i' \circ d_j') | b_i' \varepsilon V_1', d_j' \varepsilon V_2'\} \\ &= \bigcup \{g(f_k') | f_k' \varepsilon V_1' \circ V_2'\} \\ &= g^*(V_1' \circ V_2'). \end{aligned}$$

Thus $\mathcal{S}^*_{A'}(A)$ is a subsemigroup of $\mathcal{S}^*(A)$, and g* is a homomorphism. We wish to show that g* is 1-1. Suppose $g^*(V_1') = g^*(V_2')$. Choose a triple $b_1' \varepsilon V_1'$, and let $\{b\} = g(\{b_1'\})$. Then there is a triple $b_2' \varepsilon V_2'$ such that $\{b\} = g(\{b_2'\})$. If $b = (s,w,t)$, $b_1' = (\phi(s),x_1',\phi(t))$ and

$b'_2 = (\phi(s), x'_2, \phi(t))$, where $w = h_s(x'_1) = h_s(x'_2)$.

Then, by the lemma, $x'_1 = x'_2$, so $b'_1 = b'_2$ and $V'_1 \subseteq V'_2$. The symmetric argument gives $V'_1 = V'_2$ so that g^* is 1-1, and hence g^* is an isomorphism between $\mathcal{S}^*(A')$ and $\mathcal{S}^*_{A'}(A)$. Q.E.D.

In particular, if A and A' are both strong n-state automata,

$$\mathcal{S}^*(A) \cong \mathcal{S}^*_A(A') < \mathcal{S}^*(A') \cong \mathcal{S}^*_{A'}(A) < \mathcal{S}^*(A),$$

where $\mathcal{S}_1 < \mathcal{S}_2$ indicates that $\mathcal{S}_1$ is a proper subsemigroup $\mathcal{S}_2$. To complete our partial solution to the conjecture, we need only note that any two strong, autonomous, n-state automata are isomorphic, and hence have isomorphic $\mathcal{S}^*$-semigroups. To completely settle the conjecture it only remains to decide whether the $\mathcal{S}^*$semigroup of the unique autonomous, strong, n-state automaton can isomorphically contain the $\mathcal{S}^*$-semigroup of every other strong, n-state automaton.

REFERENCES

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2. Hedetniemi, S.T. and A.C. Fleck, "$\mathcal{S}$-semigroups of Automata," Technical Report No. 6, THEMIS Project, University of Iowa, 1970.