WORKING PAPERS OF THE

CENTER FOR RESEARCH ON SOCIAL ORGANIZATION

DEPARTMENT OF SOCIOLOGY

UNIVERSITY OF MICHIGAN

Paper # 27

April, 1967

ON THE STRUCTURE OF SHARED AWARENESS

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April, 1967

Please do not
quote Section IV
without con-
sulting the
author.

I. Introduction

This paper will discuss who thinks who thinks what, and so what, from various perspectives. First, a mathematical language will be developed for "A thinks B thinks A thinks..." type propositions. This will elucidate their interrelations and make explicit some assumptions used in dealing with them. It will begin to allow one to avoid the mental strain associated with such propositions. The next sections, on Structure and Dynamics, will discuss substantive topics, especially the power of perceived expectations over the individual. The final section, Economics, will deal with A thinks B thinks... reasoning in formal games.

The reader will notice that the mathematical sections are limited in applicability by their specialized assumptions and vocabulary, while the substantive sections tend to be essayistic. This paper may be considered to represent the two ends of a bridge whose middle part has not yet been constructed. However, it may be instructive to demonstrate the possibility of formalization in what has been one of the "softest" areas in sociology.[1] This will perhaps lead to greater clarity and thus to greater refutability for the theoretical apparatus in this area.

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1. Statements like, "Autrui n'existe qu'aux yeux du Moi; le Moi n'existe qu'aux yeux d'autrui. Si connaître autrui c'est aussi se regarder, chercher à se perdre dans l'autre, c'est encore se retrouver," (1962) have long since served their purpose. Compare Shakespeare: "Property was thus appalled,/ That the self was not the same;/ Single nature's double name/ Neither two nor one was called."

An attempt is made to provide enough informal comment to make all but the final section, which requires an elementary knowledge of game theory, accessible to the less mathematical reader; unexplained technical remarks are peripheral to the argument.

Let us orient ourselves to the study of A thinks B thinks... by means of an example. Consider the sequence of propositions:

1) A thinks he is despised.

2) B thinks A thinks he (A) is despised.

3) A thinks B thinks A thinks he (A) is despised.

4) B thinks A thinks B thinks A thinks he (A) is despised.

. . . .

("Thinks" is always used to mean "would assent to if given ample opportunity to reflect." It does not mean, "is actively engaged in thinking about." "Believes" will sometimes be used as a synonym for "thinks".) As we progress along this sequence the propositions become simultaneously harder to understand and more unnatural.[2] Several questions are raised.

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2. Maucorps and Bassoul (1962, p. 47) speak of an "effort de gymnastique mentale." It is perhaps related to the strain associated with assessing one's subjective probabilities. There is a similar "jeu de miroirs" for these: how certain am I of my degree of belief for...? Combinations are possible: How certain are you that I believe it is more likely than not that you believe...? I suggest the nature and consequences of this strain as a rewarding subject for research; and that it depends not merely on complexity but involves important Gestalts. A mathematical model for subjective probability which took strain systematically into account would be interesting.

1) Do they remain logically meaningful? Yes. This is best perceived by rewriting the propositions as follows, so they do not get longer and longer:

1) A thinks he is despised.

2) B thinks "1)" is true.

3) A thinks "2)" is true.

4) B thinks "3)" is true.

. . . o c o o o . o o

2n-1) A thinks "2n-2)" is true.

2n) B thinks "2n-1)" is true.

o . o . o . . . o . o

Now it seems quite unreasonable to say, e.g., that statements 1 through 8 are meaningful but statements 9 and on are meaningless. In fact, it is clear that all that is necessary to prove that all these statements are meaningful is to postulate: if A names a person and "x" is a meaningful proposition, then the proposition, "A thinks "x" is true," is meaningful. This postulate appears indisputable, and will be assumed throughout.

2) Are propositions far along in our sequences (These will be called "long" propositions.) ever empirically valid? Yes. As a rather exaggerated example, let A and B be two mathematicians who have enjoyed a long and intimate professional relationship. Our statements are:

1) A thinks $2 + 2 = 4$.

2) B thinks A thinks $2 + 2 = 4$.

. o . . o . o o o . o o

Clearly the whole infinite set of statements are valid.
Indeed, there is a name for this situation. The belief that
2 + 2 = 4 is said to be shared, or taken for granted between
them, or a matter of common opinion. Common opinion will be
discussed extensively below.

3) Given the answers to 1) and 2), why are long
propositions so rarely used? I don't really know. Among
the reasons are probably a) their difficulty, b) that in
practice they may be implied by other kinds of propositions,
and c) that information about them may be difficult to obtain.
(This is partly for normative and practical reasons, such as
regulate asking questions like, "What do you really think
about me?") Finally, information about long propositions
may be of little practical use. But the example in the final
section of this paper shows that a law of diminishing returns
does not necessarily hold.[3]

3. Thomas J. Scheff (1967, p. 37) adduces the image of the
    details in the endless chain of reflections in two
    opposing mirrors coalescing into a formless blur and
    suggests that empirical instances of what I call common
    opinion might be hard to find. On the contrary, long
    propositions which are believed but are not common opinion
    (or are not thought to be, etc.) should be rare; common
    opinion, itself, is very common.

II. Language

Basic Terminology.--The objects of belief are pro-
positions, which will be considered as determined by English
sentences. Propositions form a Boolean algebra. This means
that if x and y stand for any two propositions: "x and y"
(written $x \cap y$), "x or y or both" (written $x \cup y$), and not-x
(written $\bar{x}$) are also propositions, and "and", "or", and "not"
obey the usual rules.[4] Two logically equivalent propositions
will be considered the same (written =). So, for example,
we have $x \cap y = y \cap x$, which implies that "Jack is tall and
Jill is small," means the same as "Jill is small and Jack is
tall."

There is a finite set of "person operators", A_1,
$A_2, \ldots, A, B, C, \ldots,$ functions which transform propositions into
(usually other) propositions. A will also be used as a
generic (variable) person operator. "Ax" means "Person A
thinks 'x' is true (at a given time)."[5] If x is a proposition
Ax is also: This is a restatement of the meaningfulness postulate
in the Introduction.

Note that a simplifying idealization already has
been made. To illustrate this, define a = "The sun rose today."
and b = "The sun rose today and the tenth digit of pi is 3."

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4. Logic will always be sacrificed to perspicuity in the use
   of quotation marks.
5. Use of the same symbols for both persons and person oper-
   ators should not cause confusion. "Epistemic operators"
   is a good name for all the operators that can be generated
   from person operators by the methods described below.

These are logically equivalent (say, in terms of a verifiability criterion) so a = b. Then Aa = Ab, implying that anyone who believes one of these propositions must believe the other--absurd. This distortion is allowed in the belief that actors' problems in logical reasoning usually can be ignored in the analysis of social interaction. Rather, opinions vary mainly because of differences in data and perhaps in induction. So the effect of the assumption that everyone is logical (and so must know all logical facts like those about decimals in pi) should not ordinarily be serious.[6]

Rationality Laws.--Let t be the unique logically true proposition and f = $\bar{t}$. Rationality (consistency)

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6. Two kinds of idealization are represented by the statements, "The average man in 5'9" tall," and "The average family has 2.8 children." The present model is of the latter kind and must be applied with due caution--it, so to speak, ignores air resistance.

    The criticism, "Men are not computers, and thus models that treat them as rational are helpless to approach the more interesting phenomena in their lives," is not only false but shallow. "For social control is more homogeneous, more consistent from person to person than individual, rational control. It permits a smooth predictability in the affairs of men." (Slater, 1963, p. 346; this is an excellent discussion.) People try to reason about what (A thinks B thinks) in struggles for power, triangles of love, and in comedies and tragedies. Two major empirical studies have dealt with troubled marriages (Laing et al., 1966) and the awareness of dying (Glaser and Strauss, 1965) respectively. Of course, in "crises of awareness" people do not self-consciously use mathematical reasoning, but neither do they self-consciously use Euclidean geometry in getting around.

requirements are formulated as laws for person operators.[7]

(1)   At = t

(2)   Af = f

(3) $A(x \cap y) = Ax \cap Ay$

     For example, the third one reads: 'A believes x and y' means the same as 'A believes x and A believes y.' Notice that "or" cannot be substituted for "and". The first and second idealize the actors as ones for whom it is logically impossible to be mistaken about logic.

     Write $\leq$ for "logically implies." We now derive some theorems from the laws.

     (a): $Ax \leq A\overline{x}$. If A believes x then A cannot believe not-x. Since (a) is a statement of logical truth it is believed by all the actors, according to law (1). The proofs of most of the theorems are in the Appendix.

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7.   No attempt has been made to be exhaustive in listing these laws and the ones given later. For example: "For some x, $Ax \neq x$," is a rather trivial law independent of those stated here.

Hintikka (1962) has a related but much more complex set of rules. Mine are just complicated enough to derive interesting theorems about combinations of person operators. The reader interested in subtleties of the relevant epistemology on the individual level is referred to his book.

