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An Approach To Nonlinear Feedback Control With Applications to Robotics[1]

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Abstract

A control problem involving a mechanical system with generalized coordinates $q \in R^m$ is considered. The error in tracking a desired input $y^d \in R^p$ is $e = E(q, y^d) \in R^m$. If E satisfies simple conditions, it leads to a nonlinear control law which assures $e(t) \to 0$ as $t \to \infty$. The law is robust in that small changes in it do not produce large steady-state errors or loss of stability. The theory presents a unified framework for treating a number of problems in the control of mechanical manipulators. This is illustrated by an example.

1. Introduction

In this paper we consider the control of systems described by dynamic equations of the form

$$M(q(t))\ddot{q}(t) + F(q(t), \dot{q}(t)) = P\,u(t)\,, \tag{1.1}$$

$$y(t) = G(q(t))\,, \tag{1.2}$$

Where: $q(t), u(t) \in R^m$; $y(t) \in R^k$, $k \geq m$; P, $M(q)$ are nonsingular square matrices. The most obvious application is in robotics [1]. The rigid body equations of motion for a mechanical manipulator with dc motor drives can be described by (1.1) if: $m =$ number of joints in the kinematic chain (usually, $m, = 6$), $q(t) =$ vector of generalized coordinates (usually, joint angles and displacements), $M(q) =$ generalized inertia matrix, $F(q, \dot{q}) =$ vector of equivalent forces due to gravitational, centrifugal, Coriolis and back-emf effects, $u(t) =$ vector of control voltages for the motors.

Through feedback control it is desired to have the output $y(t)$ follow closely a desired motion $y^d(t)$. In the case of a mechanical manipulator y may be specified in various ways: $y = p$, where $p \in R^3$ is the position of the end effector; $y = (p, \Theta) \in R^6$, where $\Theta \in R^3$ is a vector of Euler angles specifying the orientation of the end effector; $y = (p, n, s, a) \in R^{12}$, where $n, s, a \in R^3$ are unit vectors specifying the orientation of end effector axes in terms of their direction cosines. The specific form of $G(q)$ depends on the choice of y and the configuration of the manipulator [1]. The inverse of G is denoted by $G^\dagger$, i. e., if $y = G(q)$ has a solution q it is denoted by $G^\dagger(y)$. There may be multiple solutions [1], but for each of them $G^\dagger$ usually exists on an open set in R^k.

In manipulator problems the tracking error may be measured in several ways. Four examples are:

$$E_1(q, y^d) \;=\; G^\dagger(y^d) - q\,, \tag{1.3}$$

$$E_2(q, y^d) \;=\; y^d - G(q)\,, \tag{1.4}$$

$$E_3(q, y^d) = G^d(y^d) - G(q) \,, \tag{1.5}$$

$$E_4(q,y^d) = \begin{bmatrix} p^d - p \\ \frac{1}{2}(n \times n^d + s \times s^d + a \times a^d) \end{bmatrix} \tag{1.6}$$

The measure E_1 describes the most common situation, where servo performance is measured in terms of the joint angles [1]. E_2 measures the output error directly and is appropriate when $y = p$ or $y = (p, \Theta)$. By choosing G^d to be the map which carries (p, n, s, a) into (p, Θ), E_3 allows E_2 to be extended to the case where $y = (p, n, s, a)$. A problem with E_2 and E_3 is the lack of uniform behavior with respect to the Euler angles. As the Euler angle singularities are approached, E_2 and E_3 become extremely sensitive to angular errors. The measure E_4 applies to $y = (p, n, s, a)$, which is a function of q, and uses vector cross products to measure angular errors. This avoids the sensitivity problem and Euler angle singularities associated with E_2 and E_3. The cross product expression is discussed further in [2].

Just as there are different ways of measuring tracking errors, there are different conceptual approaches to feedback system design. The most common approach for manipulators is to use joint coordinates. Given $y^d(t)$, the desired joint coordinates $q^d(t)$ are computed from $G^{\dagger}(y^d(t))$. Then servos are designed so that components of $q(t)$ track corresponding components of $q^d(t)$. Often the design is based on single-input, single-output models for each joint, although corrections for varying inertias and gravitational loads are sometimes introduced [1]. Alternatively, $\dot{q}^d(t)$ may be computed and $\dot{q}(t)$ controlled through the use of rate servos. This is called resolved-rate control [3] and has certain advantages in reducing the complexity of path computation and control, though precise position control is difficult to achieve.

