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COMPUTER TECHNIQUES FOR THE EVALUATION
OF DETECTOR PERFORMANCE

by

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ABSTRACT

In many situations, the theory of signal detectability lacks a connection between the theoretical aspects and the practical implementation of a detector. This condition exists because there are no general techniques for evaluating the detector performance. This report contains a collection of available techniques which have been adapted for the computer evaluation of a large class of detectors.

In addition to the classical approximations, the Pearson system of frequency curves is integrated into the computer programs. The Pearson system yields a closed-form approximation to the detector performance, based on the moments of the signal and noise distribution functions. Such closed-form approximations enable the user to evaluate the effects of signal-to-noise ratio, detector nonlinearities, and filter bandwidth.



INTRODUCTION TO THE EVALUATION PROBLEM

1.1 Introduction

This report is concerned with several techniques feasible for the computer evaluation of receivers which are to detect signals in the presence of additive noise. The general problem studied is the evaluation of a fixed-time detection receiver which tests a sample function of a stochastic process and decides whether the sample function is noise alone or contains a signal component as well as noise. A block diagram of the receiver complex is shown in Fig. 1.

The symbol N represents the null hypothesis that the received sample function $y(t)$ is noise alone, while SN represents the hypothesis that $y(t)$ is a sample function of signal plus noise:

$$\begin{aligned} N: y(t) &= n(t), \\ SN: y(t) &= n(t) + x(t). \end{aligned} \tag{1.1}$$

The symbol $x(t)$ denotes a sample function of a stationary stochastic process, \mathbf{X} , which describes the ensemble of transmitted signals, while $n(t)$ denotes a sample function from a stationary random process, N , describing the noise ensemble. Moreover, it will always be assumed that the random processes \mathbf{X} and N are statistically independent processes.

Although \mathbf{X} is the ensemble of transmitted signals, the set of signals may contain only one known element. This one-element set gives rise to "signal-known-exactly" (SKE) detection problems. Another class of detection problems arises when the signal is known exactly, but the characteristics of the signal are randomly disturbed by the channel. The channel's influence on the signal can be considered as some mapping from the signal space to a channel space which has a stochastic description. In this event, the space \mathbf{X} will represent the ensemble of channel-disturbed signals instead of the transmitted signals. This provision for a dual role of the transmission space, \mathbf{X} , allows a unified treatment of random or non-random signal detection and evaluation problems.

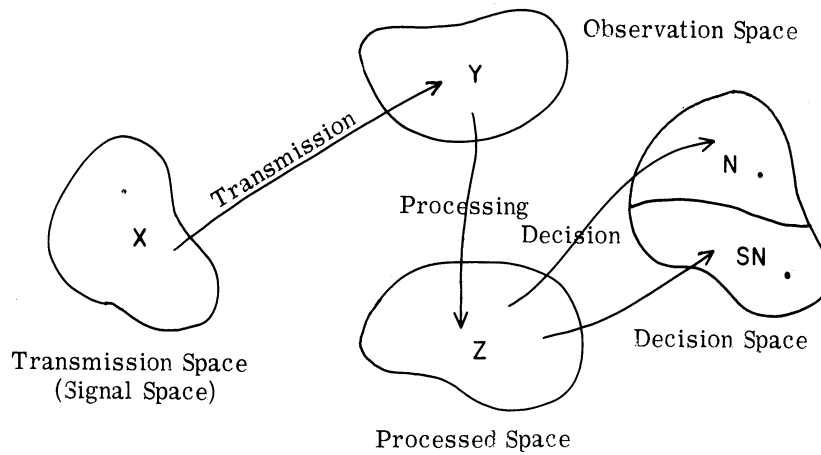
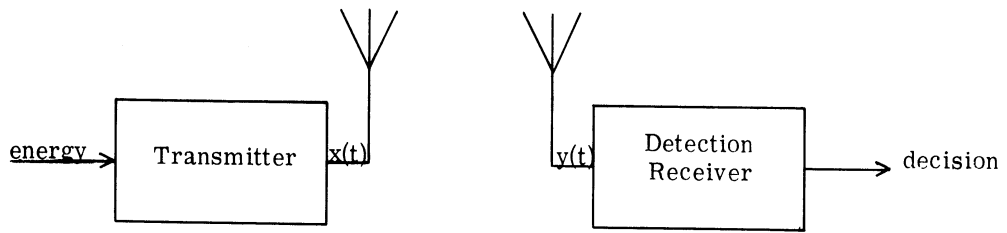


Fig. 1. Transmitter-receiver complex

The function of the receiver is to perform a statistical test on the received sample function $y(t)$ and to make a decision (presence or absence of signal) based on this test in such a manner that some index of performance is optimized. The selected index determines the statistical test the receiver must perform, and thus, the general nature of the receiver.

1.2 Observation Space Y

It is assumed, in this report, that Y is the set of integrable functions defined over the interval of time, $[0, T]$, with the property that any sample function $y(t)$ has a Fourier series expansion with vanishingly small coefficients outside a frequency band of width W . One can then choose $2WT$ time samples of $y(t)$ such that

$$y(t) = \sum_{k=1}^{2WT} y(t_k) \phi_k(t), \quad (1.2)$$

where the functions $\sqrt{2W}\phi_k(t)$, $k = 1, 2, \dots, 2WT$, form a complete orthonormal set on the interval $[0, T]$ with respect to the elements of Y (Ref. 1). Moreover, the coefficients of the expansion are assumed to be statistically independent. Any element of Y can be considered as a point in a $2WT$ Euclidean space, with coordinates equal to the sampled values of $y(t)$. Hereafter, the vector of sampled values will be denoted by

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{2WT} \end{bmatrix} \quad (1.3)$$

1.3 Noise Space N and Signal Space X

The spaces N and X are subspaces of the observation space Y . Hence, every element in N will be denoted by a vector

$$n = \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_{2WT} \end{bmatrix}, \quad (1.4)$$

and associated with N is a probability density function (p. d. f.) denoted by $f_N(n)$. Similarly, the elements of X are denoted by

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{2WT} \end{bmatrix}, \quad (1.5)$$

with associated p. d. f. , $f_{\mathbf{X}}(x)$.

Moreover, it follows that the space \mathbf{Y} is the union of the space of the vector sum $\mathbf{X} + \mathbf{N}$ and the space \mathbf{N} , and \mathbf{Y} has associated with it the conditional density functions $f_{\mathbf{Y}}(y|\mathbf{SN})$ or $f_{\mathbf{Y}}(y|\mathbf{N})$, depending on the hypothesis under consideration.

1.4 Observation, Detection, and Evaluation Procedure

An idealized model will be assumed for the detection problem. The observer knows a priori that either noise or signal plus noise is present for the entire time interval $[0, T]$. At the end of the time interval, the observer decides which condition he thinks actually existed. Decision theory (Refs. 1, 2) is largely concerned with the choice of a receiver and a decision device which will process the observation, y , to optimize the performance index of the decision. A large class of performance criteria leads the observer to use a non-random decision device which partitions the observation space by using the likelihood ratio of the observation y . For simple-hypothesis testing, the likelihood¹ ratio satisfies performance criteria such as:

- (1) Minimum average risk (Bayes' Class),
- (2) Maximum a posteriori probability,
- (3) Neyman-Pearson Criteria,
- (4) Ideal observer.

The likelihood ratio is defined as

$$\ell(y) = \frac{f_{\mathbf{Y}}(y|\mathbf{SN})}{f_{\mathbf{Y}}(y|\mathbf{N})}, \quad (1.6)$$

where y is the $2WT$ observation vector,

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{2WT} \end{bmatrix} \quad (1.7)$$

¹ Birdsall (Ref. 3) has shown that a likelihood ratio receiver should be used under the very general condition where the observer prefers correct decisions over incorrect decisions.

and $f_Y(y|\dots)$ is the conditional probability density of the event y given the event \dots .²

The likelihood ratio is, then, a mapping of the 2WT observation space Y into a one-dimensional space where a randomized decision function is applied. This decision function partitions, according to costs and the a priori probabilities, the one-dimensional space into regions corresponding to either the decision N or SN .

One might then drop the problem with the comment, "Well, that's the best one can do." However, the question remains, "How good is that?" Indeed, Birdsall and Nolte (Ref. 4) have shown that the design of the likelihood receiver (one which performs the likelihood transformation) is not easy. The required hardware is expensive and the receiver performance may well be sensitive to slight changes introduced by hardware aging or partial ignorance of the channel or signal space statistics. Another receiver, nonoptimum in the sense of the listed performance criteria, may be cheaper to build and, in the long run, more reliable to operate. Does one gain that much by using the likelihood receiver?

The problem is complicated further when the observer insists on meeting certain detection requirements. A properly interpreted evaluation of the receiver may tell the interested party working in the assumed environment that he is unable to keep his losses below a certain point unless, for example, the transmitted signal energy is increased.

The evaluation of detection receivers is portrayed graphically by curves of the receiver operating characteristic (ROC). These curves display the detection versus false-alarm probability with the assumed environment as a parameter. Figure 2 illustrates the ROC curves for the signal-known-exactly detection problems. Peterson, et al., (Ref. 1) and Birdsall and Ristenbatt (Ref. 5) have demonstrated the significance and use of ROC curves³ as criteria for receiver performance in certain well-known detection problems.

² Note that this definition of the likelihood ratio differs from the definition of the generalized likelihood given by Middleton (Ref. 2). The generalized likelihood ratio of Middleton scales the likelihood ratio by the ratio of the a priori probabilities of the hypotheses SN and N .

³ There are several other graphical representations of the receiver evaluation, such as "betting curves" or the "operating characteristics", but these portrayals can be obtained from the ROC curves.

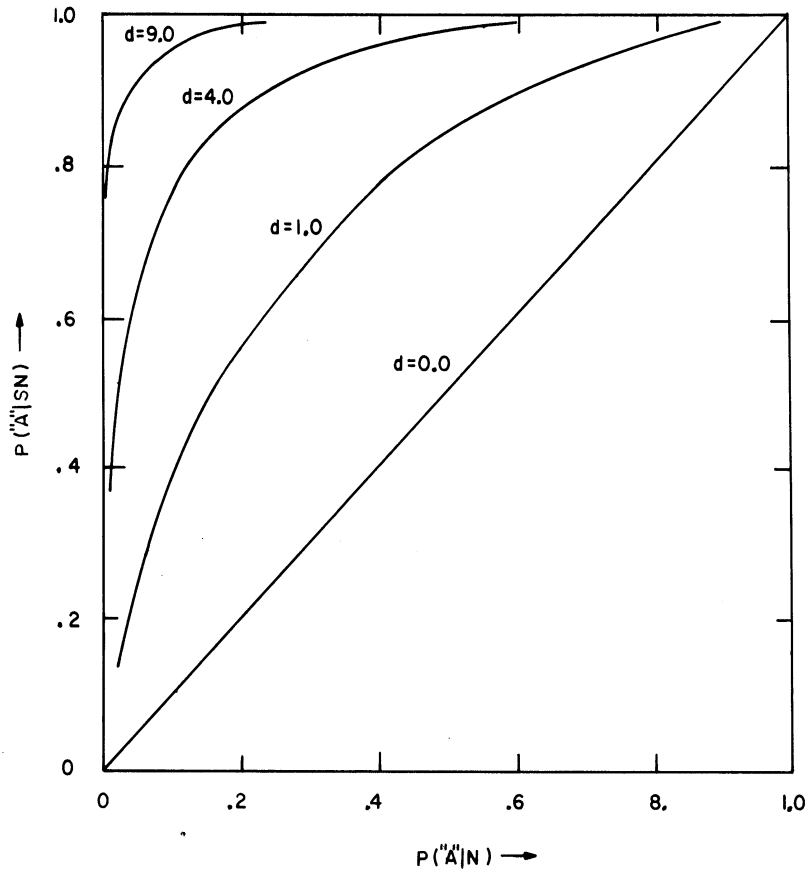


Fig. 2. Three normal ROC curves on linear paper with parameter d .

The evaluation seems to be straightforward. For the general receiver of Fig. 1, one seeks the distribution functions of the receiver output under the two hypotheses,

$$f_Z(z|N), \quad (1.8a)$$

$$f_Z(z|SN), \quad (1.8b)$$

where $z = \ell(y)$ for the likelihood receiver.

1.5 Evaluation

Two major difficulties are encountered when an analytic theory is applied to the problem of detection performance in a practical case:

The first problem is the description of the receiver in a form useful for performance evaluation. That is, a form of likelihood-ratio mapping must be deduced which is useful and suitable for further computation. Since the likelihood ratio is defined

to be

$$\ell(y) = \frac{f_Y(y|SN)}{f_Y(y|N)} \quad (1.9)$$

and the noise is considered to be additive, then

$$\ell(y) = \frac{\int f_N(y-x) f_X(x) dx}{f_Y(y|N)} \quad (1.10)$$

Except for a few singular cases, the integral of Eq. 1.10 cannot be evaluated in terms of elementary functions.

Second, a given receiver's performance, both in the environment for which it was designed and in some other environment, must be evaluated, that is, the distribution functions of the receiver output must be determined under the two hypotheses. An examination of the extensive body of literature concerning the distributions of the outputs of nonlinear devices should easily convince the reader of the potential problems faced in determining the distributions of the receiver output.

These two problems become increasingly more difficult when one realizes that, in the practical situation, one cannot start out with a "neat" closed-form expression for the Signal Space or Noise Space distributions. Instead, provisions should be made in characterizing these distributions to permit successful performance using raw data obtained from channel experiments.

As has already been discussed in Section 1.4, evaluation is an integral part of the practical application of detection theory. In fact, evaluation is the link between the theory and practice. This report will introduce several techniques in an attempt to forge this link.

A CLASSICAL APPROACH

2.1 Introduction

Before proceeding, we introduce the notation used throughout this report. A subsequent part of this chapter deals with some elementary statistical ideas, while the latter part of the chapter introduces some new views on the design of a likelihood-ratio receiver.

2.2 Moments

(a) Given a function $g(x)$ of a random variable x , where x has a p. d. f. denoted by $f_x(s)$, the k -th moment of g is defined by:

$$\alpha_k(g) \triangleq \int_{\mathbf{X}} [g(s)]^k f_x(s) ds. \quad (2.1)$$

In particular, for $g(x) = x$, the first moment of g ,

$$\alpha_1(g(x)) = \alpha_1(x), \quad (2.2)$$

is the mean of the random variable x .

(b) Given a function $g(x)$ of a random variable x , where x has a p. d. f., $f_x(s)$, the k -th central moment of g is defined by

$$\mu_k(g) \triangleq \int_{\mathbf{X}} [g(s) - \alpha_1(g)]^k f_x(s) ds. \quad (2.3)$$

It follows from the linearity property of integrals that

$$\mu_1 = 0. \quad (2.4)$$

The second central moment of g is called the variance of g and is denoted by the symbol $\sigma^2(g)$, where

$$\sigma^2(g) \triangleq \mu_2(g) = \alpha_2(g) - \alpha_1^2(g). \quad (2.5)$$

2.3 Characteristic Function

The characteristic function of a random variable x is defined by

$$M_x(it) \triangleq \int_{\mathbf{X}} e^{its} f_x(s) ds. \quad (2.6)$$

$M_x(it)$ is the Fourier transform of the p. d. f. of the random variable x . Since, by definition,

$$f_x(s) \geq 0 \text{ for } s \in \mathbf{X}, \quad (2.7)$$

$$\int_{\mathbf{X}} f_x(s) ds = 1, \quad (2.8)$$

and since the Fourier transform of an absolutely integrable function exists (Ref. 6), then the characteristic function always exists. The inverse transform of the characteristic function is then the p. d. f. of the random variable x ; that is,

$$f_x(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} M_x(it) e^{-its} dt. \quad (2.9)$$

Given a function $g(x)$ of a random variable x , the characteristic function of g is

$$M_g(it) = \int_{\mathbf{X}} e^{itg(s)} f_x(s) ds, \quad (2.10)$$

so that the p. d. f. of g can be obtained from the characteristic function,

$$f_g(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itg} M_g(it) dt. \quad (2.11)$$

If the integrand of Eq. 2.10 is expanded in a Maclaurin series, and the integration is performed term-by-term, then,

$$M_g(it) = 1 + \alpha_1(g) (it) + \dots + \alpha_n(g) \frac{(it)^n}{n!} + \dots \quad (2.12)$$

Whenever the p. d. f. of g has moments all of order less than n , the characteristic function can be expanded in a Maclaurin series with a remainder of order n . Moreover, if a density function has moments of all order as n approaches infinity, and if Eq. 2.11 converges for some $t > 0$, then the sequence of moments $\{\alpha_k(g)\}$ defines a unique class of density functions

which can differ at most by a set of points of measure zero (see Ref. 7).