Could paradoxes arise since no propositions are excluded from consideration, even propositions which refer to the language itself? I rather think not. Mathematical paradoxes should be avoided because the Boolean algebra is only countable. Criteria of meaningfulness external to the model are assumed--this should obviate semantic paradoxes. For example, "This sentence is false," is excluded as not being or representing a proposition.

(b): $Ax \cap A(x \Rightarrow y) \leq Ay.$ [8]

(c): $A(x \Rightarrow y) \leq (Ax \Rightarrow Ay).$

(d): $(x \leq y) \leq (Ax \leq Ay).$

(e): $Ax \cup Ay \leq A(x \cup y).$

The reader may wish to translate these theorems into ordinary English and verify them intuitively.

Self-knowledge.--The following self-knowledge laws handle beliefs about one's own beliefs. Complete self-knowledge is assumed:

(4): $AAx = Ax$

(5): $A\overline{Ax} = \overline{Ax}$

(4) is translatable: what A believes he believes is by definition coextensive with what he believes.

The following are among the more interesting consequences:

(f): $\overline{A\overline{Ax}} = Ax$

(g): $A(Ax \cup Ay) = Ax \cup Ay$

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8. $x \Rightarrow y$ is read, "If x, then y," or "x only if y," or "x implies y." It means that it is false that x is true but y is false. This is all it means: there need be no especial connection between x and y.

It is easy to confuse $\leq$ and $\Rightarrow$. (If it is any help, they are related as are $=$ and $\Leftrightarrow$, the sign for "if and only if".) $x \Rightarrow y$ is formally defined as $x \cap \overline{y}$; $x \leq y$ is defined as obtaining when $(x \Rightarrow y) = t$. Whenever $x \leq y$, $x \Rightarrow y$ is always true, but not necessarily conversely. $x \leq y$ is itself a proposition, whose validity (unlike that of $x \Rightarrow y$) does not depend on which particular propositions happen to be true. So $(x \leq y) = t$ or $(x \leq y) = f$, and $(x \leq y) = A(x \leq y) = B[A(x \leq y)]$, etc. Knowledge of the language is never in question among actors. (The reader may test his understanding by verifying that $(x \Rightarrow y) \Rightarrow (Ax \Rightarrow Ay)$ may be false. Cf. theorem (d).)

In a sense, we have a three-valued logic. For any x, A can believe x, can believe not-x, or can believe neither. A more elaborate language might systematically use statements like $A_p^{(t)} x$, meaning: A believes at time t that x has probability greater than p.

(h): $A(Ax \Rightarrow x) = t.$ [9]

Of course, Ax may be true while x is false. Contrast the laws for "A knows x," meaning $Ax \cap x$, i.e., A believes x and x is true (not necessarily, "A believes x because x is true."). If I is the identity operator defined by Ix = x, and if the intersection $(M \cap N)x$ of two operators M and N is defined as $Mx \cap Nx$, then the "A knows" operator is $A \cap I$. $A \cap I$ satisfies laws (1), (2), (3), (4), but not (5): a person need not know that he doesn't know what he doesn't know. [10]

Opinions of others' opinions.--"A thinks B thinks x," means "A thinks that 'B thinks x,' is true," and equals A(Bx). AB, the composition of the person operators A and B, may be defined by (AB)x = A(Bx); the parentheses may be removed. So AB translates "A thinks B thinks"; similarly, $A \cap B$ translates "A and B think." Now arbitrarily long "polynomials" of person operators and the identity operator may be formed. E.g., $(AB)^2 \cap ABC$ means "A thinks B thinks A thinks B thinks and, also, A thinks B thinks C thinks." These polynomials satisfy laws (1), (2) and (3) (but not, in general, (4) and (5)) and therefore theorems (a) through (e). They

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9.   (h) shows that person operators are not invertible, and that there are statements that a rational believer must believe, even though they may be empirically disconfirmable. He must believe that his beliefs are true. (Cf. Hintikka, 1962.)

10. The operator I could be read, "God knows;" of course, it satisfies (1), (2), (3), (4), (5). The laws for person operators have interesting analogs in topology and logic. Thus, $A \cap I$ is an interior operator (cf. Sikorski, 1964, p. 198) and, if the postulate $Ax \leq x$ (useful at times) were added, A would be a universal quantifier (Halmos, 1962, p. 22, with the misprint in his (Q'2) corrected.)

also satisfy the distributive laws: $L(M \cap N) = LM \cap LN$ and $(L \cap M)N = LN \cap MN$. But remember $AB \neq BA$.

The empirical literature goes up to about the fourth "degree," with Maucorps and Bassoul eliciting responses like, "Je crois que...tu penses que je pense que tu t'attends à être choisi...par moi," (1962, p. 48), or Laing et al asking (1966, p. 57), "How would she think you have answered the following? 1. She loves me..."

Common opinion. -- Suppose Ax and (taking the role of the other) A believes that B is in the same cognitive position as A himself (say, because their experiences with x and with each other were symmetrical). A believes, that is, that if S is a polynomial formed from A and B, and S' is the polynomial obtained from the expression for S by interchanging A and B, then Sx and $S'x$ are true or false together. Now, A believes "Ax is true," and A thinks this symmetry exists, so A thinks Bx. A is aware that he (A) thinks Bx, so he could use symmetry again to get $A(BAx)$, and so on. It turns out that A thinks that x is a matter of common opinion (Co) between A and B or, symbolically: $ACo_{A,B}x$. Perceived symmetry, then, is one basis for common opinion.

Another approach to common opinion will also be introduced by an example. Suppose A and B are in face-to-face contact and hear from a voice whose authority is not in question, "You both believe x." Now A and B, as well as believing x, both believe that both believe it, both believe that both believe that both believe it, etc., and x becomes

common opinion. Another way to look at this is: $Co_{A,B}x$ is the most prominent solution to the "equation"; "A and B believe x and this statement." The mathematical formalization of this type of thing must be more difficult than before, due to the so-to-speak infinite operations. (The less mathematical reader should skip to the next subsection.)

It seems desirable to assume a stronger version of law (3) for person operators:

(3') If $\bigcap_{i\in\sigma} x_i$ exists, then $\bigcap_{i\in\sigma} Ax_i$ exists, and $\bigcap_{i\in\sigma} Ax_i = A\bigcap_{i\in\sigma} x_i$.

(3') reduces to (3) when σ is finite.

Theorem (i): If $\{L_i\}$, M, N satisfy 1), 2) and 3') so does $\bigcap_i L_i$, if meaningful, and so does MN. Also the distributive laws: $(\bigcap_i L_i)M = \bigcap (L_i M)$ and $M(\bigcap_i L_i) = \bigcap (ML_i)$ hold whenever $\bigcap L_i$ is defined.

If "A and B believe x and this statement," does denote a proposition, the proposition ought to satisfy $y = (A\cap B)(x\cap y)$. This, however, lacks a unique solution ($y = f$ is a solution.) But there is a greatest solution (with respect to the partial order of the Boolean algebra), whose reasonableness is supported by the fact that it is thus a solution with maximal likelihood of being true.

Theorem (j): If $y = v\cap My$, M satisfies (3') and $\bigcap_{i=0}^{\infty} M^i v$ is defined, then $y = \bigcap_{i=0}^{\infty} M^i v$ is the greatest solution. ($M^0 = I$ by definition.)

There seems no reason not to postulate:

$$(6) \quad \bigcap_{i=0}^{\infty} (\bigcap_j A_j)^i = Ck_{A_1, A_2, \ldots} \text{ is always defined.}[10a]$$

Applying (j) to the previous discussion with $v = (A \cap B)x$, $M = A \cap B$, we obtain $Co_{A,B} = \bigcap_{i=1}^{\infty} (A \cap B)^i$ --just what A and B would learn from the authoritative voice above.

Properties of shared awareness.--We have formally defined Co and Ck (common knowledge, or true common opinion) and we proceed to delineate some of their properties. For example, common opinion is what everybody thinks is common knowledge or, equivalently, what it is common knowledge that everybody thinks. This is part of theorem (k):

$Co = CkP = PCk = CoP = PCo$, $Ck = Co \cap I = Ck(P \cap I) = (P \cap I)Ck$,

where P is the intersection of the person operators involved ("Everybody thinks"). Co and Ck each satisfy laws 1), 2), 3') (stronger than 3) and 4), but not 5). The most interesting result is perhaps that $Co^2 = Co$. If something is common opinion it is common opinion that it is common opinion. E.g., not only is it Co that flat-earthers are crackpots but it is Co that everybody believes this. Co and Ck share this "idempotence" property (law (4)) with person operators but not with intersections and compositions of person operators. Public opinion, then, acts in a special way like an individual consciousness, and can be considered as a generalized Other.

10a. If S is a polynomial in person operators, $\bigcap_{i=0}^{\infty} S^i$ is probably always meaningful as well, but sometimes hard to interpret. Is there a scenario where Sx is true if and only if $S = \bigcap_{i=1}^{\infty} (ABC)^i$ for some σ? Systematic postulates for the meaningfulness of various propositions about symmetry would probably be desirable.