A more sophisticated approach to joint-coordinate control is the "inverse problem" or "computed torque" technique [4,5,6,7]. In this approach $q^d(t)$, $\dot{q}^d(t)$, $\ddot{q}^d(t)$ are computed and the following control law is used

$$u(t) = P^{-1}(M(q(t))(\ddot{q}^d(t) - K_P(q^d(t)-q(t)) - K_D(\dot{q}^d(t) - \dot{q}(t))) + F(q(t), \dot{q}^d(t))). \quad (1.7)$$

Assuming there are no modelling errors, this gives

$$\ddot{e}(t) = K_P e(t) + K_D \dot{e}(t),\ e(t) = q^d(t) - q(t) = E_1(q(t), y^d(t)). \quad (1.8)$$

By proper choice of the matrices K_P, K_D the solutions of (1.8) are asymptotically stable and the tracking error e(t) tends to zero. More complex variants of this procedure, which cause y(t) to track $y^d(t) = p^d(t)$ or $y^d(t) = (p^d(t), n^d(t), s^d(t), a^d(t))$, are described in [2,8]. At present the use of computed-torque methods has practical limitations because of the complexity of the function $M(q)\,\gamma + F(q, \dot{q})$. Methods for reducing the computational complexity and the use of special processors is the subject of research [9-12].

Another concern with computed-torque methods, which has received little attention, is inaccuracy of the feedback mechanization. In general, the basis for the control law is an idealized model of the manipulator. When the "idealized" control law is used with an actual plant the validity of the scheme is subject to doubt. Thus theoretical results concerning the robustness of computed-torque methods are of interest.

The purpose of this paper is to examine in a unified framework the computed-torque method for a variety of error measures. We begin, in Section 2, by stating a general control problem and giving conditions (Theorem 2.1) such that nonlinear feedback produces an error equation of a form similar to (1.8). This gets at the main idea of the computed-torque method without using the complex formulas associated with specific applications. Section 3 treats perturbations in the feedback law. It is shown that the situation is robust: steady-state errors are small and stability is maintained if errors in the feedback law are sufficiently small. Section 4 applies the preceding theory to the manipulator control problem considered in [2] where E_4 is the error measure. A different control law is obtained. While it is more complex than the one obtained in [2], it has the advantage that error convergence is assured for large errors. Approximations to the control law are investigated in some detail and it is shown that the simplest one leads to the control law described in [2]. Section 5 reviews the general value and applicability of the results.

2. The General Problem

The following notation is used: $x \in R^n$ is an n-tuple of real numbers which is usually written as a column vector, x' is the corresponding row vector, linear operators $Q : R^n \to R^p$ are written $x \mapsto Qx$ where $Q \in R^{p \times n}$ is a real p by n matrix, when $Q \in R^{n \times n}$ has an inverse it is written Q^{-1}, $I_n \in R^{n \times n}$ is the identity matrix, bilinear operators $Q : R^n \times R^m \to R^p$ are written $(x,y) \mapsto Q[x][y]$ and (since the ith component of $Q[x][y]$ can be written $x' Q_i y$, $Q_i \in R^{n \times m}$, $i = 1, \ldots, p$) is designated by $Q \in R^{p \times n \times m}$. Let $X \subset R^n$ be open and assume $f: X \to R^p$ is C^2 (twice continuously differentiable). Then for all $x \in X$, $Df(x) \subset R^{p \times n}$ denotes the first derivative of f and $D^2 f(x) \in R^{p \times n \times n}$ denotes the second derivative of f. If $X \subset R^n$, $Y \subset R^m$ and $f : X \times Y \to R^p$ is C^2, similar notations apply. Specifically, $D_1 f(x, y) \in R^{p \times n}$ and $D_2 f(x, y) \in R^{p \times m}$ are respectively the first derivatives of f with respect to the first and second arguments of f, while the second derivatives are:

$D_1^2 f(x, y) \in R^{p \times n \times n}$, $D_2^2 f(x, y) \in R^{p \times m \times m}$,

$D_1 D_2 f(x, y) \in R^{p \times n \times m}$, $D_2 D_1 f(x, y) \in R^{p \times m \times n}$

and $\quad D_2 D_1 f(x,y)[\alpha][\beta] = D_1 D_2 f(x,y)[\beta][\alpha]$.

