THE UNIVERSITY OF MICHIGAN

INDUSTRY PROGRAM OF THE COLLEGE OF ENGINEERING

SMALL LIQUID OSCILLATIONS IN MOVING CIRCULAR AND ELLIPTIC CYLINDRICAL CONTAINERS

Ernst C. Glauser

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy in the
University of Michigan
Department of Civil Engineering
1966

December, 1966

IP-763

Doctoral Committee:

Professor Bruce G. Johnston, Chairman
Professor Glen V. Berg
Professor Samuel K. Clark
Associate Professor Ivor K. McIvor
Professor Victor L. Streeter

ACKNOWLEDGMENTS

The author is deeply indebted to Professor Bruce G. Johnston, Chairman of his doctoral committee, for suggesting the topic of this dissertation and for the help and encouragement during its preparation. He is also very grateful for the suggestions and the advice given by all other members of his doctoral committee.

The author wishes to express his appreciation to the Horace H. Rackham School of Graduate Studies of The University of Michigan for granting a pre-doctoral Fellowship under which most of the research was carried out and to the Computing Center of the University of Michigan for the use of the IBM 7090 computer in calculating and plotting the numerical results.

The reproduction of the thesis was undertaken by the College of Engineering Industry Program at The University of Michigan. For this valuable assistance the author is sincerely grateful.

TABLE OF CONTENTS

TABLE OF CONTENTS (Continued)

TABLE OF CONTENTS (Continued)

LIST OF TABLES

LIST OF ILLUSTRATIONS

LIST OF ILLUSTRATIONS (Continued)

LIST OF SYMBOLS

$A_i^{(m)}$	i-th coefficient in the expansions of the cosine-elliptic Mathieu functions ce_m and Ce_m, Equations (91a), (92a)
$B_i^{(m)}$	i-th coefficient in the expansions of the sine-elliptic Mathieu functions se_m and Se_m, Equations (91b), (92b)
C	Volume occupied by the liquid in the position of equilibrium
$Ca_{m,n}$	Equation (182) } Integrals involving modified Mathieu functions $Ce_m(\xi, q_{m,n})$
$Cb_{m,n}$	Equation (183)
$Cc_{m,n}$	Equation (184)
$Cd_{m,n}$	Equation (185)
Ce_m	Modified Mathieu function of the first kind and of order m, one solution to Equation (74)
$Cf_{m,n}$	Fourier coefficient corresponding to the cosine-elliptic mode type $\varphi_{m,n}$, Equation (172)
$Cq_{m,n}$	Surface integral and numerator in the expression for $Cf_{m,n}$, Equation (176)
$Cr_{m,n}$	Surface integral and a part in the denominator of the expression for $Cf_{m,n}$, Equation (177)
$F(x)$	Function of the independent variable x
$F_i(\phi_i)$	Functional which is minimized by the Stokes-Zhukovskii potential ϕ_i, Equation (108)
F_n	n-th coefficient in the Fourier expansion of the auxiliary function ψ in the case of a circular cylindrical container, Equation (137)
G	Axis of rotation in the system presented in Figures 1 and 11
I_c	Moment of inertia of the solidified liquid with surface S with respect to the axis of rotation through its center of gravity S_c

I_c^*	Dimensionless $I_c/\pi\ \rho\ a^5$
$I_{i,j}$	(i,j)-th component of the inertia tensor of a rigid body
I_w	Moment of inertia of the container W with respect to the axis of rotation through its center of gravity S_w
J_m	Bessel function of the first kind and of order m
$\vec{K}$	Body force per unit mass on a liquid particle
$[K]$	Resistance matrix, Equation (222)
$[M]$	Inertia matrix, Equation (222)
M_c	Equivalent moment of inertia of the liquid with rigid surface S with respect to the axis of rotation through its center of gravity S_c
$M_{i,j}$	(i,j)-th component of the equivalent inertia tensor of an ideal, incompressible liquid completely filling a closed, rigid container, Equation (112)
O	Origin of the Cartesian coordinate system (x,y,z)
$\overline{O}$	Origin of the Cartesian coordinate system $(\overline{x},\overline{y},\overline{z})$
P	Liquid particle or point in the system (x,y,z)
Q_i	Modal constant, Equations (248), (252), (256) and Tables III to XIII of the appendix.
R_i	Modal constant, Equations (249), (253), (257) and Tables III to XIII of the appendix
S	Surface or area of the liquid surface in the position of equilibrium
S_c	Center of gravity of the liquid with surface S, Figure 11
S_w	Center of gravity of the container W
$Sa_{m,n}$	Equation (197) } Integrals involving modified Mathieu functions $Se_m(\xi, q_n)$
$Sb_{m,n}$	Equation (198) } Integrals involving modified Mathieu functions $Se_m(\xi, q_n)$
$Sc_{m,n}$	Equation (199) } Integrals involving modified Mathieu functions $Se_m(\xi, q_n)$
$Sd_{m,n}$	Equation (200) } Integrals involving modified Mathieu functions $Se_m(\xi, q_n)$

Se_m	Modified Mathieu function of the first kind and of order m, one solution to Equation (74)
$Sq_{m,n}$	Surface integral involving the sine-elliptic mode type $\varphi_{m,n}$, result is given in Equation (201)
$Sr_{m,n}$	Surface integral involving the sine-elliptic mode type $\varphi_{m,n}$, result is given in Equation (202)
T	Axis of rotation in the system of Figures 1 and 11
T	Kinetic energy
T_c	Kinetic energy of the liquid in the system of Figure 11, Equation (206)
U_i	Value of a basic integral of φ_i appearing in the equations of motion (221), Equation (223)
U_i^*	Dimensionless $U_i/\pi\ a^4$
V	Potential energy
V_c	Potential energy of the liquid in the system of Figure 11, Equation (213)
V_i	Value of a basic integral of φ_i appearing in the equations of motion (221), Equation (224)
V_i^*	Dimensionless $V_i/\pi\ a^3$
W	Container, wetted walls or area of the wetted walls of the container
W_i	Value of a basic integral of φ_i appearing in the equations of motion (221), Equation (225)
W_i^*	Dimensionless $W_i/\pi\ a^2$
X,Y,Z	Components of $\vec{K}$ in the system (x,y,z)

a	Radius of a circular cylindrical and principal axis in x-direction of an elliptic cylindrical container
a_m	Characteristic number corresponding to the cosine-elliptic Mathieu function $ce_m(\eta,q)$
b	Principal axis in y-direction of an elliptic cylindrical container
b_m	Characteristic number corresponding to the sine-elliptic Mathieu function $se_m(\eta,q)$
c	Length parameter in the definition of the elliptic cylindrical coordinates, Equations (54)
ce_m	Ordinary Mathieu function of the first kind and of order m, one solution to Equation (73)
d	Separation constant introduced in Equations (67) and (68)
d	Distance between axis T and surface S in the system of Figure 11
f	Separation constant introduced in Equations (44) and (63)
f_c	Distance between center of gravity S_c and axis T in the system of Figure 11
f_c^*	Dimensionless length parameter f_c/a
f_w^*	Dimensionless length parameter f_w/a
g	Acceleration of gravity = 386.4 in/sec^2
h	Depth of the liquid in containers with flat bottom
h_i	Directional derivative of the Stokes-Zhukovskii potential ϕ_i in the direction of the unit normal $\vec{\nu}$
i,j	Integer numbers mostly used in indices
$k_{m,n}$	$\sqrt{q_{m,n}}$ given in Tables III to VII and IX to XIII of the appendix
ℓ	Length of the rods connecting the axis T and G in the system of Figure 11

ℓ	x-component of the unit normal $\vec{\nu}$
m	Integer number mostly used in indices
m	y-component of the unit normal $\vec{\nu}$
m_c	Mass of liquid of volume C
m_w	Mass of the container W
n	Integer number mostly used in indices
n	z-component of the unit normal $\vec{\nu}$
p	Pressure in the liquid
p_i	Generalized coordinate corresponding to the liquid mode φ_i, defined in Equation (208)
q	Parameter in the canonical forms of Mathieu's differential equations, Equations (73) and (74)
q_i	Generalized coordinate in Lagrange's equations, Equation (220)
$q_{m,n}$	Eigenvalues of q for liquid oscillations in elliptic cylindrical containers, introduced in Equation (75)
r	Radial coordinate in the system of circular cylindrical coordinates (r,θ,z)
s	Parameter in the canonical forms of Mathieu's differential equations
s(t)	Displacement function of point G in the system of Figures 1 and 11
se_m	Ordinary Mathieu function of the first kind and of order m, one solution to Equation (73)
t	Time
u	x-component of the liquid velocity $\vec{v}$
$\vec{u}$	Generalized velocity vector with the components $u_1, u_2, .., u_6$
$\vec{v}$	Velocity of a liquid particle in P (x,y,z)
v	y-component of the liquid velocity $\vec{v}$

v_i	A_{2i+3}/A_{2i+1}, Equation (86a)
$\vec{v}_o$	Velocity of the point O of a rigid body
v_{ox}, v_{oy}, v_{oz}	Components of $\vec{v}_o$ with respect to the system (x,y,z)
w	z-component of the liquid velocity $\vec{v}$
w_1	$\sqrt{q}\, e^{-\xi}$, Equation (93)
w_2	$\sqrt{q}\, e^{\xi}$, Equation (94)
x	Independent variable
x,y,z	Coordinates in a Cartesian coordinate system which is fixed with the container W
$\bar{x},\bar{y},\bar{z}$	Coordinates in a Cartesian coordinate system fixed in space
z	Coordinate in the Cartesian, circular and elliptic cylindrical coordinate system

$\alpha(t)$, $\beta(t)$	Generalized coordinates for the system in Figure 11
γ	Dimensionless parameter a/b
$\mathring{\delta}$	Derivative of the function $\delta(t)$ with respect to t
δ'	Derivative of the function $\delta(t)$ with respect to the dimensionless time $\tau = \sqrt{\frac{g}{a}}\, t$
ϵ	Dimensionless parameter h/a
ζ	Function of the free liquid surface
η	Coordinate in the elliptic cylindrical coordinate system (ξ,η,z)
θ	Coordinate in the circular cylindrical coordinate system (r,θ,z)
$\varkappa$	Dimensionless parameter m_w/m_c
λ_c	Dimensionless parameter M_c/I_c
$\lambda_{i,j}$	Ratio $M_{i,j}/I_{i,j}$
$\mu_{m,n}$	n-th zero of the function $\frac{d}{dx} J_m(x)$
$\vec{\nu}$	Unit outward normal to the container walls W
σ_i	Dimensionless frequency $\omega_i^2 \frac{a}{g}$ of the i-th mode φ_i in the infinitely deep circular or elliptic cylindrical container. Numerical values are given in Tables III to XIII of the appendix
τ	Dimensionless time $\sqrt{\frac{g}{a}}\, t$
ϕ	Velocity potential
$\phi_i, \phi_2, ..\phi_6$	Stokes-Zhukovskii potentials corresponding to the generalized velocities u_1, u_2, .., u_6
ϕ_c	Stokes-Zhukovskii potential of the liquid with surface S in the system of Figure 11 with respect to the axis of rotation through the cneter of gravity S_c
φ_i	i-th natural mode of ϕ

$\varphi_1, \varphi_2, \varphi_3$	Functions of one variable only derived from φ by the separation of variable procedure
$\varkappa$	Dimensionless parameter I_w/I_c
ψ	Auxiliary function used in the evaluation of the Stokes-Zhukovskii potentials, Equations (126) and (161)
ψ_1, ψ_2, ψ_3	Functions of one variable only, derived from ψ by the separation of variable procedure
ω_i	Frequency of the i-th natural mode φ_i
ω_i^*	Dimensionless frequency parameter $\omega_i\sqrt{\frac{a}{g}}$
$\vec{\omega}_o$	Angular velocity of a rigid body with respect to the point O
$\omega_{ox}, \omega_{oy}, \omega_{oz}$	Components of $\vec{\omega}_o$ with respect to the system (x,y,z)

I. INTRODUCTION

1.1 Initiation of the Research

The author's interest in the problem of liquid oscillations in moving containers arose with his study of the dynamic behavior of hot metal ladles and their contents. Ladles are huge containers, used in the steel producing industry to carry up to 400 tons of molten steel. When the ladle is carried by a moving crane, it is part of a dynamic system which, when simplified for an analytical investigation, will assume the configuration given in Figure 1. The ladle W is engaged by the hooks of the crane G in the trunnion axis T.

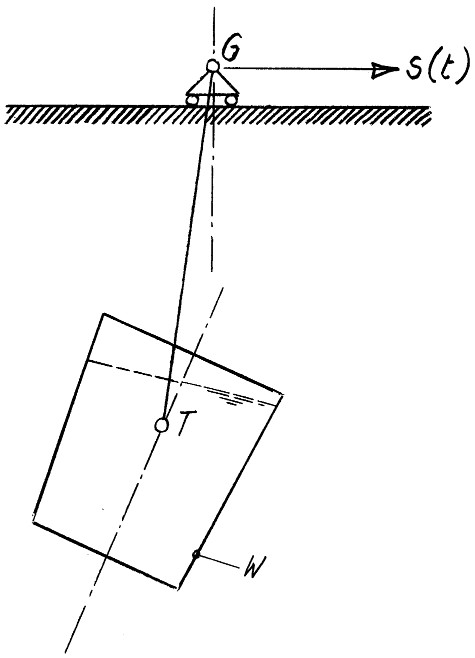

Figure 1
Simplified Ladle System

The motion of the crane along the craneway is characterized by the displacement function s(t). The following characteristics of the system and the motion will determine the analytical tools to be used for its investigation.

- The molten steel in the ladle has a temperature of about 5250°C at which its kinematic viscosity is $.825 \cdot 10^{-6}$ m^2 sec^{-1} compared to the value of $1.01 \cdot 10^{-6}$ m^{2}sec^{-1} for water at a temperature of 20°C. The liquid is practically incompressible.

- The ladle walls and bottom consist of heavy steel plates with welded seams, protected against the molten steel by a thick lining of fire resistant brick.
- The motion always starts from rest and the acceleration, but more important, the rate of change in acceleration of the crane is always very small, thus causing only small oscillations when the system is stable.
- The friction in the hinges T and G may safely be neglected.

The problem may therefore be simplified to the investigation of small oscillations of a conservative system with a rigid container partially filled with an ideal, incompressible liquid.

It was soon relaized in the course of the analytical work that in order to give a more general and more widely applicable solution, very fundamental problems had to be solved first. There exist hardly any attempts to solve the general liquid oscillation problem in elliptic cylindrical containers, which is the obvious generalization of the simple circular cylindrical case. The eigenvalues, eigenfunctions and inertia properties had first to be found to an extent necessary to solve the general forced oscillation problem. It was therefore decided to put the investigation into a much broader framework, obtaining results which may be applied for a wide class of similar problems.

1.2 Review of the Research on Dynamics of Liquids in Moving Containers

With the exception of some recent and highly theoretical papers (12), investigations on dynamics of liquids in moving containers deal with an ideal, incompressible liquid. The problem becomes an application of the classical theory of hydrodynamics, which has been treated by Lamb (13) in a most authoritative manner. There seem to be very few problems for which he did not give at least some hints as to possible ways of solution. Moreover, with the exception of a few studies of longitudinally excited (7) or rotating circular cylindrical containers, all theoretical studies assumed the liquid motion to be irrotational with respect to an internal frame of reference. Since the fluid is acted upon by a rotation free force field, it can be shown (16) that the motion of rigid tank boundaries cannot produce rotational flow if the flow was initially irrotational. The investigations assume, furthermore, small surface slopes, small displacements and small velocities. Despite these restrictive conditions, the solution for more general container shapes under both small translational and rotational oscillations may face tremendous numerical difficulties.

This is why the first attempts to consider containers partially filled with liquid in dynamic systems replace the liquid by a mechanical analog, spring-mass systems or physical pendulums, or tried to make some intuitive assumptions regarding the motion of the liquid. This is the way civil engineers found the response of the liquid in water tanks subjected to earthquakes (8, 9). The same crude approach was used in the aircraft industry and later in the design of liquid propellant rockets to investigate the influence of fuel sloshing on flight stability.

Exact solutions to the linearized hydrodynamic differential equations have only been found for a few very regular container shapes. The liquid oscillations in circular cylindrical containers under small translational and rotational oscillations seem to have been extensively investigated (2, 3). For this case, the assumptions of the linear hydrodynamic theory have frequently been experimentally justified (7, 10). Abramson (1) reports excellent correlation between theorectical and experimental results for the case of steady-state horizontal oscillations.

In recent years, some excellent Russian papers have appeared, covering different aspects of the liquid dynamics in moving containers, (4, 16, 17, 18). These have treated the problem from a very general and highly mathematical point of view and have provided much inspiration and guidance for the author's research.

1.3 Investigational Procedure

This research involves a systematic investigation of liquid oscillations in moving circular and elliptic cylindrical containers as a basis for developing a solution of a more complex dynamic system of which the container is a part. The equations of motion are based on the assumption of an ideal, incompressible liquid, potential flow and surface waves of small amplitude and small slopes. Following Poincaré, the displacement of the liquid from the equilibrium position is expanded in a certain series of functions with time dependent coefficients and the problem is reduced to the solution of an infinite set of equations of second order. This is a classical method in the investigation of continuous systems and it is a very powerful tool in this case because this certain series of functions, the eigenfunctions and associated eigenvalues can be obtained by a separation of variable procedure.

The solution for the circular cylindrical container is well known and its derivation is included because it is a limiting case of the solution for the elliptic cylindrical container, thus furnishing a check on the results. The author believes that the oscillation problem for systems involving elliptic cylindrical containers is solved here for the first time and is the main contribution of this dissertation.

A main characteristic of this approach is the partition of the liquid motion into two parts: (1) the motion if the surface is replaced by a rigid lid, and (2) the wave motion at the surface. The first contribution can be derived from the Stokes-Zhukovskii potentials which are completely determined by the shape of the container alone. They are the basis used to compute the components of the equivalent inertia tensor for the liquid completely enclosed in a rigid container. The second contribution, the wave motion, is then expanded into an infinite series with time-dependent coefficients with respect to the complete orthogonal set of modes of free vibration.

The research has led to the following results:

- Dimensionless tables and curves for the equivalent moment of inertia of an ideal, incompressible liquid completely enclosed in an elliptic cylindrical container with respect to rotations about the two principal axes through its center of gravity.
- Tables of modal constants which permit the evaluation of the contribution of the n-th mode in the system of second order differential equations of motion for a certain class of dynamic systems having elliptic cylindrical containers partially filled with liquid.
- As an example, response curves for the ladle system in Figure 11 are presented in a dimensionless form.

II. FREE OSCILLATIONS OF AN IDEAL INCOMPRESSIBLE LIQUID IN A VESSEL AT REST

2.1 Fundamental Equations

In the following, the classical, linearized equations for the small oscillations of a liquid are derived with the help of Hamilton's principle.

Consider an ideal, incompressible liquid enclosed in a fixed shell (Figure 2). It is assumed that the free surface S in the position of equilibrium coincides with the plane determined by the x- and y-axis of the Cartesian coordinate system. C designates the volume occupied by the liquid in the position of equilibrium and W stands for the wetted walls of the container.

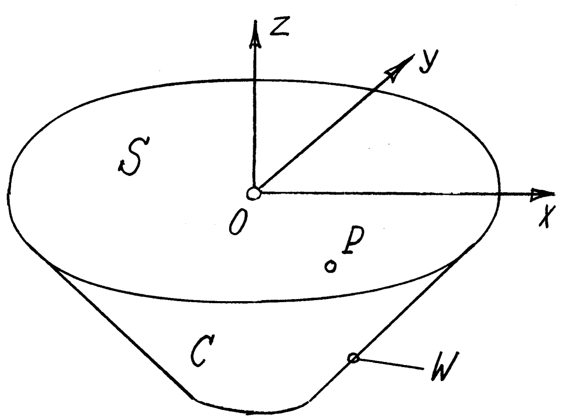

Figure 2
General Container

Any liquid particle P at the point (x,y,z) at the time t is described by:

Velocity of the liquid particle:	$\vec{v} = \vec{v}(u, v, w)$
Body force per unit mass:	$\vec{K} = \vec{K}(X, Y, Z)$
Pressure:	p
Density:	ρ

(u, v, w) are the components of $\vec{v}$ and (X, Y, Z) the components of $\vec{K}$ in the coordinate system (x, y, z).

Since the gravity g is the only body force acting and the liquid is assumed to be incompressible, $\vec{K} = \vec{K}\,(0,\ 0,\ -g)$ and ρ = const.

Finally, with the equation for the free surface, $z = \zeta\,(x,\ y,\ t)$, the kinetic and the potential energy of the liquid are given by the formulae (Ref. 13, Art. 174):

$$T = \frac{\rho}{2} \int_C \vec{v}^{\,2}\, dC \qquad (1)$$

$$V = \frac{\rho \cdot g}{2} \int_S \zeta^2 dS \qquad (2)$$

Hamilton's principle can be stated in the form

$$\delta \int_{t_1}^{t_2} (T - V)\, dt = 0 \qquad (3)$$

which says that the actual path followed by a dynamical process is such as to make the integral of the function $(T - V)$ stationary.

The liquid enclosed in volume C is subject to the following conditions:

- Impermeability of the shell, e.g. no velocity perpendicular to the wall.

$$\vec{\nu} \cdot \vec{v} = 0, \text{ for } P \in W \qquad (4)$$

$\vec{\nu}$ is the unit outward normal to the container walls.

- Continuity

$$\nabla \vec{v} = 0, \text{ for } P \in C \qquad (5)$$

The condition at the free surface is a consequence of Equation (5). To avoid a flow through the free surface, a liquid particle has to satisfy the condition

$$w = u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} + \frac{\partial \zeta}{\partial t}, \text{ for } P \in S \qquad (6)$$

For the assumed small oscillations with small surface slopes, this simplifies to

$$w = \frac{\partial \zeta}{\partial t}, \text{ for } P \in S \qquad (7)$$

It is usual to impose the condition of irrotationality on the motion of the liquid. This is not necessary for this linearized problem. It has been proven in Ref. 16, pp. 242, that the vortical component of the velocity vector $\vec{v}$ remains constant if second order terms are neglected. The pressure and the shape of the free surface can be determined with the same degree of accuracy by the irrotational component alone, which can be found in this linearized problem independently of the vortical component. The condition

$$\vec{v} = \nabla \phi \qquad (8)$$

where ϕ is the velocity potential, will give a first degree accuracy of the problem, no matter whether the motion is actually rotational or irrotational.

Applying Hamilton's principle to the energy expressions (1) and (2) yields

$$0 = \delta \int_{t_1}^{t_2} \left[\int_C \vec{v}^{\,2} \, dC - g \int_S \zeta^2 \, dS \right] dt$$

$$0 = \int_{t_1}^{t_2} \left[\int_C \vec{v} \, \delta\vec{v} \, dC - g \int_S \zeta \, \delta\zeta \, dS \right] dt$$

which after introducing the velocity potential ϕ becomes

$$0 = \int_{t_1}^{t_2} \left[\int_C \nabla\phi \nabla\delta\phi \, dC - g \int_S \zeta \, \delta\zeta \, dS \right] dt \tag{9}$$

Green's theorem is used to transform the volume integral of Equation (9) into an intergral over the free surface S. Using in addition the continuity condition, Equation (5), and the condition at the container walls, Equation (4), the following relation is obtained:

$$\int_C \nabla\phi \nabla \, \delta\phi \, dC = \int_S \phi \frac{\partial\delta\phi}{\partial z} \, dS \tag{10}$$

Together with the surface condition, Equation (7), which is used in the form

$$\frac{\partial\delta\phi}{\partial z} = \frac{\partial\delta\zeta}{\partial t} \text{, for } P \in S \tag{11}$$

Equation (9) can be rewritten

$$0 = \int_{t_1}^{t_2} \int_S \left(\phi \frac{\partial\delta\zeta}{\partial t} - g \, \zeta \, \delta\zeta \right) dS dt \tag{12}$$

Integrating Equation (12) by parts and using the isochronism of the variations, leads to :

$$0 = \int_{t_1}^{t_2} \int_S \left(\frac{\partial \phi}{\partial t} + g\,\zeta \right) \delta\zeta \; dS dt \qquad (13)$$

Due to the arbitrariness of the variations, Equation (13) is satisfied only when

$$0 = \frac{\partial \phi}{\partial t} + g\,\zeta, \text{ for } P \in S \qquad (14)$$

This important equation gives the relations between the surface function ζ and the velocity potential ϕ. It can be used, together with Equation (8), to give the governing differential equation, the boundary conditions and the energy relations in terms of the velocity potential ϕ only.

Differential equation and boundary conditions:

From Equations (5) and (8) : $\nabla^2 \phi = 0$, for $P \in C$ (15)

From Equations (4) and (8) : $\vec{\nu} \nabla \phi = 0$, for $P \in W$ (16)

From Equations (14), (7) and (8) : $\dfrac{\partial^2 \phi}{\partial t^2} + g \dfrac{\partial \phi}{\partial z} = 0$, for $P \in S$ (17)

Energy relations:

From Equations (1) and (8) : $T = \dfrac{\rho}{2} \displaystyle\int_C (\nabla \phi)^2 \, dC$ (18)

From Equations (2) and (14) : $V = \dfrac{\rho}{2g} \displaystyle\int_S \left(\frac{\partial \phi}{\partial t} \right)^2 dS$ (19)

To study the question of natural oscillations of the liquid, it is assumed that

$$\phi(x,y,z,t) = \varphi(x,y,z)\ \sin\omega t \tag{20}$$

φ is the natural mode and ω the natural frequency of liquid oscillations.

If Equation (20) is introduced into Equations (15) to (19), corresponding relations in terms of the natural mode φ and the natural frequency ω can be derived.

Differential equation and boundary conditions:

$$\nabla^2 \varphi = 0, \text{ for } P \in C \tag{21}$$

$$\vec{\nu}\,\nabla \varphi = 0, \text{ for } P \in W \tag{22}$$

$$\frac{\partial \varphi}{\partial z} - \frac{\omega^2}{g}\varphi = 0, \text{ for } P \in S \tag{23}$$

Extreme values for energies:

$$T_{max} = \frac{\rho}{2} \int_C (\nabla \varphi)^2\ dC \tag{24}$$

$$V_{max} = \frac{\rho \cdot \omega^2}{2g} \int_S \varphi^2\ dS \tag{25}$$

Equating Equations (24) and (25) yields the Rayleigh quotient

$$\frac{\omega^2}{g} = \frac{\int_C (\nabla \varphi)^2\ dC}{\int_S \varphi^2\ dS} \tag{26}$$

which has many advantageous properties in numerical computations. These properties are discussed extensively in the literature, e.g. Reference 22, pp. 486 ff. While the numerical procedures which are based on Rayleigh's principle may be the only approach to the general liquid oscillation problem, there exist a few very special cases where the solution can be obtained by rigorous integration of the boundary value problem, Equations (21) to (23).

2.2 Some Properties of the Solution

2.2.1 Solvability of the Problem

It has been proven (Reference 16, pp. 268 ff.) that the problem which is described by Laplace's differential equation, Equation (15), and the boundary conditions, Equations (16) and (17), has the following properties:

- For the motion of a liquid about the equilibrium position in a finite container there exist natural oscillations, e.g. solutions of the form of Equation (20).
- The eigenvalues ω_n^2 are positive, of finite multiplicity and form a sequence increasing without bound.
- The eigenfunctions φ_n form a sequence which is complete in S.