We conclude this section by indicating two phenomena that may arise where more than two persons are involved. First, it is possible to have $(Ck_{A,B} \cap Ck_{B,C} \cap Ck_{A,C} \cap \overline{Ck_{A,B,C,}}) x$ -- suppose B and C are a quarreling couple who have each secretly confided to A the topic of their quarrel.

Second, suppose D learns y, tells E, E tells this (including the conversation with D) to F, F tells all this back to D, D talks to E again and tells all this. We have $Ck_{D,E} Ck_{F,D} Ck_{E,F} Ck_{D,E} y$. Since there is always something new we never get to the stopping place of $Ck_{D,E,F}$. But instead, suppose E tells D, truthfully, about y and that he will tell all to F. Then (after sufficient time has elapsed) $z = Ck_{D,E} Ck_{E,F} (y \cap z)$ is satisfied by what is known, and the maximal solution is $Ck_{D,E,F} y$. So in this fashion triads with pairwise communication can achieve common knowledge.

III.   Structure

In the previous section a language was developed for who knows who knows what.   Now the language can be used to formally discuss the distribution of knowledge within a culture.

One might begin a taxonomy with "common sense", defined here as that which is common opinion between typical strangers in a culture, or among everyone who is a full participant in the culture: $Cs = Co_{A_1, A_2, \ldots}$.   Probably, assumed common sense tends to be ontogenetically and phylo-genetically prior to self-consciously private opinion, as awareness precedes self-awareness.

In the mathematical model logical tautologies are automatically assumed part of common sense; more generally, rules of language may be assumed so. Also there is a range of natural science facts which are common sense, e.g., "The earth is round." The example is one which it is common knowledge was once not common sense, and so has a certain prominence which most examples of common sense lack.   There is usually no point in communicating common sense, and one may lack self-conscious awareness of it, as the invisible obvious.   In fact, utterance of common sense propositions is often associated with impropriety, since it tends im-plicitly but pointedly to bring into focus the relation be-tween the speakers. During the "Initial Calibration" phase of an encounter, when the participants are 'tuning in' on

each other[11]. the interpersonal relation must be handled; it seems required then that propositions, if they be uttered at all, be common sense--presumably, to leave the participants free to concentrate on the situation. Even here the blatantly obvious, like the Fijian greeting, "You are awake," (Parsons, 1914) is improper unless extremely conventional. "It's thundering," as chat about the weather, might elicit a response like, "I'm not deaf."

Common sense may also be appropriate during psychiatric examinations (indicative of why it is usually inappropriate otherwise)[12]. Even here there is a place for tact. The taboo against expressing common sense is dysfunctional for science but continues to influence scientists. It is only partly circumvented by the use of mathematical or stylistic adornment, or by illustration with data not generally available. Perhaps this partly explains why common sense itself has not received the study it deserves.

A fascinating region of common sense is what everybody is expected to know about himself. This is an important aspect of the culture's definition of the role of the individual. Define $A*x$, "A knows about x," as $(A \cap I)x \cup (A \cap I)\bar{x}$. Then let us say that a function $x_j(A_i)$ from individuals to propositions is "biographical" if:

$$Cs \bigcap_i A_i^* \; x_j(A_i)$$

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11. See Pittenger, et al., (1960).
12. "...he may be given ... questions to ascertain if he can give the dates of well-known events," (Luria, 1966, p. 310.)

An example of a biographical function is "A's name is John," for it is Cs that everybody knows whether or not his own name is John.[13] Biographical functions generally seem to refer to the individual's history of placement in social categories: e.g., sex, military service record, number of children, but there are a number of interesting types of divergence from the demographic items on a questionnaire. There is knowledge contingent on relationship's, e.g., "wife's eye color," and that which is private in various senses: age at first coitus, or presence of toothache. A's biography might be defined as the set of true $x_j(A)$. Goffman (1963) has vividly described the problems entailed by deviant biographies.

Intimacy of interpersonal relationships is measured by degree of common knowledge of biographies. Acquaintances tend to know only those parts of each others' biographies which it is common sense are not secret[14]—except sometimes when their encounter is understood as fleeting. For closer relationships there are prescriptive norms for biographical knowledge--e.g., of the physical location of close kin.

More specialized roles than that of individual are also associated with fields of knowledge. The charisma of expertise is related to the Durkheimian sacredness of the individual in an organic society, in that everyone (if not

13. The propositions A knows about form a Boolean algebra, as do the biographical functions.
14. Cf. Simmel, (1964). I am diffusely indebted to Simmel with respect to both content and method.

defined as sick) is the expert on himself.

As Plato envisaged, the logical stratification of knowledge is linked with social stratification, although in a subtle and complex manner. Vertical (logical implication) relations among propositions are much more multidimensional than those among persons. An intermediate order is provided by what might be called "common knowledge implication", or cki: "x cki y" obtains when it is common knowledge that one who knows about x also knows about y. cki allows one to draw far more extensive conclusions than mere logic would justify.[15] Thus the cki relation is basic to impression management. Consider x above y if x cki y. Then, e.g. in sociology, knowing the true names of most of the little communities that have been studied is rather high, and accrediting, while ignorance of the identity of Talcott Parsons is very low and discrediting.

More generally, roles are stratifiable by knowledge, e.g., it is common sense that a doctor knows about what a medical technician knows about. In formal organizations, though, it is often sufficient that the superordinate merely know what the subordinate knows about. Stratified knowledge can also be that of other members of a group -- one can define the insider,

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15. Compare and contrast the "logic" used in balance theory, as in Abelson and Rosenberg (1958). cki is a quasi-order.

the gossip, etc.

For simplicity, I have been talking of common sense rather than of perceived common sense. Of course, the content of this cultural apparatus varies among individuals, like the idiolects of a language. I think the forms are more con- stant than the content--and that this is, itself, a matter of common sense, enabling empathy somewhat to transcend cultural barriers.

IV. Dynamics

How does knowledge change over time? Are there equilibria in the distribution of knowledge and, if so, how are they maintained? These are what I would call dynamic questions.

Acquisition of knowledge.--New knowledge can be acquired by perception. Vision is particularly interesting because of this common sense property: if B sees A look at B, then A sees B look at A. From this and a few simpler properties one can demonstrate that eye contact leads to common knowledge of the presence of the interactants. It is no coincidence that eye contact is of considerable emotional and normative significance.

Common knowledge always seems to be built up from previous common knowledge. Suppose we are on a picnic. I know that if it starts raining this will be common knowledge among us. For rain is a public event. It is in the class of events for which it is and (it is Ck) it remains common knowledge that every member of a group will know about such events when they happen. And it follows from theorem (1), $Ck(x \Rightarrow Px) = Ck(x \Rightarrow Ck\ x)$, that it is Ck that public events will be Ck when they happen. The public location for events is a vital taxonomic category, as illustrated by its utility for research (cf. Lofland, 1966).

Most important, knowledge comes from communication. Normally, face-to-face utterances are public for the par-

ticipants. And, normally, through communication knowledge becomes common knowledge.[16] The possible abnormal ramifications are very extensive. When a question is asked, even taking communication channels for granted, all that may be certain is that the questioner believes it is in his interest to ask the question. For (recalling Descartes) we can say that even if the questioner was only concerned to give that impression, it would still be in his interest to ask the question. On the other hand, the questioner may know the answer already, or he may not, and if he knows it he may think the other does not know he knows it or not, and if..., etc.

"Abnormal" equilibria.--Of greatest interest are equilibria short of the "normal" equilibrium of common knowledge, such as that of pluralistic ignorance. The notion of pluralistic ignorance does not uniquely specify a cognitive situation, but includes cases where everybody thinks a given way, but everybody thinks everybody else thinks that the opposite is common knowledge: $Px \cap \bigcap_{i \neq j} A_i A_j Ck\bar{x}$. Drinking practices are the topic of a classic sociological example. Using theorem (m): $\bigcap_{i \neq j} A_i A_j Ck = Ck \bigcap_{i \neq j} A_i A_j$, we see that there is a so-to-speak hidden but real consensus in this situation of pluralistic ignorance--a consensus as to other's expectations for oneself. Paradoxically, pluralistic ignorance is

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16. Compare the consideration of communication as a process of equalizing information by Newcomb et al, (1965) and the derivation of Co from symmetry, above.

stable because of this consensus. E.g., one hides one's drinking behavior not so much because one thinks other's practices are different but because one thinks others' expectations for one's own practices differ from the reality. For the community to achieve consensus on drinking, this consensus on expectations would have to be temporarily destroyed.

Another schema for equilibrium short of Ck is that of the shy lovers. The fact of the lack of avowal is common knowledge and has its own common sense implications, which are: $Ck[(\overline{a} \lor \overline{Ab}) \cap (\overline{b} \lor \overline{Ba})]$ where a = "A loves B" and b = "B loves A". (Note that who and whether one loves is more or less a part of one's biography.) A might soliloquize: "She has said nothing because she doesn't love me, or if not so, it must be because she doesn't realize I love her. But that would show she lacks intuitive understanding of me."

The pressure of expectations.--The extent of irrational avoidance of potentially discrediting situations like those discussed above is probably responsible for a great deal of private suffering, social conformity and political alienation. It cries for explanation. Whence the mysterious power of consensus (indeed, of society) over the individual?