The general problem consists of the equations of motion (1.1) and an error measure $E(q, y^d)$. The error e(t) and its derivatives are given by

$$e(t) = E(q(t), y^d(t)), \tag{2.1}$$

$$\dot{e}(t) = \overset{\circ}{E}(q(t), y^d(t), \dot{q}(t), \dot{y}^d(t)),$$

$$\ddot{e}(t) = \overset{\circ\circ}{E}(q(t), y^d(t), \dot{q}(t), \dot{y}^d(t), \ddot{q}(t), \ddot{y}^d(t)),$$

where the functions $\overset{\circ}{E}$ and $\overset{\circ\circ}{E}$ are defined by

$$\overset{\circ}{E}(q, y, \dot{q}, \dot{y}) = D_1E(q, y)\dot{q} + D_2E(q, y)\dot{y} \tag{2.2}$$

$$\overset{\circ\circ}{E}(q, y, \dot{q}, \dot{y}, \ddot{q}, \ddot{y}) = D_1^2E(q, y,)[\dot{q}]\,[\dot{q}] \ +D_2D_1E(q, y)[\dot{y}][\dot{q}] \ + \tag{2.3}$$

$$D_1E(q, y)\ddot{q} \ +D_2^2E(q, y)[\dot{y}][\dot{y}] \ +D_1D_2E(q, y)[\dot{q}][\dot{y}] \ +D_2E(q, y)\ \ddot{y}\ .$$

As a first step in the solution of the problem we seek a control law of the form

$$u(t) = K(q(t), y^d(t), \dot{q}(t), \dot{y}^d(t), v(t), \ddot{y}^d(t)) \tag{2.4}$$

such that when u is substituted into (1.1)

$$\ddot{e}(t) = v(t). \tag{2.5}$$

That is

$$\overset{\circ\circ}{E}(q, y, \dot{q}, \dot{y}, M^{-1}(q)(PK(q, y, \dot{q}, \dot{y}, v, \ddot{y}) - F(q, \dot{q})), \ddot{y}) \ \equiv \ v\ . \tag{2.6}$$

Having achieved (2.5), various choices for $v(t)$ suggest themselves. For example, a PID (proportional, integral, derivative) controller has the form

$$v(t) = K_PE_P(q(t), y^d(t)) + K_Iz(t) + K_DE_D(q(t), y^d(t), \dot{q}(t), \dot{y}^d(t))\ , \tag{2.7}$$

$$\dot{z}(t) = E_I(q(t), y^d(t))\ , \tag{2.8}$$

$$\text{where} \qquad E_P = E_I = E\ ,\ E_D = \overset{\circ}{E}\ . \tag{2.9}$$

Using (2.7), (2.8) together with (2.5) gives a differential equation for the error of the form

$$\dot{w}(t) = Aw(t),\ w(t) = \begin{bmatrix} z(t) \\ e(t) \\ \dot{e}(t) \end{bmatrix},\ A = \begin{bmatrix} 0 & I_m & 0 \\ 0 & 0 & I_m \\ K_I & K_P & K_D \end{bmatrix} \ \text{when} \ \ K_I \neq 0\ , \tag{2.10}$$

$$\dot{w}(t) = Aw(t),\ w(t) = \begin{bmatrix} e(t) \\ \dot{e}(t) \end{bmatrix},\ A = \begin{bmatrix} 0 & I_m \\ K_P & K_D \end{bmatrix} \ \text{when} \ \ K_I = 0\ . \tag{2.10'}$$

If $K_P, K_I, K_D \in R^{m\times m}$ are chosen appropriately A is a stability matrix and $e(t) \to 0$.

Conditions for the existence of K are formulated in the simple theorem which follows. Let $E \subset R^m \times R^k$ be open set. In a robotics problem E would be chosen so that $(q, y) \in E$ avoids singularities of $E(q, y)$ having to do with the geometry and coordinate systems associated with the manipulator. Let $Q = \{q:(q, y) \in E\}$, the projection of E on R^m. The assumptions

$$M: Q \to R^{m\times m},\ F: Q \times R^m \to R^m \quad \text{are} \quad C^1 , \tag{A.1}$$

$$M(q),\ q \in Q \quad \text{and} \quad P \in R^{m\times m} \quad \text{are nonsingular} \tag{A.2}$$

$$E : E \to R^m \quad \text{is} \quad C^3 , \tag{A.3}$$

$$D_1E(q, y),\ (q, y) \in E \ , \quad \text{is nonsingular,} \tag{A.4}$$

lead to the desired result.

Theorem 2.1. Suppose (A.1)-(A.4) are satisfied. Then $K : E \times (R^m \times R^k)^2 \to R^m$ given by

$$K(q, y, \dot q, \dot y, v, \ddot y) = P^{-1}(M(q)(D_1E(q, y))^{-1} (v - \overset{oo}{E}(q, y, \dot q, \dot y, \ddot q, \ddot y) + D_1E(q, y)\ddot q) + F(q, \dot q)) \tag{2.11}$$

is C^1 and causes (2.6) to be satisfied for all $(q, y) \in E$ and $(\dot q, \dot y), (v, \ddot y) \in R^m \times R^K$

Proof: verify by substituting (2.11) into (2.6).