2.4 Cumulants or Semi-Invariants

The cumulant function of a random variable x is defined by

$$C_x(it) \triangleq \ln M_x(it). \quad (2.13)$$

Using the Maclaurin expansion for $\ln(1+Z)$ where

$$Z = \sum_{k=1}^{\infty} \frac{\alpha_k(x) (it)^k}{k!} = M_x(it) - 1, \quad (2.14)$$

then

$$C_x(it) = \sum_{\nu=1}^{\infty} \frac{K_{\nu}(x) (it)^{\nu}}{\nu!} + o(t_k^{\nu}), \quad (2.15)$$

where $K_{\nu}(x)$ is defined as the ν -th cumulant of x .

Formally, one can obtain the relation between K_{ν} and $\alpha_1, \dots, \alpha_{\nu}$ by equating the coefficients of (2.1.2) to the coefficients in an expansion

$$\exp C_x(it) = \exp \sum_{\nu} K_{\nu} \frac{(it)^{\nu}}{\nu!}. \quad (2.16)$$

For example:

$$\begin{aligned} K_1(x) &= \alpha_1(x), \\ K_2(x) &= \mu_2(x), \\ K_3(x) &= \mu_3(x), \\ K_4(x) &= \mu_4(x) - 3\mu_2^2(x). \end{aligned} \quad (2.17)$$

The use of the characteristic and cumulant functions facilitates the calculation of moments for the sum of N independent random variables. For example, letting

$$y = \sum_{j=1}^N x_j, \quad x_j \text{ independent}, \quad (2.18)$$

then

$$M_y(it) = \int_{\mathbf{X}_1} \int_{\mathbf{X}_2} \dots \int_{\mathbf{X}_N} f_{\mathbf{X}}(s_1, s_2, \dots, s_N) ds_1 ds_2 \dots ds_N. \quad (2.19)$$

Since the x_j are independent,

$$f_{\mathbf{X}}(s_1, s_2, \dots, s_N) = \prod_{j=1}^N f_{x_j}(s_j), \quad (2.20)$$

the characteristic function of y is given by

$$M_y(it) = \prod_{j=1}^N M_{x_j}(it). \quad (2.21)$$

The cumulant function of y is given by

$$C_y(it) = \ln M_y(it) = \sum_{j=1}^N C_{x_j}(it). \quad (2.22)$$

Therefore, the ν -th cumulant of y equals the sum of the ν -th cumulants of x_j ($j = 1, \dots, N$),

$$K_{\nu}(y) = \sum_{j=1}^N K_{\nu}(x_j). \quad (2.23)$$

In particular, if the x_j are drawn from the same distribution,

$$K_{\nu}(y) = NK_{\nu}(x_1). \quad (2.24)$$

Thus, if the moments of x_i are known, one can obtain the moments of y using the addition property of the cumulant function as the intermediate step in the calculation.

2.5 Central Limit Theorem

Define a normal probability density function with zero mean and unit variance

$[N(0, 1)]$ as

$$\phi(x) \triangleq (2\pi)^{-\frac{1}{2}} e^{-x^2/2}. \quad (2.25)$$

and a normal distribution function

$$\Phi(x) \triangleq \int_{-\infty}^x \phi(u) du. \quad (2.26)$$

If a probability density function $f(x)$ depends on a parameter n and if $f(\frac{x-m}{\sigma})$ converges to $\phi(x)$ as n approaches infinity, then x is said to be asymptotically normal $N(m, \sigma)$, which means only that (Ref. 7):

$$\lim_{n \rightarrow \infty} P(m + a\sigma < x < m + b\sigma) = \Phi(b) - \Phi(a). \quad (2.27)$$

Rather general conditions, based on Lindeberg's Theorem, for the sum of a sequence of independent variables $\{x_n\}$ to be asymptotically normal are given by Feller (Ref. 8). However, for this report, the more restricted conditions of Liapunoff (Ref. 7) are adequate. Liapunoff's conditions for a sum of independent random variables, $y = \sum x_k$, to be asymptotically normal $N(m, \sigma)$ require that

$$\lim_{n \rightarrow \infty} \left(\frac{\gamma}{\sigma} \right) = 0, \quad (2.28)$$

where

$$\gamma = \left[\sum_{k=1}^n E(|x_k - m_k|)^3 \right]^{1/3}, \quad (2.29)$$

$$\sigma = \left(\sum_{k=1}^n \sigma_k^2 \right)^{1/2}, \quad (2.30)$$

and m_k and σ_k^2 are the mean and variance of x_k , respectively.

In the more restricted case, where the x_k are independent samples of the same distribution, it is sufficient for the x_k to have finite second moments (Ref. 9) for $y = \sum x_k$ to be asymptotically normal.

As a warning, it should be pointed out that even for very large n , the approximation to $\sum_{k=1}^n x_k$ by a normal density function is poor on the tails of the distribution.⁴

2.6 Classic Series

In this section, two series useful for the approximation of probability density functions in terms of the normal function and its derivatives are given.

2.6.1 Gram-Charlier A Series

Define

$$\phi^n(x) \triangleq \frac{d^n}{dx^n} \phi(x), \quad (2.31)$$

where

$$\phi(x) = (2\pi)^{-1/2} \exp \left\{ -\frac{x^2}{2} \right\}. \quad (2.32)$$

⁴ For an interesting discussion of this point with respect to binomial and Poisson approximation by the normal function, see Ref. 10.

Then,

$$\begin{aligned}\phi^1(x) &= -x \phi(x), \\ \phi^2(x) &= (x^2 - 1) \phi(x), \\ &\vdots \\ \phi^{(n)}(x) &= (-1)^n H_n(x) \phi(x),\end{aligned}\tag{2.33}$$

where $H_n(x)$ is the n -th Hermite polynomial (Refs. 11, 12).

The sequence of functions $\{H_0, H_1, \dots\}$ form a bi-orthogonal set in $(-\infty, \infty)$ with the sequence $\{\phi^{(0)}, \phi^{(1)}, \dots\}$; that is, $\phi^{(n)}(x)$ and $H_n(x)$ are:

1. real and continuous on the entire line,
2. no one of them identically zero on the real line,
3. $\int_{-\infty}^{\infty} \phi^{(n)}(x) H_m(x) dx = (-1)^m m \delta_{mn}$ ⁵

That is, the set of functions $\left\{ \frac{H_m(x)}{n!} \right\}$ forms an orthonormal set with respect to a weighting function $\phi(x)$ in $(-\infty, \infty)$.

It is then possible⁶ to expand an arbitrary p. d. f. into a series of the form

$$f_x(x) = \sum_{k=0}^{\infty} C_k \phi^{(k)}(x),\tag{2.34}$$

where C_k are the coefficients of a generalized Fourier Series (Ref. 13), that is,

$$C_k = \frac{(-1)^k}{k!} \int f_x(x) H_k(x) dx.\tag{2.35}$$

Instead of expanding $f_x(x)$, consider the expansion of the density function for a standard random variable

$$z = \frac{x - \alpha_1(x)}{\sigma(x)}\tag{2.36}$$

so that $\alpha_1(z) = 0$ and $\sigma^2(z) = 1$ (See (2.1) and (2.5)). Then,

$$f_z(z) = \sum_{k=0}^{\infty} d_k \phi^{(k)}(z),\tag{2.37}$$

⁵ 1 for $m = n$, $\delta_{mn} = 0$, otherwise.

⁶ Sufficient conditions for the convergence of the series to $f(x)$ are given by Fisher (Ref. 11) and Cramer (Ref. 7).

where

$$d_k = \frac{(-1)^k}{k!} \int f_z(z) H_k(z) dz . \quad (2.38)$$

Substitution of the Hermite Polynomials into (2.8) reveals that

$$\begin{aligned} d_0 &= 1, \\ d_1 &= d_2 = 0. \end{aligned} \quad (2.39)$$

Thus,

$$f_z(z) = \phi(z) + \sum_{k=3}^{\infty} d_k \phi^{(k)}(z), \quad (2.40)$$

where from (2.3), it is apparent that the d_k are sums of the central moments of $f_z(z)$ (central moments since $\alpha_1(z) = 0$). Using the transformation theory of random variables (Refs. 9, 14), one obtains

$$f_x(x) = \frac{1}{\sigma} \phi(z) + \sum_{k=3}^{\infty} d_k \phi^{(k)}(z) , \quad (2.41)$$

where z is now given by Eq. 2.6. Moreover, since

$$\mu_k(z) = \frac{\mu_k(x)}{\sigma^k(x)} , \quad (2.42)$$

the d_k coefficients can be expressed in terms of the central moments of $f_x(x)$. In particular,

$$\begin{aligned} d_3 &= \frac{-\mu_3(z)}{3!} = -\left(\frac{\mu_3(x)}{\sigma^3(x)}\right) \left(\frac{1}{3!}\right) , \\ d_4 &= \frac{(\mu_4(z) - 3)}{4!} = \left(\frac{\mu_4(x)}{\sigma^4(x)} - 3\right) \left(\frac{1}{4!}\right) , \\ d_5 &= \frac{(-\mu_5(z) + 10\mu_3(z))}{5!} = \left(\frac{-\mu_5(x)}{\sigma^5(x)} + 10\frac{\mu_3(x)}{\sigma^3(x)}\right) \left(\frac{1}{5!}\right) . \end{aligned} \quad (2.43)$$

The expansion of (2.34) is primarily of theoretical interest since, in practice, one uses a truncated series. In this case, the C_k are the coefficients which give a least-mean-square fit (Ref. 13) for the series approximation. It has been found, in practice, that for moderately skewed distributions, the first few terms in the expansion yield a satisfactory fit to the distribution function (Ref. 10).

Let y be a sum of n independent variates from the same distribution,

$$y = \sum_{i=1}^n x_i. \quad (2.44)$$

Since y is a sum of variates, depending on a parameter n , one expects y to be asymptotically normal by the Central Limit Theorem. The Gram-Charlier expansion for the standard variable

$$z = \frac{y - n \alpha_1(x)}{n \sigma(x)} \quad (2.45)$$

is

$$f_z(z) = \sum C_k \phi^{(k)}(z). \quad (2.46)$$

The coefficients C_k can now be related to the central moments of a single variate, x_1 , through the additive property of the cumulants. Defining a standard cumulant as

$$\lambda_k = \frac{K_k(x)}{\sigma^k(x)}, \quad (2.47)$$

then the first seven coefficients of (2.46) are

$$\begin{aligned} c_0 &= 1, \\ c_1 &= c_2 = 0, \\ c_3 &= -\lambda_3 n^{-1/2} / 3!, \\ c_4 &= \lambda_4 n^{-1} / 4!, \\ c_5 &= \lambda_5 n^{-3/2} / 5!, \\ c_6 &= (\lambda_6 n^{-2} + 10 \lambda_3^2 n^{-1}) / 6!. \end{aligned} \quad (2.48)$$

An examination of Eq. 2.48 shows that the natural sequence of ordering is not the most efficient in terms of the parameter n (c_6 has a term of order n^{-1} while c_5 is of order $-3/2$).

2.6.2 Edgeworth Series

An alternative series, proposed by Edgeworth (Ref. 15), groups the terms of the Gram-Charlier series according to the power of the parameter n ,

$$\begin{aligned}
& 0, \\
& 0, 3, \\
& 0, 3, 4, 6, \\
& 0, 3, 4, 6, 5, 7, 9.
\end{aligned} \tag{2.49}$$

Hence, a three-term approximation for f_z is

$$f_z(z) = \phi(z) - \left[\frac{1}{3!} \frac{\lambda_3}{n^{1/2}} \phi^{(3)}(z) \right] + \left[\frac{1}{4!} \frac{\lambda_4}{n} \phi^{(4)}(z) + \frac{10}{6!} \frac{\lambda_3^2}{n} \phi^{(6)}(z) \right]. \tag{2.50}$$

Since the λ_k are constants for any distribution function $f_x(x)$, Eq. 2.50 is an asymptotic expansion demonstrating the convergence of $f_z(z)$ to the Normal function in terms of the parameter n . If the λ_k are replaced by the standard central moments of z , Eq. 2.50 can be written in the compact notation of Eq. 2.37

$$f_z(z) = \phi(z) - \left[\frac{1}{3!} \mu_3(z) \phi^{(3)}(z) \right] + \left[\frac{1}{4!} (\mu_4(z) - 3) \phi^{(4)}(z) + \frac{10}{6!} \mu_3^2(z) \phi^{(6)}(z) \right], \tag{2.51}$$

where

$$\mu_k(z) = \frac{\mu_k(y)}{\sigma^k(y)} \tag{2.52}$$

and

$$f_y(y) = \frac{1}{\sigma(y)} f_z \left(\frac{y - \alpha_1(y)}{\sigma(y)} \right). \tag{2.53}$$

2.7 Application of Hermite Polynomial Expansion

One useful application of the Hermite polynomial expansion is in the determination of a series expansion for the likelihood ratio. If independent sampling of the received waveform and independent signal space samples are assumed, the likelihood ratio equation 1.9 can be factored into the product of 2WT integrals

$$l(y) = \frac{\prod_{i=1}^{2WT} \int_{x_i} f_Y(y_i | x_i) f_X(x_i) dx_i}{\prod_{i=1}^{2WT} f_Y(y_i | N)}. \tag{2.54}$$

Expanding $f_Y(y_i | x_i)$ in a Taylor Series, and using the fact that the noise is additive, one obtains

$$f_{\mathbf{Y}}(y_i | x_i) = \sum \frac{a_k x_i^k}{k!}, \quad (2.55)$$

where

$$a_k = (-1)^k \left. \frac{d^k f_{\mathbf{Y}}(u)}{d u^k} \right|_{u = y_i}. \quad (2.56)$$

Thus, for a general noise density function,

$$f_{\mathbf{Y}}(y_i | \text{SN}) = \sum_{k=0}^{\infty} \frac{a_k(y_i) \alpha_k(x_i)}{k!}, \quad (2.57)$$

where $\alpha_k(x_i)$ is the k -th moment of the i -th component of the signal space. If $f_{\mathbf{Y}}(u)$ is the normal density function, then

$$a_k = (-1)^k \phi^{(k)}(y_i) = H_k(y_i) \phi(y_i). \quad (2.58)$$

Substituting (2.58) into (2.57) and dividing through by $f_{\mathbf{Y}}(y_i | N)$ which equals $\phi(y_i)$, yields

$$\ell(y_i) = \frac{f_{\mathbf{Y}}(y_i | \text{SN})}{f_{\mathbf{Y}}(y_i | N)} = \frac{\phi(y_i) \sum_{k=0}^{\infty} \frac{\alpha_k(x_i) H_k(y_i)}{k!}}{\phi(y_i)}. \quad (2.59)$$

Thus, it follows that

$$\ell(y_i) = \sum_{k=0}^{\infty} \frac{\alpha_k(x_i) H_k(y_i)}{k!}. \quad (2.60)$$

The expression in Eq. 2.60 is a series expansion for the nonlinear transformation which the receiver must perform on the input in order to process the reception in an optimum manner. This expansion is given in terms of the Hermite polynomials and the moments of the signal sample space.

A different expansion of $f_{\mathbf{Y}}(y_i | x_i)$ can be made. If this density function is expanded about the expected value of the i -th signal components $\alpha_1(x_i) \triangleq m_i$, then

$$f_{\mathbf{Y}}(y_i | x_i) = \sum \frac{a_k (x_i - m_i)^k}{k!}, \quad (2.61)$$

where, again assuming normal additive noise,

$$a_k = (-1)^k \left. \frac{d^k \phi(u)}{d u^k} \right|_{u = y_i - m_i} = H_k(y_i - m_i) \phi(y_i - m_i). \quad (2.62)$$

This leads to a likelihood ratio

$$\ell(y_i) = e^{y_i m_i} e^{-m_i^2/2} \sum_{k=0}^{\infty} \mu_k(x_i) H_k(y_i - m_i), \quad (2.63)$$

where $\mu_k(x_i)$ is the central moment of the i -th component of the signal.