We will begin by examining some current theories. Here it should be noted that the focus is not on their considerable heuristic usefulness but on their causal validity. Then a different theory will be presented.

The usual contemporary explanation is ego-psychological and runs something like this: individuals need an adequate self-image in order to function, but must validate their self-concept through others. Thus they are extremely sensitive to the threat of being labelled unworthy--as is shown by their egos' thick padding in rituals of deference and patterns of defense.[17] One difficulty with this hypothesis is that it seems to inadequately account for the special role of consensus (common knowledge of agreement) in social pressure. This criticism does not apply to Scheff's (1967, p. 36) suggestion:

> "The collective representations are felt as power-
> ful exterior constraints because each individual
> agrees, recognizes that his neighbors agree,... and
> so on indefinitely. Although he agrees (or
> disagrees) with the sentiment, it is also something
> beyond his power to change, or even completely
> explore. The potentially endless mirror reflections
> of each of the others' recognitions is felt as
> something utterly final. From this formulation it
> follows that each actor feels the presence of the
> collective representation with a sense of exterior-
> ity and constraint,...."

Scheff seems to imply that collective representations have their power because the propositions involved are so manifold and complex. But the implicit psychology of awe before the infinite is ad hoc. Achilles will beat the tortoise in spite of Zeno's reflections.

It is instructive to compare Durkheim's (1951, p. 307) treatment of collective representations in Book III, Ch. 1

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17. Lofland, (1966) provides an excellent exposition of such a model in a section entitled The Fragile Nature of Man.

of <u>Suicide</u>:

> "Usually when collective tendencies or passions
> are spoken of, we tend to regard these expressions
> as mere metaphors and manners of speech with no
> real signification but a sort of average among a
> certain number of individual states. They are not
> considered as things, forces <u>sui generis</u> which
> dominate the consciousness of single individuals.
> None the less, this is their nature, as is
> brilliantly[5] shown by statistics of suicide. The
> individuals making up a society change from year
> to year yet the number of suicides is the same so
> long as the society itself does not change."

Unfortunately, Durkheim did not understand the meaning of

statistical regularity. He did not realize the profound

significance of the small but definite amount of irregularity

that would remain in his data if social variables (and

reporting errors) were controlled for. For suppose that χ^2

values then came out significantly <u>lower</u> than would be

expected by chance. Durkheim would be literally correct and

our conception of the nature of society would be profoundly

different from what it is. Then it would indeed be useful

to conceive of a social suicide quota and a suprapersonal

entity responsible for that quota.

Durkheim constructed an untenable sociology to account

for what seems to be the power of social forces. But forms

of the collective representations concept have remained use-

ful enough (for qualitative data, though) that psychology has

been distorted to preserve it. Society still dominates the

individual, but not because society is strong but because the

individual is weak.

Both the contemporary hypotheses that have been

presented overemphasize the role of cognitive antecedents in action. Western culture and historical and contemporary sociology are rife with myths about this: Hamlet's resolution was supposed to be sicklied o'er by the pale cast of thought, but there is a widespread belief that belief in free-will is necessary not to be paralyzed. (Compare the earlier belief that belief in God is necessary not to do evil.) "In every higher kind of production a person needs to understand and believe in himself," (Cooley, 1956, p. 224); cf. Simmel's (1964, p. 310) discussion of the Lebenslüge. The centrality of expectations in contemporary action theory is consistent with this pattern (vide Parsons, 1964, Ch. I). Most of this is the ultimate rationalization--that there is any reason (cf. Ryle, 1949). In particular, individuals may rationalize that they conform because they fear what others will think (Discrediting rationalizations are safest from challenge), but there may be only a correlation. And sensitivity, real or ostensible, to others' opinions does not mean that a funda-mental need is involved. A propos to the first hypothesis discussed, there are sociological reasons for treating people as having sensitive egos, whether they have them or not.[18]

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18. There is perhaps something like a natural law, resulting from interpersonal comparisons of utility, that worth as an ethical object is proportional to range and sensitivity of feelings. (Vide Andersen's (1835) simulation of status-validation ceremonies.) At any rate, this law has some validity in this culture, and we will distort the feelings if necessary to match the ascribed worth. Thus the neurotic will be denied range and the psychotic or criminal will be denied sensitivity. A person would wish to be almost anything but a clod, except a vegetable. So it is no wonder that deference rituals treat people as if sensitive and intense.

A fourth hypothesis is suggested by clinical data. Analysis will center on the dyad--an already complex level.[19] I see no reason to believe that the social inhibitions of neurotics in therapy differ importantly from those of average persons except in degree (and not even with respect to degree in the middle stages of successful therapy). So by examining what seems to help these people we may obtain insights on the nature of socialization. There is good evidence that therapy can be effected by reciprocal inhibition of anxiety responses to difficult social situations. This is done by repeatedly encouraging assertive responses to subjectively very mild forms of the aversive stimuli. (Salter (1961); Wolpe and Lazarus (1966)). This suggests that the apparent power of collective representations is due to conditioned anxiety responses to the associated behavioral stimuli rather than to something more intrinsic. What accounts for the existence and stability of these responses?

Let me sketch the hypothesis by means of a scenario. Suppose Ego asks a question which he fears may be rather silly to Alter who is explaining something to him. Much that is more concrete than Ego's being "redefined" may happen. Alter's reaction may discomfit him by confirming his fears; it will be, literally, bad news, and therefore produce an

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19. My impression is that the presence of an audience tends to amplify the crucial positive feedback effects hypothesized. This is generally true in the particular situations discussed. In them, further, there may be Ck that one actor represents society's expectations.

emotional reaction. In this culture he will have the problem
of managing the expression of affect superimposed on his inter-
actional task--under penalty of being perceived as (Here,
rather than directly, the threat of "unworthiness" seems to
come most into play.) inappropriately demeaned. As the flow
of interaction is disrupted, Alter will tend to be frustrated
and indicate resentment toward Ego which will further fluster
him, etc.[20] During this period of readjustment unpleasant
uncertainty about propositional polynomials further heightens
the tension. Thus, in a word, embarassment is unpleasant,
and expectations are conformed to, though not so much because
of a fear of being thought unworthy as because of a conditioned
and perhaps unconscious aversion to "losing one's cool".

The argument given does not seem sufficient to account
for the intensity or tenacity of aversion to potentially
embarassing situations. Here I think that factors militating
against the extinction of social inhibitions are crucial--
particularly, the ubiquity in social interaction of the
positive feedback indicated above. In the précis of behavior
therapy, the subjective extreme mildness of the aversive
stimulus was crucial. With inanimate objects one can come,
literally or figuratively, a few inches closer each time
until all fear has vanished, never being exposed to even
moderate discomfort. When it is a matter of human relation-

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20. Cf. Goffman (1961) on "flooding out", a related type of
     contagion.

ships something like this is much more difficult--for it is
hard to make them vary gradually and continuously. The inter-
active adjustment of reciprocal expectations that occurs
whenever something happens suggests that a relationship will
not possess even temporary stability except at a discrete
set of equilibrium positions. I believe studies of the pro-
cesses of courtship and divorce support this.

Objective discussion of the relationship might
solve this stabilization problem but often tends to defeat
its purpose. Also, there are norms in this society against
discussing relationships (primary or secondary). With most
people unused to discussing most relationships their first
attempts would be likely to be embarrassing. All this tends
to maintain the status quo.

Aspects of the sociology of sexual impotence form
a simplified paradigm for the discussion (embarrassment,
positive feedback, discontinuity). (It is simplified because
the common sense link between affect and behavior is tighter
than in cases more typical of social action.) One type of
therapy involves a partner with full knowledge of it and use
of deliberate restraint in order to move gradually to full
sexual activity. (Wolpe and Lazarus, 1966). This illustrates
what I mean by gradual continuous variation and the difficulties
in arranging this. Imagine an attempt to directly adopt this
technique to extinguish inhibition against asking one's boss
for a raise!

To summarize: the endless mirrors of mutual

expectations amplify by positive feedback the emotions associated with unconventional interaction, and give the field of interaction patterns a character of discontinuity. This discontinuity makes everyday life unable to extinguish the aversive conditioning contingent on the disruption of consensus. Questions remain unasked, knowledge remains unshared, and the status quo is maintained.

To test this theory it will be necessary to make it more elaborate and precise. Because of the special prominence it gives to cultural regulation of affective behavior, cross-cultural inquiry should be useful in differentially evaluating it.

The Ostensible.--One more topic must be alluded to: mutual pretense, or the open secret: there is common knowledge but tacit collusion to act as if the contrary of the proposition were true--e.g., as if the patient were not dying (Glaser and Strauss, 1965). It then is common knowledge that it ostensibly is common knowledge that the patient is not dying. When a proposition is ostensibly valid it means that interaction is carried on as if that proposition were true. It is, of course, common sense that there is a typical relation of correspondence between reciprocal knowledge patterns and interaction patterns. So interaction patterns can be classified and structured by means of the knowledge patterns in typical correspondence, whether or not this knowledge actually obtains. Ground-rules for interaction are described by ostensible propositions. This conceptualization suggests the research tasks of describ-

ing the structures of ostensible worlds and comparing them
with common sense worlds, and both with the real world.
Typically, ostensibly everything is for the best in the best
of all possible worlds.[21]

21. Hence "cooling out" procedures are possible. See
    Bittner (1967) on how police apprehend the mentally
    ill for illustration.