2.2.2 Orthogonality of the Eigenfunctions

Under the assumption that all eigenvalues ω_n^2 are positive and distinct, the orthogonality of the corresponding eigenfunctions φ_n can be proved quite easily. Let φ_m and φ_n be two modes associated with the eigenvalues ω_m^2 and ω_n^2. Both modes of course satisfy Equation (21)

$$\nabla^2 \varphi_m = 0, \text{ for } P \in C \tag{27}$$

$$\nabla^2 \varphi_n = 0, \text{ for } P \in C \tag{28}$$

Green's theorem leads to:

$$\int_S \left(\varphi_m \frac{\partial \varphi_n}{\partial z} - \varphi_n \frac{\partial \varphi_m}{\partial z} \right) dS = 0 \tag{29}$$

But at the surface, both functions have to satisfy Equation (23). This leads to the two relations:

$$\frac{\omega_m^2 - \omega_n^2}{g} \int_S \varphi_m \ \varphi_n \ dS = 0 \qquad (30)$$

$$\frac{g}{\omega_m^2 \omega_n^2} \ (\omega_m^2 - \omega_n^2) \int_S \frac{\partial \varphi_m}{\partial z} \ \frac{\partial \varphi_n}{\partial z} \ dS = 0 \qquad (31)$$

For $m \neq n$, ω_m^2 is according to the assumption different from ω_n^2, therefore

$$\int_S \varphi_m \ \varphi_n \ dS = 0 \qquad (32)$$

$$\int_S \frac{\partial \varphi_m}{\partial z} \ \frac{\partial \varphi_n}{\partial z} \ dS = 0 \qquad (33)$$

2.2.3 Criterion for the Comparison of the Lowest Natural Frequency in Different Containers

Reference 16 gives on page 241 a criterion which has proved to be extremely useful in checking numerical results. Consider two different containers with equal liquid surfaces S but where the walls W_1 of the first container completely envelop the walls W_2 of the second container (Figure 3). Let φ be some function which satisfies the Laplace equation in C_1 and the surface condition, Equation 23, in S. Since $C_1 > C_2$, the following inequality holds for φ

$$\int_{C_1} (\nabla \varphi)^2 \ dC_1 > \int_{C_2} (\nabla \varphi)^2 \ dC_2 \qquad (34)$$

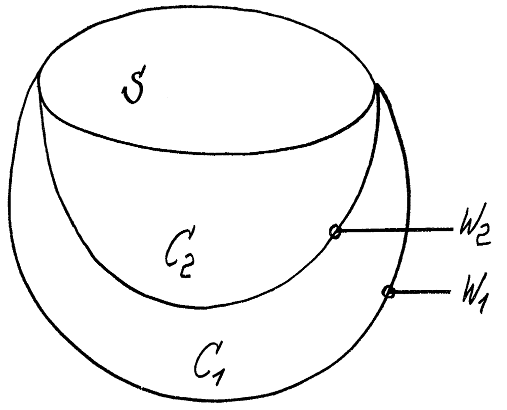

Figure 3
Two Enveloping Containers

Let $\left(\omega^{(1)^2}, \varphi^{(1)}\right)$ and $\left(\omega^{(2)^2}, \varphi^{(2)}\right)$ correspond to the lowest mode in the two containers. Equation (26) must be satisfied in both cases.

$$\frac{\omega^{(1)^2}}{g} = \frac{\int_{C_1} \left|\nabla\varphi^{(1)}\right|^2 dC_1}{\int_S \varphi^{(1)^2} dS} \qquad (35)$$

$$\frac{\omega^{(2)^2}}{g} = \frac{\int_{C_2} \left|\nabla\varphi^{(2)}\right|^2 dC_2}{\int_S \varphi^{(2)^2} dS} \qquad (36)$$

Since $\omega^{(2)^2}$ is a minimum, the right hand side of Equation (36) must become bigger when $\varphi^{(2)}$ is replaced by any other admissible function. Though $\varphi^{(1)}$ does not satisfy the boundary condition in W_2, it is according to the general theory, Reference 6, pp. 398 and 461, an admissible function because the variational principle is of the free boundary condition type. Therefore,

$$\frac{\omega^{(2)^2}}{g} \leq \frac{\int_{C_2} \left|\nabla\varphi^{(1)}\right|^2 dC_2}{\int_S \varphi^{(1)^2} dS} \qquad (37)$$

Using the inequality (34), the right hand side of inequality (37) becomes the one of Equation (35), which means

$$\omega^{(2)} < \omega^{(1)} \qquad (38)$$

The result can be formulated as follows:

If two containers with the same free liquid surface are such that the walls of the first can completely envelop the walls of the second, then the corresponding lowest natural frequency will be greater in the container whose volume is larger.

2.3 Solution for the Circular Cylindrical Container

This solution is described by Lamb in Reference 13, Art. 191. It is given as a basis for later developments.

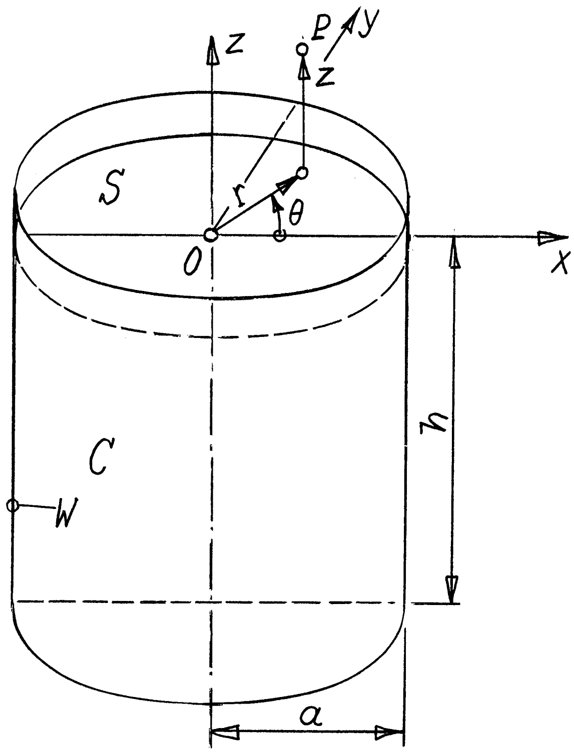

Figure 4

Circular Cylindrical Container

Consider a cylindrical shell of radius a, filled with an ideal, incompressible liquid up to the depth h (Figure 4). The problem will be discussed in circular cylindrical coordinates, which are assumed as indicated in Figure 4. The natural modes of oscillation have to satisfy Equations (21), (22) and (23) which are given below with respect to circular cylindrical coordinates.

$$r^2 \frac{\partial^2 \varphi}{\partial r^2} + r \frac{\partial \varphi}{\partial r} + \frac{\partial^2 \varphi}{\partial \theta^2} + r^2 \frac{\partial^2 \varphi}{\partial z^2} = 0, \text{ for } P \in C \qquad (39)$$

$$\frac{\partial \varphi}{\partial r} = 0, \text{ for } P \in W \qquad (40)$$

$$\frac{\partial \varphi}{\partial z} - \frac{\omega^2}{g} \varphi = 0, \text{ for } P \in S \qquad (41)$$

Using the separation of variable procedure with

$$\varphi(r, \theta, z) = \varphi_1(r)\, \varphi_2(\theta)\, \varphi_3(z) \tag{42}$$

Equation (39) becomes

$$r^2 \frac{\varphi_1''}{\varphi_1} + r \frac{\varphi_1'}{\varphi_1} + \frac{\varphi_2''}{\varphi_2} + r^2 \frac{\varphi_3''}{\varphi_3} = 0 \tag{43}$$

Equation (43) requires that all the ratios (φ_i''/φ_i) are constants. In particular

$$\frac{\varphi_3''}{\varphi_3} = f^2 \tag{44}$$

The solution to Equation (44) which satisfies condition (40) is

$$\varphi_3(z) = \cosh f(z + h) \tag{45}$$

whereas condition (41) leads to the relation

$$\omega^2 = g \cdot f \cdot \tanh\,(f \cdot h) \tag{46}$$

In a similar way

$$\frac{\varphi_2''}{\varphi_2} = -\, m^2 \tag{47}$$

Since Equation (47) determines the number of nodal diameters of some particular mode shape, it is obvious that m has to be a positive integer $m = 0, 1, 2, \ldots$ Equations (44) and (47) reduce Equation (43) to the parametric form of Bessel's differential equation

$$r^2 \varphi_1'' + r\, \varphi_1' + (f^2 r^2 - m^2)\, \varphi_1 = 0 \tag{48}$$

which in this case has a solution of the form:

$$\varphi_1(r) = J_m(f \cdot r) \tag{49}$$

The remaining boundary condition, Equation (40), requires

$$\frac{d}{dr}\varphi_1(r) = \frac{d}{dr}J_m(f \cdot r) = 0, \text{ for } r = a \tag{50}$$

If $x = \mu_{m,n}$ are the zeros in the derivative of the m-th order Bessel function satisfying

$$\frac{d}{dx}J_m(x) = 0 \tag{51}$$

then the corresponding separation constant f is simply

$$f_{m,n} = \frac{\mu_{m,n}}{a} \tag{52}$$

The indices of the eigenvalues are chosen such that $m = 0, 1, 2, \ldots$ corresponds to the order of the Bessel function or the number of the nodal diameters while $n = 1, 2, \ldots$ numbers the positive zeros to Equation (51) in increasing order.

If finally the separated functions are put together according to Equation (42), the expression for the (m,n)-th natural mode is

$$\varphi_{m,n} = J_m(\mu_{m,n} \cdot \frac{r}{a}) \cos m\,\theta \cosh\left(\mu_{m,n}\frac{z+h}{a}\right) \tag{53}$$

2.4 Solution for the Elliptic Cylindrical Container

This solution is presented by N. W. McLachlan as an illustration for the application of Mathieu functions, Reference 14, Arts. 16.20 ff. The method of solution is very similar to that used for the circular cylindrical container, which will be a limiting case in the procedure.

The Laplace equation can be separated when elliptic cylindrical coordinates are introduced, which are related to the Cartesian system in the following way:

$$\begin{aligned} x &= c \cosh \xi \cos \eta \\ y &= c \sinh \xi \sin \eta \\ z &= z \end{aligned} \qquad (54)$$

The corresponding coordinate surfaces are:

Right elliptic cylinders:

$$\frac{x^2}{c^2 \cosh^2 \xi} + \frac{y^2}{c^2 \sinh^2 \xi} = 1 \qquad (55)$$

Right hyperbolic cylinders:

$$\frac{x^2}{c^2 \cos^2 \eta} - \frac{y^2}{c^2 \sin^2 \eta} = 1 \qquad (56)$$

Planes parallel to the x-y plane:

$$z = z \qquad (57)$$

Figure 5 shows the lines of intersection between the surfaces of Equation (55) and Equation (56) with the plane z = z. The fundamental system of equations, Equations (21), (22) and (23), when written with respect to elliptic cylindrical coordinates takes the form:

$$\frac{1}{c^2(\sinh^2\xi + \sin^2\eta)}\left(\frac{\partial^2\varphi}{\partial\xi^2} + \frac{\partial^2\varphi}{\partial\eta^2}\right) + \frac{\partial^2\varphi}{\partial z^2} = 0, \text{ for } P \in C \qquad (58)$$

$$\frac{\partial\varphi}{\partial\xi} = 0, \text{ for } P \in W \qquad (59)$$

$$\frac{\partial\varphi}{\partial z} - \frac{\omega^2}{g}\varphi = 0, \text{ for } P \in S \qquad (60)$$

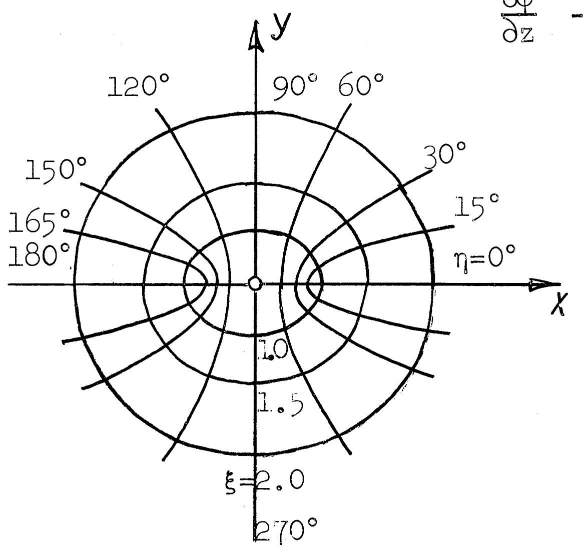

Figure 5

Elliptical Coordinates ξ, η

If it is again assumed that a solution of the form

$$\varphi(\xi, \eta, z) = \varphi_1(\xi)\,\varphi_2(\eta)\,\varphi_3(z) \qquad (61)$$

exists, then Equation (58) becomes:

$$\frac{1}{c^2(\sinh^2\xi + \sin^2\eta)}\left(\frac{\varphi_1''}{\varphi_1} + \frac{\varphi_2''}{\varphi_2}\right) + \frac{\varphi_3''}{\varphi_3} = 0, \text{ for } P \in C \qquad (62)$$

The solution for φ_3 is exactly the same as in the case of a circular cylindrical container. This means

$$\frac{\varphi_3''}{\varphi_3} = f^2 \tag{63}$$

$$\varphi_3(z) = \cosh f(z + h) \tag{64}$$

and

$$\omega^2 = g \cdot f \cdot \tanh (f \cdot h) \tag{65}$$

With the separation constant f, Equation (62) takes the form

$$\frac{\varphi_1''}{\varphi_1} + f^2c^2 \sinh^2 \xi + \frac{\varphi_2''}{\varphi_2} + f^2c^2\sin^2 \eta = 0 \tag{66}$$

which requires that

$$\frac{\varphi_2''}{\varphi_2} + f^2c^2 \sin^2 \eta = d \tag{67}$$

$$\frac{\varphi_1''}{\varphi_1} + f^2c^2 \sinh^2 \xi = -d \tag{68}$$

where d is some constant. Equations (67) and (68) are equivalent to

$$\varphi_2'' + \left(\frac{f^2c^2}{2} - d - \frac{f^2c^2}{2} \cos 2\eta\right)\varphi_2 = 0 \tag{69}$$

$$\varphi_1'' - \left(\frac{f^2c^2}{2} - d - \frac{f^2c^2}{2} \cosh 2\xi\right)\varphi_1 = 0 \tag{70}$$

Using the substitutions

$$\frac{f^2 \cdot c^2}{2} - d = s \tag{71}$$

$$\frac{f^2 \cdot c^2}{4} = q \tag{72}$$

the canonical forms of Mathieu's differential equations are obtained.

$$\varphi_2'' + (s - 2q \cos 2\eta)\,\varphi_2 = 0 \qquad (73)$$

$$\varphi_1'' - (s - 2q \cosh 2\xi)\,\varphi_1 = 0 \qquad (74)$$

Equation (73) is simply called "Mathieu's Differential Equation" while Equation (74) is usually known as the "Modified Mathieu Differential Equation". Both equations, however, are very closely related due to the fact that Equation (73) takes the form of Equation (74) simply by putting $i\eta$ for η.

There are two conditions from which the parameters s and q can be determined. The first is the very obvious requirement that $\varphi_2(\eta)$ must have a period of 2π. This leads to a functional dependency $s = s(q)$ which is well known in the form of the Characteristic Numbers (Ref. 14, Art. 3.25). The other condition is derived from Equation (59), requiring a zero derivative of $\varphi_1(\xi)$ at the boundary $\xi = \xi_B$ of the container. In fact, there exists an unbounded set of distinct, positive eigenvalues $q_{m,n}$ and corresponding frequencies

$$\omega_{m,n}^2 = \frac{2g}{c}\sqrt{q_{m,n}}\ \tanh\left(\frac{2h}{c}\sqrt{q_{m,n}}\right) \qquad (75)$$

which satisfy these conditions.

The properties of the solutions to Equations (73) and (74) are extensively discussed in Reference 14 upon which the subsequent work will draw heavily.

It is sufficient for this investigation to select from the various possible solutions to Mathieu's differential equations only the following two cases:

Cosine-Elliptic Case:

$$\varphi_{2m+1,n} = Ce_{2m+1}\left(\xi, q_{2m+1,n}\right) ce_{2m+1}\left(\eta, q_{2m+1,n}\right) \cosh \frac{2}{c} \sqrt{q_{2m+1,n}}\,(z+h) \tag{76}$$

Sine-Elliptic Case:

$$\varphi_{2m+1,n} = Se_{2m+1}\left(\xi, q_{2m+1,n}\right) se_{2m+1}\left(\eta, q_{2m+1,n}\right) \cosh \frac{2}{c} \sqrt{q_{2m+1,n}}\,(z+h) \tag{77}$$

Equation (76) represents the class of natural modes which are symmetric with respect to the (x-z) plane and antisymmetric with respect to the (y-z) plane. Similarly Equation (77) represents the modes, symmetric to the (y-z) and antisymmetric to the (x-z) plane. These functions are fundamental for liquid oscillation problems where the nature of the system dictates a motion with the same symmetry properties.

The most basic problem is now the evaluation of the eigenvalues $q_{2m+1,n}$. The selected numerical procedure for the solution of this problem will be presented in the next section.

2.5 The Numerical Evaluation of Mode Frequencies

2.5.1 Circular Cylindrical Container

The evaluation of mode frequencies for this case involves the solution of the transcendental equation

$$\frac{d}{dx} J_m(x) = 0 \qquad (78)$$

Solutions $\mu_{m,n}$ are tabulated to some extent in the literature, e.g. Reference 11. If additional values are needed, a rapidly convergent Newton-Raphson procedure can easily be developed with the help of the following relations:

$$J_{m+1}(x) = \frac{2m}{x} J_m(x) - J_{m-1}(x) \qquad (79)$$

$$\frac{d}{dx} J_m(x) = \frac{1}{2} \left[J_{m-1}(x) - J_{m+1}(x) \right] \qquad (80)$$

This means that derivatives of Bessel functions can be expressed in terms of Bessel functions of the same kind and higher order Bessel functions can be computed from lower order functions.

The mode frequencies can be computed from $\mu_{m,n}$ with the relation

$$\omega^2_{m,n} = \mu_{m,n} \frac{g}{a} \tanh \left(\mu_{m,n} \frac{h}{a} \right) \qquad (81)$$

The applications in Chapters III and IV will only require solutions to Equation (78) for m = 1. The lowest 20 values of $\mu_{1,n}$ are given for completeness in column SIGMA in Table IIX of the Appendix.

2.5.2 Elliptic Cylindrical Container

The determination of eigenvalues for the two cases given in Equations (76) and (77) is discussed in this section. Since these two cases will lead to relations which are very similar, each will be given the same number with the letter a attached when the equation corresponds to the cosine-elliptic case and the letter b when it corresponds to the sine-elliptic case.

2.5.2.1 Characteristic Numbers and Coefficients

As indicated in Section 2.4 the solution to Equation (73) must be periodic, with period 2π. An expansion of the solution into a Fourier series will impose this condition.

$$ce\,(\eta,q) = \sum_{i=0}^{\infty} A_{2i+1} \cos\,(2i+1)\eta \tag{82a}$$

$$se\,(\eta,q) = \sum_{i=0}^{\infty} B_{2i+1} \sin\,(2i+1)\eta \tag{82b}$$

If these series expansions are made to satisfy Mathieu's differential equation, Equation (73), identically, the following recurrence relations are obtained. They are given in the form in which they were used in numerical computations.

$$\frac{A_3}{A_1} = \frac{s - 1}{q} - 1$$

$$\frac{A_5}{A_3} = \frac{s - 3^2}{q} - \frac{1}{\frac{A_3}{A_1}}$$

- - - - - - - - - - - - -

$$\frac{A_{2m+3}}{A_{2m+1}} = \frac{s - (2m+1)^2}{q} - \frac{1}{\frac{A_{2m+1}}{A_{2m-1}}} \qquad (83a)$$

$$\frac{A_{2m+3}}{A_{2m+1}} = \frac{q}{s - (2m+3)^2 - \frac{A_{2m+5}}{A_{2m+3}} q}$$

- - - - - - - - - - - - - - - - - -

$$\frac{A_{2i+1}}{A_{2i-1}} = \frac{q}{s - (2i+1)^2 - \frac{A_{2i+3}}{A_{2i+1}} q}$$

- - - - - - - - - - - - - - - - - - $m = 0, 1, 2, \ldots$

The corresponding relations between the coefficients B_{2i+1} of the sine-elliptic case are the same except for the first equation

$$\frac{B_3}{B_1} = \frac{s - 1}{q} + 1 \qquad (83b)$$

These recurrence relations are satisifed for every value of q by the characteristic numbers, which are continuous functions in q.

$$s = a_{2m+1}(q) \qquad (84a)$$

$$s = b_{2m+1}(q) \qquad (84b)$$

These functions are qualitatively plotted for m=0, 1, 2 in Figure 6.

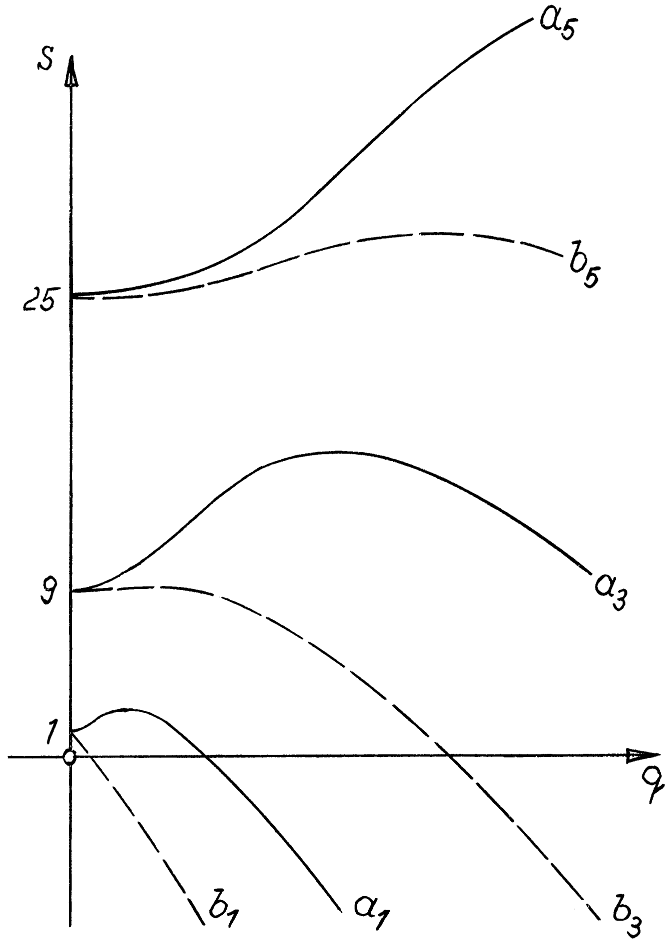

Figure 6
Characteristic Curves
$a_{2m+1}(q)$ and $b_{2m+1}(q)$

The integer m=0, 1, 2, . . . designates the order of the Mathieu function and its significance can be visualized by considering the solution of Equation (73) for q = 0. Then

$$ce_{2m+1}(\eta,0) = \cos(2m+1)\eta, \; a_{2m+1}(0) = (2m+1)^2 \qquad (85a)$$

$$se_{2m+1}(\eta,0) = \sin(2m+1)\eta, \; b_{2m+1}(0) = (2m+1)^2 \qquad (85b)$$

The roots of Equations (83) can be found by a trial and error procedure. In order to explain the algorithm, Equations (83a) are rewritten with

$$v_i = \frac{A_{2i+3}}{A_{2i+1}} \tag{86a}$$

and under the assumption that s_1 is a good approximation for a_{2m+1}

$$\begin{aligned}
v_o &= \frac{s_1 - 1}{q} - 1 \\
v_1 &= \frac{s_1 - 3^2}{q} - \frac{1}{v_o} \\
&-\ -\ -\ -\ -\ -\ -\ -\ -\ - \\
v_m^{(1,1)} &= \frac{s_1 - (2m+1)^2}{q} - \frac{1}{v_{m-1}} \\
v_m^{(1,2)} &= \frac{q}{s_1 - (2m+3)^2 - v_{m+1}\, q} \\
&-\ -\ -\ -\ -\ -\ -\ -\ -\ -\ -\ -\ -\ -\ -\ - \\
v_i &= \frac{q}{s_1 - (2i+3)^2 - v_{i+1}\, q} \\
&-\ -\ -\ -\ -\ -\ -\ -\ -\ -\ -\ -\ -\ -
\end{aligned} \tag{87a}$$

To make this infinite algorithm finite, it has to be started by putting some v_i equal to zero. If a second pair of values $v_m^{(2,1)}$ and $v_m^{(2,2)}$ are obtained from another approximation $s_2 = s_1 + \Delta s$, then a better approximation s_3 can be expected from a simple linear interpolation.

$$s_3 = s_1 + \Delta s \, \frac{v_m^{(1,1)} - v_m^{(1,2)}}{v_m^{(1,1)} - v_m^{(1,2)} - v_m^{(2,1)} + v_m^{(2,2)}} \tag{88}$$

The iteration procedure, which was put together on the basis of this equation, converged to the desired characteristic number a_{2m+1} (b_{2m+1}) except for some small regions of mainly higher values of q, where it converged to the next higher or the next lower characteristic number, no matter how good the original approximation was. This problem was solved by arbitrarily decreasing m in Equations (87) by one in the first case and increasing by one in the second case.

When the recurrence relations, Equations (83), are satisfied, they determine at the same time the coefficients of the trigonometric expansions, Equations (82), up to some constant factor, which has to be determined from a normalization convention. It has become common practice to normalize according to the condition

$$\frac{1}{\pi} \int_0^{2\pi} ce_{2m+1}^2 (\eta,q)d\eta = 1 \tag{89a}$$

$$\frac{1}{\pi} \int_0^{2\pi} se_{2m+1}^2 (\eta,q)d\eta = 1 \tag{89b}$$

which by using the expansions of Equations (82) leads to

$$\sum_{i=0}^{\infty} \left[A_{2i+1}^{(2m+1)}\right]^2 = 1 \tag{90a}$$

$$\sum_{i=0}^{\infty} \left[B_{2i+1}^{(2m+1)}\right]^2 = 1 \tag{90b}$$

This normalization avoids the possibility that coefficients may become infinite if one coefficient happens to become zero.

The characteristic numbers and the coefficients are tabulated extensively in Reference 19. But to avoid interpolation in tabular values and to insure that the characteristic numbers and the coefficients were available to an extent required by the following applications, the outlined procedure was programmed for automatic computation.

2.5.2.2 Series Expansions of the Mathieu Functions

The expansions of two solutions for the ordinary Mathieu differential equation, Equation (73), have already been given. They are, if the order of the functions is included in the notation:

$$ce_{2m+1}(\eta,q) = \sum_{i=0}^{\infty} A_{2i+1}^{(2m+1)} \cos\,(2i+1)\eta \tag{91a}$$

$$se_{2m+1}(\eta,q) = \sum_{i=0}^{\infty} B_{2i+1}^{(2m+1)} \sin\,(2i+1)\eta \tag{91b}$$

Due to the indicated close relation between the two Mathieu differential equations, the corresponding solution to the modified equation can immediately be given.

$$Ce_{2m+1}(\xi,q) = \sum_{i=0}^{\infty} A_{2i+1}^{(2m+1)} \cosh(2i+1)\xi \tag{92a}$$

$$Se_{2m+1}(\xi,q) = \sum_{i=0}^{\infty} B_{2i+1}^{(2m+1)} \sinh(2i+1)\xi \tag{92b}$$

It is proven in Reference 14, Art. 3.22 and 3.23, that all four series and the p-th derivatives obtained by term by term differentiation are absolutely and uniformly convergent in any closed interval of the real argument. Nevertheless only Equations (91) can be used in the given form for numerical computations. With the accuracy of an ordinary electronic computer, series (92) will become unstable for arguments q which are still of great practical interest. The literature gives, however, series expansions which have far better convergence properties. In this investigation, expansions into Bessel function products were selected. With

$$w_1 = \sqrt{q}\, e^{-\xi} \tag{93}$$

$$w_2 = \sqrt{q}\, e^{\xi} \tag{94}$$

Equations (92) are written in the form:

$$Ce_{2m+1}(\xi,q)$$

$$= - \frac{ce_{2m+1}(0,q)ce'_{2m+1}\left(\frac{\pi}{2},q\right)}{\sqrt{q}\left(A_1^{(2m+1)}\right)^2} \sum_{i=0}^{\infty}(-1)^i A_{2i+1}^{(2m+1)}\left[J_i(w_1)J_{i+1}(w_2)+J_{i+1}(w_1)J_i(w_2)\right] \tag{95a}$$

$$Se_{2m+1}(\xi,q)$$

$$= \frac{se'_{2m+1}(0,q)se_{2m+1}\left(\frac{\pi}{2},q\right)}{\sqrt{q}\left(B_1^{(2m+1)}\right)^2} \sum_{i=0}^{\infty}(-1)^i B_{2i+1}^{(2m+1)}\left[J_i(w_1)J_{i+1}(w_2)-J_{i+1}(w_1)J_i(w_2)\right] \tag{95b}$$

2.5.2.3 Characteristic Equations and Eigenvalues

The condition at the boundary $\xi = \xi_B$, Equation (59), leads immediately to the characteristic equation for this problem.

$$\frac{d}{d\xi}\, Ce_{2m+1}(\xi,q) = 0, \text{ for } \xi=\xi_B \tag{96a}$$

$$\frac{d}{d\xi}\, Se_{2m+1}(\xi,q) = 0, \text{ for } \xi=\xi_B \tag{96b}$$

To each of these modified Mathieu functions of order (2m+1) corresponds an unbounded set of distinct, positive eigenvalues q, which satisfy the characteristic equation (96). They are arranged in increasing order and characterized by the number n = 1, 2, 3, . . .