For special cases the formula for K becomes less formidable. For example, consider $E = E_1$ where in (1.3) $G^\dagger$ is defined on Y and $E = G^\dagger(Y) \times Y$. Then it follows from (2.11) that

$$K(q, y, \dot q, \dot y, v, \ddot y) = P^{-1}(M(q)(-v+D^2G^\dagger(y)[\dot y][\dot y] + DG^\dagger(y)\ddot y) + F(q, \dot q)) \tag{2.12}$$

By setting $\ddot q^d = D^2G^\dagger(y^d)\ [\dot y^d][\dot y^d] + DG^\dagger(y^d)\ddot y^d$ and using (2.7) with (2.9) and $K_I = 0$, the system (1.7), (1.8) is obtained. Using $E = E_2$ with G defined on Q and $E = Q \times G(Q)$ gives

$$K(q, y, \dot q, \dot y, v, \ddot y) = P^{-1}(M(q)(DG(q))^{-1}\ (-v-D^2G(q)[\dot q][\dot q] + \ddot y) + F(q, \dot q)) . \tag{2.13}$$

For specific applications in robotics $DG(q)$ and $D^2G(q)$ may become quite complex. See [8] for some details when $y = G(q) = p \in R^3$. The case $E = E_4$ is considered in Section 4.

3. Errors in the Feedback Law

The overall control law of the preceding section is determined by the functions K, E_P, E_I , E_D and the matrices K_P, K_I, K_D. Different functions, $\bar{K}$, $\bar{E}_P$, $\bar{E}_I$, $\bar{E}_D$ are used in an actual implementation because of errors due to computations, approximations and the inaccurate modelling of M, F, P, and E.

In this section the effects of such implementation errors are considered. To simplify the analysis it is assumed that $\dot{y}_d(t)$ is constant. Effects of implementation errors on the accuracy of the equilibrium solution and system dynamics in the neighborhood of the equilibrium solution are studied. The underlying perturbation theory has been used previously in a similar context [13].

The differential equations for the perturbed closed-loop system are obtained by substituting $\bar{K}$, $\bar{E}_P$, $\bar{E}_I$, $\bar{E}_D$ for K, E_P, E_I, E_D in (1.1), (2.4), (2.7), (2.8). This gives

$$\dot{x}(t) = f(x(t)),\ x(t) = \begin{bmatrix} x(t) \\ q(t) \\ \dot{q}(t) \end{bmatrix}, \tag{3.1}$$

where

$$f(\begin{bmatrix} z \\ q \\ \dot{q} \end{bmatrix}) = \begin{bmatrix} \bar{E}_I(q,\ y^d) \\ \dot{q} \\ M^{-1}(q)(Pu-F(q,\ \dot{q})) \end{bmatrix}, \tag{3.2}$$

$$u = \bar{K}^o(q,\ y^d,\ \dot{q},\ v),$$

$$v = K_P\bar{E}_P(q,\ y^d) + K_I z + K_D\bar{E}_D^o(q,\ y^d,\ \dot{q}).$$

Here, and in what follows, the superscript o denotes the substitution which results from $\dot{y}^d(t) \equiv \ddot{y}^d(t) \equiv 0$. For example, $K^o(q,\ y^d,\ \dot{q},\ v) = K(q,\ y^d,\ \dot{q},\ 0,\ v,\ 0\ )$. While it is not indicated explicitly in (3.1), f depends on y^d and $\bar{K}^o$, $\bar{E}_P$, $\bar{E}_I$,$\bar{E}_D^o$.

In general, the equilibrium solution of (3.1), $x(t) \equiv \bar{x}$ will differ from the ideal : $z(t) \equiv \dot{q}(t) \equiv 0$, $q(t) \equiv q^*$, $E(q^*,\ y^d) = 0$. Moreover, the system matrix for the linearized equations at $\bar{x}$, $Df(\bar{x})$, will have characteristic roots which differ from the ideal characteristic roots $\lambda_i(A)$, $i = 1, \ldots, 3m$. These deviations are characterized precisely by the theorem which follows.

First, it is necessary to introduce appropriate measures of functional closeness and other notation. For an open set $\chi \subset R^n$ let $\gamma(\chi, R^p)$ denote the set of all C^1 functions from χ into R^p. Let $|\,.\,|_n$ and $|\,.\,|_p$ be norms for R^n and R^p. These norms generate norms for $A \in R^{p \times n}$ and $f \in \gamma(\chi, R^p)$: $||\,A\,||_{pn} = \max \{\,|Ax|_p : x \in R^n, |x|_n = 1\}$, $||f||_{\gamma(\chi, R^p)} = \sup \{\,|f(x)|_p, \, ||Df(x)||_{pn} : x \in \chi\}$. Thus functions $f, \bar{f} \in \gamma(\chi, R^p)$ are close if and only if both the functions and their derivatives are close. Hereafter, subscripts are omitted from the norms when the meaning is clear from context.