V. Economics[22]

We return to a formal perspective. Knowledge can be translated into rational action through the more or less common sense principle of maximizing expected utility. Game theory studies situations where the utilities of multiple decision-makers interlock; it has been assumed that the form of the game is common knowledge. In this final section of the paper we analyze a simple type of case where this restriction is relaxed. The normative theory of the 2-person 0-sum game will be extended to this case, and an algorithm for solving such generalized games will be derived and applied to an example.

Let two rational players play a game, M, whose payoffs depend on a parameter vector h with given probability distribution. M is 0-sum for each value of h. The distribution of h and the rationality of the players is common knowledge.

Let the players be A and B, and let C be a variable standing for either. C's knowledge of h may be briefly described by a proposition written (Ch = v), where $v = (\ldots,v_i,\ldots)$ is the vector with $v_i = 1$ if C knows the value of component i of h and $v_i = 0$ otherwise.

The distribution of knowledge in this situation is given by a function e(S,C): e(S,C) = v if S(Ch = v) is true, where S is formed from A, B, and I by composition. (The set of such S will be hereafter referred to as the semigroup.)

22. This section is a revision of Friedell (1966).

For instance, e(ABA,B) = (1,0) means that A thinks B thinks
A thinks that B knows the first component of h but not the
second.

Find rational strategies and the value of M, given
e. (Some restriction will later be placed on e to avoid
certain infinite regresses.)[23]

Decisions given the opponent's decision.--In solving
a game of the type given, C will judge the appearance of the
situation to D (C's opponent), calculate D's strategy on this
basis, and act accordingly. Given that C believes D will
choose a strategy D* from a set z, we may, guided by conven-
tional game theory, postulate a set of rational strategies,
m_C, for C. m_C may be obtained from the class of all possible
strategies for C by 1) eliminating those which depend on
parameters C does not know, 2) eliminating those which maximize
expected utility against no D* in z, and 3) eliminating those
which do not maximize C's minimum on the chance that he may be
wrong about D's decision. (C does not assume he may be mis-
taken about D's knowledge, just as C does not assume this in
conventional game theory.)

m_C, then, depends on e(C,C) and e(C,D) as well as
on z, and we may write: (α) $C[(Ch = x) \cap (Dh = y) \cap (D*\varepsilon z)]$
$\leq [C*\varepsilon m_C (x,y,z)]$. (Fortunately, m_C tends to have just
one member.) Note that this inequality translates opinion (on
the left side) to action (on the right side). (α) is not
empirically refutable--an apparent counterinstance would only
prove that the rationality condition used here were not

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23. Harsanyi (1962) treats a somewhat related problem.

satisfied. Thus the inequality (α) is equal to t in the formal system, and hence is common knowledge.

Now we can find E's optimal strategies (E = A or E = B) given only e and ETC(D*ϵx) for some T in the semigroup. For ET may be prefixed to each side of (α) to obtain a result of the form ET(C*ϵ z'), and this process may be iterated.

<u>Decisions where the opponent's decision is unknown.</u>

1) If the situation is a conventional game, each player knows the distribution of knowledge about the parameter. This is equivalent to the condition on e: $e(I,C) = e(S,C)$ for all C and S. Define g as the set of such e.

2) It is assumed, in accordance with conventional game theory, that $E(e \epsilon g) \leqslant$ (E plays maximin in the game specified by g.)

3) It will now be shown how E's strategies can be derived, given only that there exists a semigroup element T such that $e(ET,C) = e(ETS,C)$ for all S and C.

4) Define $e^I = e$; define the function e^E by: $e^E(S,C) = e(ES,C)$; if e^W is defined, define e^{WE} as $(e^W)^E$. Then it can be shown by induction that e^T is defined and $e^T(S,C) = e(TS,C)$ for any T and S.

5) Write as \$e the statement: "e obtains." $\$e \leqslant E(\$e^E)$ follows from consideration of the definition of e^E.

6) Lemma: $\$e \leqslant S(\$e^S)$ for any S in the semigroup (proved in the Appendix).

7) Suppose for some T, $e(T,C) = e(TS,C)$ for all S and C. Then $e^T \epsilon g$ --for $e^T(I,C) = e(T,C) = e(TS,C) = e^T(S,C)$.

8) Now suppose, as in 3), e(ET,C) = e(ETS,C). Then $e^{ET} \varepsilon$ g (from 7)), and ET(e^{ET}) (from 6)). If T = I, substitute e^E for e in 2) to obtain that E plays maximin in the game specified by e^E. If T = WF (F = A or F = B), then EW(F(e^{ET})), so EW (F plays maximin in the game specified by e^{ET}). Now the discussion in the previous subsection may be applied to determine E's strategies.

9) If z_A and z_B are the sets in which A* and B* are finally determined to lie, the value V of M will satisfy:

$$\max_{\substack{A* \varepsilon\, z_A \\ B* \varepsilon\, z_B}} E_h P \geq V \geq \min_{\substack{A* \varepsilon\, z_A \\ B* \varepsilon\, z_B}} E_h P,$$ where $E_h P$ is A's expected utility.

10) A final weak assumption may be used to simplify the notation: (Ch = x) = C(Ch = x). Then for all S and C, e(SC,C) = e(S,C) = e(SC), where here e is newly defined as a function of one variable (e(I) is not defined).

Some consequences.--1) Suppose A knows what B is thinking: that is, e(AS) = e(S) for all S ≠ I. Then e^A = e, and $e \Rightarrow A(e)$. A will know the actual situation and hence B's decision. To know the whole truth, obviously enough, is best of all for a rational player.

2) It is easy to show that to gain information about the parameter does not hurt a player even when the fact of this gain is common knowledge (although he may be disadvantaged if he does not know that the other knows about his new information). The question is posed whether these two conditions on V as a function V(e) of e are the only

powerful conditions that can be demonstrated.

An example.--The reader may find a simple illustration of $V(e)$ interesting. Consider the game, G:
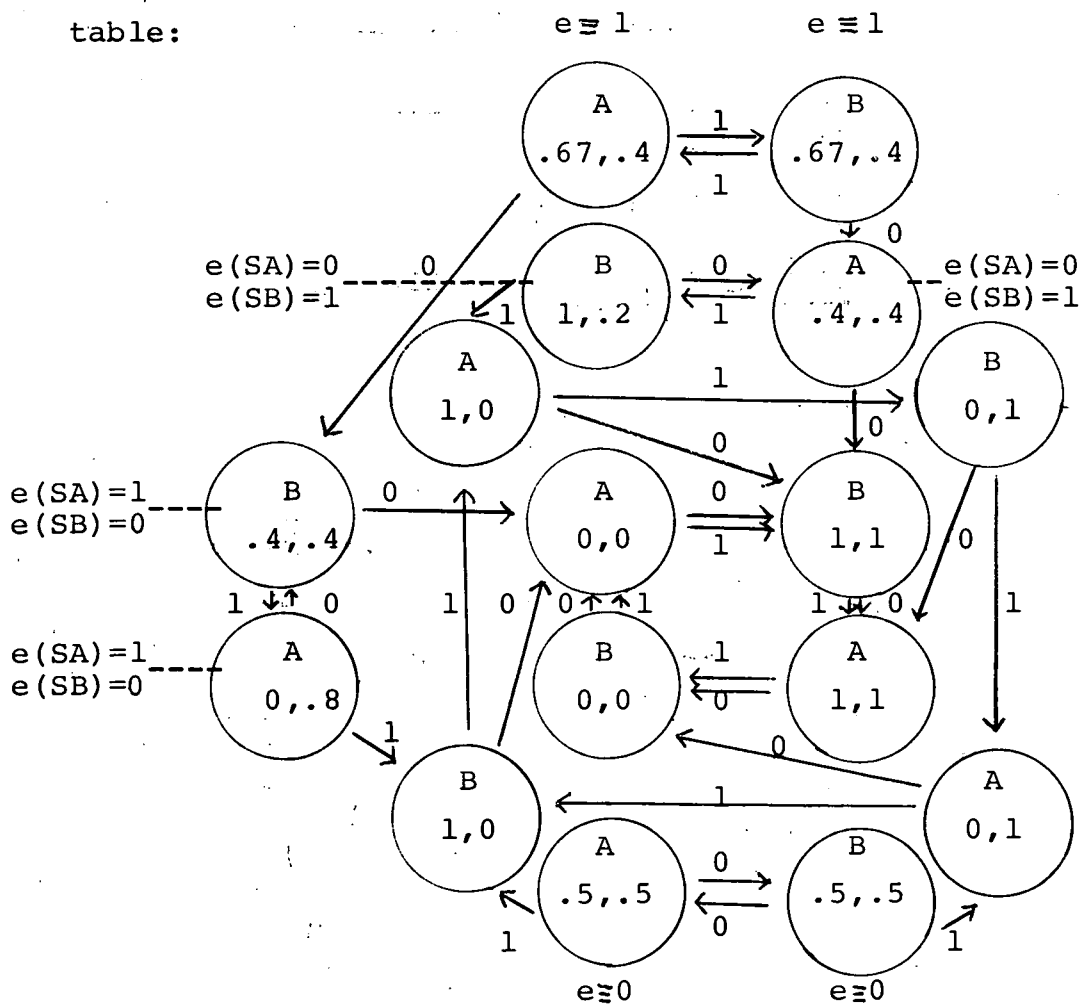
$$
\begin{array}{cc}
 & B \\
A & \begin{array}{cc} w & -1 \\ -1 & 1 \end{array}
\end{array}
\text{, where prob } (w = 0) = \text{prob } (w = 2) = .5.
$$