Taking the natural logarithm of both sides to obtain the log-likelihood ratio,

$$L(y_i) \approx m_i y_i - \frac{m_i^2}{2} + \ln \{ 1 + \mu_2(x_i) H_2(y_i - m_i) + \dots \}. \quad (2.64)$$

Equation 2.64 can be interpreted as follows: The first two terms correspond to the log-likelihood receiver for signal-known-exactly in white Gaussian noise. The $\ln\{\}$ corresponds to a perturbation of the SKE case. An approximate receiver design for this expansion is shown in Fig. 3.

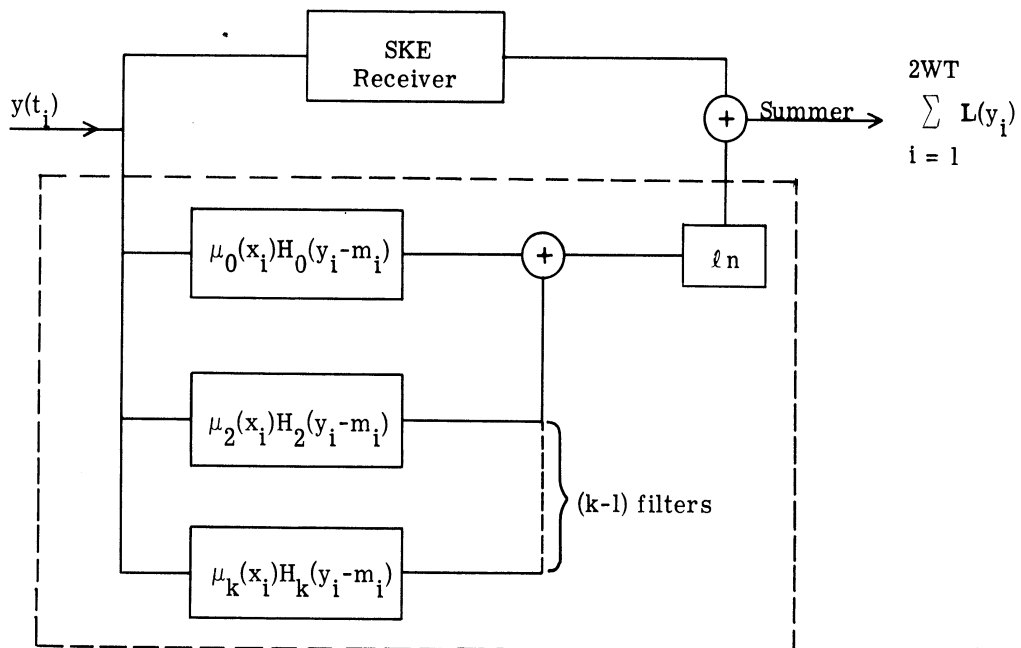


Fig. 3. Approximation to a log-likelihood ratio receiver.

The receiver enclosed by the dashed lines is the k-th order correction to the "signal-known-exactly" case. If such a receiver were implemented, several simplifications could be made. First, a time-varying bias equal to the expected signal amplitude "m", could be inserted at the input to the correction section. The (k - 1) filters are then fixed nonlinearities with time-varying scale factors⁷ adjusted according to the central moments of the signal distribution. The nonlinearity which is now fixed may be described by a polynomial whose degree equals the index on the H functions. Thus, to use this receiver in a signal environment other than the original one would require merely the adjustment of the bias and (k - 1) scale factors while the nonlinearities remain the same. The advantage of this type of receiver over a likelihood receiver with a single nonlinearity is its adaptability to other environments. In particular, the design of the likelihood receiver (single nonlinearity) is based on the hypothesis of a priori distributions, while the receiver of Fig. 3 allows the operator to adjust the parameters (bias and scaling) to maximize signal detectability. The obvious disadvantage is the requirements of (k - 1) nonlinearities instead of only one. However, the nonlinearities are polynomial functions, which facilitate their implementation or on-line processing by a computer.

2.8 Application of Edgeworth Series

Consider the correlator as a receiver for signal detection. The output z^* is given by

$$z^* = \sum_{i=1}^n y_i s_i . \quad (2.65)$$

If the output is biased and then scaled, the new output z is given by

$$z = \frac{z^* - m}{\sigma} = \sum_{i=1}^n \frac{(y_i - m_i)}{\sigma} , \quad (2.67)$$

but (2.67) is the exact form to be used in an Edgeworth series.

In order to use the Edgeworth series, one need calculate only the central moments of z^* under hypotheses N and SN. Since it is assumed that the distributions of the samples x_i and n_i are known and hence their moments, the computer is programmed to

⁷The scale factors are constant if the signal space statistics are stationary.

use the moments $\alpha_k(x_i)$ and $\alpha_k(n_i)$ as input data. These moments are then converted to cumulants and added. The central moments of z^* are computed from the cumulants of the sum $\sum y_i s_i$, and are used in the expression for the coefficients of two Edgeworth expansions for $f(z|N)$ and $f(z|SN)$.

In order to evaluate the receiver, the integral of the Edgeworth expansion is needed;

$$F(\eta|N) = \int_{-\infty}^{\eta} f(z|N) dz, \quad (2.68)$$

$$F(\eta|SN) = \int_{-\infty}^{\eta} f(z|SN) dz, \quad (2.69)$$

where the probability of false alarm is given by $1 - F(\eta|N)$ and the probability of detection is given by $1 - F(\eta|SN)$.

Either (2.68) or (2.69), which are conditional distribution functions of z , may be expressed as

$$F(z \leq \eta) = \sum_{\substack{k \\ \text{k unordered}}} C_k \Phi^{(k)}(\eta), \quad (2.70)$$

where

$$\Phi^{(k)}(\eta) = \int_{-\infty}^{\eta} \phi^{(k)}(x) dx \quad (2.71)$$

or

$$\Phi^{(k)}(\eta) = \phi^{(k-1)}(\eta) - \phi^{(k-1)}(-\infty). \quad (2.72)$$

Since

$$\phi^{(k-1)}(-\infty) = 0,$$

then

$$\Phi^{(k)}(\eta) = \phi^{(k-1)}(\eta) = (-1)^{k-1} H_{k-1}(\eta) \phi(\eta). \quad (2.73)$$

Therefore, the only integration involved in the computation is of $\Phi^{(0)}(\eta)$, which is a standard computer routine. The integral of the remaining terms is replaced by the Hermite polynomials and the $\phi(x)$ function.

In practice, one really wants to approximate the distribution functions as closely as possible, since these distributions functions yield the receiver evaluation. Fortunately, every intuitive and mathematical idea on convergence of the Edgeworth series to $f(z)$ likewise applies to the convergence of the Edgeworth series in Eq. 2.70 to the distribution function $F(\eta|\dots)$.

The Edgeworth Series provides a tool for the evaluation of a detection receiver when the receiver output moments are known. Thus, the Edgeworth Series answers the second problem posed in Section 1.5. Although the Hermite polynomial expansion, given in Section 2.7, yields a novel physical realization of the log-likelihood receiver, Eq. 2.64 is not a convenient analytic representation of the log-likelihood ratio mapping. That is, the form of Eq. 2.64 is unsuitable for the computation of the receiver output moments which are needed in the Edgeworth Series. Thus, the first problem posed in Section 1.5 remains unanswered.

THE PEARSON SYSTEM

3.1 Introduction

The material in this chapter presents the background for the approximation of the likelihood ratio mapping by a closed form expression. The closed form expression can be used to approximate the moments of the likelihood ratio receiver output.

The likelihood ratio is the ratio of the signal-plus-noise p. d. f. to the noise p. d. f. The signal-plus-noise p. d. f. cannot usually, be expressed in terms of elementary functions. The approximation to the likelihood ratio replaces the signal-plus-noise p. d. f. by a Pearson frequency curve. The Pearson System of Frequency Curves, developed by Karl Pearson (Refs. 16-18), approximates raw statistical data with a closed form expression. The Pearson curve which does the approximation has the property that its first four moments ($\alpha_1, \dots, \alpha_4$) are the same as the moments of raw data.

In the discussion of Gram-Charlier and Edgeworth Series, credibility was given to the approximation used by some "acceptable" applied mathematics and by appealing to one's intuitive belief in the Central Limit Theorem. In the case of the Pearson System, no such mathematical justification can be given. However, having made such a strong statement a rationale is now presented for using his work in the evaluation of detection receivers.

3.2 Approach I

There are two approaches to the Pearson System. Each one gives some insight into the final results. Pearson was concerned with fitting raw statistical data with curves. He sought a system of curves which would allow him freedom in choosing the skewness (measure of the nonsymmetric deviation about the mean) and the range⁸ of the resulting curve. He found, in many cases, a fundamental relationship between the slope and amplitude of the density function. For example, consider the symmetric ($p = q$)-point binomial with a polygon

⁸ Here, range implies the range of the random variable or, alternatively, the domain of the probability density function.

drawn through its ordinates as shown in Fig. 4.

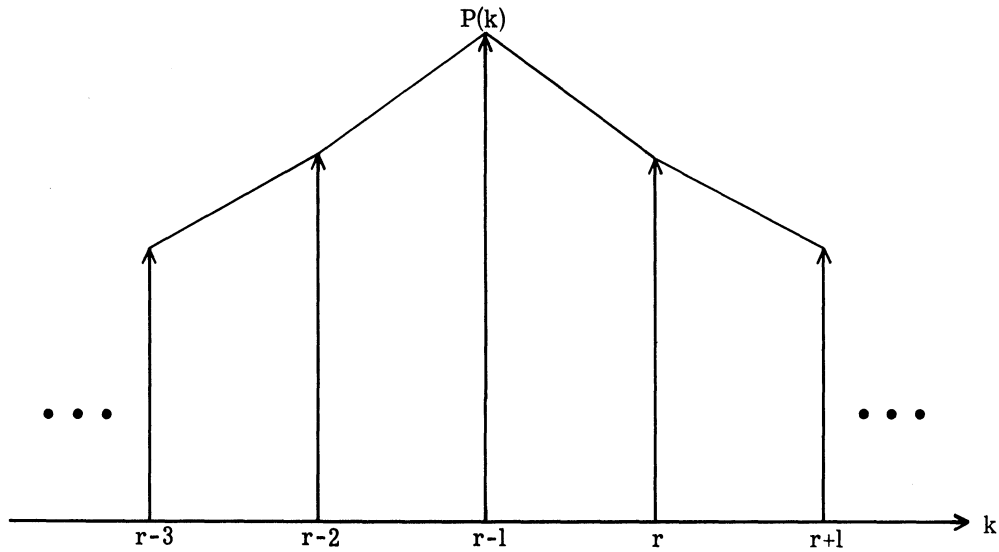


Fig. 4. Symmetric point binomial polygon.

The normalized first difference of this density function is given by

$$\frac{P(r+1) - P(r)}{\frac{1}{2} [P(r+1) + P(r)]} = \frac{\text{mean abscissa}}{\text{variance}}, \quad (3.1)$$

which may be interpreted as

$$\frac{\text{"slope of polygon"}}{\text{mean ordinate}} = - \frac{2 \text{ mean abscissa}}{2\sigma^2}. \quad (3.2)$$

Similarly, consider the normal density function,

$$f(x) = (2\pi\sigma^2)^{-1/2} e^{-x^2/2\sigma^2}. \quad (3.3)$$

Then

$$\frac{f'(x)}{f(x)} = \frac{-2x}{2\sigma^2}. \quad (3.4)$$

Comparing Eq. 3.4 with Eq. 3.2, the two density functions are seen to have a close geometric relation which is independent of the size of the binomial distribution.

As a second example of the relation between a density function and its derivative, Pearson found that the skew point binomial ($p \neq q$) had the same slope relation as the curve:

$$f(x) = \left(1 + \frac{x}{a}\right)^{\gamma a} e^{-\gamma x}, \quad (3.5)$$

where γ and a are given by

$$\gamma = \frac{2}{(p-q)}, \quad a = \frac{2pq(n+1)}{(p-q)}. \quad (3.6)$$

The result of Pearson's work was a differential equation representing the slope to amplitude relation of a large class of symmetric and nonsymmetric density functions, having finite and infinite range. The Pearson differential equation is

$$\frac{df(x)}{dx} = \frac{x+a}{b_0 + b_1x + b_2x^2} f(x). \quad (3.7)$$

Any density function satisfying this differential equation is said to belong to the Pearson class. This class includes the Normal, Chi-Squared, Poisson, Rayleigh and Maxwell-Boltzmann probability density functions, etc. Pearson's curves are divided into three main groups or types, according to their use:

- (1) Type I - Skewed density functions of limited range.
- (2) Type IV - Skewed density functions of unlimited range.
- (3) Type VI - Skewed density function of semilimited range.

Subtypes of these main types cover the symmetric cases as transition curves. There are nine subtypes giving a total of twelve Pearson type curves.

3.3 Approach II

Before proceeding, it is well worth the time to examine the second approach to Pearson's differential equation.⁹ Consider some of the more obvious characteristics of the probability density function found in practice.

⁹ See Elderton [19] for further discussion of the Pearson System.

- (a) If x_0 and x_1 denote the end points of the range, then $f(x_0)$ and $f(x_1)$ equal zero.
- (b) The p. d. f. is unimodal. Thus, the derivative of the p. d. f. equals zero at some point within the range $[x_0, x_1]$.
- (c) The p. d. f. has k-th order smoothness at the end points of the range. That is, the first k derivatives of the p. d. f. at x_0 and x_1 equal zero.

Figure 5 shows a typical unimodal probability function with the above three properties. A density function which satisfies the differential equation,

$$\frac{df(x)}{dx} = \frac{(x+a)}{g(x)} f(x) , \quad (3.8)$$

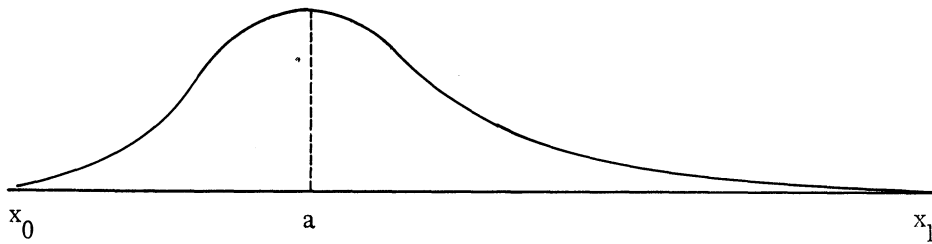


Fig. 5. A unimodal density function.

certainly satisfies the conditions (a), (b) and (c) above. In fact, by leaving $g(x)$ general, one can vary the conditions on $f(x)$ (e. g. , $g(x_0) = 0$ so that $f'(x_0) \neq 0$). If $g(x)$ is expanded in a Maclaurin series retaining only the first three terms, Pearson's differential equation results:

$$\frac{df(x)}{dx} = \frac{(x+a) f(x)}{b_0 + b_1x + b_2x^2} \quad (3.9)$$

Pearson must have realized the advantages of this approach for he suggests in a later paper (Ref. 18) the differential equation,

$$\frac{1}{f(x)} \frac{df(x)}{dx} = \frac{a_0 + a_1x + a_2x^2}{b_0 + b_1x + b_2x^2 + b_3x^3} \quad (3.10)$$

This differential equation allows data from higher-order hypergeometric distributions to be fitted, in particular, bimodal density functions. An example of a p. d. f. satisfying Eq. 3.10 is the Halsted density function (Ref. 20),

$$f(x) = x^k e^{bx} e^{cx^2}, \quad x \geq 0. \quad (3.11)$$

Nevertheless, little has been published on the types arising from the more general differential equation (3.10) because the available data do not effectively illustrate the solutions of the equation.