It is also necessary to restrict attention to neighborhoods of the ideal equilibrium. Let $\eta(\varepsilon, \xi) = \{y \in R^m : |\,y-\xi\,| < \varepsilon\}$ and assume

$$\eta(\varepsilon, q^*) \times \eta(\varepsilon, y^d) \subset E\,. \tag{3.3}$$

Define

$$\gamma^o(\varepsilon) = \gamma(\eta(\varepsilon, q^*) \times \eta(\varepsilon, y^d), R^m), \tag{3.4}$$

$$\gamma^i(\varepsilon) = (\eta(\varepsilon, q^*) \times \eta(\varepsilon, y^d) \times (\eta(\varepsilon, 0))^i, R^m),\ i = 1, 2. \tag{3.5}$$

Finally, for $A \in R^{n \times n}$ let $\lambda_i(A)$, $i = 1, \ldots, n$, be the characteristic roots of A.

<u>Theorem 3.1.</u> Assume: (A.1) – (A.4) are satisfied; $(q^*, y^d) \in E$ and $E(q^*, y^d) = 0$; $\mathrm{Re}\lambda_i(A) < \sigma < 0$, $i=1, \ldots, 3m$, where A is given by (2.10); $K^o, \bar{K}^o \in \gamma^2(\varepsilon)$; $E_P, E_I, \bar{E}_P, \bar{E}_I \in \gamma^o(\varepsilon)$; E_D^o, $\bar{E}_D^o \in \gamma^1(\in)$; K^o is determined from (2.11) and E_P, E_I, E_D^o from (2.9) by appropriately restricting the functional domains. Given any $\varepsilon > 0$ satisfying (3.3), there exist $\delta_K, \delta_P, \delta_I, \delta_D > 0$ such that $||\,\bar{K}^o - K^o\,|| < \delta_K$, $||\bar{E}_P - E_P\,|| < \delta_P$, $||\bar{E}_I - E_I\,|| < \delta_I$, $||\bar{E}_D^o - E_D^o\,|| < \delta_D$ imply (3.1) has a unique equilibrium solution $z(t) \equiv \bar{z}$, $q(t) \equiv \bar{q}$, $\dot{q}(t) \equiv 0$ such that $|E(\bar{q}, y^d)| < \varepsilon$ and $|\bar{z}| < \varepsilon$. Furthermore, it is possible to choose $\delta_K, \delta_P, \delta_I, \delta_D$ so that $\mathrm{Re}\lambda_i(Df(\bar{x})) < \sigma$, $i=1, \ldots, 3m$ where $\bar{x} = (\bar{z}, \bar{q}, 0)$.

Remark 3.1. The theorem shows that both the accuracy of the equilibrium solution and the dynamic behavior of the linearized closed-loop system can be maintained if $\overline{K} - K$, $\overline{E}_P - E_P$, $\overline{E}_I - E_I$, $\overline{E}_D - E_D$ are sufficiently small in the indicated way. In practice the δ_K, δ_P, δ_I, δ_D and the norms are difficult to evaluate, so the theorem should be viewed as a qualitative rather than quantitative result.

Remark 3.2. The choice of the tolerances δ_K, δ_P, δ_I, δ_D depends on y^d. Thus, the tolerances may be smaller for some y^d than others. The same can be said for σ and ε. The tolerances are likely to be larger if σ = max $\{\lambda_i(A) : i = 1, \ldots, 3m\}$, and ε are larger.

Remark 3.3. The condition $Re\lambda_i(Df(\overline{x})) < \sigma < 0$, $i = 1, \ldots, 3m$, implies that the equilibrium point $\overline{x}$, of the nonlinear system (3.1), is isolated and asymptotically stable (see [14]). The rapidity of approach to the equilibrium point depends on σ, which can be influenced by the choice of K_P, K_I, K_D.

Remark 3.4. The theorem could be generalized to account for errors in K_P, K_I, K_D. Usually such errors are insignificant.

Remark 3.5. Note that $\overline{q}$ depends only on $\overline{E}_I(\overline{q}, y^d) = 0$. The integral control automatically corrects for errors produced by $\overline{K}$, $\overline{E}_P$, $\overline{E}_D$. Generally this necessitates $\overline{z} \neq 0$. To assure $E(\overline{q}, y^d) = 0$ it is sufficient to require $\overline{E}_I(q, y^d) = 0$ for all $(q, y^d) \in E$ such that $E(q, y^d) = 0$.