This game is interesting because the apparently symmetrical change from $e \cong 0$ to $e \cong 1$ give A a disadvantage (of .07). $V(e)$ is most intelligibly presented by a computational diagram and table:

Legend: $(P_{C0}, P_{C2}) = C*$; P_{Ci} is the probability that C will choose the first row or column when $w = i$. $\{(P_{D0}, P_{D2})\} = m_D(x, y, \{(P_{C0}, P_{C2})\})$. The value of y is not needed here. A node associated with a function $e \, \varepsilon \, g$ indicates an optimal strategy for that game.

Values of V

B*

V	0,0	0,1	.4,.4	.5,.5	.67,.4	1,0	1,.2	1,1
0,0	1	0	.2	0	-.07	0	-.2	- 1
0,.8	.2	1.2	.2	.2	-.07	-.8	-.6	.2
0,1	0	1.5	.2	.25	-.07	- 1	-.7	.5
.4,.4	.2	.2	.04	0	-.07	-.2	-.2	-.2
.5,.5	0	.25	0	0	-.07	-.25	-.2	0
.67,.4	-.07	-.07	-.07	-.07	-.07	-.07	-.07	-.07
1,0	0	- 1	-.2	-.25	-.07	.5	.3	-.5
1,1	- 1	.5	-.2	0	-.07	-.5	-.2	1

A* is the row label to the left of the table.

The computational procedure will be indicated by
an example. Suppose that knowing the value of w is "cheating",
and suppose that in fact no one is cheating, but A thinks
B thinks A is cheating because A observes B frowning, and B
thinks A thinks B is cheating because B observes A scrutinizing
him. (Actually, the frown and the stare are reinforcing
each other.) Since each trusts the other, neither thinks the
other is cheating. In short, e(ABA) = e(BAB) = 1, e = 0 for
other arguments. $e^{ABAB} \leqslant g$, so ABA(B plays maximin in the
game specified by $e \cong 0$). Enter the table at B's $e \cong 0$ node
to find ABA(B plays (.5,.5)). Now since e(ABA) = 1 follow
the "1" arrow to node (A, 0,1). So AB(A plays (0,1)). Continuing,
A(B plays (0,0)) and, finally, A plays (0,0). Similarly

we obtain: B plays (1,1). From the table, the value of the game is -1, a worst outcome for A, even though the cognitive situation is symmetric.

A could get .2 if it were Ck that he alone knew the value of w. But A gets his maximum of 1.5 if he knows w and can correctly deduce that B will play (0,1). B will play (0,1) if it is Ck that he alone knows w, except that $t(BA) = 1$ (i.e., B thinks A is peeking). So if A wants to cheat most profitably he should sell B ostensibly exclusive knowledge of w and then cheat, but do it just clumsily enough that B catches him but does not realize A realizes this.

Conclusion.--Let us return to the question discussed in the Introduction of the rarity of long propositions. It would be nice if there were a convergence law like: for any $\varepsilon > 0, \exists n \ni |V(e_1) - V(e_2)| < \varepsilon$ whenever e_1 and e_2 differ only for an argument of the form $S(AB)^n T$. But nothing like this holds for game G; we have seen that it can be quite profitable for an actor to induce another to believe a polynomial proposition of high degree, but this may require much adroitness. Perhaps we have a clue as to the common lack of independent significance of these propositions. The general question arises: is there a structure of props necessary or sufficient to induce a reasonable man to believe a given such proposition, and if so--what is it? Findings on this topic might perhaps be directly useful only to confidence men and espionage agencies, but valuable sidelights might be shed on the geometry and furniture of the social world.

Appendix

(Proofs)

Theorem (a): $Ax \leq \overline{A\overline{x}}$

Proof: $t = \overline{f} = \overline{Af} = \overline{A(x \cap \overline{x})} = \overline{Ax \cap A\overline{x}} = \overline{Ax} \cup \overline{A\overline{x}} = (Ax \leq \overline{A\overline{x}})$.

Theorem (b): $Ax \cap A(x \Rightarrow y) \leq Ay$

Proof: $Ax \cap A(x \Rightarrow y) = A(x \cap (x \Rightarrow y)) = A(x \cap y) = Ax \cap Ay \leq Ay$.

Theorem (d): $(x \leq y) \leq (Ax \leq Ay)$

Proof: $(x \leq y)$ means $(x \cap y = x)$, so $Ax = A(x \cap y) = Ax \cap Ay$, and $(Ax \leq Ay)$.

Theorem (e): $Ax \cup Ay \leq A(x \cup y)$

Proof: $x \leq x \cup y$, so $Ax \leq A(x \cup y)$; similarly, $Ay \leq A(x \cup y)$. So $Ax \cup Ay \leq A(x \cup y)$.

Theorem (f): $A\,\overline{A\overline{x}} = Ax$

Proof: $A\,\overline{A\overline{x}} = \overline{A\overline{x}} = Ax$.

Theorem (g): $A(Ax \cup Ay) = Ax \cup Ay$

Proof: $A(Ax \cup Ay) = A(\overline{\overline{Ax \cup Ay}})$ (from (f)) $= A(\overline{\overline{Ax} \cap \overline{Ay}}) = A\,\overline{Ax} \cap A\,\overline{Ay} = \overline{\overline{Ax}} \cap \overline{\overline{Ay}} = Ax \cup Ay$.

Theorem (h): $A(Ax \Rightarrow x) = t$

Proof: $t = \overline{Ax} \cup Ax = Ax \cup A\,\overline{Ax} \leq A(x \cup \overline{Ax})$ (from (e)) $= A(Ax \Rightarrow x) \leq t$.

An associativity lemma (used below): If $z = \bigcap\limits_{x \varepsilon \bigcup T_j} x$ is defined and,

for each j, $x_j = \bigcap\limits_{x \varepsilon T_j} x$ is defined, then $\bigcap\limits_j x_j$ exists and

equals z.

Proof: 1) $z \leqq x_j$ for every j (obvious),

2) If $y \leqq x_j$ for every j, then $y \leqq x$ for every x; hence $y \leqq z$.

Thus z is the greatest lower bound of the x_j.

Theorem (i): Satisfaction of laws (1) and (2) is obvious. We

first show: If for all i, M_i satisfies (3'), and if $\bigcap\limits_i M_i$

exists, then $\bigcap\limits_i M_i$ satisfies (3').

Proof: Assume $\bigcap x_k$ exists.[1]

$(\bigcap\limits_i M_i)(\bigcap\limits_k x_k)$ then exists and equals $\bigcap\limits_i [M_i \bigcap\limits_k x_k] = \bigcap\limits_i [\bigcap\limits_k M_i x_k] =$

$\bigcap\limits_{i,k} M_i x_k$, from the infinite associative law (Sikorski, 1964,

p. 59). Now we show $\bigcap\limits_{i,k} M_i x_k$ is the g.l.b. of $\left\{(\bigcap\limits_i M_i) x_k\right\} =$

$\left\{\bigcap\limits_i (M_i x_k)\right\}$. In the lemma, let $x_j \to \bigcap\limits_i M_i x_k$, $z \to \bigcap\limits_{i,k} M_i x_k$;

then $\bigcap\limits_k (\bigcap\limits_i M_i x_k)$ exists and equals $\bigcap\limits_{i,k} M_i x_k$. So

$\bigcap\limits_k [(\bigcap\limits_i M_i) x_k] = (\bigcap\limits_i M_i)(\bigcap\limits_k x_k)$, q.e.d.

We next show: If M_1, M_2 satisfy (3'), then $M_1 M_2$ does.

Proof: If $\bigcap x_i$ exists, $\bigcap M_2 x_i$ and $\bigcap M_1 (M_2 x_i)$ do. Now,

$M_1 M_2 (\bigcap x_i) = M_1 (\bigcap M_2 x_i) = \bigcap M_1 M_2 x_i$.

We finally turn to the distributive laws: $M(\bigcap L_i) = \bigcap M L_i$

is a restatement of (3'); $(\bigcap L_i) M = \bigcap (L_i M)$ follows from

the definition of composition.

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1. Incidentally, any countable Boolean algebra is incomplete (Sikorski, 1964, p. 66).

<u>Theorem (j)</u>: If M satisfies (3') and $z = \bigwedge_{i=0}^{\infty} M^i v$ is defined, then z is the greatest value for y that satisfies $y = v \wedge My$.