3.4 The Pearson System

Pearson's differential equation,

$$\frac{df(x)}{dx} = \frac{(x+a)f(x)}{b_0 + b_1x + b_2x^2}, \quad (3.12)$$

is integrated to yield

$$\ln f(x) = \int \frac{(x+a)}{b_0 + b_1x + b_2x^2} dx. \quad (3.13)$$

Although the actual form of the density curve depends on the integral of Eq. 3.13, the three main types of Pearson's system depend on the pole locations of the integrand. The three main types are classified according to the location of the zeros of the quadratic in the integrand denominator. If the roots are real and of opposite sign, the density function is Type I. For complex roots, the density function is Type IV, while for real roots of the same sign, the density function is Type VI. The coefficients a , b_0 , b_1 , b_2 can be related to the moments of the random variable¹⁰ so that a criterion to determine the Pearson Type can be based on the moments of the distribution. The criterion, K ,¹¹ determines the location of the quadratic roots and thus implies the choice of the Pearson Type.

The three main Pearson Types¹² and the criterion K follow.

¹⁰ See Appendix A.

¹¹ See Appendix B.

¹² See Appendix C.

Type I - Limited range, skew, roots real and of opposite sign

$$f(x) = f_0 \left(1 + \frac{x}{a_1}\right)^{\nu a_1} \left(1 - \frac{x}{a_2}\right)^{\nu a_2},$$

$$- a_1 \leq x \leq a_2,$$

$$K < 0. \quad (3.14)$$

Type IV - Unlimited range, skew, complex conjugate roots

$$f(x) = \frac{f_0 e^{-\nu \tan^{-1}\left(\frac{x}{a}\right)}}{\left[1 + \left(\frac{x}{a}\right)^2\right]^m},$$

$$-\infty \leq x \leq \infty,$$

$$0 < K < 1, \quad (3.15)$$

Type VI - Semilimited range, skew, roots are real and of same sign

$$f(x) = f_0 \frac{(x-a)^{m_1}}{(x)^{m_2}},$$

$$a \leq x \leq \infty,$$

$$1 < K. \quad (3.16)$$

In Types IV and VI, the "a" has a different meaning from the "a" in the differential equation.

The criterion

$$K = \frac{\beta_1(\beta_2 + 3)^2}{4(4\beta_2 - 3\beta_1)(2\beta_2 - 3\beta_1 - 6)}, \quad (3.17)$$

where

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3}, \quad \beta_2 = \frac{\mu_4}{\mu_2^2}. \quad (3.18)$$

Some of the more interesting subtypes occur for the transition points $K = 0$, $K = 1$ and also for the magnitude of K large.

3.5 Errors and Rationale

Since the Pearson approximation to a probability density function is based only on the first four moments, there is no limit to the possible error involved. Clearly, an infinite number of different density functions yields the same first four moments. Under the Pearson system, all of these density functions would have the same classification.

When Pearson was fitting data, he was able to measure the mean square error of his approximation and compare this error with the errors from other approximations. But, in the evaluation problem, the moments are used to fit unknown distributions (if the distributions were known, it would not be necessary to use the approximation techniques). Thus, no error estimate can be found directly. However, one could re-evaluate the receiver using an approximating differential equation of the form:

$$\frac{1}{f(x)} \frac{df}{dx} = \frac{x + a}{b_0 + b_1x + b_2x^2 + b_3x^3} \quad (3.19)$$

Since the f of Eq. 3.19 is based on the first five moments, the evaluation using this class of approximation functions when compared with the evaluations from the Pearson class would determine an estimate of the error due to the omission of the fifth moment.

In spite of the inherent problems in estimating the errors incurred when using the Pearson system for receiver evaluations, several strong practical reasons are in its favor. The evaluation of the receiver if not exactly, then, at least in the Pearson sense, allows insight into the environmental effects on the signal detectability. When the Pearson system is used, the results both include those of the usual normal approximation (first two moments) and allow for the qualities of skewness and kurtosis. Moreover, in practice, the probability density functions should be based on the statistical data obtained from channel measurements. The Pearson system is ideal for fitting these data with a closed form expression. To expect the reliability of the data to be high enough to estimate the first four moments of the distributions with a small probable error is asking quite a bit of the channel experiments. To ask more seems impractical, at the present time, from an equipment point of view. Indeed, the Pearson system can be criticized for its reliance on the accuracy

of the fourth moment as well as the first moment while, in practice, the estimate of the first moment is far more reliable than that of the fourth moment.

RECEIVER EVALUATION

4.1 Introduction

The techniques for the approximation of probability density functions introduced in Sections 2 and 3 will now be applied to a broad class of receivers. A unified treatment of this class will be presented through the introduction of the general receiver. In Sections 4.2 - 4.5, it will be assumed that the density functions of the observation space are known; Section 4.6 will discuss the evaluation of detection receivers when this is not the case.

4.2 General Receiver

A fixed-time receiver maps the observation vector, y , into the real line. The general receiver is a detection receiver which maps the components of y , via a zero memory nonlinearity, g , into the random variables, z_i , and then sums the random variables, z_i , to give the output Z , (see Fig. 6). The output Z , at time $t = T$, is compared against a threshold, η . If $Z > \eta$, one asserts that signal-plus-noise was present while if $Z \leq \eta$, one asserts noise alone was present.

The evaluation of this type of receiver is straightforward. Since

$$Z = \sum z_i \quad , \quad (4.1)$$

where

$$z_i = g_i(y_i) \quad , \quad (4.2)$$

and the y_i are independent samples, then Z is the sum of $2WT$ independent samples from the same population (either N or SN). The moments of z_i follow from their definition

$$\alpha_k(z_i) = \int_Y g^k(y_i) f(y_i) dy_i \quad , \quad (4.3)$$

where

$$f(y_i) = f(y_i|N) \quad (4.4)$$

if the moments of z_i under the noise hypothesis are desired, while

$$f(y_i) = f(y_i|SN) \quad (4.5)$$

when the moments of z_i under the SN hypothesis are sought.

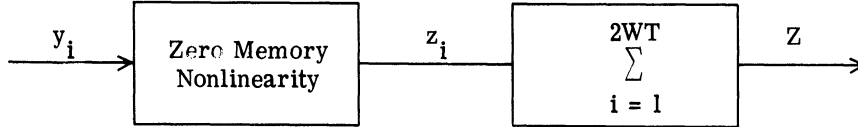


Fig. 6. General receiver block diagram.

Using the additive property of the cumulants of independent random variables, one can calculate the general receiver output moment sets,

$$\alpha(Z|N) = \left\{ \alpha_k(Z|N) \right\}, \quad k = 1, 2, \dots, \ell, \quad (4.6)$$

$$\alpha(Z|SN) = \left\{ \alpha_k(Z|SN) \right\}, \quad k = 1, 2, \dots, \ell. \quad (4.7)$$

At this point, one can use either a Pearson fit or an Edgeworth series to approximate the receiver output probability density functions. If a Pearson fit is to be made, it is sufficient to use the first four moments ($\ell = 4$) while the Edgeworth series can use more than four moments ($\ell \geq 4$).

Once the density functions are obtained, the false alarm and detection probabilities follow from

$$P(A|N) = 1 - \int_{\eta}^{\infty} f(Z|N) dz, \quad (4.8)$$

$$P(A|SN) = 1 - \int_{\eta}^{\infty} f(Z|SN) dz. \quad (4.9)$$

As explained in Section 2.8, the Edgeworth series evaluation can be reduced to the evaluation

of one integral and a number of Hermite polynomials. The Pearson evaluation requires only one integration procedure which, in general, must be carried out by some quadrature method since no standard routines are known for the integration of the Pearson curves over subintervals of the random variable range.

4.3 Examples of the General Receiver

Some examples of the general receiver are:

4.3.1 Correlator:

$$g_i(y_i) = x_i y_i , \quad (4.10)$$

$$\alpha_k(z_i) = (x_i)^k \alpha_k(y_i) , \quad (4.11)$$

$$\alpha_k(z_i|N) = (x_i)^k \alpha_k(n_i) ; \quad (4.12)$$

$$\alpha_k(z_i|SN) = \sum_{j=0}^k (x_i)^{2k-j} \alpha_j(n_i) \binom{k}{j} . \quad (4.13)$$

A fuller development of the correlator and its evaluation is given in Section 2.8.

4.3.2 Square-Law Detector.

$$g_i(y_i) = y_i^2 , \quad (4.14)$$

$$\alpha_k(z_i|N) = \alpha_{2k}(n_i) , \quad (4.15)$$

$$\alpha_k(z_i|SN) = \alpha_{2k}(y_i|SN) . \quad (4.16)$$

The square-law detector is used for incoherent detection where the epoch of the signal is not known. The detection is based on the increase of observed energy when signal is present as compared to the noise-alone energy.

4.3.3 Clipper Crosscorrelator. (Figure 7)

$$\begin{aligned} g(y_i) &= 1 \text{ for } y_i > K \\ &= -1 \text{ for } y_i \leq K , \end{aligned} \quad (4.17)$$

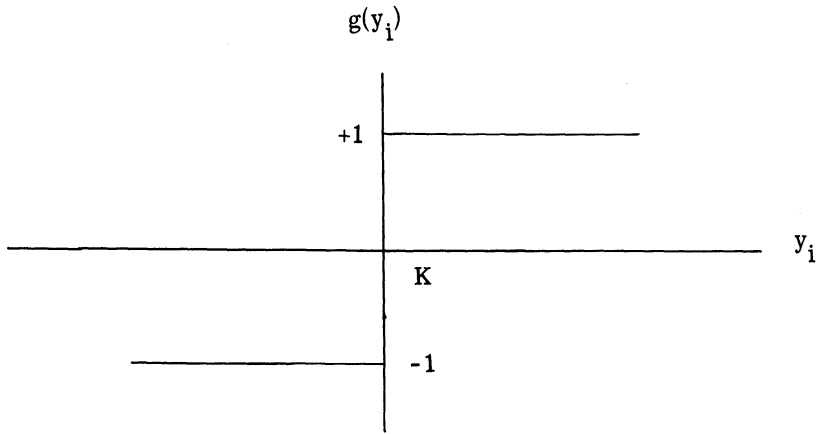


Fig. 7. Clipper crosscorrelator nonlinearity.

$$\begin{aligned} \alpha_k(z_i|N) &= 1 \text{ for } k \text{ even} \\ &= 1 - 2 P(K|SN) \text{ for } k \text{ odd ,} \end{aligned} \quad (4.18)$$

$$\begin{aligned} \alpha_k(z_i|SN) &= 1 \text{ for } k \text{ even} \\ &= 1 - 2 P(K|SN) \text{ for } k \text{ odd ,} \end{aligned} \quad (4.19)$$

where

$$P(K|\dots) = \int_{-\infty}^K f(y_i|\dots) dy_i . \quad (4.20)$$

If, instead, the nonlinear function

$$\begin{aligned} g(y_i) &= 1 \text{ for } y_i > K \\ &= 0 \text{ for } y_i \leq K \end{aligned} \quad (4.21)$$

is used, then

$$\alpha_k(z_i) = 1 - P(K|\dots) \text{ for all } k . \quad (4.22)$$

Thus, the z_i are samples of a two-point, skew binomial distribution. The output Z is then

the number of successes in 2WT Bernoulli trials with probability

$$p = 1 - P(K|\dots) \quad (4.23)$$

of success. It is well known (Ref. 8) that Z has a binomial distribution with expectation, (2WT)p and variance (2WT) p(1-p). When such a receiver is evaluated by the Pearson technique, the resultant distributions are continuous-curve approximations to the skew point binomial distribution (see Section 3.2).

4.4 Log-Likelihood Ratio Receiver

Instead of the general receiver of the preceding section, consider the optimum Bayes detector, the likelihood-ratio receiver given by

$$\ell(y) = \frac{f(y|SN)}{f(y|N)} \quad (4.24)$$

Independence of the observations in N and SN permits one to write the likelihood ratio of the total observation as

$$\ell(y) = \prod_{i=1}^{2WT} \ell_i(y_i) = \prod_{i=1}^{2WT} \frac{f(y_i|SN)}{f(y_i|N)} \quad (4.25)$$

Since a decision based on $\ell(y)$ is optimum in a Bayes sense, a decision based on a monotone function of $\ell(y)$ is also optimum in a Bayes sense. In particular, a decision based on the logarithm of the likelihood ratio is optimum. Denoting the log-likelihood ratio by L,

$$L(y) = \ln \ell(y) \quad (4.26)$$

then from Eq. 4.25

$$L(y) = \sum_{i=1}^{2WT} L(y_i) = \sum_{i=1}^{2WT} \ln \left[\frac{f(y_i|SN)}{f(y_i|N)} \right] \quad (4.27)$$

Now, if the zero memory nonlinear function of the general receiver is given by

$$g(y_i) = L(y_i) = \ln \frac{f(y_i|SN)}{f(y_i|N)} \quad (4.28)$$

it is evident that the optimum Bayes detector can be represented by the general receiver with the nonlinearity set equal to the log-likelihood ratio of the observation.

The evaluation of the Bayes detector can thus be performed in the same manner as described in Section 4.2 with the following exception. In Section 4.3, the moments of z_i were directly related to the moments or the distribution function of the observation component y_i . When a log-likelihood ratio nonlinearity is used, such a relation is not generally available. Hence, the formal expectation of Eq. 4.3 must be performed with a quadrature method. At the same time, the Bayes detector imposes a structure on the processing such that the output p. d. f. under signal-plus-noise hypothesis is related to the output p. d. f. under the noise-alone hypothesis by the likelihood ratio;¹³ that is,

$$f(L|SN) = e^L f(L|N) . \quad (4.29)$$

Hence, in theory, one need only evaluate the p. d. f. under the noise hypothesis without concern for the signal-plus-noise moments. Although this is theoretically feasible, Pearson fits, in practice, have the potential of diverging when multiplied by a term e^L . To expand on this statement, consider Eq. 4.29 and the conditions under which $f(L|SN)$ remains a member of the Pearson class.

$$\frac{d f(L|SN)}{dL} = f(L|SN) \left[1 + \frac{P(L)}{Q(L)} \right] , \quad (4.30)$$

where

$$\frac{P(L)}{Q(L)} = \frac{d f(L|N)}{dL} \frac{1}{f(L|N)} . \quad (4.31)$$

It is evident from the form of Eq. 4.30 that the maximum degree of $Q(L)$ is one and thus, the class of allowable p. d. f. under noise is restricted to the form

$$\frac{df}{dL} = \frac{L + a}{b_0 + b_1 L} f(L) . \quad (4.32)$$

¹³ See Appendix E.

Therefore, if $f(L|N)$ is not a member of the restricted class defined by Eq. 4.32, $f(L|SN)$ will not be in the Pearson class.

4.5 Postintegration Optimum Processing

It is worthwhile to point out that threshold detection of the general receiver output is not, in general, the optimum Bayes procedure. Instead of threshold detection, one should form the likelihood ratio of Z and base a decision on the likelihood ratio; i. e., to optimize the Bayes performance of a receiver with preprocessing, the observer should treat Z as the observation instead of $y(t)$. The generalized receiver then can be considered as a part of the transmission channel which maps the $2WT$ Y space into the real line. The optimum Bayes detector for preprocessing by the generalized receiver is shown in Fig. 8.

If the generalized receiver is a likelihood ratio receiver, then further processing of Z cannot improve the observer's decision. Intuitively, it follows that the likelihood ratio of the likelihood ratio must equal the likelihood ratio.¹⁴

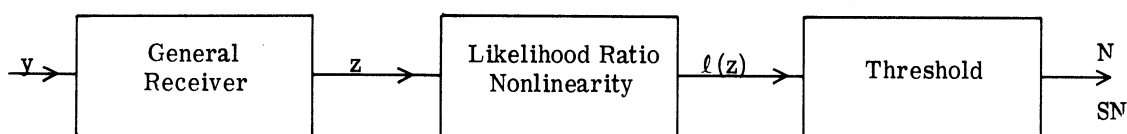


Fig. 8. Optimum post-integration processor.