Remark 3.6. The theorem applies when $K_I = 0$. It is only necessary to define A by (2.10.1) and eliminate all reference to $z(t)$, $\overline{z}$, δ_I and $\overline{E}_I$.

Remark 3.7. If $K_I = 0$ the conditions for $E(\overline{q}, y^d) = 0$ are more severe than in Remark 3.5. Suitable condition are: $\overline{K}(q, y^d, 0, 0, 0, 0) = K(q, y^d, 0, 0, 0, 0)$, $\overline{E}_P(q, y^d) = E_P(q, y^d)$, $\overline{E}_D(q, y^d, 0, 0) = E_D(q, y^d, 0, 0)$ for all $(q, y^d) \in E$ such that $E(q, y^d) = 0$.

The proof of Theorem 3.1 follows from the following theorem which is a simple rewording of results found in Chapter 16 of [14].

Theorem 3.2. Suppose $\chi \subset R^n$ is an open set and $g \in \gamma(\chi, R^n)$. For $x^* \in \chi$ assume $g(x^*) = 0$ and $Dg(x^*)$ is nonsingular. Then for any $\varepsilon > 0$ satisfying $\eta(\varepsilon, x^*) \subset \chi$ there exists $\delta > 0$ such that $|| \bar{g} - g ||_{\gamma(\eta(\varepsilon, x^*), R^n)} < \delta$ implies $\bar{g}(x) = 0$ has a unique solution $x = \bar{x}$ satisfying $|\bar{x} - x^*| < \varepsilon$. Furthermore, if $\mathrm{Re}\lambda_i(Dg(x^*)) < \sigma$, $i = 1, \ldots, n$, it is possible to choose δ so that in addition $\mathrm{Re}\lambda_i(D\bar{g}(\bar{x})) < \sigma$, $i = 1, \ldots, n$.

To apply this theorem the state x in (3.1) is related to $w = (z, e, \dot{e})$ by

$$w = T(x) = T(\begin{bmatrix} z \\ q \\ \dot{q} \end{bmatrix}) = \begin{bmatrix} z \\ E(q, y^d) \\ D_1E(q, y^d)\dot{q} \end{bmatrix} . \tag{3.6}$$

For all $z, \dot{q} \in R^m$, $(q, y^d) \in E$ it follows from (A.4) that $DT(x)$ is nonsingular. Thus by the implicit function theorem (3.1) can be written in terms of w by

$$\dot{w}(t) = g(w(t)) , \tag{3.7}$$

where: $g: \chi \to R^{3m}$ is C^1, $\chi \subset R^{3m}$ is open, $0 \in \chi$ and

$$g(w) = (DT(T^{-1}(w)))^{-1} f(T^{-1}(w)) . \tag{3.8}$$

Clearly, g depends on $\bar{K}, \bar{E}_P, \bar{E}_I, \bar{E}_D$. When $\bar{K} \neq K$, $\bar{E}_P \neq E_P$, $\bar{E}_I \neq E_I$, $\bar{E}_D \neq E_D$ we indicate this by writing $g = \bar{g}$; when there are no errors it follows from Section 2 that $g(w) = Aw$. Because A is nonsingular (this follows from $\lambda_i(A)<0$, $i = 1, \ldots, 3m$), $g(w) = Aw = 0$ implies $w = x^* = 0$. By choosing $\delta_K, \delta_P, \delta_I, \delta_D$ appropriately it follows that $|| \bar{g} - g ||<\delta$ for any $\delta > 0$. Thus Theorem 3.2 gives the equilibrium results of Theorem 3.1 The characteristic root result follows from the fact that $Dg(T(\bar{x}))$ and $Df(\bar{x})$ are similar matrices.

4. An Example

In this section a mechanical manipulator with error measure E_4 is considered. The feedback law given by (2.4), (2.7)-(2.9), (2.11) is determined. The form of the law is examined and its relationship to the control law obtained in [2] is discussed. Because of limited space, it is necessary to simplify notation and omit details from the following developments.

Let $y = (p, \pi)$ where $\pi = (n, s, a)$. Then (1.2) can be written as

$$p = G_p(q)\ , \ \pi = G_\pi(q), \tag{4.1}$$

where $G_p : R^6 \to R^3$, $G_\pi : R^6 \to R^9$. Let $\omega(t), \omega^d(t) \in R^3$ denote the angular velocity of the sets (n, s, a), (n^d, s^d, a^d). Then there exist $H_p(q), H_\omega(q), \overset{o}{H}_p(q, \dot{q})$, $\overset{o}{H}_\omega(q, \dot{q}) \in R_{3\times 6}$ such that

$$\dot{p} = H_p(q)\dot{q}, \ \ \ddot{p} = H_p(q)\ddot{q} + \overset{o}{H}_p(q, \dot{q})\dot{q}, \tag{4.2}$$

$$\omega = H_\omega(q)\dot{q}, \ \ \dot{\omega} = H_\omega(q)\ddot{q} + \overset{o}{H}_\omega(q, \dot{q})\dot{q}\ . \tag{4.3}$$

See, for example, [2].

