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Scientific Report

EFFECT OF MOTION ON THE ALTITUDE DISTRIBUTION OF ATMOSPHERIC DENSITY

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ABSTRACT

The effect of motion on the altitude distribution of atmospheric density has been determined. It has been shown, in particular, that for vertical waves moving with increasing velocity along its direction of propagation and for $\frac{m}{kT} v^2 << 1$, the density is given by

$$\rho = \rho_o \exp - \int \left\{ \frac{mg}{kT} + \frac{1}{T}\frac{\partial T}{\partial z} + \frac{m}{kT}\left(\frac{mg}{kT} + \frac{1}{T}\frac{\partial T}{\partial z}\right) v^2 \right\} dz$$

where v is the velocity of the waves. Other terms have the usual significances.

The three terms on the right-hand side correspond to density variation due to gravity, temperature, and the motion in the atmosphere respectively. The relative magnitudes of the three terms at 100 km altitude has been obtained and it has been found that for v = 22 m/sec the variation of density due to the above type of motion is one-tenth of that due to temperature. The vertical components of the wind (Edwards, H. D., et al., JGR 68, 3021 (1963), and the dominant gravity waves (Midgley, J. E. and H. B. Liemohn, JGR 71, 3729 (1966)), are 6 m/sec and 1 m/sec, respectively, and are small to affect the altitude distribution of atmospheric density at 100 km.

The distortions of trails of long-enduring meteors (Liller and Whipple, 1954; Greenhow and Neufeld, 1959), and chemiluminescent vapor released from rockets (Rosenberg and Edwards, 1964; Kochanski, 1964), and barometric oscillations of atmospheric pressure are certain illustrations of motions in the atmosphere. Winds blow through the atmosphere with speeds which may approach 100 m sec^{-1} or more and having wind shears of about 0.04 sec^{-1}. Also, the atmosphere is subjected to tides having 24-hour and 12-hour periods of which the semidiurnal component is stronger. In this note the effect of motion on the altitude distribution of atmospheric density is considered.

For a motion through the atmosphere, the momentum equation and the equation of continuity should hold. Neglecting the effects of viscosity and the rotation of the earth, the former can be written as

$$-\rho \frac{D\vec{v}}{Dt} = \rho\vec{g} + \vec{\nabla}p$$

or

$$-\left(\rho \frac{\partial \overline{v}}{\partial t} + \rho\vec{v}\cdot\vec{\nabla}\vec{v}\right) = \rho\vec{g} + \vec{\nabla}p$$

where

ρ = atmospheric density

p = atmospheric pressure

$\overline{v}$ = velocity

In the steady state

$$-\rho(\vec{v}\cdot\vec{\nabla})\vec{v} = \rho\vec{g} + \vec{\nabla}p$$

$$-\rho.\left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}\right)(\vec{i}v_x+\vec{j}v_y+\vec{k}v_z)$$

$$= \rho\vec{k}g + \left(\vec{i}\frac{\partial p}{\partial x} + \vec{j}\frac{\partial p}{\partial y} + \vec{k}\frac{\partial p}{\partial z}\right) \qquad (1)$$

Therefore for a constant motion the density distribution, as expected, is not affected. The equation of continuity, given by

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \rho \vec{v} = 0$$

becomes in the steady state

$$\vec{\nabla} \cdot \rho \vec{v} = 0, \quad \text{or} \quad \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = 0 . \tag{2}$$

Broadly speaking, the effect of motion on the atmospheric density profile can be obtained by considering four specific cases.

Case 1. Horizontal waves along x or y direction with increasing velocity along their direction of motion, for example

$v_x = v \neq 0$, $v_y = v_z = 0$; $\frac{\partial v_x}{\partial x} \neq 0$, other derivatives are zero.

Applying the continuity and momentum equations we have

$$\frac{\partial}{\partial x}(\rho v_x) = 0 \quad \text{or} \quad \rho \frac{\partial v}{\partial x} + v \frac{\partial \rho}{\partial x} = 0$$

and

$$-\vec{i} \rho v \frac{\partial v}{\partial x} = \rho \vec{k} g + \vec{i} \frac{\partial p}{\partial x} + \vec{j} \frac{\partial p}{\partial y} + \vec{k} \frac{\partial p}{\partial z}$$

Therefore

$$-\rho v \frac{\partial v}{\partial x} = \frac{\partial p}{\partial x} \quad \text{or} \quad v^2 \frac{\partial \rho}{\partial x} = \frac{\partial p}{\partial x} \tag{3}$$

$$\frac{\partial p}{\partial y} = 0 \tag{4}$$

$$\frac{\partial p}{\partial z} = -\rho g \tag{5}$$

Therefore in addition to the hydrostatic equation (5), Eqs. (3) and (4) are obtained. Equation (4) shows that the pressure or density along the y-direction remains constant.