To find the roots of the characteristic equations, the Mathieu functions are expanded according to Equations (95). Considering the fact that the proportionality factor of these expansions never vanishes for $q > 0$ (Reference 14, Art. 3.30, Reference 15, pp. 134), which is quite obvious from physical considerations, the characteristic equations which are suitable for numerical computations take the form

$$\begin{aligned} 0 = &\sum_{i=0}^{\infty} (-1)^i A_{2i+1}^{(2m+1)} \left[-w_1 J_i'(w_1) J_{i+1}(w_2) + w_2 J_i(w_1) J_{i+1}'(w_2) \right] \\ &+ \sum_{i=0}^{\infty} (-1)^i A_{2i+1}^{(2m+1)} \left[-w_1 J_{i+1}'(w_1) J_i(w_2) + w_2 J_{i+1}(w_1) J_i'(w_2) \right] \end{aligned} \tag{97a}$$

$$\begin{aligned} 0 = &\sum_{i=0}^{\infty} (-1)^i B_{2i+1}^{(2m+1)} \left[-w_1 J_i'(w_1) J_{i+1}(w_2) + w_2 J_i(w_1) J_{i+1}'(w_2) \right] \\ &- \sum_{i=0}^{\infty} (-1)^i B_{2i+1}^{(2m+1)} \left[-w_1 J_{i+1}'(w_1) J_i(w_2) + w_2 J_{i+1}(w_1) J_i'(w_2) \right] \end{aligned} \tag{97b}$$

where w_1 and w_2 are of course taken at the boundary $\xi = \xi_B$.

Though these equations are incomparably more difficult than the characteristic equation for the circular cylindrical container, similar methods may be used to find the roots. Equations (79) and (80) are again fundamental to reduce higher order Bessel functions and derivatives of Bessel functions. Once the right sides of Equations (97) can be calculated with reasonable speed and accuracy, the Method of False Position (Regula Falsi) may be employed to find the roots.

The actual numerical evaluation of the roots to the characteristic equations (97) is fundamental in the discussion of liquid oscillations in elliptic cylindrical containers. It turned out that the solution of this problem was at the same time the most difficult and the most time-consuming part of the whole investigation. To simply evaluate the right hand sides of Equations (97) for an arbitrary value of q, the expansion coefficients have first to be determined. They result as a by-product in the iteration for the characteristic values. In order to insure that the characteristic value is obtained which belongs to a certain order of the Mathieu function, this iteration has to be started with a very good approximation. This approximation might be the characteristic value belonging to a closely adjacent parameter q. This dictates that the search for the roots has to proceed in small steps starting from $q = 0$, for which the characteristic value can be immediately given, to higher values of q. In each step, the characteristic value, the expansion coefficients and the sum of the infinite series in Equations (97) has to be determined. Only when a change of sign in the right hand side occurs within one step, the Regula Falsi procedure may be applied to find the zero. The iteration was performed until the root $k_{2m+1,n} = \sqrt{q_{2m+1,n}}$ has been obtained with a relative precision of at least 10^{-5}.

This procedure became more and more difficult for higher values of q and higher order (2m+1) of the functions. In every case it was carried out until the accuracy requirement could not be satisfied any further.

The roots $k_{2m+1,n}$ are tabulated in column K of the Tables III to VII for the sine-elliptic case and in Tables IX to XIII for the cosine-elliptic case.

The frequencies which correspond to the eigenvalue or root of the characteristic equation of order (2m+1,n) can now be obtained from the following relation

$$\omega^2_{2m+1,n} = \frac{2g}{a} \sqrt{q_{2m+1,n}} \cosh \xi_B \tanh \left(\frac{2h}{a} \sqrt{q_{2m+1,n}} \cosh \xi_B \right) \quad (98)$$

III. THE STOKES - ZHUKOVSKII PROBLEM

It has already been shown by Stokes and Zhukovskii that a closed container completely filled with an ideal and incompressible liquid, whose absolute motion is irrotational, is dynamically equivalent to some rigid body. This means that its dynamic properties can be completely described by its inertia tensor.

3.1 The Stokes - Zhukovskii Potentials

Consider a closed container of arbitrary shape, completely filled with an ideal, incompressible liquid (Figure 7). The container with the volume C and the walls W is moving with respect to the system $(\bar{x}, \bar{y}, \bar{z})$ with the velocity $\vec{v}_o$ and the angular velocity $\vec{\omega}_o$ which have in the body coordinate system (x, y, z) the components (v_{ox}, v_{oy}, v_{oz}) and $(\omega_{ox}, \omega_{oy}, \omega_{oz})$. $\vec{r}$ is a position vector in the system (x, y, z) and $\vec{\nu}(\ell, m, n)$ is again the unit outward normal to the container walls W.

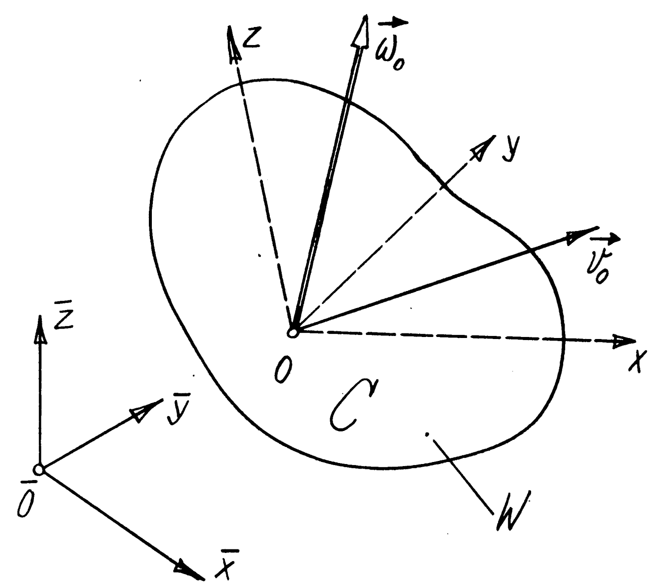

Figure 7
Motion of a Completely Filled Container

The velocity potential ϕ has of course to satisfy the Laplace equation.

$$\nabla^2\phi = 0, \text{ for } P \in C \qquad (99)$$

and the boundary condition

$$\frac{\partial\phi}{\partial\nu} = \vec{v}_o\vec{\nu} + (\vec{\omega}_o \times \vec{r})\vec{\nu}\ , \text{ for } P \in W \tag{100}$$

Since Equation (100) can be trivially rewritten in the form

$$\frac{\partial\phi}{\partial\nu} = v_{ox}\ell + v_{oy}m + v_{oz}n + \omega_{ox}(yn-zm) + \omega_{oy}(z\ell-xn) + \omega_{oz}(xm-y\ell), \text{ for } P \in W \tag{101}$$

it is obvious that the assumption

$$\phi = v_{ox}\phi_1 + v_{oy}\phi_2 + v_{oz}\phi_3 + \omega_{ox}\phi_4 + \omega_{oy}\phi_5 + \omega_{oz}\phi_6 \tag{102}$$

introduces functions ϕ_i which depend on the shape of the container only. According to an accepted terminology, they are called the Stokes-Zhukovskii potentials. They are harmonic and satisfy the boundary conditions

$$\frac{\partial\phi_1}{\partial\nu} = \ell\ , \quad \frac{\partial\phi_2}{\partial\nu} = m\ , \quad \frac{\partial\phi_3}{\partial\nu} = n \tag{103}$$

for pure translations and

$$\begin{aligned} \frac{\partial\phi_4}{\partial\nu} &= yn - zm \\ \frac{\partial\phi_5}{\partial\nu} &= z\ell - xn \\ \frac{\partial\phi_6}{\partial\nu} &= xm - y\ell \end{aligned} \tag{104}$$

for pure rotations.

The Stokes-Zhukovskii potentials for pure translation can immediately be given. It is e.g.

$$\frac{\partial \phi_1}{\partial \nu} = l \frac{\partial \phi_\ell}{\partial x} + m \frac{\partial \phi_1}{\partial y} + n \frac{\partial \phi_1}{\partial z} \tag{105}$$

which identically satisfies the boundary conditions (103) when

$$\frac{\partial \phi_1}{\partial x} = 1, \quad \frac{\partial \phi_1}{\partial y} = 0, \quad \frac{\partial \phi_1}{\partial z} = 0 \tag{106}$$

Using a similar reasoning for ϕ_2 and ϕ_3 gives

$$\phi_1 = x, \quad \phi_2 = y, \quad \phi_3 = z \tag{107}$$

The determination of the Stokes-Zhukovskii potentials for the case of rotations is far less trivial and may even become an extremely difficult problem in more general cases. But here, the Rayleigh-Ritz procedure may again effectively be applied since the potentials ϕ_i minimize the functional

$$F_i(\phi_i) = \int_C (\nabla \phi_i)^2 \, dC - 2 \int_W \phi_i h_i \, dW \tag{108}$$

where h_i is the value $\frac{\partial \phi_i}{\partial \nu}$ in W, Equations (103) and (104). In fact

$$\delta F_i(\phi_i) = 2 \int_C \nabla \phi_i \nabla \delta \phi_i \, dC - 2 \int_W \delta \phi_i h_i \, dW$$

which with the help of Green's theorem leads to

$$\delta F_i(\phi_i) = -2 \int_C \delta \phi_i \nabla^2 \phi_i \, dC + 2 \int_W \delta \phi_i \frac{\partial \phi_i}{\partial \nu} \, dW + 2 \int_W \delta \phi_i h_i \, dW$$

$\delta F_i(\phi_i)$ vanishes identically for arbitrary variations only when ϕ_i is harmonic and satisfies the boundary conditions of the problem.

3.2 The Equivalent Inertia Tensor

To be able to write the following relations in a more compact way, Equation (102) is rewritten in the form

$$\phi = \sum_{i=1}^{6} u_i \phi_i \tag{109}$$

u_i can be considered as components of the generalized velocity vector $\vec{u}$.

The expression for the kinetic energy is

$$T = \frac{\rho}{2} \int_C (\nabla \phi)^2 \, dC \tag{110}$$

which is transformed by Green's theorem into

$$T = \frac{\rho}{2} \int_W \phi \frac{\partial \phi}{\partial \nu} \, dW \tag{111}$$

Introducing Equation (109) into the energy expressions yields a relation of the following form

$$T = \frac{1}{2} \sum_{i=1}^{6} \sum_{j=1}^{6} u_i \, u_j \, M_{ij} \tag{112}$$

M_{ij} is a component of the equivalent inertia tensor and given by the two equivalent expressions

$$M_{ij} = \rho \int_C (\nabla \phi_i)(\nabla \phi_j) \, dC \tag{113}$$

$$M_{ij} = \rho \int_W \phi_i \frac{\partial \phi_j}{\partial \nu} \, dW \tag{114}$$

This inertia tensor with its 36 components can be simplified very considerably. Equation (113) establishes immediately the symmetry of the tensor.

$$M_{ij} = M_{ji} \tag{115}$$

Since the gradients of the Stokes-Zhukovskii potentials for translatory motion, Equation (107), are mutually orthogonal unit vectors, it can be concluded from Equation (113) that

$$M_{11} = M_{22} = M_{33} = \rho C \tag{116}$$

$$M_{12} = M_{13} = M_{23} = 0 \tag{117}$$

Consider now as an example the component

$$M_{14} = \rho \int_W x(yn - zm)\, dW$$

which can be written as

$$M_{14} = \rho \int_W xy \frac{\partial z}{\partial \nu}\, dW - \rho \int_W xz \frac{\partial y}{\partial \nu}\, dW$$

Both terms in this equation are zero according to Green's theorem. This means $M_{14} = 0$ and from similar considerations

$$M_{14} = M_{25} = M_{36} = 0 \tag{118}$$

Consider next

$$M_{15} = \rho \int_W x(z\ell - xn)\, dW$$

which again is rewritten as

$$M_{15} = \rho \int_W xz \frac{\partial x}{\partial \nu}\, dW - \rho \int_W x^2 \frac{\partial z}{\partial \nu}\, dW$$

Green's theorem transforms both terms into

$$\rho \int_W xz \frac{\partial x}{\partial \nu}\, dW = \rho \int_C z\, dC$$

$$\rho \int_W x^2 \frac{\partial z}{\partial \nu}\, dW = 2\rho \int_C z\, dC$$

Hence

$$M_{15} = -\rho \int_C z\, dC$$

Similar calculations lead to

$$M_{24} = \rho \int_C z dC$$

The results can be summarized as follows:

$$\begin{aligned} M_{24} &= - M_{15} = \rho \int_C z dC \\ M_{35} &= - M_{26} = \rho \int_C x dC \\ M_{16} &= - M_{31} = \rho \int_C y dC \end{aligned} \tag{119}$$

All components given by Equations (119) vanish if the origin 0 of the coordinate system (x, y, z) coincides with the centre of gravity of the liquid.

With this result, the analogy with a rigid body as originally stated by Stokes and Zhukovskii is complete. The expression for the kinetic energy, Equation (112), reduces now to the well-known quadratic form

$$T = \frac{\rho}{2} C \sum_{i=1}^{3} u_i^2 + \frac{1}{2} \sum_{i=4}^{6} \sum_{j=4}^{6} u_i u_j M_{ij} \tag{120}$$

For comparison purposes it is appropriate to introduce λ_{ij} which stands for the ratio between the component M_{ij} of the equivalent inertia tensor and I_{ij} of the rigid body inertia tensor.

3.3 Solution for the Circular Cylindrical Container

This solution was given by De Veubeke in Reference 21. Since his work proved to be of very fundamental value for the derivations in the next section, it is briefly presented here.

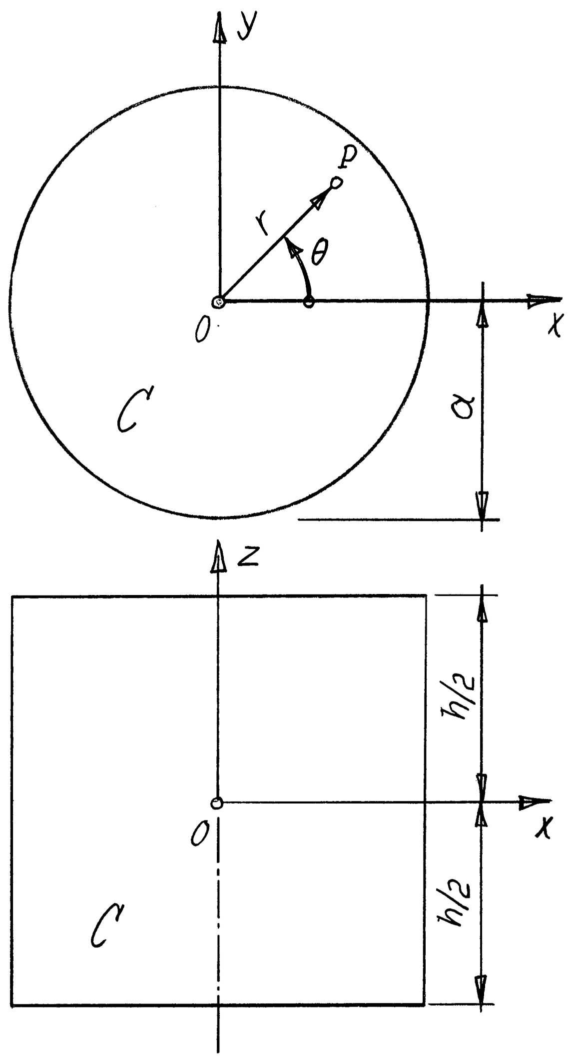

Figure 8

Closed Circular Cylindrical Container

Consider the closed circular cylindrical container of height h and radius a in the reference system (x,y,z) as shown in Figure 8. Because of the symmetry of revolution with respect to axis z, $\phi_6=0$. Therefore

$$M_{46} = M_{56} = M_{66} = 0 \qquad (121)$$

Due to the symmetry of $\nabla\phi_4$ and antisymmetry of $\nabla\phi_5$ with respect to the (y-z) plane,

$$M_{45} = 0 \qquad (122)$$

The only components which therefore remain to be determined are M_{44} and M_{55} which are obviously equal.

The first problem is the determination of the Stokes-Zhukovskii potential ϕ_5. It has to satisfy the following conditions:

- in the liquid C

$$r^2 \frac{\partial^2 \phi_5}{\partial r^2} + r \frac{\partial \phi_5}{\partial r} + \frac{\partial^2 \phi_5}{\partial \theta^2} + r^2 \frac{\partial^2 \phi_5}{\partial z^2} = 0 \tag{123}$$

- on the cylindrical surface r = a, where $l = \cos\theta$, $m = \sin\theta$, $n = 0$

$$\frac{\partial \phi_5}{\partial r} = z \cos\theta \tag{124}$$

- on the flat ends $z = \pm \frac{h}{2}$, where $l = m = 0$, $n = \pm 1$

$$\frac{\partial \phi_5}{\partial z} = -r \cos\theta \tag{125}$$

The boundary conditions suggest the following assumption for ϕ_5:

$$\phi_5 = \left[z\, r + \psi\,(r,z)\right] \cos\theta \tag{126}$$

If this assumption is introduced into the Laplace equation and the boundary conditions, it can be found that the auxiliary function $\psi(r,z$ has to satisfy the following conditions:

$$r^2 \frac{\partial^2 \psi}{\partial r^2} + r \frac{\partial \psi}{\partial r} + r^2 \frac{\partial^2 \psi}{\partial z^2} - \psi = 0, \text{ for } P \in C \tag{127}$$

$$\frac{\partial \psi}{\partial r} = 0, \text{ for } r = a \tag{128}$$

$$\frac{\partial \psi}{\partial z} = -2r, \text{ for } z = \pm \frac{h}{2} \tag{129}$$

Using again the separation of variable approach with

$$\psi(r,z) = \psi_1(r)\, \psi_2(z) \tag{130}$$

Equation (127) becomes

$$r^2 \frac{\psi_1''}{\psi_1} + r \frac{\psi_1'}{\psi_1} + r^2 \frac{\psi_2''}{\psi_2} - 1 = 0 \tag{131}$$

This leads to the two separated differential equations

$$\psi_2'' - f^2 \psi_2 = 0 \tag{132}$$

$$r^2 \psi_1'' + r \psi_1' + (r^2 f^2 - 1)\, \psi_1 = 0 \tag{133}$$

where f is the separation constant. The appropriate solution to Equation (133) is

$$\psi_1(r) = J_1(fr) \tag{134}$$

The boundary condition (128) requires

$$\psi_1'(a) = J_1'(fa) = 0 \tag{135}$$

which yields the eigenvalues $f_{1,n} = \mu_{1,n}/a$. The solution to Equation (132) which corresponds to $f_{1,n}$ and which satisfies the symmetry requirement indicated by condition (129) is

$$\psi_2(z) = \sinh\left(\mu_{1,n} \frac{z}{a}\right) \tag{136}$$

The general solution to Equation (127) which satisfies condition (128) and is antisymmetric with respect to the (x,y) - plane is therefore

$$\psi = \sum_{n=1}^{\infty} F_n\, J_1\left(\mu_{1,n} \frac{r}{a}\right) \sinh\left(\mu_{1,n} \frac{z}{a}\right) \tag{137}$$

The expansion coefficients F_n have to be determined with the remaining condition

$$\frac{\partial \psi}{\partial z} = -2r, \text{ for } z = \frac{h}{2} \tag{138}$$

Hence

$$-2r = \frac{1}{a} \sum_{n=1}^{\infty} F_n \mu_{1,n} \cosh\left(\frac{\mu_{1,n}}{2} \frac{h}{a}\right) J_1\left(\mu_{1,n} \frac{r}{a}\right) \tag{139}$$

The coefficient F_n can now be determined according to the general theory of Fourier - Bessel series, Reference 20, pp. 221. If for simplicity x is substituted for $\frac{r}{a}$, it is obtained:

$$F_n = - \frac{2a^2}{\mu_{1,n} \cosh\left(\frac{\mu_{1,n}}{2} \frac{h}{a}\right)} \frac{\int_o^1 x^2 J_1 (\mu_{1,n} x)\, dx}{\int_o^1 x J_1^2 (\mu_{1,n} x)\, dx} \tag{140}$$

The Lommel integrals appearing in Equation (140) can be expressed in terms of Bessel functions of the same kind and order. It is according to Reference 21, pp. 29,

$$\int_o^1 x^2 J_1 (\mu_{1,n} x)\, dx = \frac{1}{\mu_{1,n}^2} J_1 (\mu_{1,n}) \tag{141}$$

and according to Reference 20, pp. 218,

$$\int_o^1 x J_1^2 (\mu_{1,n} x)\, dx = \frac{1}{2} \left(1 - \frac{1}{\mu_{1,n}^2}\right) J_1^2 (\mu_{1,n}) \tag{142}$$

After substituting Equation (141) and (142) into Equation (140), the expansion coefficient F_n is obtained in the form

$$F_n = - \frac{4a^2}{\mu_{1,n} (\mu_{1,n}^2 - 1) \cosh\left(\frac{\mu_{1,n} h}{2a}\right) J_1 (\mu_{1,n})} \tag{143}$$

which finally completely determines the Stokes-Zhukovskii potential ϕ_5.

$$\phi_5 = \left[zr + \sum_{n=1}^{\infty} F_n J_1 (\mu_{1,n} \frac{r}{a}) \sinh (\mu_{1,n} \frac{z}{a})\right] \cos \theta \tag{144}$$

Equation (114) is now used to determine M_{55} of the equivalent inertia tensor.

$$M_{55} = \rho \int_W \phi_5 (zl - xn)\, dW \tag{145}$$

M_{55} is built up of two parts, the contribution of the cylindrical surface $(M_{55})_1$, and the contribution of the flat ends $(M_{55})_2$.

- Cylindrical surface r = a, where l = cos θ and n = 0

$$(M_{55})_1 = \rho \int_0^{2\pi} \int_{-\frac{h}{2}}^{+\frac{h}{2}} \left[z\, a^2 - \sum_{n=1}^{\infty} \frac{4\, a^3 \sinh\left(\mu_{1,n} \frac{z}{a}\right)}{\mu_{1,n}(\mu_{1,n}^2 - 1)\cosh\left(\frac{\mu_{1,n}\, h}{2a}\right)} \right] z \cos^2\theta\, d\theta\, dz$$

$$(M_{55})_1 = \pi\, \rho\, a^5 \left[\frac{1}{12}\left(\frac{h}{a}\right)^3 - 8 \sum_{n=1}^{\infty} \frac{\frac{1}{2}\left(\frac{h}{a}\right)\mu_{1,n} - \tanh\left(\frac{\mu_{1,n}\, h}{2a}\right)}{\mu_{1,n}^3\,(\mu_{1,n}^2 - 1)} \right] \qquad (146)$$

- Flat ends $z = \pm \frac{h}{2}$, where l = 0 and n = $\pm$ 1.

Symmetry in the integrand of Equation (145) with respect to the (x,y) - plane allows simply to double the contribution of the end $z = + \frac{h}{2}$

$$(M_{55})_2 = -2\rho \int_0^{2\pi} \int_0^{a} r^2 \cos^2\theta \left[\frac{h}{2}\, r - \sum_{n=1}^{\infty} \frac{4a^2 \tanh\left(\frac{\mu_{1,n}\, h}{2a}\right) J_1\left(\mu_{1,n} \frac{r}{a}\right)}{\mu_{1,n}(\mu_{1,n}^2 - 1)\, J_1(\mu_{1,n})} \right] d\theta\, dr$$

The evaluation of this integral involves again the application of Equation (140). The result is:

$$(M_{55})_2 = \pi\, \rho\, a^5 \left[-\frac{1}{4}\left(\frac{h}{a}\right) + 8 \sum_{n=1}^{\infty} \frac{\tanh\left(\frac{\mu_{1,n}\, h}{2a}\right)}{\mu_{1,n}^3(\mu_{1,n}^2 - 1)} \right] \qquad (147)$$

Consider now again Equation (139) which after substitution of F_n by the expression in Equation (143) becomes

$$2r = \sum_{n=1}^{\infty} \frac{4a}{\mu_{1,n}(\mu_{1,n}^2 - 1)\, J_1(\mu_{1,n})} J_1\left(\mu_{1,n} \frac{r}{a}\right) \qquad (148)$$

Multiplying both sides by $r^2 dr$ and integrating between 0 and a immediately furnishes the sum of the following infinite series:

$$8\sum_{n=1}^{\infty}\frac{1}{\mu_{1,n}^{2}(\mu_{1,n}^{2}-1)} = 1 \tag{149}$$

Adding both contributions to M_{55} considering Equation (149) leads to the final result

$$M_{55} = \pi\ \rho\ a^{5}\left[\frac{1}{12}\left(\frac{h}{a}\right)^{3} - \frac{3}{4}\left(\frac{h}{a}\right) + 16\sum_{n=1}^{\infty}\frac{\tanh\left(\frac{\mu_{1,n}\ h}{2a}\right)}{\mu_{1,n}^{3}(\mu_{1,n}^{2}-1)}\right] \tag{150}$$

It is of interest to note that the corresponding term in the inertia tensor of the rigid body with the same geometrical dimensions would be

$$I_{55} = \pi\ \rho\ a^{5}\left[\frac{1}{12}\left(\frac{h}{a}\right)^{3} + \frac{1}{4}\left(\frac{h}{a}\right)\right] \tag{151}$$

3.4 Solution for the Elliptic Cylindrical Container

Consider the closed elliptic cylindrical container of height h, major axis a and minor axis b in the reference system (x,y,z) as shown in Figure 9. It can be immediately concluded from the symmetries of the container that

$$M_{45} = M_{46} = M_{56} = 0 \qquad (152)$$

The remaining three components, however, do not vanish and they can be determined. It has to be pointed out here that in view of the application in Chapter IV where ω_{oz} will be zero, only M_{44} and M_{55} will be evaluated below. The solution to this problem will therefore not be complete.