The error measure is written

$$E(q, y^d) = \begin{bmatrix} E_p(p, p^d) \\ E_\pi(\pi, \pi^d) \end{bmatrix}, \ E_p(p, p^d) = p^d - p\ , \tag{4.4}$$

$$E_\pi(\pi, \pi^d) = \frac{1}{2}(n \times n^d + s \times s^d + a \times a^d)\ .$$

Straightforward differentiation shows that

$$\overset{o}{E}_p(p, p^d, \dot{p}, \dot{p}^d) = \dot{p}^d - \dot{p}\ , \tag{4.5}$$

$$\overset{o}{E}_\pi(\pi, \pi^d, \omega, \omega^d) = L'(\pi, \pi^d)\omega^d - L(\pi, \pi^d)\omega\ , \tag{4.6}$$

$$\overset{oo}{E}_p(p, p^d, \dot{p}, \dot{p}^d, \ddot{p}, \ddot{p}^d) = \ddot{p}^d - \ddot{p}\ , \tag{4.7}$$

$$\overset{oo}{E}_\pi(\pi, \pi^d, \omega, \omega^d, \dot{\omega}, \dot{\omega}^d) = L'(\pi, \pi^d)\dot{\omega}^d + \tag{4.8}$$

$$\overset{o}{L}'(\pi, \pi^d, \omega, \omega^d)\omega^d - L(\pi, \pi^d)\dot{\omega} - \overset{o}{L}(\pi, \pi^d, \omega, \omega^d)\omega$$

Here $L(\pi, \pi^d)$, $\overset{o}{L}(\pi, \pi^d, \omega, \omega^d) \in R^{3\times 3}$, the $'$ denotes matrix transpose and

$$L = -\frac{1}{2}(N^d N + S^d S + A^d A) \tag{4.9}$$

$$\overset{\circ}{L} = -\frac{1}{2}(N^d\dot{N} + \dot{N}^dN + S^d\dot{S} + \dot{S}^dS + \dot{A}^dA + A^d\dot{A}) \tag{4.10}$$

where N^d, S^d, A^d, $\dot{N}^d$, $\dot{S}^d$, $\dot{A}^d$, N, S, A, $\dot{N}$, $\dot{S}$, $\dot{A} \in R^{3\times3}$ are skew symmetric and are obtained from n^d, s^d, a^d, $\dot{n}^d$, $\dot{s}^d$, $\dot{a}^d$, n, s, a, $\dot{n}$, $\dot{s}$, $\dot{a}$. For example,

$$N = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}, \quad n = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}. \tag{4.11}$$

The derivative vectors such as $\dot{n}$ are given in a simple way; for example, if ω has components ω_1, ω_2, ω_3 then $\dot{n}_1 = \omega_2 n_3 - n_2\,\omega_3$, $\dot{n}_2 = \omega_3 n_1 - n_3\omega_1$, $\dot{n}_3 = \omega_1\, n_2 - n_1\omega_2$.

Substituting (4.2), (4.3) into (4.5)-(4.8) and omitting the arguments of the functions gives

$$D_1E = \begin{bmatrix} -H_q \\ -LH_\omega \end{bmatrix}, \quad \overset{\circ}{E} = \begin{bmatrix} \dot{p}^d - H_p\dot{q} \\ L'\omega^d - LH_\omega\dot{q} \end{bmatrix}, \tag{4.12}$$

$$\overset{\circ\circ}{E} - D_1E\ddot{q} = \begin{bmatrix} \ddot{p}^d - \overset{\circ}{H}_p\dot{q} \\ L'\dot{\omega}^d + \overset{\circ}{L}'\omega^d - L\overset{\circ}{H}_\omega\dot{q} - \overset{\circ}{L}H_\omega\dot{q} \end{bmatrix}$$

When the formulas (4.2) are substituted into (2.9) and (2.11), the expressions for K, E_P, E_I, E_D are obtained. If E is chosen to avoid "geometric singularities", it can be verified that assumptions (A.1)-(A.4) are satisfied. Thus, Theorem 2.1 applies. Moreover, implementation errors can be tolerated in the sense of Theorem 3.1. We now examine further the matrices L and $\overset{\circ}{L}$ which appear in the feedback law.