When the generalized receiver is not a monotonic function of the likelihood ratio, one must compute the likelihood ratio of Z in a form suitable for further computation. If the Edgeworth series is used to approximate $f(Z|\cdot\cdot)$, an unwieldy number of coefficients and polynomials must be carried along. However, when a Pearson fit to $f(Z|\cdot\cdot)$ is used, the two p. d. f. 's are given in closed form. These two density functions are all that is needed for the remaining part of the evaluation. At this point, the same procedure which is given in

¹⁴ See Appendix E.

Section 4.4 is carried out using the p. d. f. 's, $f(Z|SN)$ and $f(Z|N)$, with the modification of setting $2WT$ equal to one. The ROC curve obtained is optimum for the class of receivers restricted to preprocessing by a generalized receiver. The performance in this situation can then be compared to the performance of the generalized receiver with threshold detection (no post-integration processing) and the optimum receiver of Section 4.4 where the likelihood ratio operates directly on the observation space Y .

4.6 Receiver Evaluation from Raw Data

This section will discuss the problem of design and evaluation of the receiver when the density functions on the observation space Y are not given, but instead either the channel measurements or the moments of the density functions are available. If the data available are the channel measurements, these can be converted into estimates of the density functions' moments. Using these moments a Pearson fit is made to the two density functions and their ratio is taken. This ratio is the Pearson approximation to the likelihood ratio non-linearity for a single component of the observation vector y . It is evident that the post-integration detector of Section 4.5 is an example of the Pearson receiver (approximation to likelihood ratio). Once the approximation is obtained, the procedure of Section 4.4 is followed using the Pearson nonlinearity.

4.7 Computational Aids

As mentioned in Section 4.4, the calculation of the output moments of a likelihood ratio receiver requires a quadrature routine. This can be partially circumvented by use of either of the following theorems.

Theorem 1 The k -th moment of the likelihood ratio under the signal-plus-noise hypothesis equals the $(k+1)$ moment of the likelihood ratio under the noise-alone hypothesis,

$$\alpha_k [\ell(y)|SN] = \alpha_{k+1} [\ell(y)|N] \quad . \quad (4.33)$$

The proof of this theorem is given in Ref. 1. Its application to the post-integration processing allows computation of all of the required signal-plus-noise moments by the computation of one extra noise moment. Use of the theorem is limited since the log-likelihood receiver of Section 4.4 requires the moments of the log-likelihood ratio. The following theorem can be used in place of Theorem 1.

Theorem 2 The k-th moment of the log-likelihood ratio under the signal-plus-noise hypothesis is related to all of the moments greater than k-1 of the log-likelihood ratio under the noise-alone hypothesis. This relation is given by

$$\alpha_k(L|SN) = \sum_{j=k}^{\infty} \frac{\alpha_j(L|N)}{(j-k)!} . \quad (4.34)$$

Proof:

By definition,

$$\alpha_k(L|SN) = \int L^k(y) f(y|SN) dy , \quad (4.35)$$

but, since

$$f(y|SN) = f(y|N) = e^L f(y|N) , \quad (4.36)$$

then

$$\alpha_k(L|SN) = \int L^k(y) e^L f(y|N) dy . \quad (4.37)$$

Expanding e^L in a Maclaurin series and integrating term-by-term, it follows that

$$\alpha_k(L|SN) = \sum_{j=k}^{\infty} \frac{\alpha_j(L|N)}{(j-k)!} . \quad (4.38)$$

When the noise distribution is such that the moments of $\alpha_j(L|N)$ are both easy to compute and growing slowly, the use of Theorem 2 can save computing time.

4.8 Computer Evaluation

A library of receiver-evaluation programs is available. These programs have been tested in several well-known detection problems where analytic results were available for comparison. In nearly all of the test programs, the evaluation results completely agreed with the analytic results. The conclusion, drawn from the test results is that the programs are operational; that is, the quadrature methods and program logic yield evaluations with negligible error.

As yet, no statement can be made about the quality of the approximation tech-

niques when the Pearson system is used. Whenever the test result deviated negligibly from the analytic results, one noted that the probability density functions in the test problem were, indeed, members of the Pearson class, and thus, the application of Pearson's techniques yields exact results.

When the density functions in the test problem were not members of the Pearson class, the approximation to the receiver nonlinearity was often excellent over a large range of the observation variable. An example of the receiver approximation (see Ref. 21) is

Fig. 9, where the log-likelihood ratio nonlinearity curves computed by both the convolution

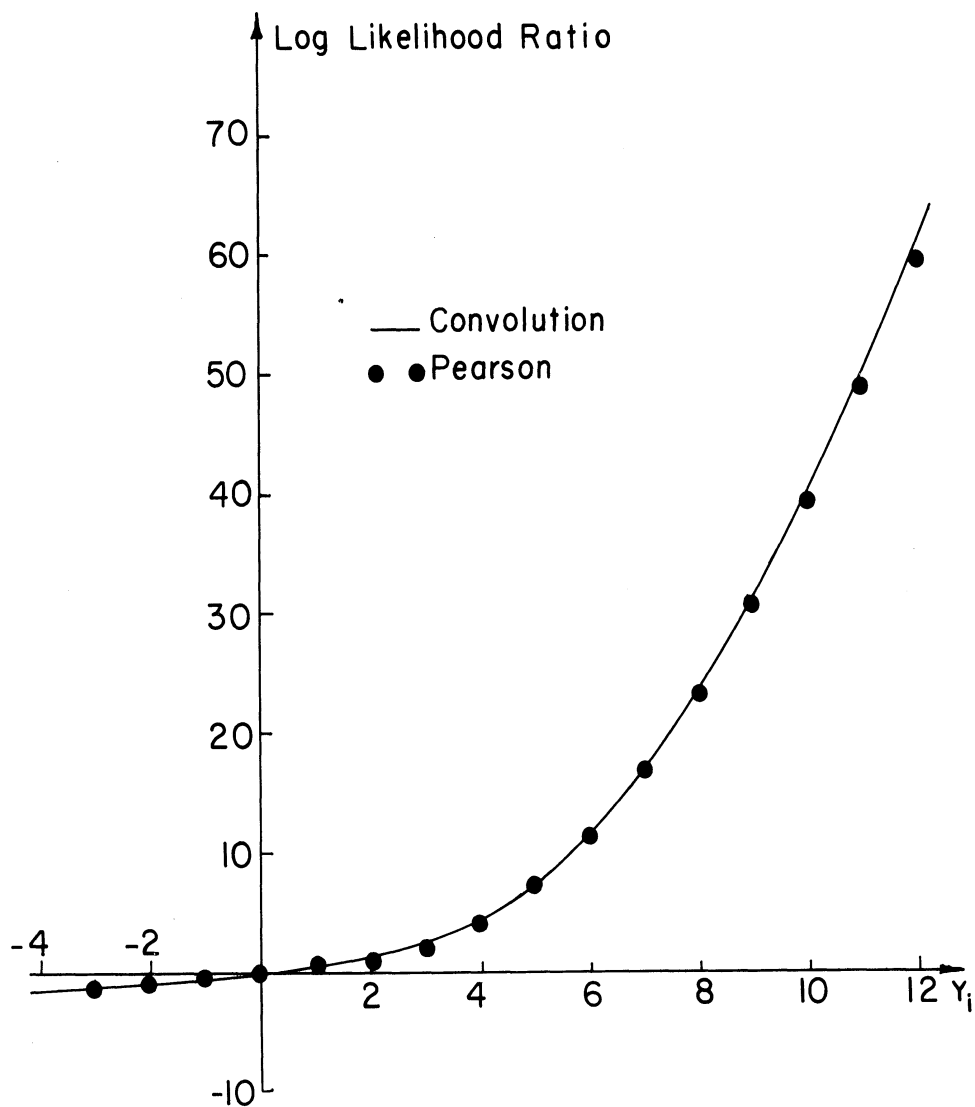


Fig. 9. Log-likelihood ratio receiver nonlinearity.

and Pearson techniques, the former yielding the theoretical result, are shown. Although the Pearson approximation is good over the range $-4 < y_i < 12$, the receiver moments calculated from this nonlinearity may require as good a fit as shown in Fig. 9 over even a larger range. The question of the range of the approximation is academic in the practical problem. In the test problems, the random variables have a range, $-\infty < x_i < \infty$, but with measured data, the range is well bounded. Thus, the question of the quality of the approximations must be deferred until channel data are available for testing purposes.

RECEIVER-EVALUATION COMPUTER LIBRARY

This section describes the programs available in the receiver-evaluation library. The programs are written in MAD (Michigan Algorithm Decoder) language - a computer language developed by personnel at the University of Michigan Computing Center. However, comprehension of any of the computer languages, such as FORTRAN, should enable the reader to follow the program description.

Each program listed will contain the following information:

- (1) Name and general description of program
- (2) Required Subroutines (external functions)*
- (3) Program Entry - Calling routine and explanation of required routine arguments
- (4) Output
- (5) Program purpose and method

Any routine called by the program falls into one of three classes:

- (a) Internal Function - the called routine is defined and listed as a part of the program
- (b) External Function - the called routine is a subroutine which belongs to the library of receiver-evaluation programs. As such, the routine is listed in the number (2) category of program information and a description of the called routine can be found in this chapter.
- (c) Computer Subroutine - the called routine belongs to the computer library system, called MESS (Michigan Executive Subroutine System). Typical routines in this category include the transcendental functions, subroutines for dating the program, etc.

5.1 Moment-Conversion Subroutine (ALTOMU) (Figure 10)

This program computes the central moments of a sum of random variables

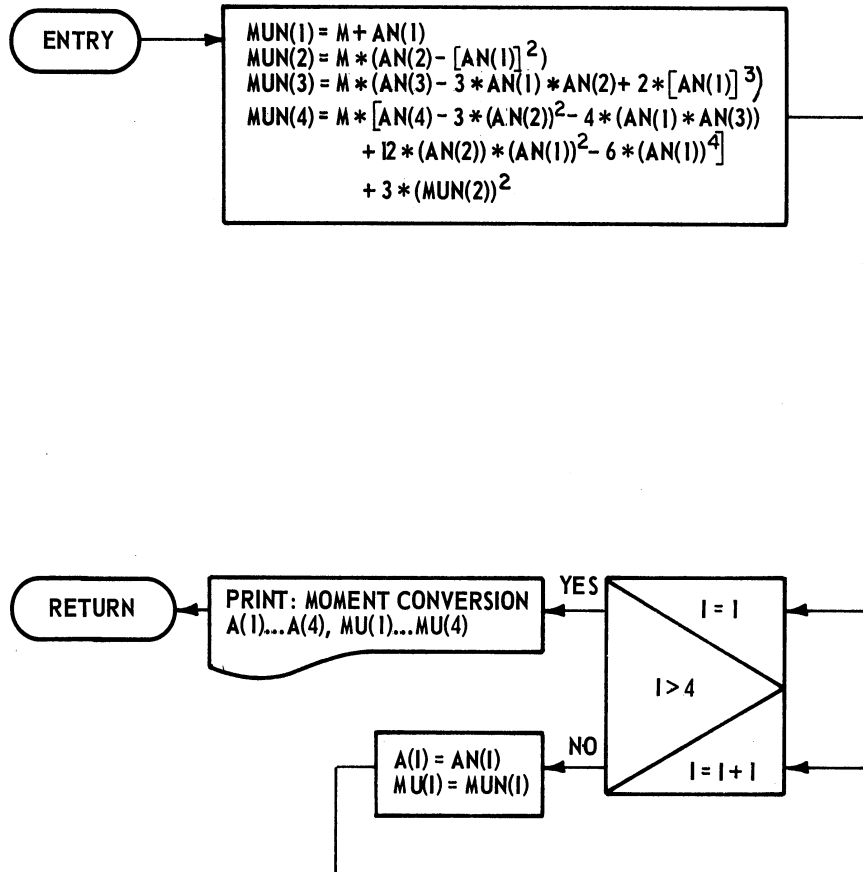


Fig. 10 Moment conversion subroutine (ALTOMU).

drawn from the same population. The program can be used to compute the output moments of linear filters.

(2) Required Subroutines - none.

(3) Execute ALTOMU. (AN, MUN, M)

AN - the location of a vector of the first four noncentral moments of a single variate in the sum.

MUN - the storage location of the computed central moments of the sum in the main program.

M - the number of random variables in the sum.

(4) Output - the function returns MUN to the main program and prints out the vectors AN and MUN.

(5) Method - the program computes MUN from AN using the additive property of the cumulants to express the relation between MUN and AN for any given M.

Note that for M equal to one, the program converts the alpha moments to mu moments.

This program is limited to linear filters with unity weighting functions. A generalization of ALTOMU would allow the output moments of arbitrary linear filters to be evaluated.

5.2 Edgeworth Subroutine (EDGE) (Figure 11)

This program takes the central moments of a noise and a signal-plus-noise distribution and computes the ROC curve. The program is used to evaluate receivers whose output central moments are known.

(2) Required Subroutines - none.

(3) Execute EDGE. (MS, MN)

MS - location of the signal-plus-noise central moments.

MN - location of the noise central moments.

(4) Output - the function prints out the following data:

Y - standardized threshold axis in terms of the signal-plus-noise parameters.

X - the actual threshold level.

PDET - the detection probability at the threshold X.

PFA - the false-alarm probability at the threshold X.

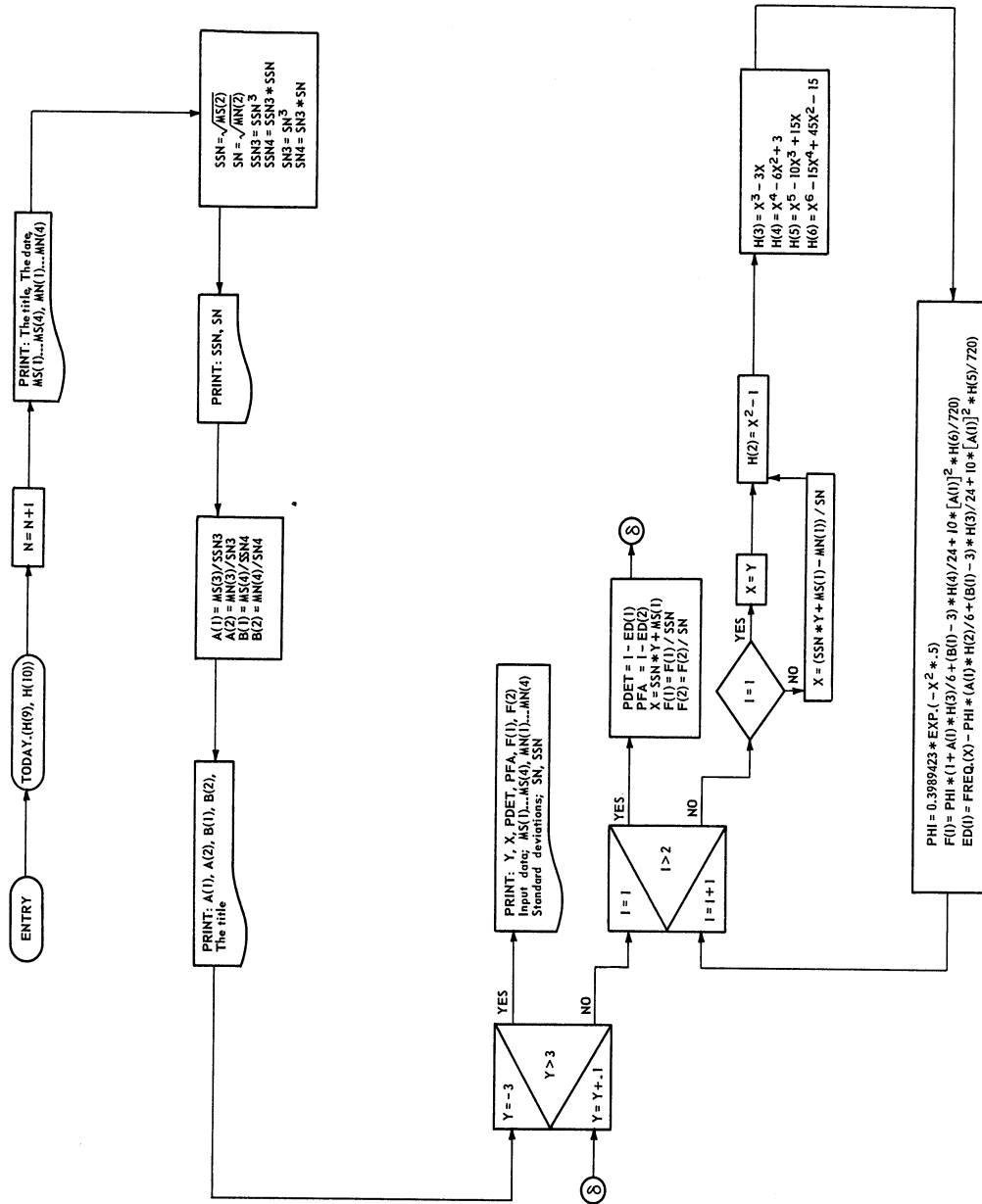


Fig. 11 Edgeworth subroutine (EDGE).