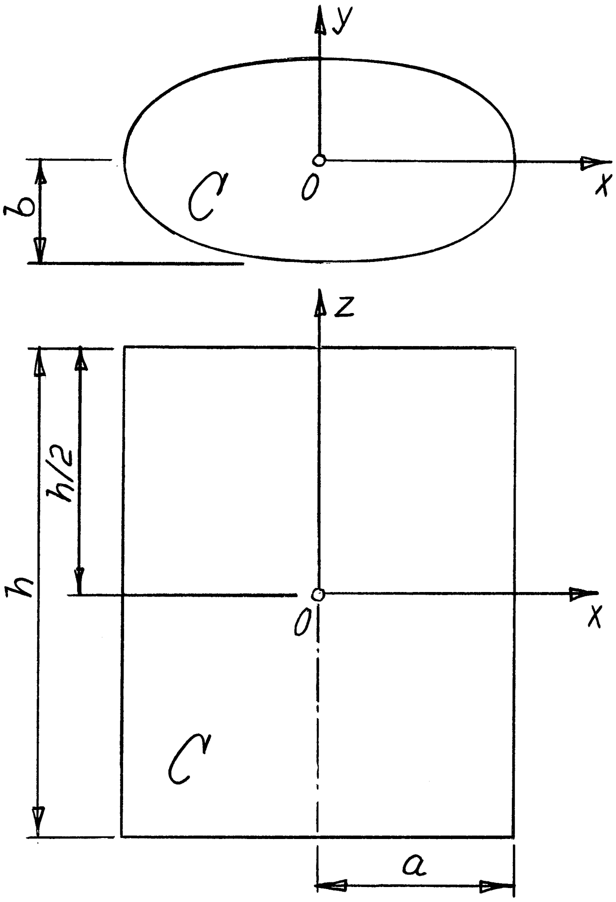

Figure 9
Closed Elliptic Cylindrical Container

The problem will be discussed in an elliptic cylindrical coordinate system, which was defined by Equations (54). Since the coordinates ξ and η do not have an immediate geometrical meaning, the needed geometrical relations have first to be expressed in terms of ξ and η. The following relations can be immediately derived from Equations (54).

- Arc length on the confocal hyperbola η

$$d\nu = \frac{c}{\sqrt{2}}\sqrt{\cosh 2\xi - \cos 2\eta}\; d\xi \qquad (153)$$

- Components of the unit outward normal $\vec{v}$ to the confocal ellipse ξ

$$l = \sqrt{2}\ \frac{\sinh \xi \cos \eta}{\sqrt{\cosh 2\xi - \cos 2\eta}} \qquad (154)$$

$$m = \sqrt{2}\ \frac{\cosh \xi \sin \eta}{\sqrt{\cosh 2\xi - \cos 2\eta}} \qquad (155)$$

- Value of ξ at the boundary of the container

$$\xi_B = \frac{1}{2} \log \frac{\frac{a}{b} + 1}{\frac{a}{b} - 1} \qquad (156)$$

It can be seen from Equation (156) that ξ_B is defined and finite only for shape ratios $\frac{a}{b} > 1$. But this will not cause more than formal difficulties since

$$M_{44}\left(\frac{a}{b}\right) = M_{55}\left(\frac{b}{a}\right) \qquad (157)$$

It suffices therefore to evaluate both M_{44} and M_{55} for shape ratios $a/b > 1$. Since the two solutions involve two different types of Mathieu functions, it is more appropriate to characterize them by the type of functions rather than by the expected result.

3.4.1 Cosine - Elliptic Case

The study of this case will lead to ϕ_5 and M_{55} for $a/b > 1$. ϕ_5 has to satisfy the following conditions in terms of elliptic cylindrical coordinates:

- in the liquid C

$$\frac{2}{c^2(\cosh 2\xi - \cos 2\eta)} \left(\frac{\partial^2 \phi_5}{\partial \xi^2} + \frac{\partial^2 \phi_5}{\partial \eta^2} \right) + \frac{\partial^2 \phi_5}{\partial z^2} = 0 \qquad (158)$$

- on the cylindrical surface $\xi = \xi_B$, where ℓ is given by Equation (154), m by Equation (155) and n = 0.

$$\frac{\partial \phi_5}{\partial \xi} = c\, z \sinh \xi \cos \eta \tag{159}$$

- on the flat ends $z = \pm \frac{h}{2}$, where ℓ = m = 0, n = $\pm$ 1

$$\frac{\partial \phi_5}{\partial z} = -\, c \cosh \xi \cos \eta \tag{160}$$

It is now assumed that a solution of the following form exists

$$\phi_5(\xi, \eta, z) = c\, z \cosh \xi \cos \eta + \psi(\xi, \eta, z) \tag{161}$$

If this assumption is introduced into Equations (158), (159) and (160), a similar but simpler set of conditions is obtained for the auxiliary function ψ.

$$\frac{2}{c^2(\cosh 2\xi - \cos 2\eta)}\left(\frac{\partial^2\psi}{\partial \xi^2} + \frac{\partial^2\psi}{\partial \eta^2}\right) + \frac{\partial^2\psi}{\partial z^2} = 0, \text{ for } P \in C \tag{162}$$

$$\frac{\partial \psi}{\partial \xi} = 0, \text{ for } \xi = \xi_B \tag{163}$$

$$\frac{\partial \psi}{\partial z} = -\, 2c \cosh \xi \cos \eta, \text{ for } z = \pm \frac{h}{2} \tag{164}$$

The separation of variable approach is used again with

$$\psi(\xi, \eta, z) = \psi_1(\xi)\, \psi_2(\eta)\, \psi_3(z) \tag{165}$$

This transforms Equation (162) into a set of three separated differential equations for the functions ψ_1, ψ_2 and ψ_3.

$$\psi_3'' - \frac{4q}{c^2}\psi_3 = 0 \qquad (166)$$

$$\psi_2'' + (s - 2q\cos 2\eta)\,\psi_2 = 0 \qquad (167)$$

$$\psi_1'' - (s - 2q\cosh 2\xi)\,\psi_1 = 0 \qquad (168)$$

s and q are the separation constants and the parameters in Mathieu's differential equations (167) and (168). It is the boundary condition (164) which suggests the solution in the class of cosine-elliptic Mathieu functions of integer order. The solution to Equation (167) and the corresponding solution to Equation (168) are therefore

$$\psi_2(\eta) = ce_{2m+1}(\eta,q) \qquad (169)$$

$$\psi_1(\xi) = Ce_{2m+1}(\xi,q) \qquad (170)$$

ψ has to satisfy boundary condition (163). This yields the eigenvalue problem

$$\frac{d}{d\xi}\,Ce_{2m+1}(\xi,q) = 0, \text{ for } \xi = \xi_B \qquad (171)$$

which has already been discussed in section 2.5.2.3. If finally the solution to Equation (166) is taken such as to satisfy the symmetry in the problem as indicated by boundary condition (164), the general solution to Equation (162) which satisfies condition (163) is:

$$\psi(\xi,\eta,z) = c^2\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} Cf_{2m+1,n}\sinh\left(\frac{2}{c}\sqrt{q_{2m+1,n}}\,z\right) ce_{2m+1}(\eta,q_{2m+1,n})\, Ce_{2m+1}(\xi,q_{2m+1,n}) \qquad (172)$$

$Cf_{2m+1,n}$ has now to be determined using the only remaining condition

$$\frac{\partial\psi}{\partial z} = -2c \cosh \xi \cos \eta, \text{ for } z = +\frac{h}{2} \tag{173}$$

which yields

$$-2c \cosh \xi \cos \eta$$

$$= c^2 \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} Cf_{2m+1,n} \frac{2}{c} \sqrt{q_{2m+1,n}} \cosh\left(\frac{h}{c} \sqrt{q_{2m+1,n}}\right) ce_{2m+1}(\eta, q_{2m+1,n})\, Ce_{2m+1}(\xi, q_{2m+1,n}) \tag{174}$$

It is justified according to Reference 14, Art. 16.20, to consider Equation (174) as a double Fourier series of the absolutely integrable function $(-2c \cosh \xi \cos \eta)$ with respect to the complete orthogonal system $ce_{2m+1}(\eta, q_{2m+1,n})\, Ce_{2m+1}(\xi, q_{2m+1,n})$ in the domain $(0 \leq \xi \leq \xi_B)(0 \leq \eta \leq 2\pi)$. This means that $Cf_{2m+1,n}$ can be given by the expression

$$Cf_{2m+1,n} = \frac{1}{\sqrt{q_{2m+1,n}} \cosh \frac{h}{c} \sqrt{q_{2m+1,n}}} \frac{Cq_{2m+1,n}}{Cr_{2m+1,n}} \tag{175}$$

where

$$Cq_{2m+1,n} = -\frac{1}{\pi} \int_0^{\xi_B} \int_0^{2\pi} \cosh \xi \cos \eta \; ce_{2m+1}(\eta, q_{2m+1,n}) Ce_{2m+1}(\xi, q_{2m+1,n}) (\cosh 2\xi - \cos 2\eta) d\xi d\eta \tag{176}$$

and

$$Cr_{2m+1,n} = \frac{1}{\pi} \int_0^{\xi_B} \int_0^{2\pi} ce^2_{2m+1}(\eta, q_{2m+1,n}) Ce^2_{2m+1}(\xi, q_{2m+1,n}) (\cosh 2\xi - \cos 2\eta) d\xi d\eta \tag{177}$$

The evaluation of $Cq_{2m+1,n}$ and $Cr_{2m+1,n}$ will lead to two different types of definite integrals, one involving ordinary and the other modified

Mathieu functions. Using the series expansion in Equation (91), the solution to the first type of integrals can immediately be given.

$$\frac{1}{\pi}\int_0^{2\pi} ce^2_{2m+1}(\eta,q_{2m+1,n})d\eta = \sum_{i=0}^{\infty}\left[A^{(2m+1,n)}_{2i+1}\right]^2 \tag{178}$$

$$\frac{1}{\pi}\int_0^{2\pi} ce^2_{2m+1}(\eta,q_{2m+1,n})\cos 2\eta d\eta = \frac{1}{2}\left[A^{(2m+1,n)}_1\right]^2 + \sum_{i=0}^{\infty}A^{(2m+1,n)}_{2i+1}\ A^{(2m+1,n)}_{2i+3} \tag{179}$$

$$\frac{1}{\pi}\int_0^{2\pi} ce_{2m+1}(\eta,q_{2m+1,n})\cos \eta d\eta = A^{(2m+1,n)}_1 \tag{180}$$

$$\frac{1}{\pi}\int_0^{2\pi} ce_{2m+1}(\eta,q_{2m+1,n})\cos \eta\ \cos 2\eta d\eta = \frac{1}{2}\left[A^{(2m+1,n)}_1 + A^{(2m+1,n)}_3\right] \tag{181}$$

It should be remembered that due to the adopted normalization convention, Equation (90a), the right side of Equation (178) will be unity.

The second type of integrals is:

$$Ca_{2m+1,n} = \int_0^{\xi_B} Ce^2_{2m+1}(\xi,q_{2m+1,n})\ d\xi \tag{182}$$

$$Cb_{2m+1,n} = \int_0^{\xi_B} Ce^2_{2m+1}(\xi,q_{2m+1,n})\ \cosh 2\xi d\xi \tag{183}$$

$$Cc_{2m+1,n} = \int_0^{\xi_B} Ce_{2m+1}(\xi,q_{2m+1,n})\ \cosh \xi\ \cosh 2\xi d\xi \tag{184}$$

$$Cd_{2m+1,n} = \int_0^{\xi_B} Ce_{2m+1}(\xi,q_{2m+1,n})\ \cosh \xi\ d\xi \tag{185}$$

If Ce_{2m+1} is represented by the expansion given in Equation (92a), a series representation of these integrals could be derived. Such a series would not be suitable at all for numerical computations. McLachlan (Reference 14, Art. 14.22) expressed the integral in Equation (183) in terms of certain derivatives with respect to the parameter q, which however, can only be evaluated by a numerical process. Since it does not seem possible to put these integrals into a rigorous mathematical form, which is at the same time suitable for numerical computations, it was finally decided to apply Simpson's rule for numerical integration. $Ce_{2m+1}(\xi,q)$ is a continuous and smooth function in the interval $0 \leq \xi \leq \xi_B$ and can be evaluated by a series with good convergence properties, Equation (95a). It can therefore be expected that the numerical integration will yield a result which is sufficiently precise for all practical applications, provided the interval $\Delta\xi$ is taken sufficiently small.

Consequently, the expressions for the modal constans Cq and Cr become:

$$Cq_{2m+1,n} = - A_1^{(2m+1,n)} Cc_{2m+1,n} + \frac{1}{2}\left(A_1^{(2m+1,n)} + A_3^{(2m+1,n)}\right) Cd_{2m+1,n} \tag{186}$$

$$Cr_{2m+1,n} = Cb_{2m+1,n} - Ca_{2m+1,n}\left\{\frac{1}{2}\left[A_1^{(2m+1,n)}\right]^2 + \sum_{i=0}^{\infty} A_{2i+1}^{(2m+1,n)} A_{2i+3}^{(2m+1,n)}\right\} \tag{187}$$

If all the results are now put together according to Equations (175), (172) and (161), the following expression for the Stokes - Zhukovskii potential ϕ_5 is obtained.

$$\phi_5 = cz \cosh\xi \cos\eta + c^2 \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{\sinh\left(\frac{2}{c}\sqrt{q_{2m+1,n}}\, z\right)}{\sqrt{q_{2m+1,n}}\cosh\left(\frac{h}{c}\sqrt{q_{2m+1,n}}\right)} \frac{Cq_{2m+1,n}}{Cr_{2m+1,n}} ce_{2m+1}(\eta, q_{2m+1,n})\, Ce_{2m+1}(\xi, q_{2m+1,n}) \tag{188}$$

ϕ_5 is the basis for the evaluation of the tensor component M_{55},

$$M_{55} = \rho \int_W \phi_5 \; (zl-xn) \; dW \tag{189}$$

which consists of the parts $(M_{55})_1$ corresponding to the cylindrical surface and $(M_{55})_2$ corresponding to the flat ends.

- Cylindrical surface $\xi = \xi_B$, where n=0 and

  $l dW = c \sinh \xi_B \cos \eta \; d\eta \; dz$

$$(M_{55})_1 = \rho \; c \sinh \xi_B \int_0^{2\pi} \int_{-\frac{h}{2}}^{+\frac{h}{2}} z \; \phi_5 \; (\xi_B, \eta, z) \cos \eta \; d\eta \; dz$$

(190)

$$\frac{(M_{55})_1}{\pi \rho c^5} = \frac{1}{24}\left(\frac{h}{c}\right)^3 \sinh 2\,\xi_B + \sinh \xi_B \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{1}{2q_{2m+1,n}} \frac{Cq_{2m+1,n}}{Cr_{2m+1,n}} A_1^{(2m+1,n)}$$

$$Ce_{2m+1}(\xi_B, q_{2m+1,n}) \left[\frac{h}{c} - \frac{1}{\sqrt{q_{2m+1,n}}} \tanh\left(\frac{h}{c}\sqrt{q_{2m+1,n}}\right)\right]$$

- Flate ends $z = \pm \frac{h}{2}$, where $l = 0$ and $n = \pm 1$

Symmetry allows again to double the contribution of the end $z = + \frac{h}{2}$

$$(M_{55})_2 = -\rho \; c^3 \int_0^{2\pi} \int_0^{\xi_B} \cosh \xi \cos \eta \; \phi_5(\xi, \eta, \tfrac{h}{2})(\cosh 2\xi - \cos 2\eta) d\xi d\eta$$

$$\frac{(M_{55})_2}{\pi \rho c^5} = -\frac{1}{16} \frac{h}{c} \left(\frac{1}{2} \sinh 4\xi_B + \sinh 2\xi_B\right) + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{\tanh\left(\frac{h}{c}\sqrt{q_{2m+1,n}}\right)}{\sqrt{q_{2m+1,n}}} \frac{Cq^2_{2m+1,n}}{Cr_{2m+1,n}} \tag{191}$$

In order to simplify these relations slightly, consider again Equation (174). Introducing Equation (175) leads to

$$- \cosh \xi \cos \eta = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{Cq_{2m+1,n}}{Cr_{2m+1,n}} ce_{2m+1}(\eta, q_{2m+1,n}) \, Ce_{2m+1}(\xi, q_{2m+1,n}) \tag{192}$$

Multiplying both sides by cosh ξ cos η (cosh 2ξ - cos 2η) $d\xi$ $d\eta$ and integrating over the ellipse with axis a, b leads immediately to

$$8 \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{Cq^2_{2m+1,n}}{Cr_{2m+1,n}} = \frac{1}{2} \sinh 4 \xi_B + \sinh 2 \xi_B \tag{193}$$

M_{55} can now be written in the final form

$$\frac{M_{55}}{\pi \rho c^5} = \frac{1}{24}\left(\frac{h}{c}\right)^3 \sinh 2 \xi_B + \sinh \xi_B \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{1}{2q_{2m+1,n}} \frac{Cq_{2m+1,n}}{Cr_{2m+1,n}} A_1^{(2m+1,n)}$$

$$Ce_{2m+1}(\xi_B, q_{2m+1,n}) \left\{ \frac{h}{c} - \frac{1}{\sqrt{q_{2m+1,n}}} \tanh\left(\frac{h}{c} \sqrt{q_{2m+1,n}}\right) \right\} + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{Cq^2_{2m+1,n}}{Cr_{2m+1,n}} \tag{194}$$

$$\left\{ \frac{\tanh\left(\frac{h}{c} \sqrt{q_{2m+1,n}}\right)}{\sqrt{q_{2m+1,n}}} - \frac{h}{2c} \right\}$$

The corresponding inertia component for the rigid body is

$$\frac{I_{55}}{\pi \rho c^5} = \frac{1}{24}\left(\frac{h}{c}\right)^3 \sinh 2 \xi_B + \frac{1}{16} \frac{h}{c} \left(\frac{1}{2} \sinh 4 \xi_B + \sinh 2 \xi_B \right) \tag{195}$$

3.4.2 Sine - Elliptic Case

This case yields ϕ_4 and M_{44} for $a/b > 1$. Since the procedure is completely analogous to the one developed in the previous section, only the results will be given.

If the eigenvalues $q_{2m+1,n}$ are the roots of the characteristic equation,

$$\frac{d}{d\xi}\ Se_{2m+1}\ (\xi,q) = 0,\ \xi = \xi_B \tag{196}$$

then with

$$Sa_{2m+1,n} = \int_0^{\xi_B} Se^2_{2m+1}(\xi,\ q_{2m+1,n})\ d\xi \tag{197}$$

$$Sb_{2m+1,n} = \int_0^{\xi_B} Se^2_{2m+1}(\xi,\ q_{2m+1,n})\ \cosh 2\xi\ d\xi \tag{198}$$

$$Sc_{2m+1,n} = \int_0^{\xi_B} Se_{2m+1}(\xi,\ q_{2m+1,n})\ \sinh\xi\ \cosh 2\xi\ d\xi \tag{199}$$

$$Sd_{2m+1,n} = \int_0^{\xi_B} Se_{2m+1}(\xi,\ q_{2m+1,n})\ \sinh\xi\ d\xi \tag{200}$$

and

$$Sq_{2m+1,n} = B_1^{(2m+1,n)} Sc_{2m+1,n} - \frac{1}{2}\left(B_3^{(2m+1,n)} - B_1^{(2m+1,n)}\right) Sd_{2m+1,n} \tag{201}$$

$$Sr_{2m+1,n} = Sb_{2m+1,n} - Sa_{2m+1,n}\left[-\frac{1}{2}\left(B_1^{(2m+1,n)}\right)^2 + \sum_{i=0}^{\infty} B_{2i+1}^{(2m+1,n)}\ B_{2i+3}^{(2m+1,n)}\right] \tag{202}$$

the expression for the Stokes - Zhukovskii potential ϕ_4 is:

$$\phi_4 = -\ c\ z\ \sinh\xi\ \sin\eta + c^2 \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{\sinh\left(\frac{2}{c}\sqrt{q_{2m+1,n}}\ z\right)}{\sqrt{q_{2m+1,n}}\ \cosh\left(\frac{h}{c}\sqrt{q_{2m+1,n}}\right)} \frac{Sq_{2m+1,n}}{Sr_{2m+1,n}}\ se_{2m+1}(\eta,q_{2m+1,n})\ Se_{2m+1}(\xi,q_{2m+1,n})$$

and finally:

$$\frac{M_{55}}{\pi \rho c^5} = \frac{1}{24}\left(\frac{h}{c}\right)^3 \sinh 2 \xi_B - \cosh \xi_B \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{1}{2q_{2m+1,n}} \frac{Sq_{2m+1,n}}{Sr_{2m+1,n}} B_1^{(2m+1,n)}$$

$$Se_{2m+1}(\xi_B, q_{2m+1,n}) \left\{ \frac{h}{c} - \frac{1}{\sqrt{q_{2m+1,n}}} \tanh \frac{h}{c} \sqrt{q_{2m+1,n}} \right\} + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{Sq^2_{2m+1,n}}{Sr_{2m+1,n}}$$

$$\left\{ \frac{\tanh \frac{h}{c} \sqrt{q_{2m+1,n}}}{\sqrt{q_{2m+1,n}}} - \frac{h}{2c} \right\} \tag{203}$$

The relation which corresponds to Equation (193) is

$$8 \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{Sq^2_{2m+1,n}}{Sr_{2m+1,n}} = \frac{1}{2} \sinh 4 \xi_B - \sinh 2 \xi_B \tag{204}$$

The corresponding inertia component for the rigid body is

$$\frac{I_{55}}{\pi \rho c^5} = \frac{1}{24}\left(\frac{h}{c}\right)^3 \sinh 2 \xi_B + \frac{1}{16} \frac{h}{c}\left(\frac{1}{2} \sinh 4 \xi_B - \sinh 2 \xi_B\right) \tag{205}$$

3.5 Numerical Results

On the basis of known eigenvalues $\mu_{1,n}$ and $q_{2m+1,n}$ which have been determined for different ratios a/b in the previous chapter, the evaluation of the corresponding equivalent moments of inertia is a straight forward numerical problem. The only question is, whether the limited number of modes on which the computation has to be based due to numerical difficulties in the evaluation of higher eigenvalues, are enough to produce a result with reasonable accuracy. It is beyond the scope of this investigation to give a rigorous answer to this question by discussing the remainder in the infinite series of Equations (194) and (203). It was, however, clearly apparent from the performed numerical calculations that in no case did the higher half of the given modes influence the required 4 digit accuracy of the result.

The results are summarized in Tables I and II of the Appendix. Values are given to four significant digits for parameters a/b and h/a which seem to be of most practical interest. As an illustration, the functions $\lambda(h/a)$ for $a/b = 2/3$, 1 and 3/2 are plotted in Figure 10.

The influence of the ellipticity (a/b) on λ is surprisingly small. There is, however, a small domain of h/a in which consideration of the deviations from the curve for the circular cylindrical container appears warranted and this is exactly the domain of most practical applications.

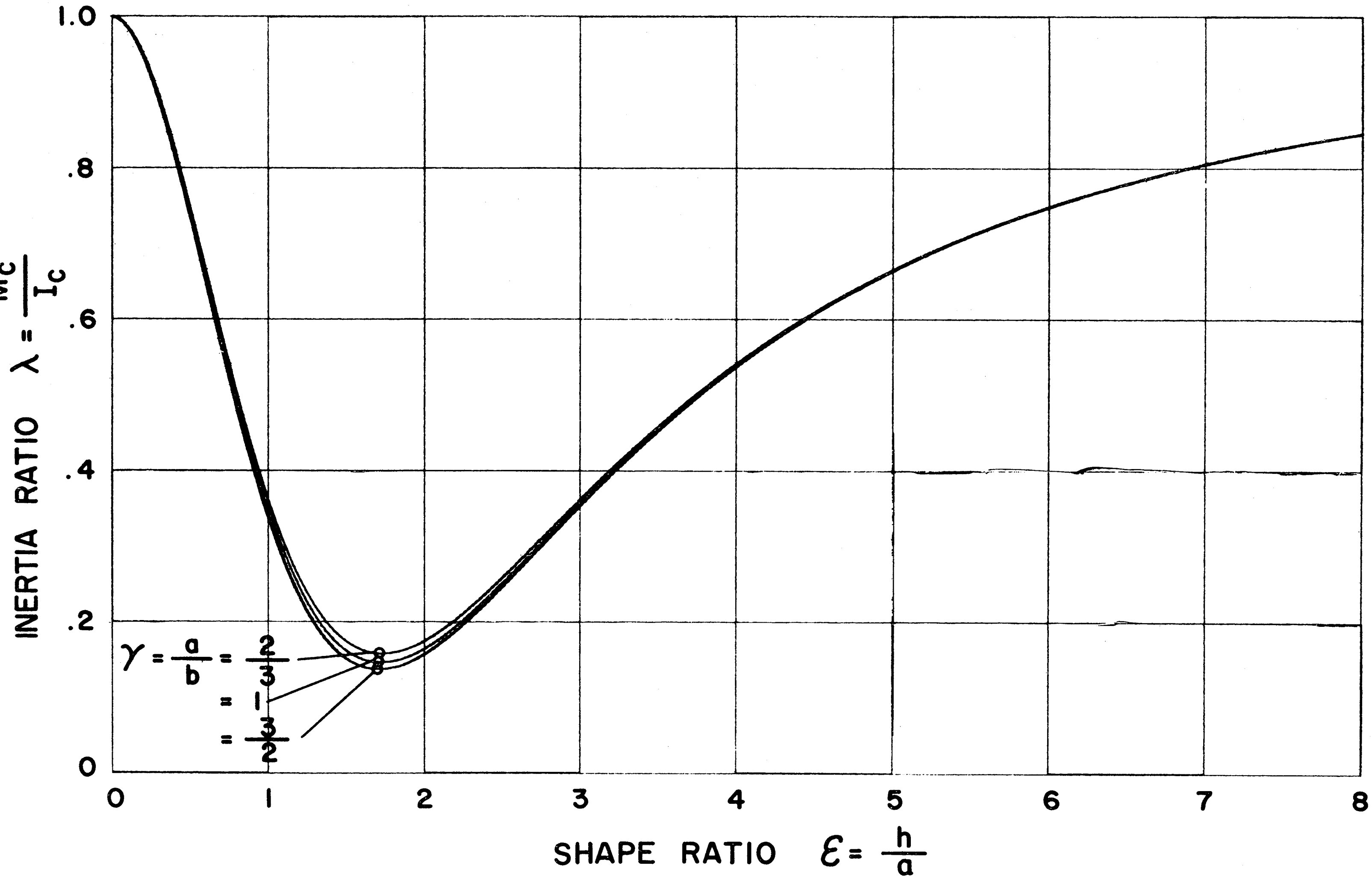

Figure 10. The Ratio $\lambda(h/a) = M_c/I_c$ for Liquid Filling Elliptic Cylindrical Containers with Shape Ratios $a/b = 2/3, 1, 3/2$

IV AN EXAMPLE FOR THE OSCILLATIONS OF A CONSERVATIVE SYSTEM WITH A LIQUID MEMBER

4.1 Dynamic System

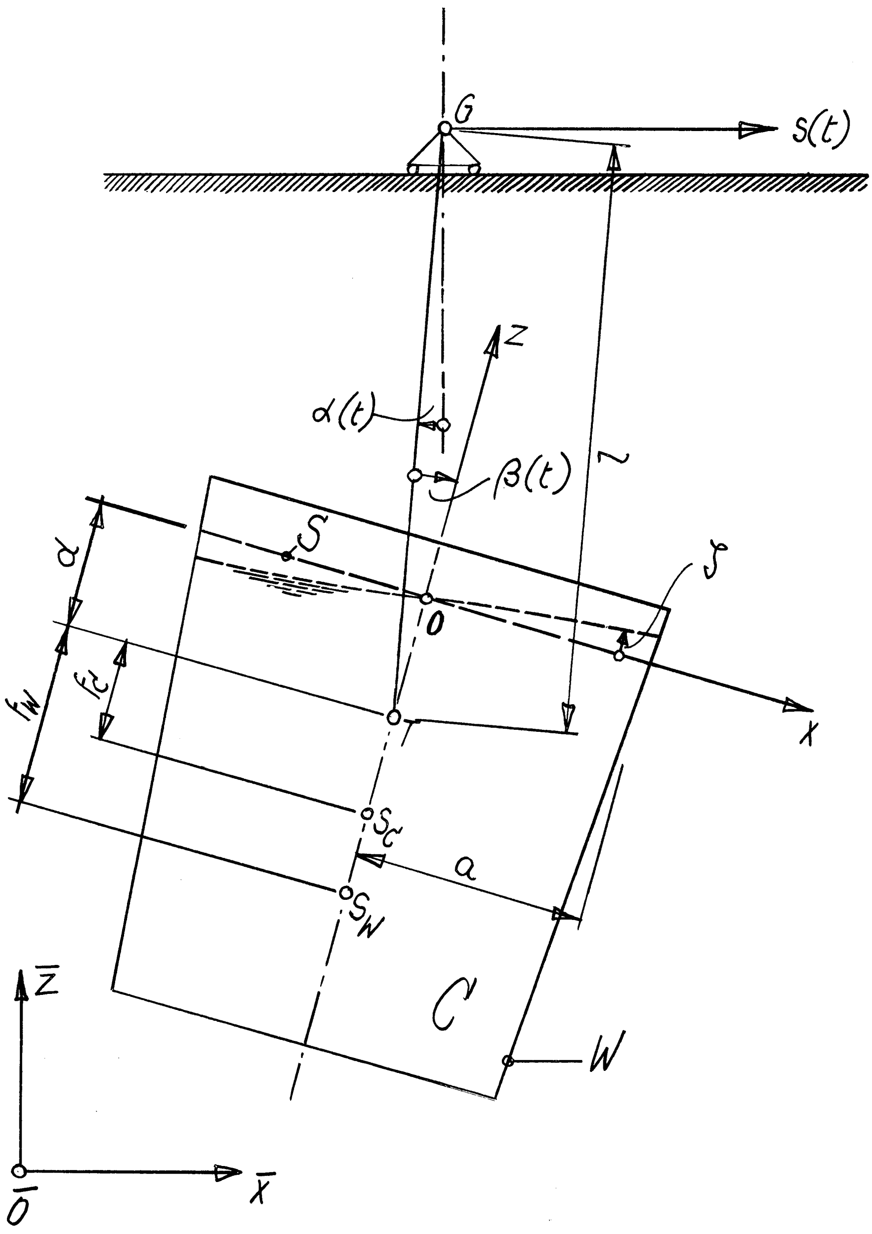

Figure 11
Dynamic System with Liquid Member

Consider the system in Figure 11. A container W, symmetric with respect to the (x,z) and (y,z) plane, is filled with an ideal, incompressible liquid, which has the free surface S in the position of equilibrium. The container is carried by the crane with axis G which moves parallel to the coordinate axis $\bar{x}$ according to some displacement function s(t). The crane and the container are connected by two weightless rods which are hinged to the rotation axis T of the container and the axis G of the crane.