To obtain the density profile let us consider the equation of state

$$p = \frac{k}{m} \rho T \tag{6}$$

where

T = atmospheric temperature

m = mean molecular mass

k = Boltzmann's constant

Equation (3) then becomes

$$v^2 \frac{\partial \rho}{\partial x} = \frac{k}{m} \left(T \frac{\partial \rho}{\partial x} + \rho \frac{\partial T}{\partial x} \right)$$

$$\left(v^2 - \frac{kT}{m} \right) \frac{\partial \rho}{\partial x} = \frac{k}{m} \rho \frac{\partial T}{\partial x}$$

$$\frac{d\rho}{\rho} = \frac{\frac{k}{m} \frac{\partial T}{\partial x}}{v^2 - \frac{kT}{m}} dx$$

$$\rho = \rho_o \exp - \int \frac{\frac{k}{m} \frac{\partial T}{\partial x}}{-v^2 + \frac{kT}{m}} dx$$

$$= \rho_o \exp - \int \frac{\frac{1}{T} \frac{\partial T}{\partial x}}{1 - \frac{m}{kT} v^2} dx$$

For $\frac{m}{kT} v^2 \ll 1$

$$\rho = \rho_o \exp - \int \frac{1}{T} \frac{\partial T}{\partial x} \left(1 + \frac{m}{kT} v^2 \right) dx \tag{7}$$

which gives the density profile along the direction of x.

Combining Eqs. (5) and (6), we obtain the altitude distribution of density, namely

$$\rho = \rho_o \exp - \int \left(\frac{mg}{kT} + \frac{1}{T} \frac{\partial T}{\partial z} \right) dz$$

Case 2. **Horizontal waves** moving along x or y direction with increasing velocity along z direction, that is $v = v_x$, $v_y = v_z = 0$; $\frac{\partial v_x}{\partial z} \neq 0$, other derivatives are zero. An example of this case is the wind in the mesosphere.

Applying the continuity equation (2), we have

$$\frac{\partial}{\partial x}(\rho v) = 0$$

$$\rho \frac{\partial v}{\partial x} + v \frac{\partial \rho}{\partial x} = 0$$

Since $v \neq 0$, $\partial\rho/\partial x = 0$ that is, ρ remains unaltered along the direction of motion. Again, Eq. (1) becomes

$$0 = \rho \vec{k} g + \left(\vec{i} \frac{\partial p}{\partial x} + \vec{j} \frac{\partial p}{\partial y} + \vec{k} \frac{\partial p}{\partial z} \right)$$

Hence

$$\frac{\partial p}{\partial x} = 0, \quad \frac{\partial p}{\partial y} = 0 \qquad (8)$$

and

$$\rho g + \frac{\partial p}{\partial z} = 0 \qquad (9)$$

Therefore, in addition to the hydrostatic equation, these equations show that the pressure or density along the x or y directions remains constant. Applying the equation of state (6), we get in this case the usual equation of altitude distribution of density, namely

$$\rho = \rho_o \exp - \int \left(\frac{mg}{kT} + \frac{1}{T} \frac{\partial T}{\partial z} \right) dz \qquad (10)$$

Case 3. Vertical waves along z direction with velocity increasing along x direction that is $v_x = v_y = 0$ and $v = v_z \neq 0$, $\partial v_z/\partial x \neq 0$

$$0 = \rho \vec{k} g + \left(\vec{i} \frac{\partial p}{\partial x} + \vec{j} \frac{\partial p}{\partial y} + \vec{k} \frac{\partial p}{\partial z} \right)$$

$$\frac{\partial p}{\partial x} = 0, \quad \frac{\partial p}{\partial y} = 0$$

$$\rho g + \frac{\partial p}{\partial z} = 0, \quad \frac{\partial p}{\partial z} = -\rho g$$

Again from the continuity equation

$$\frac{\partial}{\partial z}(\rho v) = 0$$

$$\rho \frac{\partial v}{\partial z} + v \frac{\partial \rho}{\partial z} = 0$$

Since $\partial v/\partial z = 0$ and $v \neq 0$, we have $\partial \rho/\partial z = 0$. From the equation of state (6)

$$\frac{\partial p}{\partial z} = \frac{k}{m} T \frac{\partial \rho}{\partial z} + \frac{k}{m} \rho \frac{\partial T}{\partial z}$$