F(1) - signal-plus-noise density function.

F(2) - the noise density function.

The program also prints out MS, MN, and the standard deviation of the noise, SN, and the signal plus noise, SSN.

- (5) Method - the program computes a six-term Edgeworth Series approximation to the noise (N) and signal-plus-noise (SN) probability density functions. The program then evaluates the probability of a random variable, conditional to the N or the SN hypothesis, exceeding a threshold X. The threshold, X, is varied through six standard deviations (SSN) about the mean of the signal-plus-noise distribution. The output table contains a listing of points of the ROC with a tabulated threshold parameter, X.

5.3 Correlator Program (Figure 12)

This program evaluates the correlator receiver under the assumption of independent noise and signal-plus-noise sampling.

(2) Required Subroutines

(a) ALTOMU

(b) EDGE

(3) The program requires the following input data:

(a) Two data cards (1-72) which give the user's title of the program. (Eq. , "Receiver-Evaluation Program for a Clipper Crosscorrelation").

(b) Data cards for ASN, the signal-plus-noise moments, AN, the noise moments, and M, the number of observation samples.

- (4) Output - the program print-out includes the probability of detection, the probability of false alarm, and the corresponding threshold. For details of the print-out, see the Edgeworth subroutine.

- (5) Method - the program takes the input moments, ASN and AN, and using ALTOMU, computes the output central moments of the correlator. The output central moments are fed into the Edgeworth subroutine where the ROC curve is evaluated.

This program is written for stationary noise and signal-plus-noise processes.

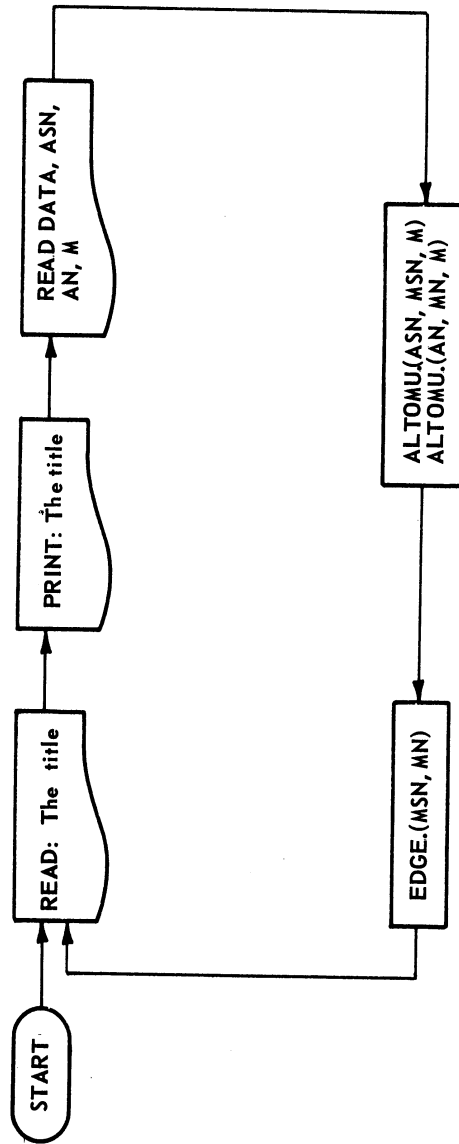


Fig. 12 Correlator receiver program.

The program is not easily generalized to nonstationary processes.

5.4 Gaussian Quadrature Subroutine (GAUINT) (Figure 13)

This program is used for numerical integration on the computer.

(2) Required Subroutines - none.

(3) AREA = GAUINT. (M, N, A, B, F,)

AREA - the storage location of the value of the integral.

M - the order of integration (number of points per subinterval).

N - the number of subintervals.

A - the lower limit of the integral

B - the upper limit of the integral

F - the integrand.

(4) Output - function returns to AREA the numerical approximation of the integral of

$F(X)$, $A \leq X \leq B$.

(5) Method - Gaussian Quadrature is a technique which chooses the minimum number of points needed to evaluate, with no error, a polynomial of degree M. For a detailed description of the method, see Kopal (Ref. 22). The program affords the user a choice of the degree of the approximation polynomial; that is, the values of M range from 2 through 16. The larger the value of M used, the more accurate the result, but the longer the computation time.

5.5 CAL-CHAN-FUN Subroutine (Figure 14)

This subroutine calculates the appropriate Pearson coordinate from a given real axis value. The program, also, evaluates the Pearson density function along the Pearson axis, and the program changes a given Pearson coordinate into the real axis value.

(2) Required Subroutines - none.

(3) The subroutine has three entries:

(a) X = CAL. (A)

A - location of real axis coordinate.

X - storage location of calculated Pearson coordinate.

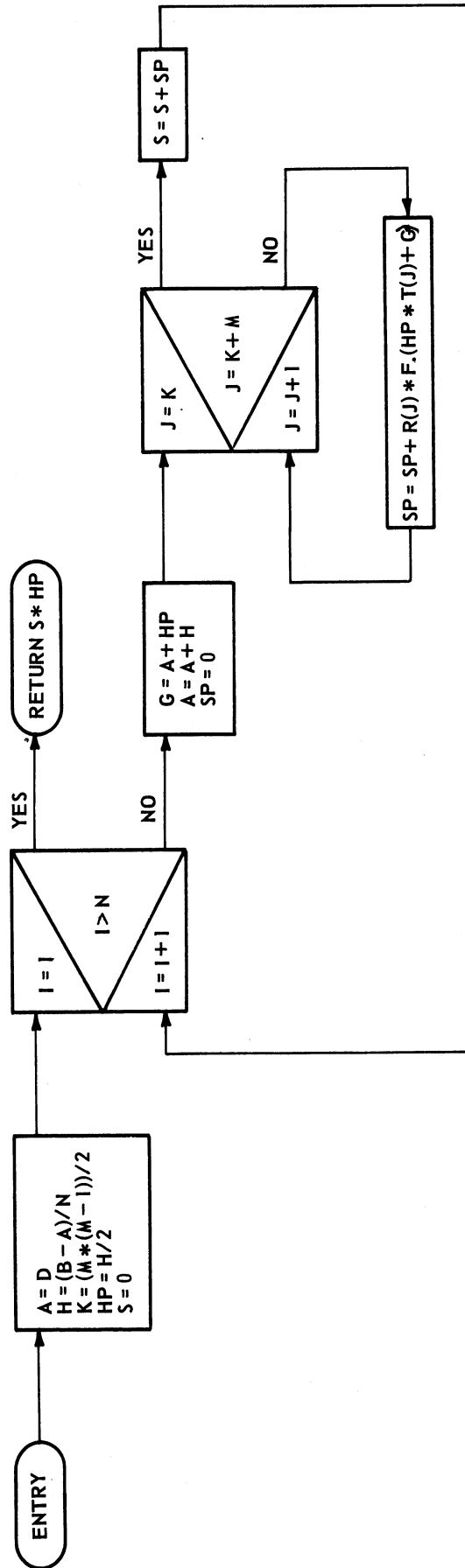


Fig. 13 Gaussian quadrature subroutine (GAUINT).

(b) X = CHAN. (A)

A - location of Pearson axis coordinate.

X - storage location of real axis value.

(c) X = FUN. (A)

A - location of Pearson axis coordinate.

X - storage location of the value of the Pearson density function, evaluated at A.

(4) Output - the subroutine returns to the calling program the computed datum of CAL-CHAN-FUN.

(5) This subroutine is needed when one uses the Pearson Frequency Curves. In Pearson System, the origin of the coordinate is related to the constants of the frequency curve. The CAL routine permits the translation of a physical threshold into the equivalent Pearson threshold. The CHAN routine is the inverse translation of the CAL routine. The FUN subroutine is used along with the output of the CAL routine. Certain error routines have been inserted into the FUN program. These error routines return control to the main calling program whenever logarithmic or exponentiation commands lead to a computer overflow or underflow. The main program then proceeds to process the next set of data.

5.6 Type A Subroutine (TYPEA) (Figure 15)

This program, using the random variable central moments, computes the Pearson constants for one of the Pearson Types 1, 2, 4, or 7. The program also computes the information used in the coordinate shifting of the CAL and CHAN routines.

(2) Required Subroutines

(a) LNGAM

(3) EXECUTE TYPEA. (RUN, LOC)

RUN - a two-element statement label array, RUN, RUN(1). RUN is the location to which the program will transfer after the Pearson curve constants are computed. RUN(1) designates the location to transfer to in the event of computer overflow when computing the constants.

LOC - a statement label to which the program will transfer in the event of an exponentiation of logarithm error return.

Note that in the Pearson Subroutine, both RUN(1) and LOC give rise to the same action.

- (4) Output - the function returns to program common the computed values of the constants for a specified Pearson type. The values of the constants are printed out.
The program also stores in the program common the values required for the translation of coordinates.
- (5) Method - the program runs through a set of logic statements which select the proper set of constants needed for the given Pearson type of curve. The constants are evaluated, if possible. Otherwise, the program prints out an error statement and returns control to either RUN(1) or LOC.

5.7 Type B Subroutine (TYPEB) (Figure 16)

This program is identical with TYPEA except the constants are evaluated for either a Pearson type 3 or 6. See TYPEA.

- (2) Required Subroutines - none.
- (3) EXECUTE TYPEB. (RUN, LOC)
See TYPEA.
- (4) Output - see TYPEA.
- (5) Method - see TYPEA.

5.8 SNINT Subroutine (SNINT) (Figure 17)

This program computes the distribution function associated with a given Pearson density function.

- (2) Required Subroutines
 - (a) CHAN
 - (b) FUN
- (3) MAX = SNINT. (D, B, X, S)
MAX - the number of elements used in the X and S arrays. Note, MAX is an integer.
D - the lower bound of the integration.
B - the upper bound of the integration.
X - array for storing the real axis values of the coordinates used in the integration.
S - array for storing the values of the distribution function.

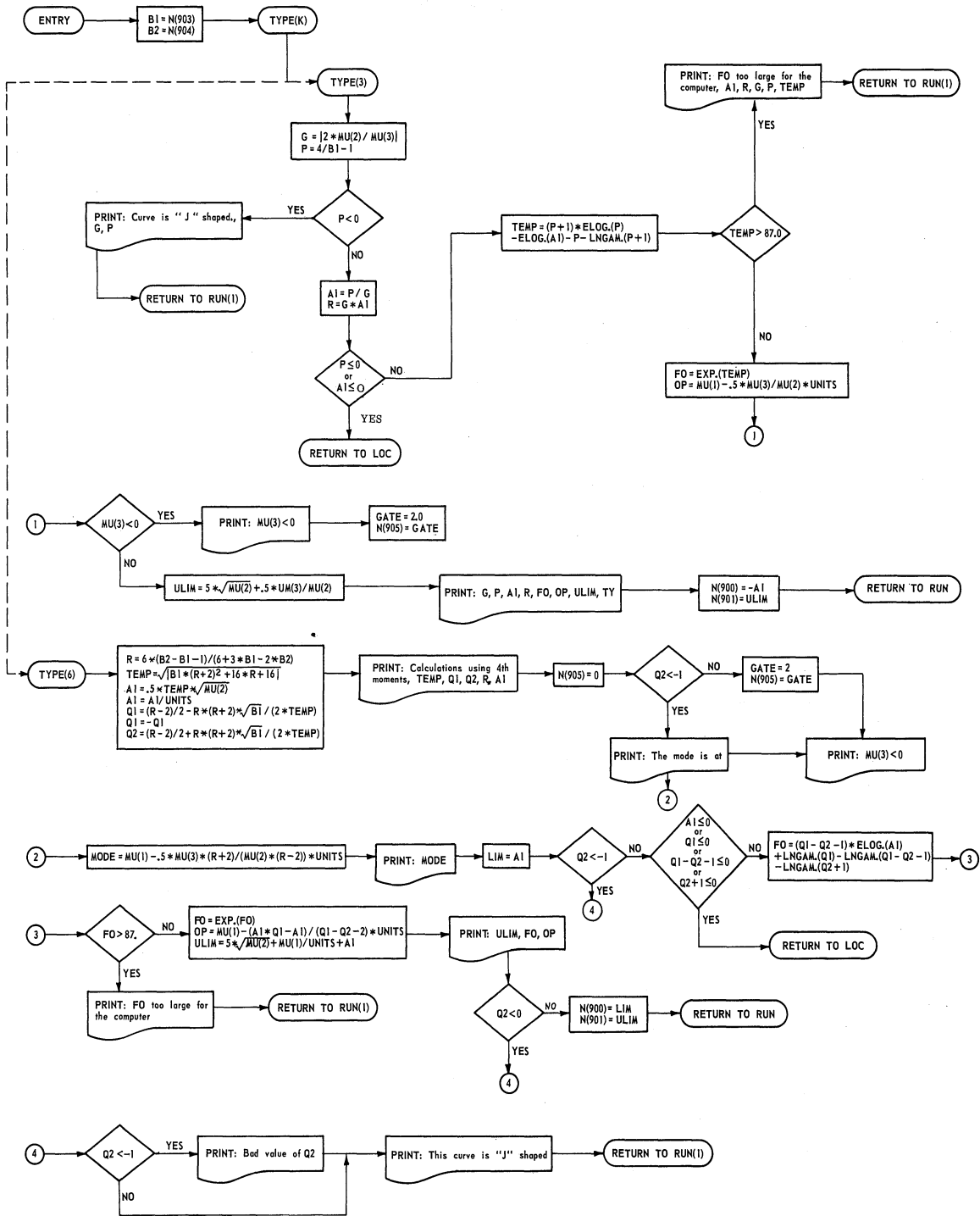


Fig. 16 TYPEB subroutine.