Besides the parameters which are explained in Figure 11, the following system properties will enter the final equations.

- m_W : Mass of the container W
- I_W : Moment of inertia of the container with respect to the axis of rotation through its centre of gravity S_W.
- m_C : Mass of the liquid C.
- M_C : Equivalent moment of inertia of the liquid with surface S with respect to the axis of rotation through its centre of gravity S_C.
- I_C : Moment of inertia of the solidified liquid with surface S with respect to the axis of rotation through S_C.
- ϕ_C : Stokes- Zhukovskii potential corresponding to M_C.
- φ_m : m-th mode of free oscillation in the container W at rest.

4.2 Solvability and Nature of the Solution

Two general theorems about the solvability and the nature of the solution for a conservative system with a finite number of members and a finite number of cavities partially filled with liquid are given in Reference 16, pp. 268 ff.

Theorem 1:

If a system consists of a finite number of conservative members and contains a finite number of cavities partially filled with a liquid and if the potential energy of the system has a minimum in the equilibrium position (e.g. is statically stable), then

- there exist natural modes in the motion of this system about the equilibrium position,
- the frequencies of these oscillations are positive numbers and form a denumerable sequence such that $\lim_{n \to \infty} \omega_n = \infty$,
- a finite number of natural modes correspond to each eigenvalue,
- the system of natural modes is complete, which means that any free motion of the system may be presented by a superposition of oscillations.

Theorem 2:

If the potential energy is not a minimum in the equilibrium position, then there is at least one negative quantity among the ω_n^2, which means that the motion is unstable.

4.3 The Equations of Motion

The main problem here is the derivation of expressions for the kinetic and potential energy of the liquid. In order to take full advantage of what has been done in Chapter III, the liquid motion is thought as being composed of two parts. The first part is the motion which would occur if the surface had been replaced by a solid cover while the second part considers the influence of the surface waves. On this basis, the expression for the kinetic energy of the liquid may be written in the form:

$$T_c = \frac{\rho}{2} \int_C \left[(\dot{\alpha} + \dot{\beta})\nabla\phi_c + (\dot{s} - \ell\dot{\alpha} - f_c(\dot{\alpha} + \dot{\beta}))\nabla x + \nabla\phi\right]^2 dC \tag{206}$$

In this expression ϕ is the potential of wave motion in the container. Expanding the integrand and integrating wherever immediately possible gives

$$\begin{aligned} T_c = & \frac{1}{2}(\dot{\alpha} + \dot{\beta})^2 M_c + \frac{1}{2}(\dot{s} - \ell\dot{\alpha} - f_c(\dot{\alpha} + \dot{\beta}))^2 m_c + \frac{\rho}{2}\int_C (\nabla\phi)^2 \, dC \\ & + \rho(\dot{\alpha} + \dot{\beta})(\dot{s} - \ell\dot{\alpha} - f_c(\dot{\alpha} + \dot{\beta}))\int_C \nabla x \nabla\phi_c dC + \rho(\dot{\alpha} + \dot{\beta})\int_C \nabla\phi_c \nabla\phi \, dC \\ & + \rho(\dot{s} - \ell\dot{\alpha} - f_c(\dot{\alpha} + \dot{\beta}))\int_C \nabla x \nabla\phi \, dC \end{aligned} \tag{207}$$

If now the potential of wave motion is expanded with respect to the natural modes of free oscillations φ_m,

$$\phi(x,y,z,t) = \sum_m \dot{p}_m(t)\, \varphi_m(x,y,z) \tag{208}$$

an assumption which is clearly justified by the completeness of the system of eigenfunctions (Section 2.2.1), the integrals encountered in Equation (207) take the following forms:

$$\int_C (\nabla \phi)^2 dC = \frac{1}{g} \sum_m \dot{p}_m^2 \, \omega_m^2 \int_S \varphi_m^2 dS \tag{209}$$

$$\int_C \nabla x \, \nabla \phi \, dC = \frac{1}{g} \sum_m \dot{p}_m \, \omega_m^2 \int_S x \, \varphi_m dS \tag{210}$$

$$\int_C \nabla \phi_c \nabla \phi \, dC = \frac{1}{g} \sum_m \dot{p}_m \, \omega_m^2 \int_S \phi_c \varphi_m dS \tag{211}$$

$$\int_C \nabla x \nabla \phi_c dC = 0 \tag{212}$$

Each of these relations was derived using the same patterns of transformations. Green's theorem was used to transform volume integrals into surface integrals, then the kinematical condition of Equation (25) replaced $\frac{\partial \varphi_m}{\partial z}$ by $\frac{1}{g} \omega_m^2 \varphi_m$ and finally because of the orthogonality of the eigenfunctions, Equation (209) became the normal form of the homogeneous quadratic in φ_m. The expression in Equation (212) simply vanishes because the centre of gravity of the liquid with surface S was chosen as the origin.

To obtain an expression for the potential energy of the liquid, the same decomposition of the liquid motion may be considered. If $\zeta(x,y,t)$ is again the free surface function, then

$$V_c = m_c g(\ell + f_c - \ell \cos \alpha - f_c \cos(\alpha+\beta)) + \frac{\rho g}{2} \int_S \zeta^2 dS - \rho g(\alpha+\beta) \int_S x \zeta dS \tag{213}$$

The unknown surface function ζ can be eliminated with the help of the surface condition given in Equation (7).

$$\frac{\partial \zeta}{\partial t} = \frac{\partial \phi}{\partial z} = \sum_m \dot{p}_m \left(\frac{\partial \varphi_m}{\partial z} \right)_{z=0} \tag{217}$$

which becomes, when it is assumed that the motion starts from rest

$$\zeta(x,y,t) = \sum_m p_m \left(\frac{\partial \varphi_m}{\partial z}\right)_{z=0} \tag{215}$$

Now the integrals encountered in Equation (213) can be given in the form

$$\int_S \zeta^2 dS = \frac{1}{g^2} \sum_m p_m^2 \omega_m^4 \int_S \varphi_m^2 dS \tag{216}$$

$$\int_S x \, \zeta \, dS = \frac{1}{g} \sum_m p_m \omega_m^2 \int_S s \, \varphi_m dS \tag{217}$$

using again Equation (25) to replace $\frac{\partial \varphi_m}{\partial z}$.

Adding finally the energies of the container, which is a simple rigid body, the total energies of the system can be given by the expressions:

$$\begin{aligned} T = & \frac{m_w}{2}\left[\dot{s} - \ell\dot{\alpha} - f_w(\dot{\alpha}+\dot{\beta})\right]^2 + \frac{I_w}{2}(\dot{\alpha}+\dot{\beta})^2 \\ & + \frac{m_c}{2}\left[\dot{s} - \ell\dot{\alpha} - f_c(\dot{\alpha}+\dot{\beta})\right]^2 + \frac{M_c}{2}(\dot{\alpha}+\dot{\beta})^2 \\ & + \frac{\rho}{2g} \sum_m \dot{p}_m^2 \omega_m^2 \int_S \varphi_m^2 \, dS \\ & + \frac{\rho}{g}(\dot{\alpha}+\dot{\beta}) \sum_m p_m \omega_m^2 \int_S \phi_c \varphi_m \, dS \\ & + \frac{\rho}{g}\left[\dot{s} - \ell\dot{\alpha} - f_c(\dot{\alpha}+\dot{\beta})\right] \sum_m p_m \omega_m^2 \int_S x \, \varphi_m \, dS \end{aligned} \tag{218}$$

$$V = m_w g\left[\ell + f_w - \ell \cos\alpha - f_w \cos(\alpha+\beta)\right]$$

$$+ m_c g\left[\ell + f_c - \ell \cos\alpha - f_c \cos(\alpha+\beta)\right]$$

$$+ \frac{\rho}{2g} \sum_m p_m^2 \omega_m^4 \int_S \varphi_m^2 \, dS$$

$$- \rho (\alpha+\beta) \sum_m p_m \omega_m^2 \int_S x\varphi_m \, dS \qquad (219)$$

Since these energies are expressed in terms of the generalized coordinates $q = \alpha, \beta, p_1, \ldots, p_m, \ldots$, Lagrange's equations can be used to derive the equations of motion of the system. Since in this case $\frac{\partial V}{\partial \dot{q}_i} = 0$ and $\frac{\partial T}{\partial q_i} = 0$, they may be used in the simplified version

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + \frac{\partial V}{\partial q_i} = 0 \; (i = 1, 2, \ldots) \qquad (220)$$

Application of Equation (220) to the energy expressions (218) and (219) leads to an infinite system of coupled linear inhomogeneous differential equations.

$\ddot{\alpha}$	$\ddot{\beta}$		$\ddot{p}_m \quad m=1,2,\ldots$
$+ m_w(\ell + f_w)^2$ $+ m_c(\ell + f_c)^2$ $+ I_w$ $+ M_c$	$+ m_w f_w(\ell + f_w)$ $+ m_c f_c(\ell + f_c)$ $+ I_w$ $+ M_c$		$- \rho(\ell+f_c)\frac{\omega_m^2}{g}\int_S x\varphi_m dS$ $+ \rho\frac{\omega_m^2}{g}\int_S \phi_c \varphi_m dS$
	$+ m_w f_w^2$ $+ m_c f_c^2$ $+ I_w$ $+ M_c$		$- \rho f_c \frac{\omega_m^2}{g}\int_S x\varphi_m dS$ $+ \rho \frac{\omega_m^2}{g}\int_S \phi_c \varphi_m dS$
			$+ \rho \frac{\omega_m^2}{g}\int_S \varphi_m^2 dS$

+ (221)

α	β		$p_m \quad m=1,2,\ldots$		$\ddot{s}$
$+m_w\, g(\ell+f_w)$ $+m_c\, g(\ell+f_c)$	$+m_w\, g\, f_w$ $+m_c\, g\, f_c$		$-\rho\omega_m^2\int_S x\varphi_m\, dS$		$+m_w(\ell+f_w)$ $+m_c(\ell+f_c)$
	$+m_w\, g\, f_w$ $+m_c\, g\, f_c$		$-\rho\omega_m^2\int_S x\varphi_m\, dS$	=	$+m_w\, f_w$ $+m_c\, f_c$
			$+\rho\frac{\omega_m^4}{g}\int_S \varphi_m^2\, dS$		$-\rho\frac{\omega_m^2}{g}\int_S x\varphi_m dS$

The equations of motion may be written in matrix notation

$$\left[M\right]\left\{\ddot{q}\right\} + \left[K\right]\left\{q\right\} = \left\{\ddot{s}\right\} \tag{222}$$

where both matrices M and K are square, symmetric and infinite. The coefficients of these matrices are given in Equations (221), omitting the symmetric part of the offdiagonal elements. These infinite matrices must be made finite in numerical computations by neglecting the contributions of higher modes. Since the cut-off point depends on the required accuracy and on the nature of the forcing function, it has to be determined in every case. But keeping in mind the restrictive assumptions of the linear theory, it is in most cases sufficient to consider only a few lower modes. After the system has been truncated, the motion can be found by using any of the well known numerical procedures for the solution of equations of the type (222). The functions $p_m(t)$ do not have an immediate physical meaning, but they may be used as a basis to compute the complete motion in the liquid at any time t. In most applications one will only be interested in the behavior of the liquid surface. Equation (215) gives the function for the free surface at any instant t.

In Equation (221), the properties of the liquid appear in the mode frequency ω_m and three different surface integrals. They have to be evaluated considering a specific container W.

4.4 Circular and Elliptic Cylindrical Container

The problem of free oscillations of liquid in these two container shapes has already been treated in Chapter II. To be able to give all coefficients in Equation (221), the three following integrals have to be determined

$$U_i = \int_S \phi_c \varphi_i dS \qquad (223)$$

$$V_i = \int_S x \varphi_i dS \qquad (224)$$

$$W_i = \int_S \varphi_i^2 dS \qquad (225)$$

In order to give tabular values, it is furthermore assumed in this section that only the surface motion in the point P (a,0,0) is of practical interest.

4.4.1 Circular Cylindrical Container

The natural modes in this container are according to the derivations in section 2.3

$$\varphi_{m,n} = J_m\left(\mu_{m,n}\frac{r}{a}\right)\cos m\theta \cosh\left(\mu_{m,n}\frac{z+h}{a}\right)$$

$$m = 0,1,2,\ \ldots \qquad (226)$$

$$n = 1,2,3,\ \ldots$$

The Stokes - Zhukovskii potential ϕ_c at the surface S can be derived from Equation (144).

$$\phi_c = \left[\frac{h}{2}r - \sum_{n=1}^{\infty}\frac{4a^2\tanh\left(\frac{\mu_{1,n}h}{2a}\right)}{\mu_{1,n}(\mu_{1,n}^2-1)J_1(\mu_{1,n})} J_1\left(\frac{\mu_{1,n}}{a}r\right)\right]\cos\theta \qquad (227)$$

Assuming now that the natural mode $\varphi_{m,n}$ is introduced into Equations

(223), (224) and (225), then it is immediately apparent that $U_{m,n}$ and $V_{m,n}$ vanish except for m = 1, while the corresponding $W_{m,n}$ vanishes as a consequence of the equations of motion. This means that only modes of the type

$$\varphi_{1,n} = J_1\left(\mu_{1,n}\frac{r}{a}\right)\cos\theta\,\cosh\left(\mu_{1,n}\frac{z+h}{a}\right) \tag{228}$$

will contribute to the motion of the system.

Using the orthogonality relation

$$\int_0^a r\,J_1\left(\mu_{1,m}\frac{r}{a}\right)J_1\left(\mu_{1,n}\frac{r}{a}\right)dr = 0,\ \text{if } m \neq n$$

and the integral relations given in Equations (141) and (142), the following relations are obtained after some obvious calculations:

$$U_{1,n} = \pi\,a^4\cosh\left(\mu_{1,n}\frac{h}{a}\right)\frac{J_1(\mu_{1,n})}{\mu_{1,n}^2}\left[\frac{1}{2}\left(\frac{h}{a}\right) - \frac{2}{\mu_{1,n}}\tanh\left(\frac{\mu_{1,n}}{2}\frac{h}{a}\right)\right] \tag{229}$$

$$V_{1,n} = \pi\,a^3\cosh\left(\mu_{1,n}\frac{h}{a}\right)\frac{J_1(\mu_{1,n})}{\mu_{1,n}^2} \tag{230}$$

$$W_{1,n} = \pi\,a^2\cosh^2\left(\mu_{1,n}\frac{h}{a}\right)\frac{\mu_{1,n}^2-1}{2}\,\frac{J_1^2(\mu_{1,n})}{\mu_{1,n}^2} \tag{231}$$

The motion of the surface in P (a,0,0) is given by the expression

$$\zeta(a,0,t) = \frac{1}{a}\sum_{n=1}^{\infty} p_{1,n}(t)\,\mu_{1,n}\,J_1(\mu_{1,n})\,\sinh\left(\mu_{1,n}\frac{h}{a}\right) \tag{232}$$

4.4.2 Elliptic Cylindrical Container

Here it is again essential to distinguish between the cosine-elliptic and the sine-elliptic case. The first case will furnish the solution for shape ratios $a/b > 1$ and the second case for ratios $a/b < 1$.

4.4.2.1 Cosine-Elliptic Case

The expression for the natural mode is given in Equation (76).

$$\varphi_{2m+1,n} = Ce_{2m+1}(\xi,q_{2m+1,n})\ ce_{2m+1}(\eta,q_{2m+1,n})\cosh\left(\frac{2}{c}\sqrt{q_{2m+1,n}}\ (z+h)\right)$$

$$m = 0,1,2\ \ldots,\ n = 1,2,3\ldots \tag{233}$$

and the epxression for the Stokes-Zhukovskii potential ϕ_c at the surface can be derived from Equation (188)

$$\phi_c = c\,\frac{h}{2}\cosh\xi\,\cos\eta + c^2\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{\tanh\left(\frac{h}{c}\sqrt{q_{2m+1,n}}\right)}{\sqrt{q_{2m+1,n}}}\frac{Cq_{2m+1,n}}{Cr_{2m+1,n}}\ ce_{2m+1}(\eta,q_{2m+1,n})$$

$$Ce_{2m+1}\ (\xi,q_{2m+1,n}) \tag{234}$$

Using the orthogonality relation

$$\int_0^{\xi_B}\int_0^{2\pi} ce_{2m+1}(\eta,q_{2m+1,n})\ Ce_{2m+1}(\xi,q_{2m+1,n})\ ce_{2i+1}(\eta,q_{2i+1,j})\ Ce_{2i+1}(\xi,q_{2i+1,j})$$

$$(\cosh 2\xi - \cos 2\eta)\ d\xi\ d\eta = 0,\ \text{if } m \neq i \text{ and } n \neq j$$

and the definitions for $Cq_{2m+1,n}$ and $Cr_{2m+1,n}$ given in Equations (176) and (177), the following expressions can be obtained after some obvious calculations:

$$U_{2m+1,n} = -\pi\,\frac{c^4}{2}\,Cq_{2m+1,n}\cosh\left(\frac{2h}{c}\sqrt{q_{2m+1,n}}\right)\left[\frac{1}{2}\left(\frac{h}{c}\right) - \frac{\tanh\left(\frac{h}{c}\sqrt{q_{2m+1,n}}\right)}{\sqrt{q_{2m+1,n}}}\right] \tag{235}$$

$$V_{2m+1,n} = -\pi\,\frac{c^3}{2}\,Cq_{2m+1,n}\ \cosh\left(\frac{2h}{c}\sqrt{q_{2m+1,n}}\right) \tag{236}$$

$$W_{2m+1,n} = \pi \frac{c^2}{2} Cr_{2m+1,n} \cosh^2\left(\frac{2h}{c}\sqrt{q_{2m+1,n}}\right) \tag{237}$$

The motion of the surface in P (a, 0, 0) is

$$\zeta(\xi_B, 0, t) = \frac{2}{c}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} p_{2m+1,n}(t)\sqrt{q_{2m+1,n}}\sinh\left(\frac{2h}{c}\sqrt{q_{2m+1,n}}\right) Ce_{2m+1}(\xi_B, q_{2m+1,n}) ce_{2m+1}(0, q_{2m+1,n}) \tag{238}$$

4.4.2.2 Sine-Elliptic Case

An exactly analogous procedure leads to the following results:

$$U_{2m+1,n} = \pi \frac{c^4}{2} Sq_{2m+1,n} \cosh\left(\frac{2h}{c}\sqrt{q_{2m+1,n}}\right)\left[\frac{1}{2}\left(\frac{h}{c}\right) - \frac{\tanh\left(\frac{h}{c}\sqrt{q_{2m+1,n}}\right)}{\sqrt{q_{2m+1,n}}}\right] \tag{239}$$

$$V_{2m+1,n} = \pi \frac{c^3}{2} Sq_{2m+1,n} \cosh\left(\frac{2h}{c}\sqrt{q_{2m+1,n}}\right) \tag{240}$$

$$W_{2m+1,n} = \pi \frac{c^2}{2} Sr_{2m+1,n} \cosh^2\left(\frac{2h}{c}\sqrt{q_{2m+1,n}}\right) \tag{241}$$

$$\zeta(\xi_B, \frac{\pi}{2}, t) = \frac{2}{c}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} p_{2m+1,n}(t)\sqrt{q_{2m+1,n}}\sinh\left(\frac{2h}{c}\sqrt{q_{2m+1,n}}\right) Se_{2m+1}(\xi_B, q_{2m+1,n})\, se_{2m+1}(\frac{\pi}{2}, q_{2m+1,n}) \tag{242}$$

4.4.3. Generaliziation of the Solutions

The three solutions to U_i, V_i and W_i show a very close similarity. It should therefore be possible by simply redefining some constant terms to obtain only one expression which then would be applicable to all three cases. If again φ_i is the i-th natural mode and σ_i the corresponding eigenvalue, then these unified expressions are:

$$U_i = \pi a^4 \cosh\left(\sigma_i \frac{h}{a}\right) Q_i \left[\frac{1}{2}\left(\frac{h}{a}\right) - \frac{2}{\sigma_i} \tanh\left(\frac{\sigma_i}{2} \frac{h}{a}\right)\right] \tag{243}$$

$$V_i = \pi a^3 \cosh\left(\sigma_i \frac{h}{a}\right) Q_i \tag{244}$$

$$W_i = \pi a^2 \cosh^2\left(\sigma_i \frac{h}{a}\right) R_i \tag{245}$$

$$\zeta(x=a,\ y=0,\ t) = \frac{1}{a} \sum_{i=1}^{\infty} p_i(t) \sigma_i \sinh\left(\sigma_i \frac{h}{a}\right) S_i \tag{246}$$

The modal constants take different forms in each of the three cases:

- circular case

$$\sigma_{1,n} = \mu_{1,n} \tag{247}$$

$$Q_{1,n} = \frac{J_1(\mu_{1,n})}{\mu_{1,n}^2} \tag{248}$$

$$R_{1,n} = \frac{\mu_{1,n}^2 - 1}{2} \frac{J_1^2(\mu_{1,n})}{\mu_{1,n}^2} \tag{249}$$

$$S_{1,n} = J_1(\mu_{1,n}) \tag{250}$$

- cosine-elliptic case for $a/b > 1$

$$\xi_B = \frac{1}{2} \log \frac{a/b + 1}{a/b - 1}$$

$$\sigma_{2m+1,n} = 2\sqrt{q_{2m+1,n}} \cosh \xi_B \tag{251}$$

$$Q_{2m+1,n} = -\frac{Cq_{2m+1,n}}{2\cosh^3 \xi_B} \tag{252}$$

$$R_{2m+1,n} = \frac{Cr_{2m+1,n}}{2\cosh^2 \xi_B} \tag{253}$$

$$S_{2m+1,n} = Ce_{2m+1}(\xi_B, q_{2m+1,n})\, ce_{2m+1}(0, q_{2m+1,n}) \tag{254}$$

- sine-elliptic case for $a/b < 1$

$$\xi_B = \frac{1}{2} \log \frac{b/a + 1}{b/a - 1}$$

$$\sigma_{2m+1,n} = 2 \sqrt{q_{2m+1,n}} \sinh \xi_B \tag{255}$$

$$Q_{2m+1,n} = \frac{Sq_{2m+1,n}}{2\sinh^3 \xi_B} \tag{256}$$

$$R_{2m+1,n} = \frac{Sr_{2m+1,n}}{2\sinh^2 \xi_B} \tag{257}$$

$$S_{2m+1,n} = Se_{2m+1} (\xi_B, q_{2m+1,n}) \, se_{2m+1}(\tfrac{\pi}{2}, q_{2m+1,n}) \tag{258}$$

Values for $\sigma_{2m+1,n}$, $Q_{2m+1,n}$, $R_{2m+1,n}$, $S_{2m+1,n}$ are given in Tables III to XIII of the appendix for different ratios a/b.