Since $\partial \rho/\partial z = 0$, $\frac{\partial p}{\partial z} = \frac{k}{m} \rho \frac{\partial T}{\partial z} = -\rho g$ or $\frac{k}{m}\frac{\partial T}{\partial z} = -g$

$$\left|\frac{\partial T}{\partial z}\right| = \frac{mg}{k}$$

$$= \frac{28.3 \mathrm{x} 1.67 \mathrm{x} 10^{-24} \mathrm{x} 949.2}{1.4 \mathrm{x} 10^{-16}}$$

$$= 33°\mathrm{K/km \ at \ 100 \ km}$$

Therefore, in order that the pressure may satisfy the hydrostatic equation while keeping the altitude distribution of density constant, a very high temperature gradient is required. As such a high gradient is not present in the atmosphere (33°K/km at 100 km) this case is of little importance.

Case 4. Vertical waves along z-direction which increase as they move, that is $v_x = v_y = 0$, $v_z = v$, $\partial v_z/\partial z \neq 0$, other derivatives are zero.

In this case the equations of continuity and momentum become

$$\frac{\partial}{\partial z}(\rho v) = 0 \quad \text{or} \quad \rho \frac{\partial v}{\partial z} + v \frac{\partial \rho}{\partial z} = 0 \tag{11}$$

and

$$-\rho\vec{k}v\frac{\partial v}{\partial z} = \rho\vec{k}g + \vec{i}\frac{\partial p}{\partial x} + \vec{j}\frac{\partial p}{\partial y} + \vec{k}\frac{\partial p}{\partial z} \tag{12}$$

Hence

$$\frac{\partial p}{\partial x} = 0, \quad \frac{\partial p}{\partial y} = 0 \tag{13}$$

and

$$-\rho v\frac{\partial v}{\partial z} = \rho g + \frac{\partial p}{\partial z} \tag{14}$$

Equation (14) reduces to the hydrostatic equation if v or its derivative in the vertical direction is zero.

Combining Eqs. (11) and (14), we obtain

$$v^2\frac{\partial \rho}{\partial z} = \rho g + \frac{\partial p}{\partial z}$$

$$= \rho\left(g + \frac{k}{m}\frac{\partial T}{\partial z}\right) + \frac{kT}{m}\frac{\partial \rho}{\partial z}$$

$$0 = \left(-v^2 + \frac{k}{m}T\right)\frac{\partial \rho}{\partial z} + \rho\left(g + \frac{k}{m}\frac{\partial T}{\partial z}\right)$$

$$\frac{\partial \rho}{\rho} = -\frac{g + \frac{k}{m}\frac{\partial T}{\partial z}}{-v^2 + \frac{k}{m}T}\,dz$$

$$= -\frac{\frac{mg}{kT} + \frac{1}{T}\frac{\partial T}{\partial z}}{1 - \frac{m}{kT}v^2}\,dz$$

For $\frac{m}{kT}v^2 \ll 1$

$$\frac{\partial \rho}{\rho} = -\left\{\frac{mg}{kT} + \frac{1}{T}\frac{\partial T}{\partial z} + \frac{m}{kT}\left(\frac{mg}{kT} + \frac{1}{T}\frac{\partial T}{\partial z}\right)v^2\right\}dz$$

Integrating

$$\rho = \rho_o \exp - \int \left[\frac{mg}{kT} + \frac{1}{T}\frac{\partial T}{\partial z} + \frac{m}{kT}\left(\frac{mg}{kT} + \frac{1}{T}\frac{\partial T}{\partial z}\right) v^2\right] dz \qquad (15)$$

If $v = 0$

$$\rho = \rho_o \exp - \int \left(\frac{mg}{kT} + \frac{1}{T}\frac{\partial T}{\partial z}\right) dz$$

Again if $\partial v/\partial z = 0$, Eq. (14) becomes

$$0 = \rho g + \frac{\partial p}{\partial z} = \rho\left(g + \frac{k}{m}\frac{\partial T}{\partial z}\right) + \frac{kT}{m}\frac{\partial \rho}{\partial z}$$

or

$$\rho = \rho_o \exp - \int \left(\frac{mg}{kT} + \frac{1}{T}\frac{\partial T}{\partial z}\right) dz$$

Therefore for both conditions $\partial v/\partial z = 0$ and $v = 0$, the altitude variation of ρ remains unaffected.