Proof: 1) $z = v \wedge Mz$. For $Mz = M \bigwedge_{i=0}^{\infty} M^i v = \bigwedge_{i=0}^{\infty} M^{i+1} v$, $v \wedge Mz = v \wedge \bigwedge_{i=1}^{\infty} M^i v = \bigwedge_{i=0}^{\infty} M^i v$ (from the associative law).

2) z is the greatest solution. For suppose $w = v \wedge Mw$; then $M^i w = M^i v \wedge M^{i+1} w$, so $M^i w \leq M^i v$. $w \leq Mw$; using (d) repeatedly we get $w \leq M^i w \leq M^i v$, if $i \geq 1$. Also $w \leq x$, so $w \leq \bigwedge_{i=0}^{\infty} M^i v$.

<u>Theorem (k)</u>: a) $\bigwedge_{i=0}^{\infty} P^i x$ is defined, so $Co = \bigwedge_{i=0}^{\infty} P^i (Px) = \bigwedge_{i=1}^{\infty} P^i x$ is defined. $Ck = Co \wedge I = \bigwedge_{i=1}^{\infty} P^i \wedge P^0 = \bigwedge_{i=0}^{\infty} P^i$.

b) Clearly, $Co = CkP$, so (1), (2) and (3') for Co follow from their validity for Ck, which will be demonstrated.

c) $P^i t = t$, so $Ck \, t = t$.

d) $P^i f = f$, so $Ck \, f = f$.

e) $Ck = \bigwedge_{i=0}^{\infty} P^i$ satisfies (3'). Apply theorem (i) with $P^i = M_i$.

f) Ck satisfies (4).

1) $Ck \, x \leq x$, so $CkCk \, x \leq Ck \, x$.

2) $\bigwedge_i (P^i x)$ exists and Ck satisfies (3'), so $\bigwedge_i Ck(P^i x)$ exists and equals $Ck(\bigwedge_i (P^i x))$. $Ck \, x \leq CkP^i x$ because $CkP^i x = \bigwedge_{j=0}^{\infty} P^j P^i x = \bigwedge_{j=i}^{\infty} P^j x \geq \bigwedge_{j=0}^{\infty} P^j x$. Now $Ck \, x \leq \bigwedge_i CkP^i x = Ck \bigwedge_i P^i x = CkCk \, x$. So $Ck \, x = CkCk \, x$.

Formulas: starting from 1): $Ck = Co \wedge I$ and 2): $Co = CkP$, and applying (3') we get $PCk = \bigwedge_{j=1}^{\infty} P^j = Co$. So $Co = CkP = PCk$. $CkP \wedge I = Ck$, from 1) and 2). Also, $Ck(P \wedge I) = (P \wedge I)Ck = Ck$, for $Ck = CkP \wedge I = Ck(CkP \wedge I)$ (from

$(4)) = CkP \cap Ck = PCk \cap Ck = (P \cap I)Ck$, and $CkP \cap Ck = Ck(P \cap I)$.

Now, $CoP = PCo = Co$, for:

1) $CoP = CkP^2 = PCkP = P^2Ck = PCo$;

2) $P^2 \leq P$ (since person operators are idempotent),

so $CoP \leq CkP^2 \leq CkP = Co$, $Ck \leq Co$, $Co = Ckp \leq CoP$,

and $CoP = Co$. (4) holds for Co (i.e., $Co^2 = Co$),

for: $Co^2 = CkPCkP = CkCkP^2 = CkP^2 = CoP = Co$.

Formal delineation of the argument from symmetry in the Common Opinion subsection:

Let S_i denote a polynomial in A and B, and let $\emptyset(S_i)$ exchange A and B in the expression for S_i. Here only we assume: $y = \bigcap_i (S_i x \leftrightarrow \emptyset(S_i)x)$ is meaningful. Show $A(x \cap y) \leq ACo\ x$.

Proof: $A(x \cap y) \leq Ay = A[\bigcap_i (S_i x \leftrightarrow \emptyset(S_i)x)] = \bigcap_i A(S_i x \leftrightarrow \emptyset(S_i)x) \leq A(S_i x \leftrightarrow \emptyset(S_i)x)$ for each i (theorem (c)) $\leq (AS_i x \Rightarrow A\emptyset(S_i)x)$.

The monomial S_i may be enumerated:

$S_0 = I$ 　　　　　　　$S_{4n-1} = (AB)^n$, $n \geq 1$

$S_1 = A$ 　　　　　　　$S_{4n+1} = (AB)^n A$, $n \geq 0$

$S_2 = B$ 　　　　　　　$S_{4n+2} = B(AB)^n$, $n \geq 0$

$S_3 = AB$ 　　　　　　$S_{4n+4} = B(AB)^n A$, $n \geq -1$.

$S_4 = BA$

$S_5 = ABA$

. . . .

Next we show by induction: $A(x \cap y) \leq AS_i x$ for every monomial S_i. We know $A(x \cap y) \leq AS_0 x$.

Suppose $A(x \cap y) \leq AS_n x$. If n is even $AS_n x = AS_{n+1} x$. If n is odd: $A(x \cap y) \leq AS_n x$ and $A(x \cap y) \leq (AS_n x \Rightarrow A\emptyset(S_n)x)$, so $A(x \cap y) \leq A\emptyset(S_n)x = AS_{n+1}x$. For, $A\emptyset(S_{4n-1}) = A\emptyset((AB)^n) = A(BA)^n = AB(AB)^{n-1} = AS_{4(n-1)+4}$ and $A\emptyset(S_{4n+1}) = A\emptyset((AB)^n A) =$

$A(BA)^n B = AB(AB)^n = AS_{4n+2}$. Now, $Ck\ x = \bigcap S_i x$, so $ACk\ x = \bigcap AS_i x$. Since we have shown $A(x \cap y) \leqslant AS_i x$, we conclude: $A(x \cap y) \leqslant ACk\ x \leqslant ACo\ x$.

Lemmas on epistemic operators: All operators used from here on are assumed to satisfy (1), (2) and (3').

Lemma: If $M \leqslant N$ and $P \leqslant Q$, then $MP \leqslant NQ$.

Proof: $MP \leqslant NP$ from the definition of $M \leqslant N$; $NP \leqslant NQ$ from $P \leqslant Q$ and theorem (d).

Lemma: If $M^2 = M \leqslant I$, $M \leqslant N$, then $M \leqslant N^i$, $i = 0, 1, \ldots$.

Proof: $M = M^i \leqslant N^i$ if $i \geqslant 1$, and the result is immediate if $i = 0$.

Lemma: If $M^2 = M \leqslant N \leqslant I$, then $MN = NM = M$.

Proof: Since $N \leqslant I$, $MN \leqslant M$ and $NM \leqslant M$. $M \leqslant N$, so $M = M^2 \leqslant NM$ and $M = M^2 \leqslant MN$. Therefore $MN = M$ and $NM = M$.

Lemma: $Ck_\alpha \leqslant CK_\sigma$, $\sigma \subseteq \alpha$.

Proof: $\bigcap_{A_i \in \alpha} A_i \leqslant \bigcap_{A_i \in \sigma} A_i$ so, from a lemma, the k-th powers also bear this inequality. Take g.l.b.'s to prove the lemma.

The following three theorems justify intuitions concerning operations on Ck operators.

Theorem: Let $L = (\bigcap_i Ck_i) \cap (\bigcap_j A_j)$, let σ be the set of all person operators involved; then $\bigcap_{k=0}^{\infty} L^k$ exists and equals Ck_σ.

Proof: We know from the lemmas: $Ck_\sigma \leqslant L^k$, so Ck_σ is a lower bound of $\{L^k\}$. Also, $L \leqslant \bigcap_{j \in \sigma} A_j$, so $L^k \leqslant (\bigcap_{j \in \sigma} A_j)^k$. Therefore any lower bound of $\{L^k\}$ is a lower bound of $\{(\bigcap_j A_j)^k\}$ and hence of Ck_σ. We conclude: $\bigcap L^k$ exists and equals Ck_σ.

<u>Theorem</u>: If for all i,j $M_{ij} \le I$ and $N = \bigcap_k (\bigcap M_{ij})^k$ exists,

let $L = (M_{11}M_{12}\ldots M_{1n_1}) \bigcap (M_{21}M_{22}\ldots M_{2n_2}) \bigcap \ldots \bigcap (M_{m1}M_{m2} \ldots$

$M_{mn_m})$. Then $\bigcap L^k$ exists and equals N.

<u>Proof</u>: Substituting I for all M_{pq} except M_{ij} we see $L \le M_{ij}$,

so $L \le \bigcap M_{ij}$.

$(\bigcap_{k,j} M_{kj})^{n_i} \le$ the i-th term in L; $\bigcap M_{kj} \le I$, so $(\bigcap M_{kj})^{\max(n_i)} \le$

$(\bigcap_{k,j} M_{kj})^{n_i}$ and $(\bigcap M_{kj})^{\max(n_i)} \le L \le \bigcap_{i,j} M_{ij}$ and $(\bigcap_{i,j} M_{ij})^{k(\max(n_i))} \le$

$L^k \le (\bigcap_{i,j} M_{ij})^k$. N, then, is a lower bound of $\{L^k\}$ and any

lower bound of $\{L^k\}$ is a lower bound of N. So $\bigcap L^k = N$, q.e.d.