4.4.4 The Equations of Motion in Dimensionless Form

Investigations of dynamic systems can be made much more general if they are carried out in a dimensionless manner. It turns out that the character of the motion of this system with a circular or elliptic cylindrical container is completely determined if the system is specified by the following parameters:

- length parameters

$$l^* = \frac{l}{a}, \; f_w^* = \frac{f_w}{a}, \; f_c^* = \frac{f_c}{a}, \; \epsilon = \frac{h}{a}, \; \gamma = \frac{a}{b} \tag{259}$$

- inertia parameters

$$\varkappa = \frac{m_w}{m_c} , \quad \chi = \frac{I_w}{I_c} \tag{260}$$

All other parameters which will enter the equations of motion can be derived. They are

$$I_c^* = \frac{I_c}{\pi \rho a^5} , \quad \lambda_c = \frac{M_c}{I_c}$$

$$U_m^* = \frac{U_m}{\pi a^4} , \quad V_m^* = \frac{V_m}{\pi a^3} , \quad W_m^* = \frac{W_m}{\pi a^2} \qquad (261)$$

$$\omega_m^* = \omega_m \sqrt{\frac{a}{g}}$$

If finally the time is transformed into the dimensionless form

$$\tau = \sqrt{\frac{g}{a}} \; t \qquad (262)$$

and if the derivative with respect to τ is denoted by a prime, the dimensionless equations of motion are, if written similarly to Equation (221):

$\ddot{\alpha}$	$\ddot{\beta}$		$\frac{p''_m}{a^2}$
$+\varkappa\frac{\epsilon}{\gamma}(\ell^*+f_w^*)^2$ $+\frac{\epsilon}{\gamma}(1^*+f_c^*)^2$ $+ I_c^* (\lambda + \chi)$	$+\varkappa\frac{\epsilon}{\gamma} f_w^*(\ell^*+f_w^*)$ $+\frac{\epsilon}{\gamma} f_c^*(\ell^*+f_c^*)$ $+ I_c^* (\lambda + \chi)$		$-(\ell^*+f_c^*)\omega_m^{*2}V_m^*$ $+\omega_m^{*2} U_m^*$
	$+\varkappa\frac{\epsilon}{\gamma} f_w^{*2}$ $+\frac{\epsilon}{\gamma} f_c^{*2}$ $+ I_c^* (\lambda + \chi)$		$-f_c^* \omega_m^{*2}V_m^*$ $+ \omega_m^{*2} U_m^*$
			$+ \omega_m^{*2} W_m$

(263)

+

α	β		$\frac{p_m}{a^2}$		$\frac{s''}{a}$
$+\varkappa\frac{\epsilon}{\gamma}(\ell^*+f_w^*)$ $+\frac{\epsilon}{\gamma}(\ell^*+f_c^*)$	$+\varkappa\frac{\epsilon}{\gamma} f_w^*$ $+\frac{\epsilon}{\gamma} f_c^*$		$-\omega_m^{*2} V_m^*$		$+\varkappa\frac{\epsilon}{\gamma}(\ell^*+f_w^*)$ $+\frac{\epsilon}{\gamma}(\ell^*+f_c^*)$
	$+\varkappa\frac{\epsilon}{\gamma} f_w^*$ $+\frac{\epsilon}{\gamma} f_c^*$		$-\omega_m^{*2} V_m^*$	=	$+\varkappa\frac{\epsilon}{\gamma} f_w^*$ $+\frac{\epsilon}{\gamma} f_c^*$
			$+\omega_m^{*4} W_m^*$		$-\omega_m^{*2} V_m^*$

4.4.5 Summary

To facilitate the application of the dimensionless equations of motion (263) to compute the response of the system to an arbitrary forcing function s"(τ)/a, a summary of all the relations needed to evaulate the coefficients in Equations (263) are given

The length parameters

$$\ell^* = \frac{\ell}{a} \, , \; f_w^* = \frac{f_w}{a} \, , \; f_c^* = \frac{f_c}{a} \, , \; \epsilon = \frac{h}{a} \, , \; \gamma = \frac{a}{b}$$

and the inertia parameters

$$\mathcal{H} = \frac{m_w}{m_c} \, , \; \mathcal{X} = \frac{I_w}{I_c}$$

depend on the specific arrangement and properties of the system. They are the basis to determine the dependent parameters

$$\xi_B = \frac{1}{2} \log \frac{1 + \gamma}{1 - \gamma} \, , \text{ for } \gamma < 1$$

$$\xi_B = \frac{1}{2} \log \frac{\gamma + 1}{\gamma - 1} \, , \text{ for } \gamma > 1 \qquad (264)$$

$$I_c^* = \frac{1}{24} \epsilon^3 \frac{\sinh 2\xi_B}{\sinh^2 \xi_B} + \frac{1}{16} \frac{\epsilon}{\sinh^4 \xi_B} \left(\frac{1}{2} \sinh 4\xi_B - \sinh 2\xi_B \right), \text{ for } \gamma < 1$$

$$I_c^* = \frac{1}{12} \epsilon^3 + \frac{1}{4} \epsilon \, , \text{ for } \gamma = 1 \qquad (265)$$

$$I_c^* = \frac{1}{24} \epsilon^3 \frac{\sinh 2\xi_B}{\cosh^2 \xi_B} + \frac{1}{16} \frac{\epsilon}{\cosh^4 \xi_B} \left(\frac{1}{2} \sinh 4\xi_B + \sinh 2\xi_B \right), \text{ for } \gamma > 1$$

Values for λ are given in Table I and II of the appendix for different values of ε and γ.

$$U_i^* = \cosh(\sigma_i \epsilon)\, Q_i \left[\frac{1}{2}\epsilon - \frac{2}{\sigma_i}\tanh\left(\frac{\sigma_i}{2}\epsilon\right)\right] \qquad (266)$$

$$V_i^* = \cosh(\sigma_i \epsilon)\, Q_i \qquad (267)$$

$$W_i^* = \cosh^2(\sigma_i \epsilon)\, R_i \qquad (268)$$

$$\omega_i^{*2} = \sigma_i \tanh(\sigma_i \epsilon) \qquad (269)$$

These relations determine all the coefficients in Equations (263). $\alpha(t)$, $\beta(t)$ and $\frac{p_i(t)}{a^2}$ can now be computed using any suitable numerical procedure. The motion of the surface point P (x=a,y=0) can now be obtained from the relation

$$\frac{\zeta(x=a,y=0,t)}{a} = \sum_i \frac{p_i(t)}{a^2}\, \sigma_i \sinh(\sigma_i \epsilon)\, S_i \qquad (270)$$

The modal constants σ_i, Q_i, R_i and S_i are given for different values of γ in Tables III to XIII of the appendix.

4.4.6 Examples for the Response of the System

In order to give some idea of the character of the response which might be expected in a specific case, Equations (263) were solved by the Runge-Kutta numerical procedure for a few cases which were designed such as to show the following effects:

- Influence of the surface waves, Figure 14.
- Effect of including higher liquid modes in the analysis, Figure 15.
- Influence of the ellipticity of the container, Figure 16.
- Behavior of the system over a longer period of time, Figure 17.

The system was subjected to the force function $\frac{\ddot{s}}{g}$ in form of a unit step function as given in Figure 12.

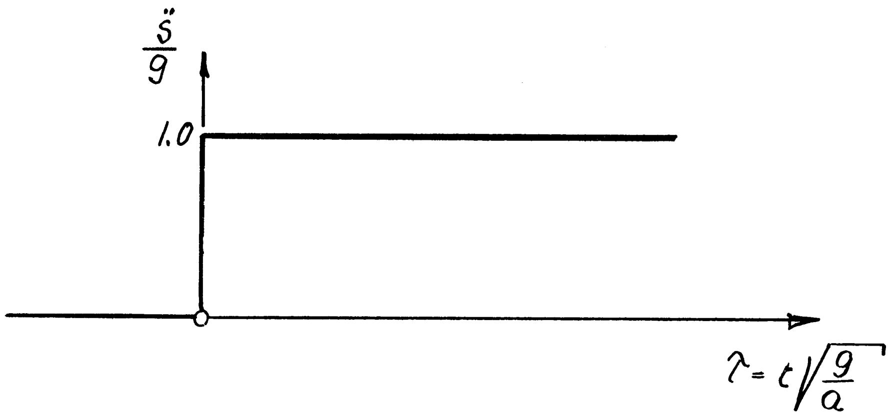

Figure 12

Force as a Unit Step Function

The following system properties are the same in all computed examples:

$\ell^* = \ell/a$	$f_w^* = f_w/a$	$f_c^* = f_c/a$	$\epsilon = h/a$	$\varkappa = m_w/m_c$	$\chi = I_w/I_c$
2.0	.4	.2	1.8	.5	.8

All the other properties are varied so as to show the intended effects.

System No.	$\gamma = \frac{a}{b}$	Number of Lower Liquid Modes Included	Duration τ of the Investigation	Response Plotted in Fig.
1	1.3	0	20.0	14
2	1.3	18	20.0	14
3	1.4	1	20.0	15
4	1.4	2	20.0	15
5	1.4	18	20.0	15
6	.6667	18	20.0	16
7	1.0	7	20.0	16
8	1.5	18	20.0	16
9	1.4	18	80.0	17

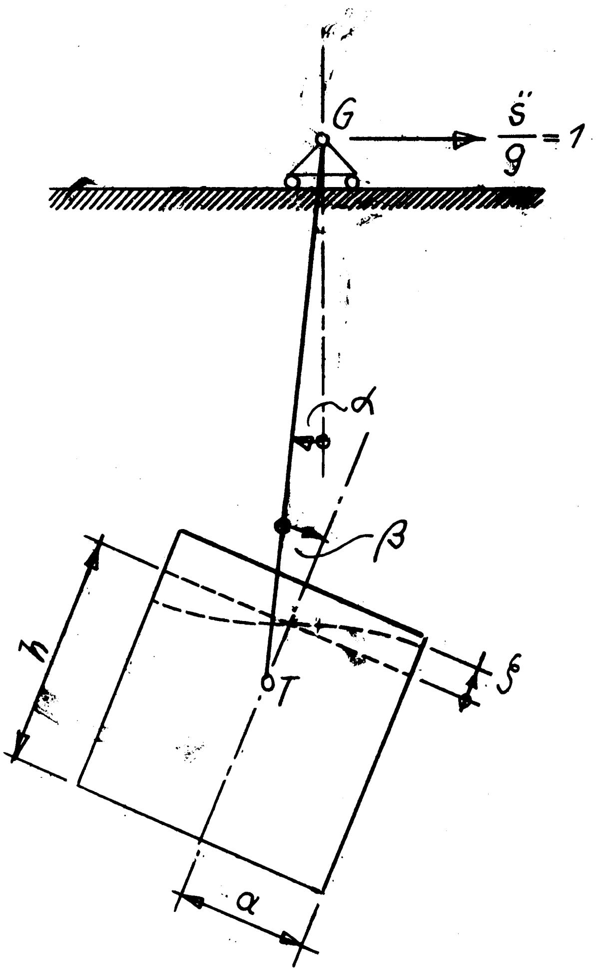

Figure 13
Significance of the Plotted Time Dependent Variables α, β, ζ/a

The responses of the time dependent variables α, β and ζ/a in Figure 13 are plotted for the 9 given dynamic systems with respect to the dimensionless time axis $\tau = \sqrt{g/a}\,t$ in Figures 14 to 17. The curves give rise to the following conclusions, which however may not be generalized to arbitrary dynamic systems. They are valid for the investigated dynamic systems and for the very simple-natured forcing function. It may be assumed that some of the observations are of quite general nature.

Figure 14 compares the response of system 1, in which the development of the surface waves is suppressed, and system 2, where this arbitrary restraint is removed. It shows clearly that the character of the response is very drastically changed by the wave motion, which is able to absorb a considerable amount of energy. Figure 15, however, demonstrates that this drastic change in the rigid body motion may only be attributed to the lowest mode. This figure compares the response of systems 3, 4 and 5, in which an increasing number of liquid modes are free to appear. It allows furthermore a visualization of the convergence of the analytical process in that it demonstrates that the influence of the second mode is about the same as the contribution of the following 16 modes together. It can therefore be concluded that the lowest mode determines the rigid body motion of the system with high accuracy, but that a sufficient number of liquid modes has to be considered, whenever the behavior of the free surface is the purpose of the investigation. Figure 16 again confirms an observation which has already been made while computing the mode frequencies and equivalent moments of inertia of a liquid in elliptic cylindrical containers. It compares the response of the systems 6, 7 and 8, which only differ in the ellipticity γ of the container. Only the wave motion shows some appreciable differences and even these are very small. The way in which the equations of motion were made dimensionless permitted a comparison of the influence of the different flow patterns in the three containers. The system properties illustrate clearly how one might consider this effect to simplify practical computations. Figure 17 shows finally the periodic energy exchange which takes place between the motion of the liquid and the motion of the rigid body.

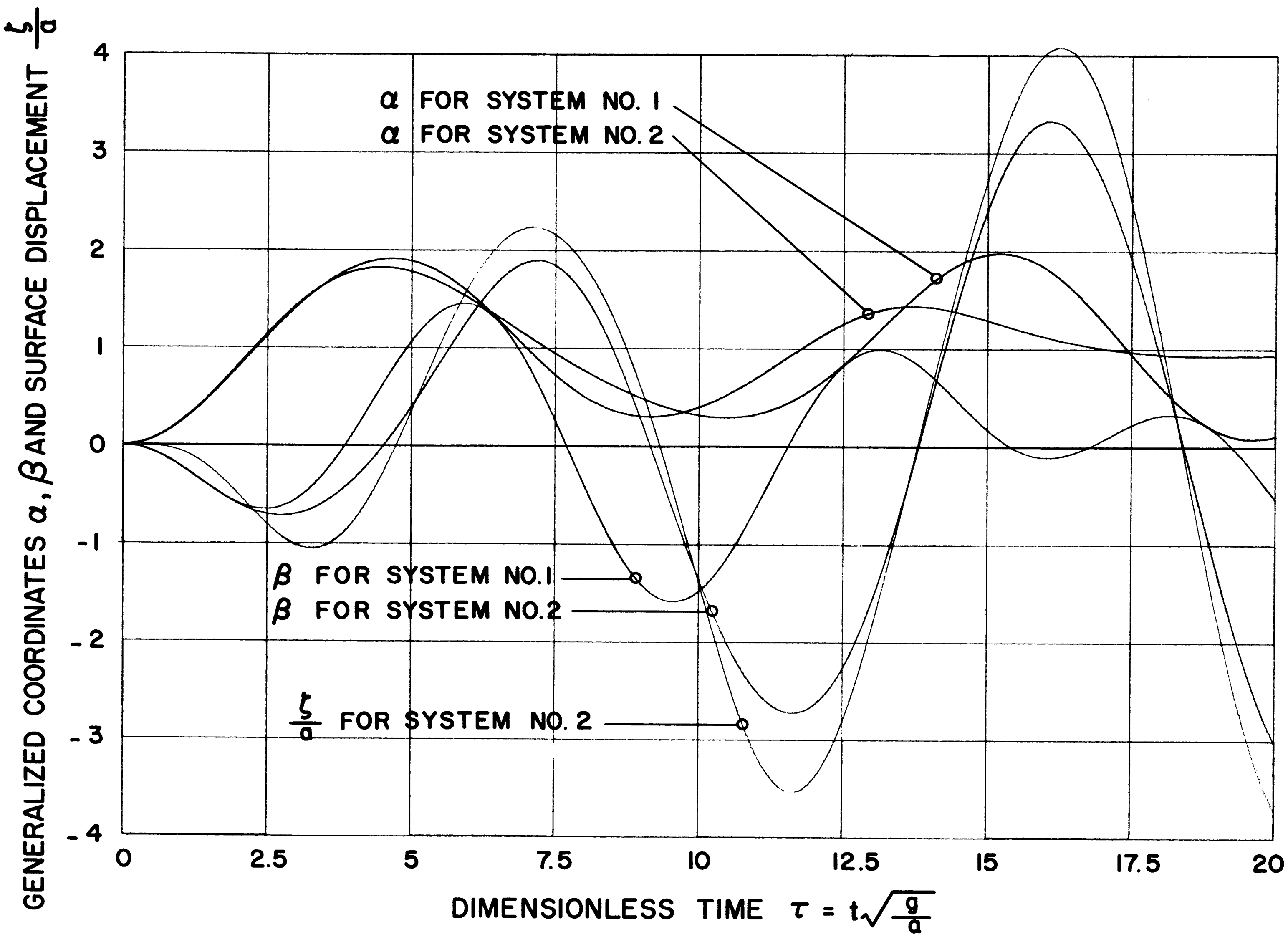

Figure 14. The Influence of the Surface Waves. Response of the Systems No. 1, where the Surface Waves are Suppressed, and System No. 2, where they are Free to Appear

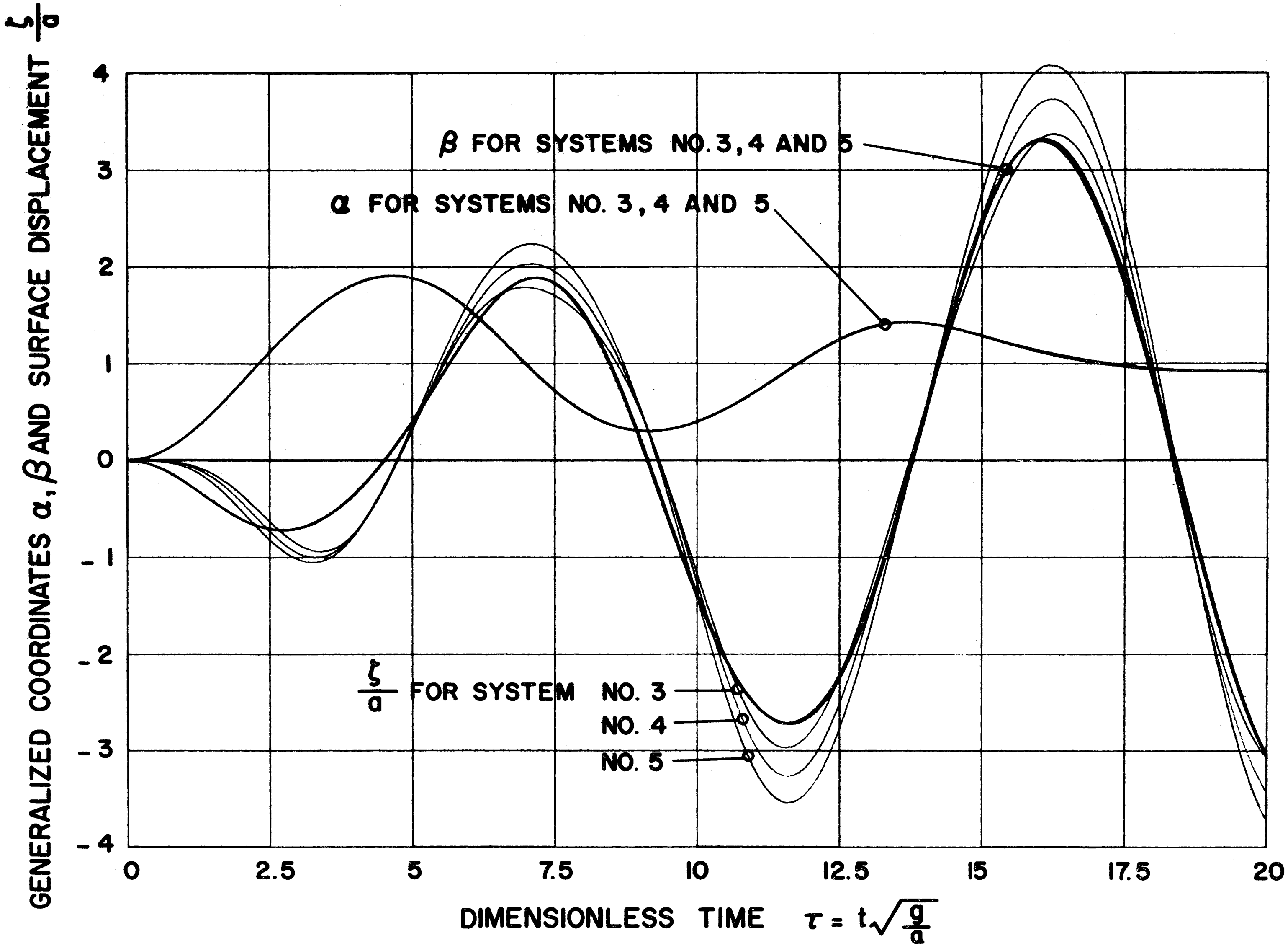

Figure 15. Effect of Including Higher Modes in the Analysis. Response of the Systems No. 3, 4 and 5, where one, two and 18 Lower Modes are Considered

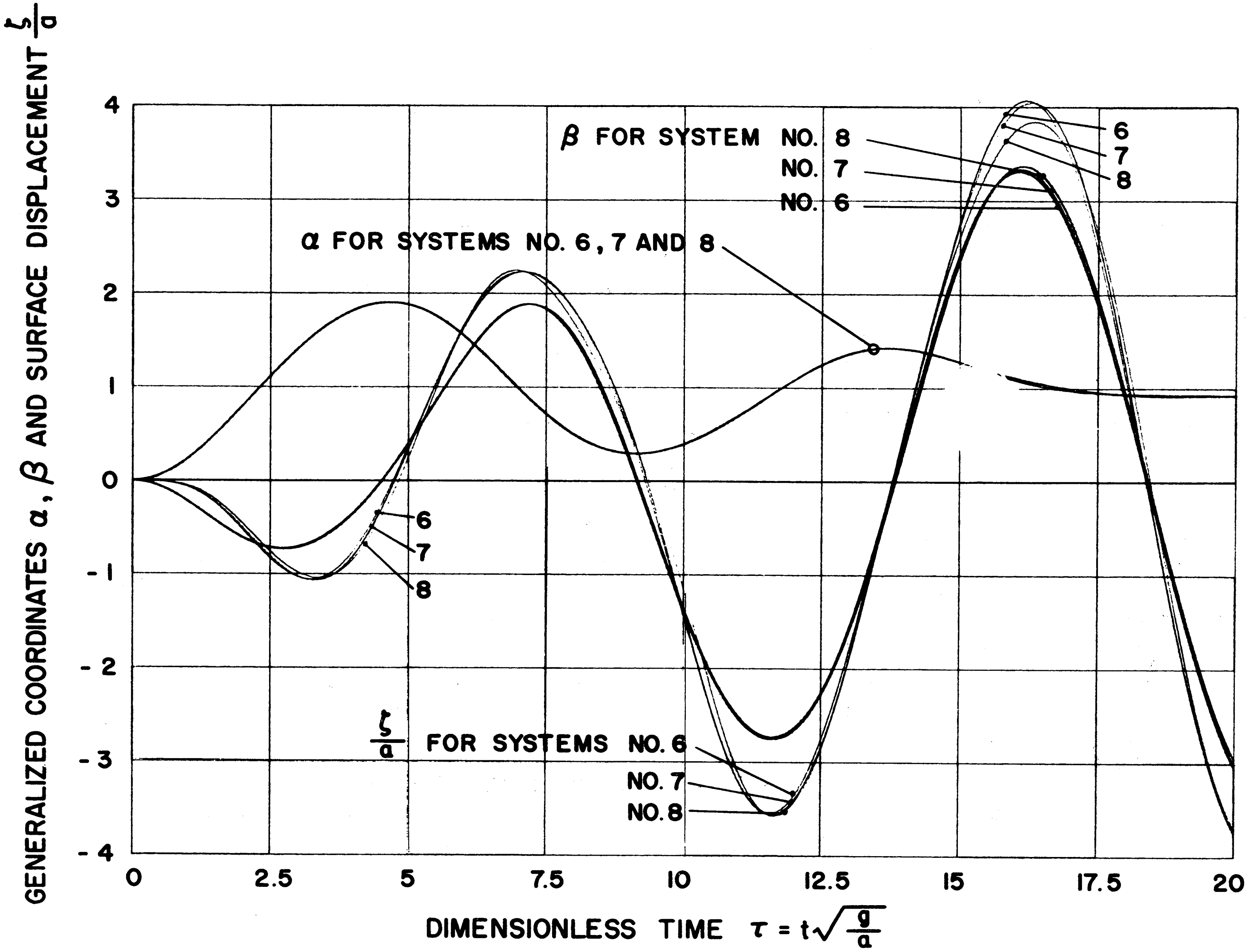

Figure 16. Influence of the Ellipticity of the Container. Response of Systems No. 6, 7 and 8, which Differ Only in the Ellipticity a/b of the Container. a/b = 2/3, 1 and 3/2

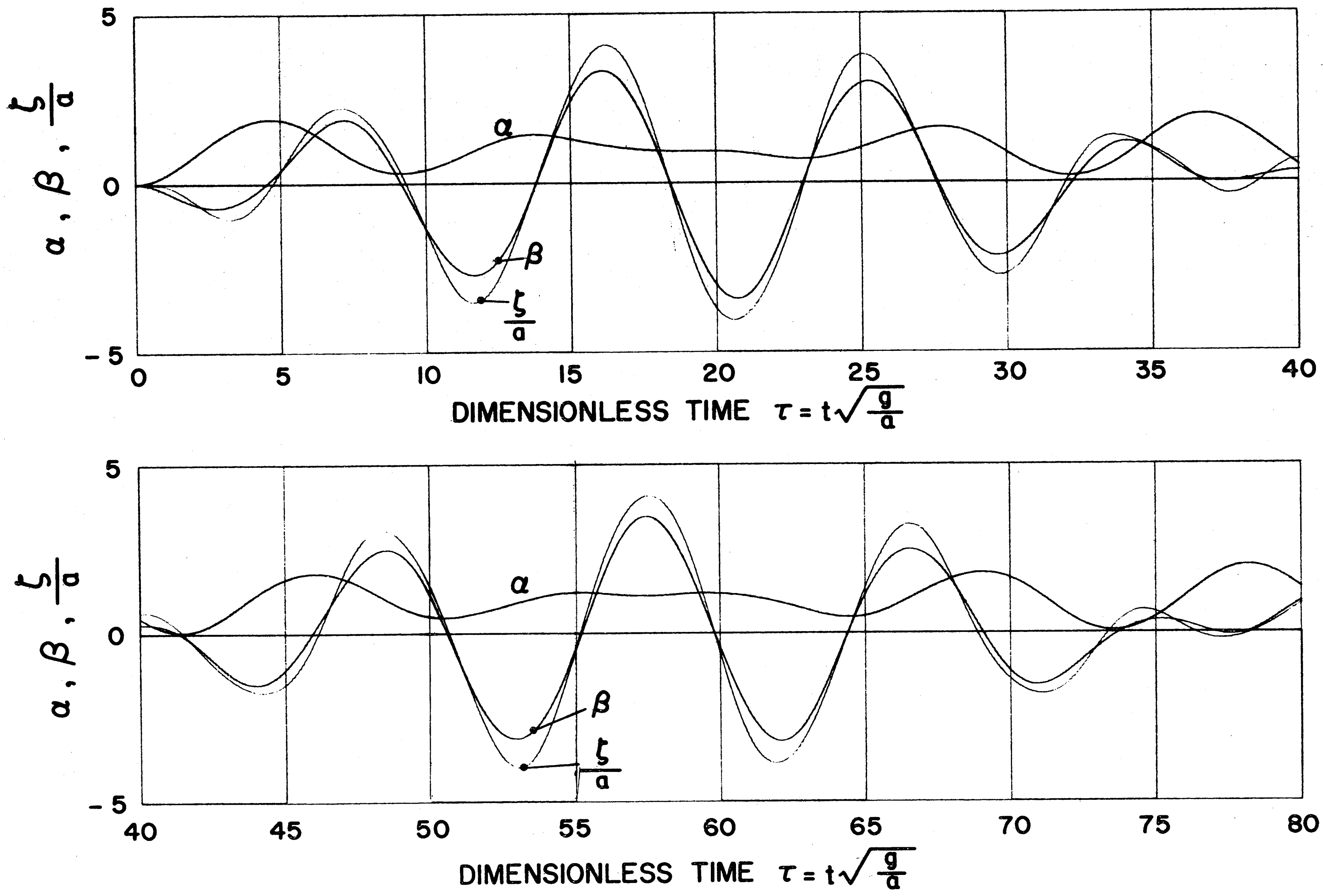

Figure 17. Response of System No. 9 over a Longer Period of Time

V SUMMARY AND CONCLUSIONS

This dissertation presents a solution of the problem of small oscillations of an ideal, incompressible liquid in moving circular and elliptic cylindrical containers. The investigation is directed toward the solution of the liquid oscillation problem in hot metal ladles but all the numerical results are nevertheless given in a completely general manner so as to be immediately applicable to similar problems.

The solution involves a classical approach in the dynamics of continuous media. The motion of the liquid is expanded in a series with time dependent coefficients with respect to the natural modes of free oscillations in the container, which represent a complete orthogonal set of coordinate functions for this problem. A clear picture of the liquid motion was obtained by separating it into two parts; (1) the motion if the free surface S is replaced by a solid lid and (2) the surface wave motion. This suggested that the solution to the forced oscillation problem could be obtained in three principal steps.

(1) The coordinate functions and corresponding eigenvalues were obtained from the solution of the free oscillation problem of liquid in containers at rest. The solution for the circular cylindrical container is very old. In the case of an elliptic cylindrical container, the problem reduces to an eigenvalue problem with two parameters. A rather involved and time-consuming numerical procedure was developed in this case to yield enough eigenvalues and eigenfunctions to handle all subsequent problems.

(2) A closed rigid container completely filled with an ideal, incompressible liquid is dynamically equivalent to some rigid body with a mass equal to the mass of the system and some moment of inertia. The motion of the liquid inside the container can be derived from the Stokes - Zhukovskii potentials, which depend on the shape of the container only. A suitable assumption reduces the Stokes - Zhukovskii problem to the same eigenvalue problem which has already been solved in the first step when only the boundary conditions at the cylindrical walls are considered. The Stokes - Zhukovskii potential is now expanded in a Fourier series with respect to the eigenfunctions and the unknown coefficients are determined from the remaining boundary conditions. The Stokes - Zhukovskii potentials provide the basis for computing the components in the equivalent inertia tensor. These operations were performed to obtain the equivalent moments of inertia of the liquid in elliptic cylindrical containers with respect to rotations about the two principal axes through their center of gravity. The results supplement the known values for the circular cylindrical container and have important technical applications.