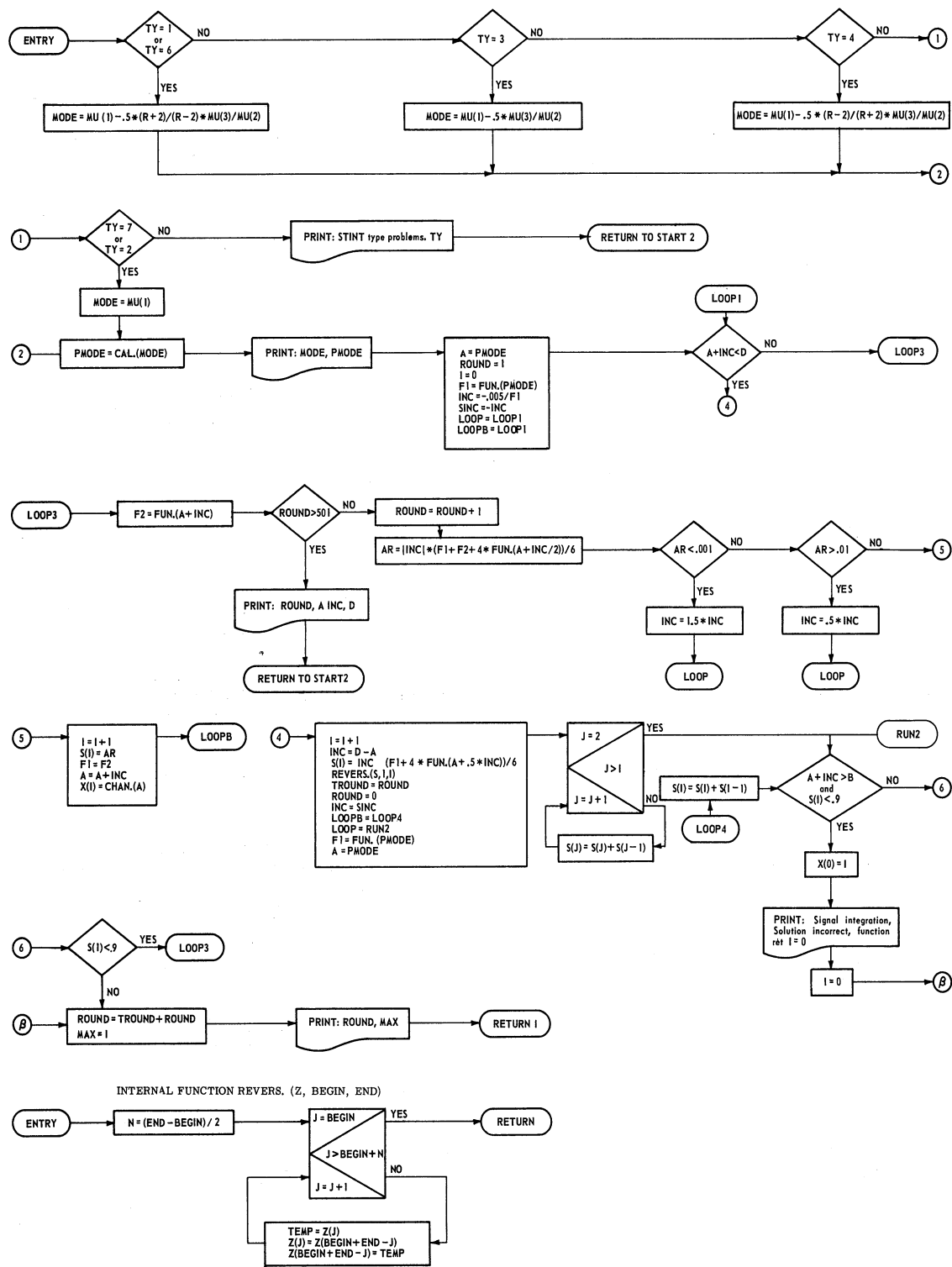


Fig. 17 Signal plus noise integration subroutine (SNINT).

- (4) Output - function returns, to the program common, the distribution function in S , the associated coordinates in X , and the integer MAX which gives the length of the X or S array. The program also prints out the integer MAX and the integer $ROUND$ which counts the number of integration steps.
- (5) Method - the $SNINT$ routine begins by computing the mode of the Pearson density function to be integrated. The integration proceeds from the mode to the lower and upper limits of the integral. The quadrature method is Simpson's rule with the length of the integration step controlled to maintain an incremental area for the step between 0.001 and 0.01. This control of the integration step permits a fast integration of regions where the distribution function is slowly changing. The interval length is changed by powers of two until the incremental area criterion is satisfied. The total number of such changes is printed out as the integer, $ROUND$. The integration stops when the distribution (total accumulated area) equals 0.9. If the upper limit, B , is reached before the area is 0.9, an error return is made. $SNINT$ is used to integrate the density functions of the signal-plus-noise hypothesis. The X array is found from the Pearson coordinates via the $CHAN$ routine. This X table is then used in the $NINT$ program.

5.9 NINT Subroutine (NINT) (Figure 18)

The $NINT$ program integrates the Pearson noise density function to give the distribution function of the noise at the coordinate values, X , determined by $SNINT$.

(2) Required Subroutines

- (a) FUN
- (b) CAL
- (c) $GAUINT$

(3) EXECUTE $NINT$. (A , B , X , N , MAX)

- A - lower limit of integration.
- B - upper limit of integration.
- X - array of coordinate values used in $SNINT$.

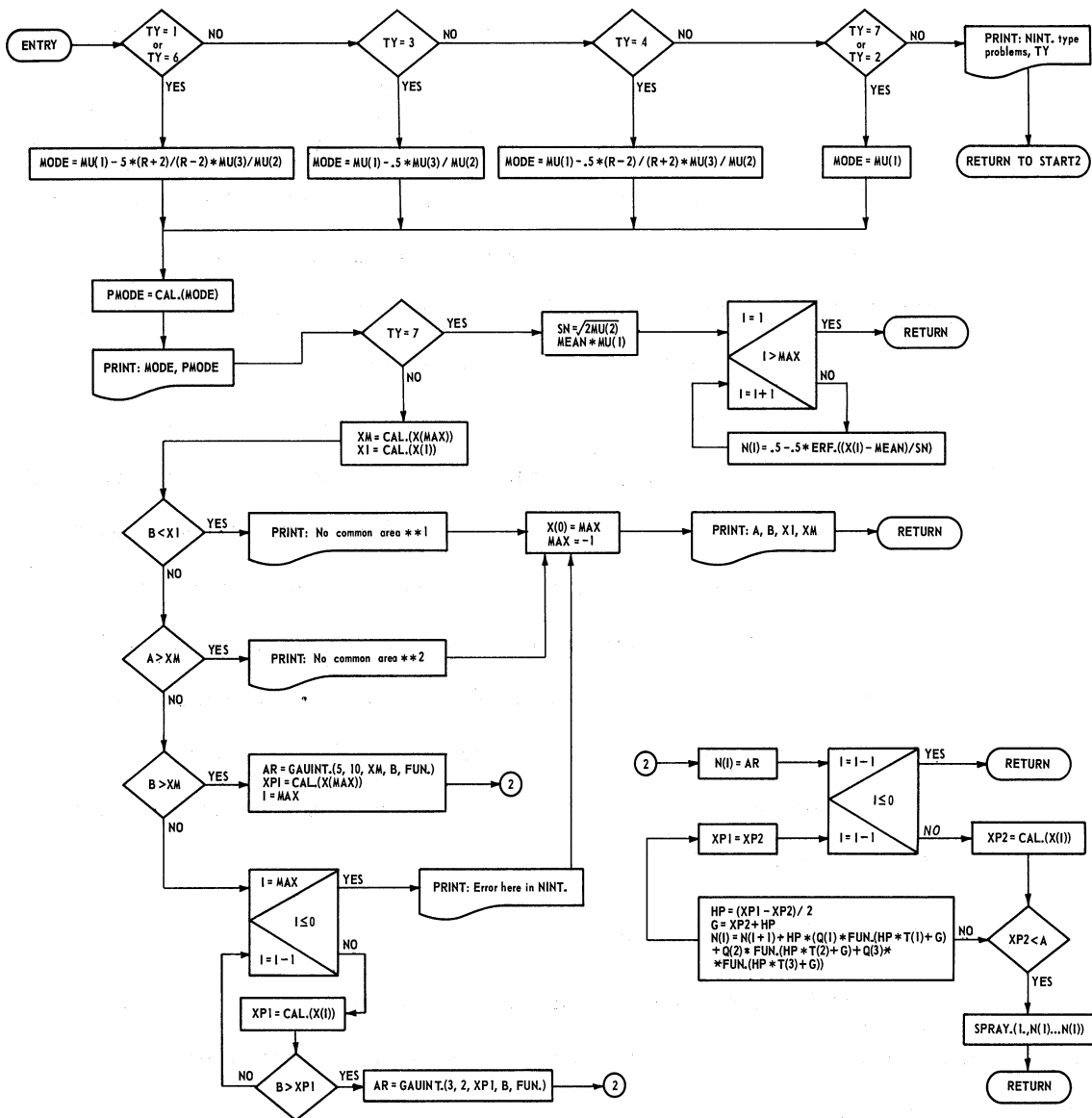


Fig. 18 Noise integration subroutine (NINT).

N - array where the complement of noise distribution ($1 - P(x|N)$) will be stored.

MAX - number of elements in X.

- (4) Output - returns to the calling program the computed values of the array N.
- (5) Method - the program evaluates the noise distribution at the values of X. The elements of X are converted, using CAL, to Pearson coordinates where the integration is performed using a three-point Gaussian quadrature (integration on tails of the density function). The values of the distribution function are stored in the array N.

After execution of the SNINT and NINT routines, the following table is stored in the computer.

$P(x SN)$	Threshold X	$1 - P(x N)$
0	D	. . .
.
.
.
.
.
.9

Since the probability of detection and the probability of false alarm are given by the complements ($1 - P(x|..)$) of the conditional distributions, the ROC curve for the receiver, along with threshold values, can be read out of computer storage. Note that the ROC curve is limited by the program to regions where the detection probability is greater than 10%. The 10% limitation is easily changed if a more general program is required.

The table uses the detection probability as its base for determining X. If one is interested in false-alarm rates, the roles of SNINT and NINT are reversed. This reversal permits the detection probability, for a fixed false-alarm range, to be evaluated.

5.10 Logarithm of Gamma Function Subroutine (LNGAM) (Figure 19)

This program computes the logarithm of the Gamma function of a given argument.

- (2) Required Subroutines - none.
- (3) X = LNGAM. (Z, LOC)

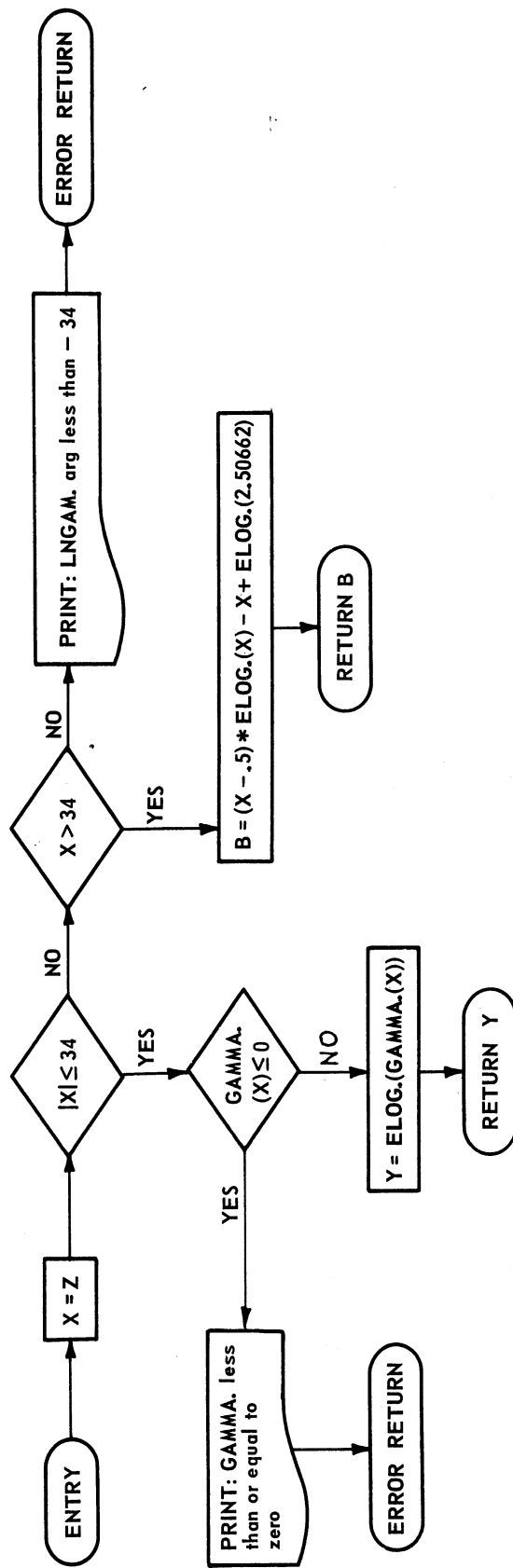


Fig. 19 Log gamma subroutine (LNGAM).

- X - the storage location of the computed answer.
 - Z - the argument of the ln-gamma function.
 - LOC - error-return location if $Z < -34$.
- (4) Output - function returns the value of $\ln \Gamma(z)$ to the location X or, when $Z < -34$, the function prints out Z.
- (5) Method - the program uses a MESS subroutine, GAMMA, for the $|Z| \leq 34$ and then computes the logarithm of the MESS output. For $Z > 34$, an asymptotic expansion of the ln Gamma function is used (Ref. 23).

5.11 Pearson Subroutines (PEARSN, PEARS2) (Figure 20)

The PEARSN subroutine takes two sets of central moments and evaluates a Pearson fit to the corresponding probability density functions. These probability density functions are integrated to give the ROC curve.

The PEARS2 program evaluates a Pearson fit to the central moment sets and stores the Pearson fit parameters in the calling program.

(2) Subroutines Needed

- (a) ALTER
- (b) CAL-CHAN-FUN
- (c) GAUINT
- (d) LNGAM
- (e) NINT
- (f) SNINT
- (g) TYPEA
- (h) TYPEB

(3) EXECUTE PEARSN. (MS, MN, BP)

EXECUTE PEARS2. (MS, MN, BP)

MS - a four-element array of the signal-plus-noise central moments

(Note: $MS(1) \triangleq a_1(SN)$).

MN - a four-element array of the noise central moments.

BP - a statement label return in the event of an error.

(4-a) PEARSN. Output - the function returns the following list to storage in program common.

- TY - the noise-alone Pearson type (integer).
- S(0) - the signal-plus-noise Pearson type.
- MAX - the number of elements in the ROC table (See NINT).
- S - an array containing the signal-plus-noise distribution function.
- N - the complement of the noise distribution, that is, $1 - P(x|N)$.
- X - an array of values, x , for which S and N are tabulated.

(4-b) PEARS2. Output - the function returns the Pearson type which fits the signal-plus-noise moment set, MS, and stores in common the parameters associated with the fit.

(5) Method - the Pearson program uses the moment set, MS, to evaluate Pearson's criterion, K which is then used to select the proper Pearson type. The constants needed for this type are evaluated by the TYPEA or TYPEB program. If the called routine is PEARS2., control is returned to the main program for additional instructions.

If the called routine is PEARSN, the program, using SNINT, evaluates the distribution function of the signal-plus-noise and stores the results in common. The program next takes the noise moment set, MN, finds the Pearson fit, and, using NINT, evaluates the complement of the noise distribution function. The function, after storing the distribution results in common, returns control to the main program.

The main program uses a PEARS2., routine to determine the approximation to the log-likelihood ratio nonlinearity. This approximation is used for the general-receiver evaluation. The PEARSN. routine is used in lieu of the Edgeworth program for the evaluation of the ROC curves.

SUMMARY

This report has introduced the evaluation problem for signal detection receivers and some computer techniques useful in the solution of the evaluation problem. A systematic method for approximating the receiver nonlinearity, such as the log-likelihood ratio, is outlined in Section 3. This method, based on the Pearson System of Frequency Curves, permits a broad class of detection problems to be considered, in contrast to the truncated Taylor series approximation (Ref. 2) which requires small input signal-to-noise ratios. The techniques of Section 3 are also applicable to raw data obtained from channel measurements and, thus, the method can be used for the receiver design and evaluation in practical detection problems.

The techniques of receiver evaluation given in this report have been applied to several theoretical models for amplitude-fading channels (Ref. 21). The results of this study, to be published in a future report, suggest the use of bandwidth spreading to increase the signal detectability for the same transmitted signal energy.

Noise and signal measurements are being collected from a practical receiver site. The receiver-evaluation methods will be applied to these data to determine the theoretical detectability of the receiver site and a comparison will be made with the actual performance. This study should provide further insight into the acoustic signal design and detection problems.