In Eq. (15), the first term under the sign of integration corresponds to the density variation due to gravity, the second term to that of temperature variation and last term due to motion in the atmosphere. To consider the relative magnitudes of these terms let us consider their values of 100 km where

$$g = 949.2 \text{ cm/sec}^2$$

$$T = 208.1°K$$

$$m = 28.3\text{x}1.6\text{x}10^{-24}\text{gm}$$

$$\frac{\partial T}{\partial z} = 2.95\text{x}10^{-5}{}^{\circ}\text{k/cm}$$

we then have

$$\frac{mg}{kT} = 1.6\text{x}10^{-6}\text{cm}^{-1}$$

$$\frac{1}{T}\frac{\partial T}{\partial z} = 1.4\text{x}10^{-7}\text{cm}^{-1}$$

$$\frac{m}{kT}\left(\frac{mg}{kT} + \frac{1}{T}\frac{\partial T}{\partial z}\right) v^2 = 2.8\text{x}10^{-15}\ v^2\ \text{cm}^{-1}\ .$$

Therefore, for v = 22 m/sec the variation of density due to the above motion will be one-tenth of that due to temperature. Information of the vertical motion in the atmosphere is meagre. From movements of vapor trails, Edward, et al., obtained that at about 100 km the vertical component of the wind velocity is 6 m/sec. Again the vertical wavelength of the dominant gravity waves is about 12 km up to about 100 km and then steadily increases with altitude. The period of these waves is about 200 minutes (Midgley, J. E. and H. B. Liemohn, JGR, 71, 3729 (1966). Therefore, the vertical velocity of the dominant gravity waves is about 1 m/sec. These velocities are too small to affect the altitude distribution of atmospheric density.

To obtain the change of velocity of atmospheric particles due to heat input and conductivity, consider the energy equation given by

$$\frac{\rho k}{(\gamma-1)m}\left(\frac{\partial T}{\partial t} + \vec{v}\cdot\vec{\nabla}T\right) = Q + \vec{\nabla}\cdot(\lambda_c\vec{\nabla}T) - p\vec{\nabla}\cdot\vec{v} \tag{16}$$

where

Q = heat production in the atmosphere
= $n(O)K(\lambda)E_o(\lambda)$ where $n(O)$ is the concentration of $O_1K(\lambda)$ absorption coefficient and $E_o(\lambda)$ ultraviolet energy flux

λ_c = thermal conductivity of the atmosphere (Nicolet, 1960)
= $1.8\text{x}10^2T^{1/2}$ where T is the abs temp (Nicolet, 1960)

γ = ratio of specific heat

Expanding the above equation, we have

$$\frac{\rho k}{m(\gamma-1)}\left(\frac{\partial T}{\partial t} + v_x\frac{\partial T}{\partial x} + v_y\frac{\partial T}{\partial y} + v_z\frac{\partial T}{\partial z}\right)$$

$$= Q + \lambda_c\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) - p\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)$$

Assuming that the velocity is directed in the vertical direction, we obtain at thermal equalibrium $\partial T/\partial t = 0$, the gradient of vertical velocity given by

$$\frac{\rho k v'}{(\gamma-1)m}\frac{\partial T}{\partial z} = Q + \lambda_c \frac{\partial^2 T}{\partial z^2} - p\frac{\partial v'}{\partial z}$$

or

$$-\frac{\partial v'}{\partial z} = \frac{1}{p}\left[\frac{\rho k v'}{(\gamma-1)m}\frac{\partial T}{\partial z} - Q - \lambda_c \frac{\partial^2 T}{\partial z^2}\right] \qquad (17)$$

To obtain the magnitude of $\partial v'/\partial z$, consider 100 km altitude where

$p = 3.1$ dynes/cm^2

$T = 208.1°K$

$\rho = 5.1 \times 10^{-10}$ g/cm^3

$m = 28.3 \times 1.67 \times 10^{-24}$ gm

$n(0) = 5 \times 10^{11}$ cm^{-3}

$k(\lambda) = 10^{-17}$ cm^2

$E_o(\lambda) = 1$ erg cm^{-2}sec^{-1}

$Q = 5 \times 10^{-6}$ g/cm^{3}sec

$\lambda_c = 1.8 \times 10^2 T^{1/2} = 1.8 \times 10^2 \times (208.1)^{1/2} = 2.6 \times 10^3$ g/cm sec deg (Nicolet)

$\gamma = 1.4$

$\partial T/\partial z = 2.95 \times 10^{-5}$ °K/cm

and

$\partial^2 T/\partial z^2 = 2.3 \times 10^{-10}$ °K/cm^2

For $v' = 1$ m/sec and 10 m/sec, $-\partial v'/\partial z$ is 1.7×10^{-6} sec^{-1} and 3.3×10^{-5} sec^{-1}, respectively and are small enough to be neglected.

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