(3) The small oscillations of a conservative system having containers partially filled with an ideal, incompressible liquid can be described by an infinite set of linear,

second order differential equations. It is demonstrated that for the ladle system with a circular or elliptic cylindrical container, the evaluation of the coefficients in these differential equations requires merely a rearrangement of some modal constants which have already been computed in the first two steps. They are therefore tabulated and may be utilized when in any dynamic system these modes are excited.

The investigation closes with the presentation of some characteristic responses of the ladle system, disturbed by a simple unit-step function. The responses show the following important characteristics, which however may not necessarily be generalized to other systems and forcing functions.

- The wave motion of the liquid changes drastically the response of the system.
- The rigid body motion is accurately determined when only the lowest liquid mode is considered, while the surface waves may be considerably affected by higher modes.
- The influence of the container ellipticity alone on the response of the system is small and it can be concluded that at least for preliminary investigations the elliptic cylindrical container may be replaced by the simpler circular cylindrical container of the same dynamic properties.

APPENDIX

1. The Equivalent Moment of Inertia of An Ideal, Incompressible Liquid Completely Enclosed in a Rigid Elliptic Cylindrical Container

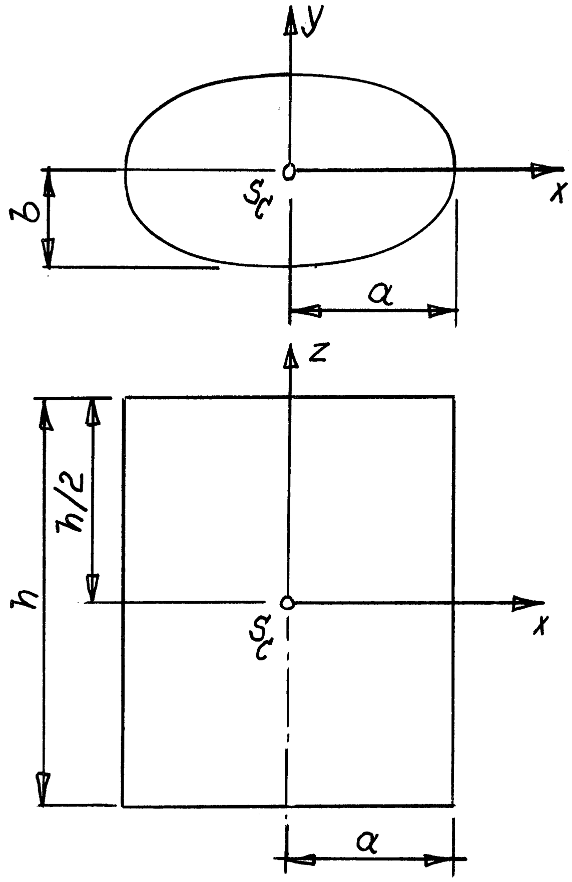

Figure 18

Reference System for the Inertia Parameter λ

Tables I and II give the dimensionless inertia parameter $\lambda = M_c/I_c$ for an ideal, incompressible liquid completely enclosed in the rigid, elliptic cylindrical container of Figure 18. λ is the ratio between the equivalent moment of inertia of the liquid M_c and the moment of inertia of the solidified liquid I_c, both with respect to rotations about the y-axis through the centre of gravity of the liquid S_c. Values of λ are given for the shape parameters $\gamma = \frac{a}{b}$ and $\epsilon = \frac{h}{a}$, which are of most practical interest. The parameter λ relates M_c with I_c, which is given by the following expressions:

$$\frac{I_c}{\pi \rho a^5} = \frac{1}{24} \epsilon^3 \frac{\sinh 2\xi_B}{\sinh^2 \xi_B} + \frac{1}{16} \frac{\epsilon}{\sinh^4 \xi_B} \left(\frac{1}{2} \sinh 4\xi_B - \sinh 2\xi_B \right), \text{for } \gamma < 1$$

$$\frac{I_c}{\pi \rho a^5} = \frac{1}{12} \epsilon^3 + \frac{1}{4} \epsilon , \text{ for } \gamma = 1 \tag{271}$$

$$\frac{I_c}{\pi \rho a^5} = \frac{1}{24} \epsilon^3 \frac{\sinh 2\xi_B}{\cosh^2 \xi_B} + \frac{1}{16} \frac{\epsilon}{\cosh^4 \xi_B} \left(\frac{1}{2} \sinh 4\xi_B + \sinh 2\xi_B \right), \text{for } \gamma > 1$$

where

$$\xi_B = \frac{1}{2} \log \frac{1 + \gamma}{1 - \gamma} , \text{ for } \gamma < 1$$

$$\xi_B = \frac{1}{2} \log \frac{\gamma + 1}{\gamma - 1} , \text{ for } \gamma > 1 \tag{272}$$

2. Eigenvalues and Modal Constants for the Natural Modes of Liquid Oscillations in Elliptic Cylindrical Containers

Tables III to XIII give a summary of constants which belong to the mode of order (2m+1,n), which proved to be of fundamental importance in problems dealing with liquid oscillations in moving elliptic cylindrical containers.

Column K contains the eigenvalues $k_{2m+1,n} = \sqrt{q_{2m+1,n}}$ of the corresponding parameter in Mathieu's differential equations. This column is omitted in the case $\gamma = a/b = 1$, where it does not have a meaning. The column SIGMA gives the dimensionless frequency parameters $\sigma_{2m+1,n}$, which are the basis to compute the mode frequencies $\omega_{2m+1,n}$ from the relation

$$\omega^2_{2m+1,n} = \sigma_{2m+1,n} \frac{a}{g} \tanh \left(\sigma_{2m+1,n} \frac{h}{a}\right) \tag{273}$$

Columns Q and R give the constant $Q_{2m+1,n}$ and $R_{2m+1,n}$, which are the basis for the evaluation of the contribution of the corresponding mode in the system of second order differential equations of motion. Column S finally gives the constants $S_{2m+1,n}$ which then are used to find the response of one particular point in the free surface.

The application of these modal constants to the investigation of a specific dynamic system is summarized in Section 4.4.5.

TABLE I

THE RATIOS LAMBDA = MC/IC FOR ELLIPTIC CYLINDRICAL CONTAINERS WITH RATIOS A/B LESS OR EQUAL TO 1

A/B	.6667	.7143	.7692	.8333	.9091	1.0000	A/B
B/A	1.5000	1.4000	1.3000	1.2000	1.1000	1.0000	B/A
H/A							H/A
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0
.2	.9511	.9511	.9510	.9509	.9509	.9508	.2
.4	.8264	.8261	.8257	.8254	.8250	.8245	.4
.6	.6640	.6632	.6623	.6614	.6604	.6593	.6
.8	.4999	.4986	.4972	.4956	.4939	.4921	.8
1.0	.3601	.3583	.3563	.3543	.3520	.3495	1.0
1.1	.3036	.3016	.2995	.2973	.2948	.2922	1.1
1.2	.2570	.2549	.2527	.2503	.2477	.2449	1.2
1.3	.2202	.2180	.2157	.2132	.2105	.2077	1.3
1.4	.1926	.1904	.1880	.1855	.1828	.1800	1.4
1.5	.1736	.1714	.1690	.1665	.1639	.1611	1.5
1.6	.1622	.1600	.1577	.1553	.1527	.1499	1.6
1.7	.1575	.1554	.1531	.1508	.1482	.1456	1.7
1.8	.1585	.1564	.1543	.1520	.1496	.1471	1.8
1.9	.1643	.1623	.1603	.1581	.1558	.1534	1.9
2.0	.1740	.1721	.1702	.1681	.1659	.1637	2.0
2.1	.1868	.1851	.1833	.1813	.1793	.1771	2.1
2.2	.2022	.2006	.1988	.1970	.1951	.1931	2.2
2.3	.2195	.2180	.2163	.2146	.2128	.2110	2.3
2.4	.2382	.2368	.2353	.2337	.2320	.2302	2.4
2.5	.2579	.2566	.2552	.2537	.2521	.2505	2.5
2.6	.2783	.2770	.2757	.2743	.2729	.2714	2.6
2.7	.2990	.2979	.2966	.2953	.2940	.2926	2.7
2.8	.3199	.3188	.3177	.3165	.3152	.3139	2.8
2.9	.3407	.3397	.3386	.3375	.3363	.3351	2.9
3.0	.3613	.3604	.3594	.3583	.3573	.3561	3.0
3.5	.4581	.4574	.4567	.4560	.4553	.4545	3.5
4.0	.5409	.5404	.5399	.5394	.5388	.5382	4.0
4.5	.6092	.6089	.6085	.6081	.6077	.6073	4.5
5.0	.6652	.6649	.6646	.6643	.6640	.6637	5.0
5.5	.7109	.7107	.7104	.7102	.7100	.7098	5.5
6.0	.7484	.7482	.7481	.7479	.7477	.7475	6.0
6.5	.7794	.7793	.7792	.7790	.7789	.7787	6.5
7.0	.8053	.8052	.8051	.8050	.8049	.8047	7.0
7.5	.8270	.8269	.8268	.8267	.8266	.8265	7.5
8.0	.8454	.8453	.8452	.8452	.8451	.8450	8.0
8.5	.8611	.8610	.8609	.8609	.8608	.8607	8.5
9.0	.8745	.8745	.8744	.8744	.8743	.8742	9.0
9.5	.8861	.8861	.8860	.8860	.8860	.8859	9.5
10.0	.8962	.8962	.8962	.8961	.8961	.8960	10.0
10.5	.9051	.9050	.9050	.9050	.9049	.9049	10.5
11.0	.9129	.9128	.9128	.9128	.9127	.9127	11.0

TABLE II

THE RATIOS LAMBDA = MC/IC FOR ELLIPTIC CYLINDRICAL CONTAINERS WITH RATIOS A/B GREATER OR EQUAL TO 1

A/B	1.0000	1.1000	1.2000	1.3000	1.4000	1.5000	A/B
B/A	1.0000	.9091	.8333	.7692	.7143	.6667	B/A
H/A							H/A
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0
.2	.9508	.9507	.9507	.9506	.9506	.9505	.2
.4	.8245	.8241	.8237	.8233	.8230	.8227	.4
.6	.6593	.6582	.6572	.6563	.6555	.6548	.6
.8	.4921	.4903	.4887	.4872	.4860	.4848	.8
1.0	.3495	.3472	.3451	.3433	.3418	.3404	1.0
1.1	.2922	.2896	.2874	.2855	.2838	.2824	1.1
1.2	.2449	.2423	.2400	.2380	.2363	.2348	1.2
1.3	.2077	.2050	.2027	.2007	.1990	.1975	1.3
1.4	.1800	.1773	.1750	.1730	.1713	.1698	1.4
1.5	.1611	.1584	.1562	.1542	.1525	.1511	1.5
1.6	.1499	.1474	.1452	.1433	.1417	.1403	1.6
1.7	.1456	.1431	.1410	.1392	.1377	.1364	1.7
1.8	.1471	.1447	.1427	.1410	.1396	.1383	1.8
1.9	.1534	.1511	.1493	.1476	.1463	.1451	1.9
2.0	.1637	.1616	.1598	.1583	.1570	.1559	2.0
2.1	.1771	.1752	.1735	.1721	.1709	.1699	2.1
2.2	.1931	.1913	.1897	.1884	.1873	.1863	2.2
2.3	.2110	.2092	.2078	.2066	.2055	.2046	2.3
2.4	.2302	.2286	.2273	.2262	.2252	.2244	2.4
2.5	.2505	.2490	.2478	.2467	.2458	.2450	2.5
2.6	.2714	.2700	.2688	.2678	.2670	.2663	2.6
2.7	.2926	.2913	.2902	.2893	.2885	.2879	2.7
2.8	.3139	.3127	.3117	.3109	.3101	.3095	2.8
2.9	.3351	.3340	.3331	.3323	.3317	.3311	2.9
3.0	.3561	.3551	.3542	.3535	.3529	.3524	3.0
3.5	.4545	.4538	.4532	.4526	.4522	.4519	3.5
4.0	.5382	.5377	.5373	.5370	.5366	.5364	4.0
4.5	.6073	.6070	.6066	.6064	.6062	.6060	4.5
5.0	.6637	.6634	.6632	.6630	.6628	.6627	5.0
5.5	.7098	.7095	.7094	.7092	.7091	.7090	5.5
6.0	.7475	.7474	.7472	.7471	.7470	.7469	6.0
6.5	.7787	.7786	.7785	.7784	.7783	.7783	6.5
7.0	.8047	.8046	.8045	.8045	.8044	.8043	7.0
7.5	.8265	.8265	.8264	.8263	.8263	.8262	7.5
8.0	.8450	.8449	.8449	.8448	.8448	.8447	8.0
8.5	.8607	.8607	.8606	.8606	.8605	.8605	8.5
9.0	.8742	.8742	.8741	.8741	.8741	.8740	9.0
9.5	.8859	.8859	.8858	.8858	.8858	.8857	9.5
10.0	.8960	.8960	.8960	.8959	.8959	.8959	10.0
10.5	.9049	.9049	.9048	.9048	.9048	.9048	10.5
11.0	.9127	.9127	.9126	.9126	.9126	.9126	11.0

TABLE III

EIGENVALUES AND MODAL CONSTANTS
FOR MODES OF THE ORDER 2*M+1,N

A/B = .6667, B/A = 1.5000

M	N	K	SIGMA	Q	R	S
0	1	1.0078	1.8028	.17721E 00	.85813E-01	.42862E 00
0	2	2.8239	5.0515	-.65435E-03	.10689E-03	-.17471E-01
0	3	4.5861	8.2039	.75963E-05	.12290E-06	.66656E-03
0	4	6.3448	11.3500	-.11963E-06	.12943E-09	-.23426E-04
0	5	8.1024	14.4939	.22031E-08	.13094E-12	.79145E-06
0	6	9.8594	17.6370	-.38533E-10	.96536E-16	-.22559E-07
1	1	1.8846	3.3713	.57502E-01	.92220E 00	-.13769E 01
1	2	3.7059	6.6293	-.12241E-02	.37973E-02	.78811E-01
1	3	5.4446	9.7396	.32099E-04	.11777E-04	-.48475E-02
1	4	7.1920	12.8654	-.80969E-06	.24144E-07	.23519E-03
1	5	8.9430	15.9977	.19871E-07	.37156E-10	-.97308E-05
2	1	2.8259	5.0552	.25878E-01	.54989E 01	.36254E 01
2	2	4.6556	8.3282	-.14284E-02	.31818E-01	-.20520E 00
2	3	6.3530	11.3646	.75585E-04	.22411E-03	.18858E-01
2	4	8.0772	14.4489	-.29488E-05	.84817E-06	-.12413E-02
2	5	9.8143	17.5564	.10231E-06	.22551E-08	.67304E-04
2	6	11.5578	20.6753	-.33582E-08	.48244E-11	-.32427E-05
2	7	13.3050	23.8007	.11733E-09	.10752E-13	.15848E-06
3	1	3.7747	6.7524	.14372E-01	.30726E 02	-.92788E 01
3	2	5.6455	10.0990	-.13881E-02	.15757E 00	.43390E 00
3	3	7.3020	13.0622	.13081E-03	.20234E-02	-.52428E-01
3	4	8.9953	16.0913	-.76884E-05	.13477E-04	.45999E-02
3	5	10.7130	19.1641	.35711E-06	.55819E-07	-.31145E-03
4	1	4.7178	8.4395	.90270E-02	.17568E 03	.23793E 02
4	2	6.6544	11.9038	-.12567E-02	.63872E 00	-.86684E 00
4	3	8.2836	14.8182	.18493E-03	.11590E-01	.11781E 00
4	4	9.9419	17.7847	-.16048E-04	.12905E-03	-.13443E-01
5	1	5.6540	10.1142	.61448E-02	.10418E 04	-.61435E 02
5	2	7.6679	13.7168	-.11077E-02	.24715E 01	.17462E 01

TABLE IV

EIGENVALUES AND MODAL CONSTANTS
FOR MODES OF THE ORDER 2*M+1,N

A/B = .7143, B/A = 1.4000

M	N	K	SIGMA	Q	R	
0	1	.8869	1.8104	.20433E 00	.12185E 00	.52232E 00
0	2	2.4891	5.0809	-.11217E-02	.33135E-03	-.30758E-01
0	3	4.0342	8.2348	.20356E-04	.91175E-06	.18168E-02
0	4	5.5757	11.3814	-.49726E-06	.22909E-08	-.98658E-04
0	5	7.1161	14.5257	.14144E-07	.55043E-11	.51374E-05
0	6	8.6560	17.6690	-.45379E-09	.13619E-13	-.26828E-06
0	7	10.1957	20.8118	.19675E-10	.53405E-16	.17498E-07
1	1	1.7144	3.4994	.59358E-01	.15225E 01	-.18530E 01
1	2	3.3341	6.8057	-.17943E-02	.10532E-01	.13046E 00
1	3	4.8565	9.9134	.76157E-04	.77124E-04	-.12347E-01
1	4	6.3867	13.0368	-.30364E-05	.37822E-06	.92893E-03
1	5	7.9205	16.1677	.12078E-06	.14942E-08	-.61653E-04
1	6	9.4561	19.3022	-.41053E-08	.37129E-11	.32123E-05
2	1	2.5890	5.2848	.26049E-01	.11814E 02	.56457E 01
2	2	4.2398	8.6544	-.18622E-02	.85922E-01	-.33735E 00
2	3	5.7279	11.6919	.15914E-03	.13099E-02	.44983E-01
2	4	7.2353	14.7689	-.10023E-04	.11872E-04	-.46094E-02
2	5	8.7553	17.8718	.54917E-06	.75423E-07	.38753E-03
2	6	10.2818	20.9876	-.28268E-07	.38670E-09	-.28952E-04
2	7	11.8118	24.1108	.14072E-08	.17197E-11	.20008E-05
2	8	13.3441	27.2386	-.66388E-10	.64642E-14	-.12651E-06
3	1	3.4621	7.0671	.14322E-01	.90532E 02	-.17020E 02
3	2	5.1745	10.5623	-.16825E-02	.46592E 00	.76526E 00
3	3	6.6377	13.5491	.24340E-03	.10707E-01	-.11809E 00
3	4	8.1160	16.5668	-.23648E-04	.16898E-03	.16045E-01
3	5	9.6171	19.6308	.17572E-05	.16756E-05	-.16907E-02
3	6	11.1305	22.7201	-.11411E-06	.12338E-07	.15145E-03
4	1	4.3268	8.8321	.89377E-02	.72524E 03	.51691E 02
4	2	6.1164	12.4851	-.14571E-02	.23106E 01	-.17438E 01
4	3	7.5758	15.4640	.30459E-03	.57876E-01	.25854E 00
4	4	9.0245	18.4213	-.44309E-04	.14479E-02	-.43952E-01
4	5	10.5027	21.4386	.44369E-05	.21928E-04	.57605E-02
5	1	5.1843	10.5823	.60536E-02	.60786E 04	-.15891E 03
5	2	7.0535	14.3979	-.12414E-02	.11821E 02	.41278E 01
5	3	8.5309	17.4137	.33603E-03	.25449E 00	-.52732E 00

TABLE V

EIGENVALUES AND MODAL CONSTANTS
FOR MODES OF THE ORDER 2*M+1,N

A/B = .7692, B/A = 1.3000

M	N	K	SIGMA	Q	R	S
0	1	.7551	1.8181	.24110E 00	.18217E 00	.65465E 00
0	2	2.1255	5.1177	-.19760E-02	.10941E-02	-.55507E-01
0	3	3.4362	8.2733	.58258E-04	.77153E-05	.52531E-02
0	4	4.7433	11.4206	-.22910E-05	.49641E-07	-.45679E-03
0	5	6.0494	14.5653	.10459E-06	.30535E-09	.38071E-04
0	6	7.3551	17.7089	-.52357E-08	.18328E-11	-.30971E-05
0	7	8.6604	20.8518	.26981E-09	.10136E-13	.23992E-06
0	8	9.9656	23.9944	-.11111E-10	.32421E-16	-.14053E-07
1	1	1.5139	3.6451	.62588E-01	.29668E 01	-.27311E 01
1	2	2.9161	7.0212	-.26186E-02	.31620E-01	.22389E 00
1	3	4.2073	10.1300	.18641E-03	.56458E-03	-.32878E-01
1	4	5.5033	13.2503	-.12314E-04	.71301E-05	.39865E-02
1	5	6.8029	16.3794	.79940E-06	.72615E-07	-.42576E-03
1	6	8.1043	19.5128	-.52260E-07	.65465E-09	.42307E-04
1	7	9.4067	22.6486	.27804E-08	.35340E-11	-.32294E-05
2	1	2.2995	5.5364	.26875E-01	.34429E 02	.10321E 02
2	2	3.7543	9.0393	-.24168E-02	.27755E 00	-.61575E 00
2	3	5.0246	12.0978	.33093E-03	.84208E-02	.11105E 00
2	4	6.2995	15.1674	-.35666E-04	.19510E-03	-.18289E-01
2	5	7.5858	18.2645	.32421E-05	.31968E-05	.24829E-02
2	6	8.8783	21.3764	-.27223E-06	.42009E-07	-.29787E-03
2	7	10.1743	24.4969	.21913E-07	.47685E-09	.32949E-04
2	8	11.4725	27.6226	-.17166E-08	.48644E-11	-.34365E-05
2	9	12.7722	30.7519	.13087E-09	.45064E-13	.34022E-06
3	1	3.0762	7.4066	.14629E-01	.41594E 03	-.39185E 02
3	2	4.6055	11.0887	-.20522E-02	.19506E 01	.16573E 01
3	3	5.8744	14.1440	.43038E-03	.64321E-01	-.28307E 00
3	4	7.1265	17.1585	-.72677E-04	.24088E-02	.58575E-01
3	5	8.3946	20.2118	.92214E-05	.62063E-04	-.10037E-01
3	6	9.6745	23.2936	-.99142E-06	.11793E-05	.14524E-02
3	7	10.9615	26.3921	.97185E-07	.18240E-07	-.18775E-03
4	1	3.8435	9.2541	.90675E-02	.53297E 04	.15041E 03
4	2	5.4510	13.1245	-.17015E-02	.14254E 02	-.47188E 01
4	3	6.7416	16.2318	.47257E-03	.37531E 00	.66548E 00
4	4	7.9789	19.2109	-.11494E-03	.18036E-01	-.14876E 00
4	5	9.2264	22.2147	.20391E-04	.70265E-03	.31447E-01
4	6	10.4909	25.2591	-.28125E-05	.19128E-04	-.54803E-02
4	7	11.7664	28.3303	.33493E-06	.40016E-06	.82637E-03
5	1	4.6041	11.0854	.58723E-02	.71997E 05	-.58265E 03
5	2	6.2849	15.1323	-.13963E-02	.11434E 03	.14114E 02
5	3	7.6114	18.3260	.47452E-03	.21252E 01	-.16163E 01

TABLE VI

EIGENVALUES AND MODAL CONSTANTS
FOR MODES OF THE ORDER 2*M+1,N

A/B = .8333, B/A = 1.2000

M	N	K	SIGMA	Q	R	S
0	1	.6056	1.8259	.29865E 00	.30210E 00	.86656E 00
0	2	1.7133	5.1657	-.36475E-02	.40233E-02	-.10440E 00
0	3	2.7608	8.3242	.18560E-03	.80783E-04	.16659E-01
0	4	3.8049	11.4722	-.12501E-04	.14956E-05	-.24605E-02
0	5	4.8480	14.6174	.97143E-06	.26406E-07	.34766E-03
0	6	5.8907	17.7612	-.82455E-07	.45340E-09	-.47856E-04
0	7	6.9332	20.9044	.74319E-08	.76574E-11	.64800E-05
0	8	7.9755	24.0471	-.70075E-09	.12849E-12	-.86952E-06
0	9	9.0178	27.1896	.73382E-10	.24798E-14	.12459E-06
1	1	1.2638	3.8107	.69313E-01	.80062E 01	-.47891E 01
1	2	2.4174	7.2887	-.38752E-02	.11542E 00	.42582E 00
1	3	3.4535	10.4126	.47222E-03	.49232E-02	-.93628E-01
1	4	4.4875	13.5302	-.55964E-04	.17911E-03	.19348E-01
1	5	5.5244	16.6568	.63559E-05	.52845E-05	-.35328E-02
1	6	6.5631	19.7886	-.71432E-06	.13647E-06	.59572E-03
1	7	7.6028	22.9233	.80241E-07	.32235E-08	-.95278E-04
1	8	8.6431	26.0598	-.90151E-08	.71208E-10	.14653E-04
1	9	9.6837	29.1974	.52996E-09	.40916E-12	-.11444E-05
2	1	1.9269	5.8098	.29179E-01	.17913E 03	.25413E 02
2	2	3.1470	9.4885	-.32319E-02	.14123E 01	-.14642E 01
2	3	4.1839	12.6149	.66543E-03	.67040E-01	.30370E 00
2	4	5.2033	15.6884	-.13085E-03	.40289E-02	-.79112E-01
2	5	6.2282	18.7786	.21706E-04	.19319E-03	.18516E-01
2	6	7.2585	21.8853	-.32199E-05	.74062E-05	-.38182E-02
2	7	8.2923	25.0021	.44979E-06	.24332E-06	.72137E-03
2	8	9.3281	28.1252	-.60584E-07	.71664E-08	-.12818E-03
2	9	10.3652	31.2523	.79612E-08	.19475E-09	.21775E-04
2	10	11.4034	34.3824	-.10177E-08	.48956E-11	-.35456E-05
3	1	2.5774	7.7713	.15728E-01	.43604E 04	-.13689E 03
3	2	3.8709	11.6712	-.25870E-02	.17729E 02	.54989E 01
3	3	4.9309	14.8672	.72791E-03	.61040E 00	-.89155E 00
3	4	5.9448	17.9242	-.20748E-03	.42407E-01	.23317E 00
3	5	6.9558	20.9726	.49988E-04	.30329E-02	-.66144E-01
3	6	7.9746	24.0444	-.99003E-05	.17115E-03	.16662E-01
3	7	8.9998	27.1354	.17237E-05	.77774E-05	-.37194E-02
3	8	10.0291	30.2389	-.27728E-06	.30206E-06	.76082E-03
3	9	11.0612	33.3508	.42257E-07	.10440E-07	-.14597E-03
3	10	12.0953	36.4686	-.61778E-08	.32941E-09	.26649E-04
3	11	13.1307	39.5906	.87086E-09	.96531E-11	-.46744E-05
3	12	14.1672	42.7158	-.12193E-09	.28086E-12	.81514E-06
4	1	3.2195	9.7071	.96864E-02	.11333E 06	.74709E 03

M	N	K	SIGMA	Q	R	S
4	2	4.5806	13.8112	-.20404E-02	.25646E 03	-.22243E 02
4	3	5.6756	17.1126	.71172E-03	.57335E 01	.28394E 01
4	4	6.7004	20.2025	-.25862E-03	.32879E 00	-.62735E 00
4	5	7.7037	23.2274	.84924E-04	.27866E-01	.18493E 00
4	6	8.7096	26.2605	-.22743E-04	.22132E-02	-.55172E-01
4	7	9.7240	29.3191	.50091E-05	.13915E-03	.14588E-01
4	8	10.7451	32.3977	-.96754E-06	.71299E-05	-.34435E-02
5	1	3.8559	11.6259	.49745E-02	.30901E 07	-.41180E 04
5	2	5.2778	15.9133	-.16052E-02	.41921E 04	.94477E 02
5	3	6.4090	19.3240	.65171E-03	.62263E 02	-.98654E 01
5	4	7.4564	22.4820	-.28167E-03	.25720E 01	.17979E 01
5	5	8.4644	25.5210	.11495E-03	.19557E 00	-.47098E 00