<u>Theorem</u>: If $M^2 = M$, $N^2 = N$ and $\bigcap_{i=0}^{\infty} (M \cap N)^i$ exists, then

$\bigcap_{i=0}^{\infty} (NM)^i$ exists and equals $N(\bigcap_{i=0}^{\infty} (M \cap N)^i M \cap I$.

<u>Proof</u>: One can show by induction: $(M \cap N)^{2i} = (N \cap I) \bigcap_{k=1}^{i-1} (MN)^k (M \cap I)$

$\bigcap (MN)^i \bigcap NM \cap N \cap M$ $(i \ge 1)$.

So, $N(M \cap N)^{2i}M = N\bigcap_{k=1}^{i-1} (MN)^k M \cap (NM)^{i+1} \bigcap NM = \bigcap_{k=1}^{i-1} (NM)^k$. This is

also true when $i = 0$, and $I \cap N(M \cap N)^{2i}M = \bigcap_{k=0}^{i+1} (NM)^k$ $(i \ge 0)$.

Now, $(M \cap N)^2 \le M \cap N$, so $(M \cap N)^{2i} \le (M \cap N)^k$, $(1 \le k \le 2i)$,

and $I \cap N(M \cap N)^{2i}M \le I \cap N(M \cap N)^k M$. This is also true when

$k = 0$. So $I \cap N(M \cap N)^{2i}M = \bigcap_{k=0}^{2i} (I \cap N(M \cap N)^k M)$, and $\bigcap_{k=0}^{i+1} (NM)^k =$

$\bigcap_{k=0}^{2i} (I \cap N(M \cap N)^k M)$.

Finally, $N(\bigcap_{i=0}^{\infty} (M \cap N)^i)M \cap I$

$= \bigcap_{i=0}^{\infty} N(M \cap N)^i M) \cap I$ (using (3'))

$= \bigcap_{i=0}^{\infty} (N(M \cap N)^i M \cap I)$ (Sikorski, 1964, p. 60)

$= \bigcap_{i=0}^{\infty} (\bigcap_{k=0}^{2i} (N(M \cap N)^k M \cap I))$ (by the associativity lemma)

$= \bigcap_{i=0}^{\infty} (\bigcap_{k=0}^{i+1} (NM)^k)$

$= \bigcap_{i=0}^{\infty} (NM)^i$ (by the infinite associative law).

Corollary: $\bigcap_{i=1}^{\infty} (Ck_{D,E} Ck_{E,F})^i$ exists and equals $Ck_{D,E,F}$

(cf. the concluding paragraph of section II).

Proof: $Ck_{D,E,F} = \bigcap_{k=0}^{\infty} (Ck_{D,E} \cap Ck_{E,F})^k$, from the first of the three theorems just proven. $\bigcap_{i=0}^{\infty} (Ck_{D,E} Ck_{E,F})^i =$ $Ck_{D,E} Ck_{D,E,F} Ck_{E,F} \cap I = Ck_{D,E} Ck_{D,E,F} Ck_{E,F} = Ck_{D,E,F}$ (from the theorem just proved and a lemma). Finally, $\bigcap_{i=1}^{\infty} (Ck_{D,E} Ck_{E,F})^i =$ $Ck_{D,E,F} Ck_{D,E} Ck_{E,F} = Ck_{D,E,F}$.

Theorem (l): $Ck(x \Rightarrow Px) = Ck(x \Rightarrow Ck\ x)$.

Proof: $Ck(x \Rightarrow Px) \leq CkP^i(x \Rightarrow Px) \leq Ck(P^i x \Rightarrow P^{i+1} x)$, so $Ck(x \Rightarrow Px) \leq Ck(x \Rightarrow P^i x)$. Now, $\bigcap P^i x$ exists, so $\bigcap (\bar{x} \cup P^i x)$ exists and equals $\bar{x} \cup (\bigcap P^i x)$ (Sikorski, 1964, p. 60); i.e., $\bigcap_{i=0}^{\infty} (x \Rightarrow P^i x) = (x \Rightarrow Ck\ x)$. Using (3'), $Ck(x \Rightarrow Ck\ x) =$ $Ck(\bigcap (x \Rightarrow P^i x)) = \bigcap Ck(x \Rightarrow P^i x)$. So $Ck(x \Rightarrow Px) \leq \bigcap Ck(x \Rightarrow P^i x) =$ $Ck(x \Rightarrow Ck\ x)$. But $(x \Rightarrow Ck\ x) \leq (x \Rightarrow Px)$, so $Ck(x \Rightarrow Px) \geq$ $Ck(x \Rightarrow Ck\ x)$. Therefore, $Ck(x \Rightarrow Px) = Ck(x \Rightarrow Ck\ x)$, q.e.d.

Theorem (m): $\bigcap_{i \neq j} A_i A_j Ck = Ck \bigcap_{i \neq j} A_i A_j$.

Proof: Let $B = \bigcap_{i \neq j} A_i A_j$. First we show $BP = PB$:

$BP = \bigcap_{i \neq j} A_i A_j A_k = \bigcap_{i \neq j \neq k} A_i A_j A_k \cap \bigcap_{j \neq k} A_j A_k = \bigcap_{i \neq j \neq k} A_i A_j A_k \cap$ $\bigcap_{i=j \neq k} A_i A_j A_k = PB$. So $BP^i = P^i B$, and $B(\bigcap P^i) =$ $(\bigcap BP^i) = (\bigcap P^i B) = (\bigcap P^i)B$.

A lemma in section V: $\$e \leq S(\$e^S)$ for any S in the semigroup.

Proof: This is true for $S = A, B, I$. Suppose $\$e \leq T(\$e^T)$; show $\$e \leq TE(\$e^{TE})$. We know $\$e^T \leq E(\$e^{TE})$, so $T(\$e^T) \leq$ $TE(\$e^{TE})$. So $\$e \leq TE(\$e^{TE})$, which is sufficient to prove the lemma by induction.

References

Abelson, R. P. and M. J. Rosenberg, "Symbolic Psycho-logic," Behavioral Science, 3, 1958, pp. 1-13.

Andersen, Hans Christian, The Princess and the Pea, 1835.

Bittner, Egon, "Police Discretion in Apprehending the Mentally Ill," Social Problems, 14, 1967, pp. 278-92.

Cooley, Charles H., Human Nature and the Social Order, Glencoe: Free Press, 1956.

Durkheim, Emile, Suicide, Glencoe: Free Press, 1951.

Friedell, Morris F., An Extension of Game Theory, 1966 (ditto).

Goffman, Erving, Stigma, Englewood Cliffs: Prentice-Hall, 1963.

Goffman, Erving, "Fun in Games," in Encounters, Indianapolis: Bobbs-Merrill, 1961.

Glaser, Barney G. and Anselm L. Strauss, Awareness of Dying, Chicago: Aldine, 1965.

Halmos, Paul R., Algebraic Logic, New York: Chelsea, 1962.

Harsanyi, J. C., "Bargaining in Ignorance of the Opponent's Utility Function," Journal of Conflict Resolution, 6, 1962, pp. 29-38.

Hintikka, Jaakko, Knowledge and Belief, Ithaca: Cornell, 1962.

Laing, R. D., et al., Interpersonal Perception, London: Tavistock, 1966.

Lofland, Lyn H., In the Presence of Strangers, Working Paper 19, Center for Research on Social Organization, University of Michigan, 1966.

Luria, Aleksandr R., Higher Cortical Functions in Man, New York: Basic Books, 1966.

Maucorps, P. H. and René Bassoul, "Jeux de miroirs et sociologie de la connaissance d'autrui," Cahiers Internationaux de Sociologie, 32, 1962, pp. 43-60.

Newcomb, Theodore M., et al, Social Psychology, New York: Holt, Rinehart and Winston, 1965.

Parsons, Elsie C., Fear and Conventionality, New York: G. P. Putnam's Sons, 1914.

Parsons, Talcott, The Social System, New York: Free Press, 1964.

Pittenger, Robert E., et al, The First Five Minutes, Ithaca: Martineau, 1960.

Ryle, Gilbert, The Concept of Mind, New York: Barnes and Noble, 1949.

Salter, Andrew, Conditioned Reflex Therapy, New York: Capricorn Books, 1961.

Scheff, Thomas J., "Toward a Sociological Model of Consensus," American Sociological Review, 32, 1967, pp. 32-46.

Sikorski, Roman, Boolean Algebras, 2nd ed., New York: Academic Press, 1964.

Simmel, Georg, "The Secret and the Secret Society," in Kurt H. Wolff (ed.), The Sociology of Georg Simmel, New York: Free Press, 1964.

Slater, Philip E., "On Social Regression," American Sociological Review, 28, 1963, pp. 339-64.

Wolpe, Joseph and Arnold A. Lazarus, Behavior Therapy Techniques, Oxford: Pergamon Press, 1966.