When $E_\pi = E_\pi(\pi, \pi^d)$ and $\Delta\omega = \omega^d - \omega$ are small the formulas for L and $\overset{\circ}{L}$ have simple approximations:

$$L = I_3 + \frac{1}{2}E_\pi + R(\pi, \pi^d), \tag{4.13}$$

$$\overset{\circ}{L} = \frac{1}{2}(\Omega^d E_\pi - E_\pi \Omega^d + \Delta\Omega) + \overset{\circ}{R}(\pi, \pi^d, \omega, \omega^d). \tag{4.14}$$

Here E^π, Ω^d, $\Delta\Omega$ are skew symmetric matrices corresponding to E_π, ω^d, $\Delta\omega$ and R, $\overset{\circ}{R}$ satisfy: $|| R(\pi, \pi^d) || < \eta |E_\pi(\pi, \pi^d)|^2$, $||\overset{\circ}{R}(\pi, \pi^d, \omega, \omega^d)|| < \eta |E_\pi(\pi, \pi^d)| \cdot (|E_\pi(\pi, \pi^d)| + |\Delta\omega|)$, where ω^d is bounded and η is sufficiently large. Thus good approximations for L and $\overset{\circ}{L}$ which are computationally simple, may be obtained by setting $R = \overset{\circ}{R} = 0$ in (4.13) and (4.14).

Now consider the even simpler approximations $L = I_3$ and $\overset{\circ}{L} = 0$. Using these substitutions in the formulas for K and E_D gives new functions $\tilde{K}$ and $\tilde{E}_D$. Note E_P and E_I remain unchanged and are equal to E. assume $y^d(t)$ is constant. Then a somewhat tedious development shows that the system (2.10) (or (2.10') if $K_I = 0$) is replaced by

$$\dot{w}(t) = Aw(t) + A_1(w(t))w(t), \tag{4.15}$$

where: $A_1 : \eta \to R^{9\times 9}$ is C^1, $\eta \subset R^9$ is a neighborhood of the origin, and $A_1(0) = 0$.

Thus the approximate control law $\tilde{K}$, E_P, E_I, $\tilde{E}_D$, maintains the original equilibrium $w = 0$ (or $z = 0$, $E(q, y^d) = 0$, $\dot{q} = 0$). Furthermore, when the system (4.5) is linearized at $w = 0$, the system (2.10) is obtained. If A is a stability matrix, this implies that the equilibrium solution of (4.15) is (locally) asymptotically stable.

The approximate control law has another interesting feature. When $K_I = 0$, $K_P = -k_2 I_6$, $K_D = -k_1 I_6$ it gives the control law of [2]. This places a new perspective on the control law in [2] and provides a rigorous proof of its local stability for $y^d(t)$ constant.

There may be good reason for using the exact control law K, E_P, E_I, E_D. The number of additional computer operations is not that large compared with the approximate law $\tilde{K}$, E_P, E_I, $\tilde{E}_D$. And, unlike (4.15), the error equation (2.10) is linear. Thus error convergence is assured even though the initial error may be large.

5. Conclusion

We have presented two rather simple results concerning the choice of feedback laws for the system (1.1) with error measure $e(t) = E(q(t), y^d(t))$. The first, Theorem 2.1 concerns the existence and form of the feedback function, K in (2.4), which gives the error equation (2.5). Once (2.5) has been obtained a variety of schemes, of which (2.7), (2.8) is a simple example, can be used to complete the overall feedback law. The second result, Theorem 3.1, shows that such designs are robust in the sense that accuracy and stability are not badly upset if implementation errors are kept sufficiently small. While Theorem 3.1 applies to the feedback law described by (2.4), (2.7)-(2.9), (2.11), it is clear from Theorem 3.2 that similar results can be obtained for more complex situations.

The main contribution of the paper is a systematic approach to feedback system design which is based on a general error expression. The value of the systematic approach is confirmed for the error expression E_4, where the resulting control law can be viewed as a generalization of the one obtained in [2]. Other error expressions have been treated with similar success by the authors.

The importance of the general solution formulas should not be overlooked. Since they are not burdened by the complex details of a specific physical problems, they allow one to focus on conceptual issues apart from the multitudinous details of implementation. For example, with integral control it is seen that high accuracy for constant (or near constant) inputs depends only on $E_I(q,y_d)$. Thus , if possible, E_I should be determined by direct physical measurement (end-effector sensors). On the other hand E_p and E_D may be determined indirectly from q and $\dot{q}$ using formulas based on a reasonably accurate model of the actual physical system.

6. References

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