APPENDIX A
PEARSON'S EQUATION COEFFICIENTS AND
RANDOM VARIABLE MOMENTS

This appendix derives the relations between the coefficients of Pearson's differential equation and the moments of the random variable. Pearson's differential equation is

$$\frac{df(x)}{dx} = \frac{(x+a)f(x)}{b_0 + b_1x + b_2x^2}, \quad x \in [L, u]. \quad (\text{A. 1})$$

If the real number L or u is unbounded, the closed interval in (A. 1) is replaced by a semi-closed or open interval. An additional constant is the set of boundary conditions,

$$f(L) = f(u) = 0 \quad (\text{A. 2})$$

Multiplying both sides of (A. 1) by $x^k [b_0 + b_1x + b_2x^2]$ and integrating the result, one obtains

$$\int [b_0x^k + b_1x^{k+1} + b_2x^{k+2}] \frac{df(x)}{dx} dx = \int [x^{k+1} + ax^k] f(x) dx. \quad (\text{A. 3})$$

Integrating the left side of (A. 3) by parts and using (A. 2) yields the following result, in terms of the random variable moments:

$$\alpha_k a + k\alpha_{k-1} b_0 + (k+1)\alpha_k b_1 + (k+2)\alpha_{k+1} b_2 = -\alpha_{k+1}. \quad (\text{A. 4})$$

Setting k equal to 0, 1, 2, 3, one obtains the system of equations,

$$\begin{bmatrix} 1 & 0 & 1 & 2\alpha_1 \\ \alpha_1 & 1 & 2\alpha_1 & 3\alpha_2 \\ \alpha_2 & 2\alpha_1 & 2\alpha_2 & 4\alpha_3 \\ \alpha_3 & 3\alpha_2 & 4\alpha_3 & 5\alpha_4 \end{bmatrix} \begin{bmatrix} a \\ b_0 \\ b_1 \\ b_2 \end{bmatrix} = - \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} \quad (\text{A. 5})$$

If α_1 is set equal to zero, which is equivalent to taking moments about the mean, the system

of (A. 5) becomes

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3\mu_2 \\ \mu_2 & 0 & 3\mu_2 & 4\mu_3 \\ \mu_3 & 3\mu_2 & 4\mu_3 & 5\mu_4 \end{bmatrix} \begin{bmatrix} a \\ b_0 \\ b_1 \\ b_2 \end{bmatrix} = - \begin{bmatrix} 0 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix} \quad (\text{A. 6})$$

Solving (A. 6), one finds,

$$a = \frac{\mu_3(\mu_4 + 3\mu_2^2)}{10\mu_2\mu_4 - 18\mu_2^3 - 12\mu_3^2},$$

$$b_0 = \frac{\mu_2(4\mu_2\mu_4 - 3\mu_3^2)}{10\mu_2\mu_4 - 18\mu_2^3 - 12\mu_3^2},$$

$$b_1 = a,$$

$$b_2 = \frac{2\mu_2\mu_4 - 3\mu_3^2 - 6\mu_2^3}{10\mu_2\mu_4 - 18\mu_2^3 - 12\mu_3^2}. \quad (\text{A. 7})$$

APPENDIX B

PEARSON'S CRITERION FOR FREQUENCY CURVES

This appendix introduces the Pearson criterion, K, and relates the criterion to the three main Pearson types.

Pearson's differential equation is

$$\frac{df(x)}{dx} = \frac{(x+a) f(x)}{b_0 + b_1 x + b_2 x^2} \quad . \quad (\text{B. 1})$$

Dividing (B. 1) by $f(x)$ and integrating, one obtains

$$\ln f(x) = \int \frac{x+a \, dx}{b_0 + b_1 x + b_2 x^2} + \text{constant} \quad . \quad (\text{B. 2})$$

Setting the denominator of the integrand equal to zero and solving for the integrand pole locations, one obtains

$$x_{1,2} = -\frac{b_1}{2b_2} \left[1 \mp \sqrt{1 - \frac{4b_0 b_2}{b_1^2}} \right] \quad . \quad (\text{B. 3})$$

Pearson defined the following parameters:

$$\beta_1 \triangleq \frac{\mu_3^2}{\mu_2^3} \quad , \quad (\text{B. 4a})$$

$$\beta_2 \triangleq \frac{\mu_4}{\mu_2^2} \quad , \quad (\text{B. 4b})$$

$$K \triangleq \frac{b_1^2}{4b_0 b_2} \quad (\text{B. 4c})$$

The parameter β_1 is a measure of the skewness of the probability density function ($\beta_1 = 0$ implies symmetry). The parameter β_2 is a measure of the density function's kurtosis or flatness, relative to a Gaussian density function ($\beta_2 = 3$). The parameter K is the Pearson

criterion. Using K , then (B. 3) is rewritten as

$$x_{1,2} = \frac{-b_1}{2b_2} \left(1 \pm \sqrt{1 - \frac{1}{K}} \right). \quad (\text{B. 5})$$

Note that, based on the results of Appendix A, K can be written in terms of the parameters β_1 and β_2 , i. e. ,

$$K = \frac{\beta_1(\beta_2 + 3)^2}{4(4\beta_2 - 3\beta_1)(2\beta_2 - 3\beta_1 - 6)} \quad (\text{B. 6})$$

An examination of (B. 5) yields the following information. If $K < 0$, the roots $x_{1,2}$ are real and of the opposite sign which is Type I. If $0 < K < 1$, the roots are complex conjugate which is Type IV while for $K > 1$, the roots are real and have the same sign which is Type VI.

APPENDIX C
PEARSON'S MAIN TYPES OF FREQUENCY CURVES

This appendix derives the main Pearson types from the differential equation and the criterion K.

Type I, $K < 0$ $-a_1 \leq x \leq a_2$

$$\ln f(x) = \int \frac{x+a}{b_2(x-x_1)(x-x_2)} dx + \text{constant}, \quad (\text{C. 1})$$

where x_1 and x_2 are given by (B. 5). Using a partial fraction expansion on the integrand of (C. 1) and integrating yields

$$f(x) = c(x-x_1)^{A_1} (x-x_2)^{A_2}. \quad (\text{C. 2})$$

When the Pearson origin is translated to the mode of the density function,

$$f(x) = f_0 \left(1 + \frac{x}{a_1}\right)^{\nu a_1} \left(1 - \frac{x}{a_2}\right)^{\nu a_2}. \quad (\text{C. 3})$$

If a_1 equals a_2 , $f(x)$ is symmetric and limited in range. This corresponds to the Pearson subtype II. The criterion for this case is given by $K = 0$, $\beta_1 = 0$, $\beta_2 \neq 3$.

Type IV, $0 < K < 1$ $-\infty < x < \infty$

Since the roots, x_1 and x_2 , are complex conjugate, (B. 2) can be written

$$\ln f(x) = \int \frac{y+c}{b_2(y^2+A^2)} dy + \text{constant}, \quad (\text{C. 4})$$

where

$$y = x + \frac{b_1}{2b_2}, \quad (\text{C. 4})$$

$$c = a - \frac{b_1}{2b_2},$$

$$A^2 = \frac{b_0}{b_2} - \frac{b_1^2}{4b_2^2}. \quad (C.5)$$

Performing the integration indicated in (C.4) yields

$$f(x) = c(y^2 + A^2)^{2b_2} e^{\frac{c}{A} \tan^{-1}\left(\frac{y}{A}\right)}, \quad (C.6)$$

which, after an origin translation, gives

$$f(x) = f_0 \frac{e^{-\nu \tan^{-1}\left(\frac{x}{a}\right)}}{\left(1 + \frac{x^2}{a^2}\right)^m}. \quad (C.7)$$

Type VI, $1 < K$ $0 < a < x$ or $a < 0 < x$

The derivation is the same as for Type I except both roots have the same sign.

Therefore,

$$f(x) = c \left(1 + \frac{x}{a_1}\right)^{\nu a_1} \left(1 + \frac{x}{a_2}\right)^{\nu a_2}. \quad (C.8)$$

A shift in the origin yields the standard Pearson form for Type VI,

$$f(x) = f_0 (x-a)^{m_1} x^{-m_2} \quad (C.9)$$

Note that in (C.7) and (C.9), the symbol "a", which occurs must be evaluated from the terms ν , A , m , or ν , m_1 , m_2 . This symbol "a" should not be confused with the "a" which appears in the Pearson differential equation.

APPENDIX D
PEARSON'S SUBTYPES OF FREQUENCY CURVES

This appendix discusses several subtypes of the Pearson system. In Appendix B, an equation for the roots of the quadratic is given in terms of the Pearson criterion, K ; namely,

$$x_{1,2} = -\frac{b_1}{2b_2} \left(1 \pm \sqrt{1 - \frac{1}{K}} \right). \quad (D. 1)$$

In addition to the three main types, a number of interesting cases arise as the transition from one main type to another occurs.

(i) $K = \pm \infty$ ($b_2 = 0$)

This criterion leads to a Type III fit given by a density function

$$f(x) = f_0 e^{-\gamma x} \left(1 + \frac{x}{a}\right)^{\gamma a}. \quad (D. 2)$$

(ii) $K = 0$ ($b_1 = b_2 = 0$)

These conditions are representative of the Gaussian density function (Type VII). In Appendix C, subtype II was introduced with the criteria $K = 0$, $\beta_1 = 0$, $\beta_2 \neq 3$.

Type VII has the criterion $K = 0$, $\beta_1 = 0$, $\beta_2 = 3$. While Type II is symmetric and finite in range, Type VII is symmetric but infinite in range. The density function is

$$f(x) = f_0 e^{-\frac{1}{2} \frac{x^2}{b_0}}. \quad (D. 3)$$

(iii) $K = 1$ (real and equal roots)

This criterion gives the Type V curve whose density function is

$$f(x) = f_0 x^{-p} e^{-\gamma/x}. \quad (D. 4)$$

The following table lists the twelve Pearson types along with their range and skewness properties.

Table I. Pearson Curves

<u>Type</u>	<u>Curve</u>	<u>Range, Skewness</u>
I	$f_0 \left(1 + \frac{x}{a_1}\right)^{\nu a_1} \left(1 - \frac{x}{a_2}\right)^{\nu a_2}$	Finite range, skewed
II	$f_0 \left(1 - \frac{x^2}{a^2}\right)^{\nu a}$	Finite range, symmetric
III	$f_0 \left(1 + \frac{x}{a}\right)^{\gamma a} e^{-\gamma x}$	Semilimited range, skewed
IV	$f_0 \left(1 + \frac{x^2}{a^2}\right)^{-m} e^{-\nu \tan^{-1}\left(\frac{x}{a}\right)}$	Unlimited range, skewed
V	$f_0 x^{-p} e^{-\frac{\gamma}{x}}$	Semilimited range, skewed
VI	$f_0 (x - a)^{m_1} x^{-m_2}$	Semilimited range, skewed
VII	$f_0 e^{-x^2/2\sigma^2}$	Unlimited range, symmetric
VIII	$f_0 \left(1 + \frac{x}{a}\right)^{-m}$	Limited range, skewed
IX	$f_0 \left(1 + \frac{x}{a}\right)^m$	Limited range, skewed
X	$f_0 e^{-\frac{x}{\sigma}}$	Semilimited range, skewed
XI	$f_0 x^{-m}$	Semilimited range, skewed
XII	$f_0 (x + a_1)^{m_1} (a_2 - x)^{m_2}$	Limited range, skewed

APPENDIX E
SOME PROPERTIES OF THE LIKELIHOOD
RATIO TRANSFORMATION

This appendix will derive certain properties which the likelihood ratio transformation induces on the conditional, probability density functions.

The likelihood ratio of an observation, (y_1, y_2, \dots, y_n) is

$$\ell(y_1, y_2, \dots, y_n) = \frac{f_Y(y_1, \dots, y_n | \text{SN})}{f_Y(y_1, \dots, y_n | \text{N})} . \quad (\text{E. 1})$$

Associated with every observation (y_1, \dots, y_n) is the image of the transformation given in (E. 1). If this image is denoted by the symbol " ℓ ", then " ℓ " is a random variable with conditional probability density functions, $f_\ell(\ell | \text{SN})$ and $f_\ell(\ell | \text{N})$. Using the characteristic function approach (see Section 2. 3), one obtains

$$f_\ell(\ell | \text{SN}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iv\ell} \int_{-\infty}^{\infty} e^{iv\ell(y_1, \dots, y_n)} f_Y(y_1, \dots, y_n | \text{SN}) dy_1, \dots, dy_n dv. \quad (\text{E. 2})$$

Substituting (E. 1) for the density function in (E. 2), one obtains

$$f_\ell(\ell | \text{SN}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iv\ell} \int_{-\infty}^{\infty} e^{iv\ell(y_1, \dots, y_n)} \ell(y_1, \dots, y_n) f_Y(y_1, \dots, y_n | \text{N}) dy_1, \dots, dy_n dv. \quad (\text{E. 3})$$

Thus,

$$f_\ell(\ell | \text{SN}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iv\ell} \left[\frac{\partial}{\partial(iv)} \int_{-\infty}^{\infty} e^{iv\ell(y_1, \dots, y_n)} f_Y(y_1, \dots, y_n) dy_1, \dots, dy_n \right] dv. \quad (\text{E. 4})$$

By definition, the inner integral of Eq. (E. 4) is the characteristic function of the likelihood transformation under the noise hypothesis. It follows from Eq. (E. 4) that

$$M_\ell(iv) \Big|_{\text{SN}} = \frac{\partial}{\partial(iv)} M_\ell(iv) \Big|_{\text{N}} . \quad (\text{E. 5})$$

A well-known property of Fourier Transformation Theory (Ref. 6) is the transformation of

differentiation into multiplication. Therefore,

$$f_{\ell}(\ell | \text{SN}) = \ell f_{\ell}(\ell | \text{N}). \quad (\text{E. 6})$$

If ℓ is considered as a random variable, then, from (E. 6),

$$\ell(\ell) \triangleq \frac{f_{\ell}(\ell | \text{SN})}{f_{\ell}(\ell | \text{N})} = \ell; \quad (\text{E. 7})$$

that is, the likelihood ratio of the likelihood ratio equals the likelihood ratio.

Let g be any mapping on the random variable ℓ with the property that the inverse of g exists and is continuously differentiable and let Z denote the image of g . Then,

$$\begin{aligned} Z &= g(\ell), \\ \ell &= g^{-1}(Z). \end{aligned} \quad (\text{E. 8})$$

Using the theory of transformations to find the probability density function of the random variable Z yields

$$\begin{aligned} f_Z(z | \text{SN}) &= f_{\ell}(g^{-1}(z) | \text{SN}) \left| \frac{\partial g^{-1}(z)}{\partial z} \right|, \\ f_Z(z | \text{N}) &= f_{\ell}(g^{-1}(z) | \text{N}) \left| \frac{\partial g^{-1}(z)}{\partial z} \right|. \end{aligned} \quad (\text{E. 9})$$

Thus, the likelihood ratio of the random variable z is

$$\ell(z) \triangleq \frac{f_Z(z | \text{SN})}{f_Z(z | \text{N})} = \frac{f_{\ell}(g^{-1}(z) | \text{SN})}{f_{\ell}(g^{-1}(z) | \text{N})} = g^{-1}(z). \quad (\text{E. 10})$$

Since, from Eq. (E. 8),

$$g^{-1}(z) = \ell, \quad (\text{E. 11})$$

the likelihood ratio of any monotone function of the likelihood ratio equals the likelihood ratio.

The existence of the inverse function is guaranteed by the monotonicity of the function.

Although the inverse function may not be differentiable everywhere, the derivative of the inverse occurs in both the numerator and denominator of (E. 10) and these nondifferentiable points are removable in the limit.

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Page

- 4 In the third line, circle the plus sign between X and N.
- 9 In the second line from the bottom of the page, replace "Eq. 2. 11" with "Eq. 2. 12."
- 13 In the first line above Eq. 2. 37, add a right parenthesis after "(2. 5)."
- 13 Footnote 5 should read " $\delta_{mn} = 1$ for $m=n$, $\delta_{mn} = 0$, otherwise."
- 15 In Eq. 2. 45, the denominator should read " $\sqrt{n} \sigma(x)$."
- 18 Equation 2. 64 should read " $L(y_1) = m_1 y_1 - \frac{m_1^2}{2} + \ln \{ 1 + \mu_2(x_1) H_2(y_1 - m_1) + \dots \}$."
- 29 After the second line, add "The Pearson System is then a tool which can be used for the design and evaluation of detection receivers which are to operate under practical conditions. A Pearson fit to the observation statistics yields a closed form approximation to the log likelihood ratio mapping which is suitable for the computation of the receiver output moments."
- 31 In Eq. 4. 8 and Eq. 4. 9, the limits on the integrals should be $-\infty, \eta$.
- 33 In Eq. 4. 22, replace " $\alpha_k(x_1)$ " with " $\alpha_k(x_1 | \dots)$."
- 41 In the tenth line from the bottom of the page, add an asterisk before "External Function."
- 72 In the fifth line from the bottom, insert the word "strictly" between "any" and "monotone."
- 72 In the fourth line from the bottom, replace the word "monotone-ness" by "strict monotonicity."