TABLE VII

EIGENVALUES AND MODAL CONSTANTS FOR MODES OF THE ORDER 2*M+1,N

A/B = .9091, B/A = 1.1000

M	N	K	SIGMA	Q	R	S
0	1	.4201	1.8336	.42233E 00	.65778E 00	.13189E 01
0	2	1.1988	5.2321	-.75425E-02	.19043E-01	-.21753E 00
0	3	1.9243	8.3985	.72774E-03	.12805E-02	.62642E-01
0	4	2.6459	11.5477	-.95408E-04	.86280E-04	-.17671E-01
0	5	3.3667	14.6933	.14308E-04	.55633E-05	.47798E-02
0	6	4.0871	17.8375	-.23285E-05	.34820E-06	-.12575E-02
0	7	4.8073	20.9809	.40030E-06	.21375E-07	.32487E-03
0	8	5.5275	24.1238	-.71450E-07	.12944E-08	-.82862E-04
0	9	6.2475	27.2664	.13084E-07	.77703E-10	.20948E-04
0	10	6.9675	30.4088	-.24324E-08	.46336E-11	-.52603E-05
0	11	7.6875	33.5510	.45091E-09	.27073E-12	.13040E-05
0	12	8.4075	36.6932	-.85423E-10	.16729E-13	-.33179E-06
0	13	9.1274	39.8352	.16033E-10	.10676E-14	.85699E-07
0	14	9.8473	42.9772	-.29934E-11	.76214E-16	-.23419E-07
1	1	.9158	3.9970	.87978E-01	.50030E 02	-.12967E 02
1	2	1.7464	7.6218	-.63944E-02	.82151E 00	.11758E 01
1	3	2.4757	10.8048	.12592E-02	.66769E-01	-.32651E 00
1	4	3.1926	13.9339	-.29555E-03	.75317E-02	.11547E 00
1	5	3.9088	17.0592	.68790F-04	.81149E-03	-.40408E-01
1	6	4.6258	20.1888	-.15473E-04	.77885E-04	.13220E-01
1	7	5.3437	23.3218	.34098E-05	.68070E-05	-.40866E-02
1	8	6.0621	26.4570	-.74268E-06	.55618E-06	.12127E-02
1	9	6.7808	29.5938	.16022E-06	.43254E-07	-.34930E-03
1	10	7.4997	32.7315	-.34134E-07	.32339E-08	.98278E-04
1	11	8.2188	35.8700	.71258E-08	.23388E-09	-.27120E-04
1	12	8.9381	39.0089	-.14061E-08	.15677E-10	.71911E-05
1	13	9.6574	42.1484	.11170E-09	.18595E-12	-.80150E-06
2	1	1.3985	6.1034	.36290E-01	.39365E 04	.12948E 03
2	2	2.2897	9.9930	-.48826E-02	.28318E 02	-.73318E 01
2	3	3.0401	13.2680	.13766F-02	.14281E 01	.14644E 01
2	4	3.7606	16.4127	-.45817E-03	.14979E 00	-.45524E 00
2	5	4.4724	19.5193	.15531E-03	.20890E-01	.17400E 00
2	6	5.1837	22.6233	-.49079E-04	.29159E-02	-.68430E-01
2	7	5.8966	25.7348	.14173E-04	.36334E-03	.25412E-01
2	8	6.6112	28.8535	-.38140E-05	.40300E-04	-.88329E-02
2	9	7.3270	31.9774	.97484E-06	.40781E-05	.29126E-02
2	10	8.0436	35.1050	-.23868E-06	.38468E-06	-.92272E-03
2	11	8.7608	38.2351	.55875E-07	.34326E-07	.28335E-03
2	12	9.4784	41.3672	-.12373E-07	.29249E-08	-.84853E-04
2	13	10.1964	44.5007	.24773E-08	.23938E-09	.24892E-04
2	14	10.9147	47.6356	-.40250E-09	.18907E-10	-.71875E-05
2	15	11.6331	50.7710	.37350E-10	.14543E-11	.20569E-05

M	N	K	SIGMA	Q	R	S
3	1	1.8701	8.1617	.19377E-01	.34541E 06	-.13203E 04
3	2	2.8164	12.2917	-.36302E-02	.12433E 04	.52005E 02
3	3	3.5952	15.6907	.12863E-02	.36756E 02	-.78827E 01
3	4	4.3315	18.9042	-.53077E-03	.25774E 01	.19153E 01
3	5	5.0484	22.0329	.22681E-03	.30179E 00	-.62412E 00
3	6	5.7572	25.1266	-.94329E-04	.46560E-01	.24560E 00
3	7	6.4648	28.2147	.36102E-04	.75995E-02	-.10308E 00
3	8	7.1740	31.3101	-.12351E-04	.11406E-02	.41857E-01
3	9	7.8854	34.4147	.38137E-05	.15192E-03	-.15935E-01
3	10	8.5985	37.5268	-.10832E-05	.18218E-04	.57199E-02
3	11	9.3128	40.6443	.27320E-06	.20084E-05	-.19585E-02
3	12	10.0280	43.7660	-.67484F-07	.20669E-06	.64650E-03
3	13	10.7439	46.8903	.15969E-07	.20046E-07	-.20780E-03
4	1	2.3354	10.1926	.11899E-01	.32405E 08	.13663E 05
4	2	3.3302	14.5343	-.27410E-02	.65185E 05	-.39686E 03
4	3	4.1363	18.0525	.11127E-02	.12209E 04	.47990E 02
4	4	4.8921	21.3511	-.52881E-03	.56987E 02	-.95314E 01
4	5	5.6224	24.5380	.26423E-03	.47069E 01	.25514E 01
4	6	6.3375	27.6591	-.13091E-03	.58271E 00	-.85204E 00
4	7	7.0447	30.7456	.61897E-04	.94168E-01	.33754E 00
4	8	7.7497	33.8224	-.26828E-04	.16886E-01	-.14640E 00
4	9	8.4558	36.9042	.10269E-04	.29023E-02	.63294E-01
4	10	9.1642	39.9959	-.42293E-05	.44744E-03	-.25919E-01
5	1	2.7967	12.2058	.69012E-02	.31844E 10	-.14288E 06
5	2	3.8350	16.7373	-.26030E-02	.38613E 07	.31862E 04
5	3	4.6663	20.3655	.91291E-03	.48955E 05	-.31652E 03
5	4	5.4409	23.7459	-.48370E-03	.16016E 04	.52952E 02
5	5	6.1862	26.9987	.26880E-03	.94830E 02	-.12074E 02
5	6	6.9130	30.1710	-.15018E-03	.86432E 01	.34292E 01
5	7	7.6273	33.2883	.81531E-04	.11032E 01	-.11627E 01
5	8	8.3338	36.3715	-.41745E-04	.18140E 00	.45903E 00

TABLE IIX

EIGENVALUES AND MODAL CONSTANTS
FOR MODES OF THE ORDER 2*M+1,N

A/B = 1.0000, B/A = 1.0000

M	N	SIGMA	Q	R	S
0	1	1.8412	.17164E 00	.11935E 00	.58187E 00
0	2	5.3314	-.12177E-01	.57794E-01	-.34613E 00
0	3	8.5363	.37506E-02	.36834E-01	.27330E 00
0	4	11.7060	-.17026E-02	.27017E-01	-.23330E 00
0	5	14.8636	.93702E-03	.21330E-01	.20701E 00
0	6	18.0155	-.57930E-03	.17621E-01	-.18802E 00
0	7	21.1644	.38725E-03	.15010E-01	.17346E 00
0	8	24.3113	-.27382E-03	.13074E-01	-.16184E 00
0	9	27.4571	.20200E-03	.11580E-01	.15228E 00
0	10	30.6019	-.15403E-03	.10392E-01	-.14424E 00
0	11	33.7462	.12062E-03	.94252E-02	.13736E 00
0	12	36.8900	-.96536E-04	.86231E-02	-.13137E 00
0	13	40.0334	.78686E-04	.79468E-02	.12611E 00
0	14	43.1766	-.65138E-04	.73688E-02	-.12143E 00
0	15	46.3196	.54644E-04	.68692E-02	.11724E 00
0	16	49.4624	-.46373E-04	.64331E-02	-.11345E 00
0	17	52.6050	.39754E-04	.60490E-02	.11001E 00
0	18	55.7476	-.34386E-04	.57082E-02	-.10687E 00
0	19	58.8900	.29981E-04	.54038E-02	.10397E 00
0	20	62.0323	-.26327E-04	.51302E-02	-.10131E 00

TABLE IX

EIGENVALUES AND MODAL CONSTANTS FOR MODES OF THE ORDER 2*M+1,N

A/B = 1.1000, B/A = .9091

M	N	K	SIGMA	Q	R	[illegible]
0	1	.3849	1.8479	.42675E 00	.81080E 00	.15729E 01
0	2	1.1402	5.4737	-.12077E-01	.71373E-01	-.32691E 00
0	3	1.8439	8.8522	.21354E-02	.19640E-01	.94679E-01
0	4	2.5573	12.2769	-.33379E-03	.28697E-02	-.12366E-01
0	5	3.2753	15.7240	.50352E-04	.27094E-03	.10907E-02
0	6	3.9943	19.1758	-.81207E-05	.21966E-04	-.85202E-04
0	7	4.7136	22.6292	.13858E-05	.16518E-05	.62607E-05
0	8	5.4331	26.0832	-.24593E-06	.11830E-06	-.44135E-06
0	9	6.1527	29.5377	.44795E-07	.81877E-08	.30194E-07
0	10	6.8723	32.9925	-.82792E-08	.55206E-09	-.20181E-08
0	11	7.5920	36.4475	.15453E-08	.36986E-10	.13431E-09
0	12	8.3117	39.9026	-.30345E-09	.27556E-11	-.99583E-11
0	13	9.0314	43.3578	.41096E-10	.10253E-12	.36958E-12
0	14	9.7511	46.8131	-.12202E-10	.20991E-13	-.75760E-13
0	15	10.4709	50.2684	-.26326E-11	.32529E-14	.11844E-13
0	16	11.1906	53.7240	.16106E-12	.19732E-15	-.73387E-15
1	1	.9149	4.3921	.74068E-01	.41806E 02	.14425E 02
1	2	1.7354	8.3314	-.81451E-02	.77277E 00	-.16136E 01
1	3	2.4412	11.7199	.22896E-02	.10677E 00	.52529E 00
1	4	3.1335	15.0435	-.85235E-03	.34289E-01	-.20304E 00
1	5	3.8349	18.4105	.25726E-03	.88036E-02	.46621E-01
1	6	4.5456	21.8227	-.59240E-04	.13080E-02	-.61900E-02
1	7	5.2602	25.2532	.12779E-04	.14861E-03	.65723E-03
1	8	5.9764	28.6915	-.27309E-05	.14806E-04	-.62728E-04
1	9	6.6935	32.1342	.58036E-06	.13560E-05	.55758E-05
1	10	7.4112	35.5797	-.12215E-06	.11680E-06	-.46981E-06
1	11	8.1293	39.0273	.25239E-07	.95911E-08	.37935E-07
1	12	8.8478	42.4763	-.50533E-08	.75933E-09	-.29659E-08
1	13	9.5664	45.9264	.95116E-09	.57946E-10	.22451E-09
1	14	10.2852	49.3773	-.15862E-09	.44045E-11	-.17038E-10
1	15	11.0042	52.8290	.16394E-10	.28303E-12	.11048E-11
2	1	1.3985	6.7137	.29217E-01	.32538E 04	.14058E 03
2	2	2.2891	10.9894	-.52637E-02	.23550E 02	-.92206E 01
2	3	3.0344	14.5673	.20894E-02	.12493E 01	.21891E 01
2	4	3.7388	17.9494	-.93098E-03	.17147E 00	-.77654E 00
2	5	4.4280	21.2578	.44598E-03	.50653E-01	.32828E 00
2	6	5.1212	24.5858	-.19170E-03	.17340E-01	-.10760E 00
2	7	5.8249	27.9642	.59136E-04	.36825E-02	.19984E-01
2	8	6.5348	31.3724	-.15453E-04	.54373E-03	-.26869E-02
2	9	7.2476	34.7944	.38197E-05	.67152E-04	.31266E-03
2	10	7.9620	38.2241	-.91019E-06	.74402E-05	-.33252E-04
2	11	8.6775	41.6589	.20818E-06	.76181E-06	.33046E-05
2	12	9.3937	45.0974	-.44305E-07	.73287E-07	-.31099E-06

M	N	K	SIGMA	Q	R	S
2	13	10.1106	48.5389	.88875E-08	.66871E-08	.27984E-07
2	14	10.8279	51.9823	-.15880E-08	.58410E-09	-.24373E-08
3	1	1.8701	8.9779	.15326E-01	.28546E 06	.14207E 04
3	2	2.8164	13.5208	-.34840E-02	.10278E 04	-.61657E 02
3	3	3.5948	17.2580	.15713E-02	.30476E 02	.10549E 02
3	4	4.3284	20.7799	-.87757E-03	.21884E 01	-.29871E 01
3	5	5.0348	24.1711	.49772E-03	.29290E 00	.11124E 01
3	6	5.7243	27.4814	-.27123E-03	.74159E-01	-.48040E 00
3	7	6.4125	30.7850	.14174E-03	.27855E-01	.19441E 00
3	8	7.1097	34.1322	-.55251E-04	.79581E-02	-.48958E-01
3	9	7.8150	37.5181	.16675E-04	.14854E-02	.81320E-02
3	10	8.5244	40.9239	-.45085E-05	.21962E-03	-.11143E-02
3	11	9.2361	44.3408	.12513E-05	.28446E-04	.13724E-03
3	12	9.9494	47.7649	-.24890E-06	.33540E-05	-.15622E-04
4	1	2.3354	11.2118	.94216E-02	.26781E 08	.14631E 05
4	2	3.3302	15.9877	-.24602E-02	.53871E 05	-.45708E 03
4	3	4.1363	19.8577	.11918E-02	.10091E 04	.60050E 02
4	4	4.8919	23.4852	-.69717E-03	.47173E 02	-.13254E 02
4	5	5.6207	26.9839	.45517E-03	.39424E 01	.40633E 01
4	6	6.3293	30.3859	-.29924E-03	.52139E 00	-.15588E 01
4	7	7.0212	33.7075	.17925E-03	.11349E 00	.67688E 00
4	8	7.7070	36.9995	-.10271E-03	.40624E-01	-.30342E 00
4	9	8.3985	40.3195	.47807E-04	.14408E-01	.99477E-01
4	10	9.0991	43.6827	-.15084E-04	.33764E-02	-.20437E-01
5	1	2.7967	13.4264	.11246E-01	.26326E 10	.15255E 06
5	2	3.8350	18.4110	-.39448E-03	.31896E 07	-.36078E 04
5	3	4.6663	22.4020	.10144E-02	.40160E 05	.38170E 03
5	4	5.4409	26.1205	-.56379E-03	.13209E 04	-.68830E 02
5	5	6.1861	29.6981	.37215E-03	.78421E 02	.17232E 02
5	6	6.9122	33.1838	-.26307E-03	.71915E 01	-.55185E 01
5	7	7.6226	36.5944	.18763E-03	.94894E 00	.21472E 01
5	8	8.3178	39.9318	-.12103E-03	.18397E 00	-.93886E 00
5	9	9.0033	43.2230	.82998E-04	.57931E-01	.43626E 00

TABLE X

EIGENVALUES AND MODAL CONSTANTS FOR MODES OF THE ORDER 2*M+1,N

A/B = 1.2000, B/A = .8333

M	N	K	SIGMA	Q	R	S
0	1	.5122	1.8533	.30566E 00	.45353E 00	.12176E 01
0	2	1.5633	5.6561	-.79932E-02	.42229E-01	-.19923E 00
0	3	2.5829	9.3454	.59102E-03	.32321E-02	.13359E-01
0	4	3.6214	13.1027	-.39259E-04	.99586E-04	-.37690E-03
0	5	4.6620	16.8679	.29880E-05	.24240E-05	.88033E-05
0	6	5.7032	20.6350	-.25016E-06	.53030E-07	-.18804E-06
0	7	6.7446	24.4030	.22285E-07	.10827E-08	.37792E-08
0	8	7.7862	28.1715	-.20639E-08	.20968E-10	-.72376E-10
0	9	8.8278	31.9404	.17460E-09	.30776E-12	.10535E-11
0	10	9.8696	35.7095	-.16156E-10	.50184E-14	-.17068E-13
0	11	10.9113	39.4786	-.82377E-11	.23991E-14	.81180E-14
1	1	1.2553	4.5417	.49627E-01	.59711E 01	.59063E 01
1	2	2.3532	8.5141	-.51000E-02	.16102E 00	-.71137E 00
1	3	3.3237	12.0257	.12316E-02	.31393E-01	.17413E 00
1	4	4.3295	15.6647	-.18050E-03	.29779E-02	-.13990E-01
1	5	5.3569	19.3819	.20145E-04	.13383E-03	.56665E-03
1	6	6.3903	23.1210	-.22191E-05	.45842E-05	-.18341E-04
1	7	7.4265	26.8700	.24596E-06	.13513E-06	.52143E-06
1	8	8.4642	30.6246	-.27431E-07	.35978E-08	-.13535E-07
1	9	9.5029	34.3829	.30708E-08	.89540E-10	.33052E-09
1	10	10.5424	38.1437	-.33587E-09	.19840E-11	-.72168E-11
2	1	1.9264	6.9700	.19350E-01	.12494E 03	.29830E 02
2	2	3.1319	11.3317	-.36695E-02	.11023E 01	-.22713E 01
2	3	4.1151	14.8890	.12051E-02	.10405E 00	.58095E 00
2	4	5.0796	18.3787	-.39866E-03	.23386E-01	-.13973E 00
2	5	6.0806	22.0005	.73092E-04	.23411E-02	.11791E-01
2	6	7.1010	25.6923	-.10491E-04	.12787E-03	-.58057E-03
2	7	8.1285	29.4101	.14271E-05	.53924E-05	.23036E-04
2	8	9.1600	33.1420	-.18876E-06	.19392E-06	-.79503E-06
2	9	10.1939	36.8828	.24469E-07	.62202E-08	.24746E-07
3	1	2.5774	9.3255	.10064E-01	.30287E 04	.15767E 03
3	2	3.8691	13.9988	-.24861E-02	.12486E 02	-.77816E 01
3	3	4.9088	17.7607	.10950E-02	.50232E 00	.15861E 01
3	4	5.8700	21.2386	-.47636E-03	.75661E-01	-.48601E 00
3	5	6.8347	24.7290	.17042E-03	.17279E-01	.10740E 00
3	6	7.8336	28.3429	-.34711E-04	.17492E-02	-.92108E-02
3	7	8.8491	32.0173	.57776E-05	.10776E-03	.51377E-03
3	8	9.8721	35.7185	-.89973E-06	.52340E-05	-.23464E-04
4	1	3.2195	11.6485	.61382E-02	.78699E 05	.85226E 03
4	2	4.5805	16.5728	-.17300E-02	.17829E 03	-.29500E 02
4	3	5.6717	20.5211	.88815E-03	.40882E 01	.45860E 01

M	N	K	SIGMA	Q	R	S
4	4	6.6710	24.1366	-.47982E-03	.28978E 00	-.12317E 01
4	5	7.6230	27.5810	.23537E-03	.57782E-01	.40523E 00
4	6	8.5899	31.0794	-.83961E-04	.12616E-01	-.80213E-01
4	7	9.5875	34.6888	.18166E-04	.12763E-02	.69344E-02
5	1	3.8559	13.9511	.59558E-02	.21394E 07	.46695E 04
5	2	5.2778	19.0959	-.12840E-02	.28968E 04	-.12069E 03
5	3	6.4086	23.1871	.68329E-03	.43369E 02	.14565E 02

TABLE XI

EIGENVALUES AND MODAL CONSTANTS
FOR MODES OF THE ORDER 2*M+1,N

A/B = 1.3000, B/A = .7692

M	N	K	SIGMA	Q	R	S
0	1	.5935	1.8577	.25099E 00	.33123E 00	.10750E 01
0	2	1.8774	5.8763	-.50651E-02	.24249E-01	-.11324E 00
0	3	3.1643	9.9043	.17080E-03	.51390E-03	.20444E-02
0	4	4.4667	13.9808	-.64363E-05	.52490E-05	-.19497E-04
0	5	5.7702	18.0611	.28639E-06	.44166E-07	.15869E-06
0	6	7.0743	22.1427	-.13997E-07	.33155E-09	-.11680E-08
0	7	8.3786	26.2251	.73651E-09	.23744E-11	.82549E-11
0	8	9.6830	30.3080	-.53217E-10	.27960E-13	-.96294E-13
0	9	10.9875	34.3911	.10216E-10	.21123E-14	.72236E-14
1	1	1.4863	4.6521	.38454E-01	.21231E 01	.37299E 01
1	2	2.7705	8.6716	-.37663E-02	.82946E-01	-.45473E 00
1	3	3.9785	12.4528	.50757E-03	.82368E-02	.41816E-01
1	4	5.2530	16.4420	-.34078E-04	.19905E-03	-.86919E-03
1	5	6.5434	20.4812	.21660E-05	.29339E-05	.11909E-04
1	6	7.8393	24.5372	-.13919E-06	.34490E-07	-.13411E-06
1	7	9.1379	28.6021	.90256E-08	.35022E-09	.13232E-08
1	8	10.4383	32.6721	-.58245E-09	.31334E-11	-.11596E-10
2	1	2.2963	7.1874	.14977E-01	.20862E 02	.13091E 02
2	2	3.6907	11.5520	-.26541E-02	.26386E 00	-.11180E 01
2	3	4.8533	15.1910	.74011E-03	.37705E-01	.23515E 00
2	4	6.0778	19.0239	-.10042E-03	.25442E-02	-.13132E-01
2	5	7.3469	22.9960	.89013E-05	.65410E-04	.29904E-03
2	6	8.6297	27.0111	-.73245E-06	.11658E-05	-.49837E-05
2	7	9.9192	31.0474	.58187E-07	.16759E-07	.68570E-07
3	1	3.0759	9.6276	.77320E-02	.24670E 03	.48168E 02
3	2	4.5874	14.3587	-.19789E-02	.12965E 01	-.27911E 01
3	3	5.7710	18.0635	.71606E-03	.97418E-01	.61083E 00
3	4	6.9399	21.7222	-.20240E-03	.15377E-01	-.95444E-01
3	5	8.1772	25.5948	.26278E-04	.75935E-03	.39380E-02
3	6	9.4427	29.5559	-.27288E-05	.20434E-04	-.96047E-04
4	1	3.8435	12.0302	.46779E-02	.31537E 04	.18209E 03
4	2	5.4476	17.0511	-.14095E-02	.86249E 01	-.72891E 01
4	3	6.6955	20.9572	.65427E-03	.30315E 00	.13406E 01
4	4	7.8412	24.5431	-.27157E-03	.47687E-01	-.33945E 00
4	5	9.0331	28.2740	.60988E-04	.53164E-02	.31919E-01
5	1	4.6041	14.4111	.34678E-02	.41816E 05	.69894E 03
5	2	6.2844	19.6703	-.10258E-02	.67871E 02	-.20450E 02
5	3	7.5975	23.7806	.56617E-03	.13645E 01	.30134E 01
5	4	8.7683	27.4451	-.28169E-03	.11627E 00	-.76312E 00
5	5	9.9178	31.0432	.10814E-03	.22238E-01	.15697E 00

TABLE XII

EIGENVALUES AND MODAL CONSTANTS
FOR MODES OF THE ORDER 2*M+1,N

A/B = 1.4000, B/A = .7143

M	N	K	SIGMA	Q	R	S
0	1	.6513	1.8613	.21753E 00	.26791E 00	.99716E 00
0	2	2.1435	6.1257	-.30371E-02	.12796E-01	-.59863E-01
0	3	3.6702	10.4885	.54778E-04	.91118E-04	.36395E-03
0	4	5.2072	14.8809	-.12690E-05	.35900E-06	-.13520E-05
0	5	6.7452	19.2759	.34943E-07	.11699E-08	.42793E-08
0	6	8.2835	23.6720	-.11154E-08	.37721E-11	-.13558E-10
1	1	1.6557	4.7315	.31697E-01	.11061E 01	.28017E 01
1	2	3.0942	8.8424	-.28609E-02	.52507E-01	-.30827E 00
1	3	4.5342	12.9574	.18899E-03	.18198E-02	.89006E-02
1	4	6.0453	17.2759	-.74036E-05	.15711E-04	-.67966E-04
1	5	7.5699	21.6327	.29028E-06	.89553E-07	.36450E-06
1	6	9.0998	26.0048	-.11568E-07	.40989E-09	-.16076E-08
1	7	10.6326	30.3851	.45835E-09	.15685E-11	.59988E-11
2	1	2.5781	7.3675	.12387E-01	.64708E 01	.77497E 01
2	2	4.1007	11.7186	-.20359E-02	.12419E 00	-.71568E 00
2	3	5.4545	15.5875	.36604E-03	.12324E-01	.73027E-01
2	4	6.9225	19.7826	-.24053E-04	.23874E-03	-.11768E-02
2	5	8.4260	24.0792	.12960E-05	.22891E-05	.10291E-04
2	6	9.9424	28.4128	-.65982E-07	.15795E-07	-.67208E-07
3	1	3.4605	9.8892	.63704E-02	.46715E 02	.22309E 02
3	2	5.1105	14.6044	-.15464E-02	.35208E 00	-.14859E 01
3	3	6.4265	18.3654	.46790E-03	.39798E-01	.27441E 00
3	4	7.8334	22.3858	-.57274E-04	.19456E-02	-.10963E-01
3	5	9.3095	26.6042	.41508E-05	.31032E-04	.15358E-03
4	1	4.3266	12.3642	.38241E-02	.37043E 03	.66256E 02
4	2	6.0948	17.4173	-.11876E-02	.13348E 01	-.31419E 01
4	3	7.4353	21.2483	.45637E-03	.88006E-01	.61497E 00
4	4	8.7766	25.0811	-.10783E-03	.97436E-02	-.63475E-01
5	1	5.1842	14.8152	.25444E-02	.30740E 04	.20000E 03
5	2	7.0480	20.1412	-.88287E-03	.62307E 01	-.68661E 01
5	3	8.4543	24.1603	.42059E-03	.20942E 00	.11785E 01
5	4	9.7538	27.8739	-.15636E-03	.30473E-01	-.22738E 00

TABLE XIII

EIGENVALUES AND MODAL CONSTANTS
FOR MODES OF THE ORDER 2*M+1,N

A/B = 1.5000, B/A = .6667

M	N	K	SIGMA	Q	R	S
0	1	.6948	1.8643	.19403E 00	.22838E 00	.94810E 00
0	2	2.3834	6.3952	-.17779E-02	.64457E-02	-.30574E-01
0	3	4.1314	11.0858	.18973E-04	.17691E-04	.72185E-04
0	4	5.8858	15.7933	-.28134E-06	.29047E-07	-.11241E-06
0	5	7.6409	20.5027	.49249E-08	.38643E-10	.14563E-09
0	6	9.3964	25.2132	-.76276E-10	.29471E-13	-.10930E-12
0	7	11.1508	29.9207	-.13979E-10	.31903E-14	.11708E-13
1	1	1.7848	4.7892	.27123E-01	.70657E 00	.23060E 01
1	2	3.3675	9.0359	-.21009E-02	.33366E-01	-.20149E 00
1	3	5.0311	13.4998	.72198E-04	.40498E-03	.19756E-02
1	4	6.7603	18.1397	-.17832E-05	.14309E-05	-.62760E-05
1	5	8.5020	22.8132	.44796E-07	.34094E-08	.14162E-07
1	6	10.2490	27.5011	-.11573E-08	.66398E-11	-.26668E-10
2	1	2.7998	7.5127	.10619E-01	.28453E 01	.53971E 01
2	2	4.4263	11.8771	-.16272E-02	.76800E-01	-.50576E 00
2	3	5.9849	16.0593	.15926E-03	.33521E-02	.19249E-01
2	4	7.6744	20.5927	-.62312E-05	.24398E-04	-.11986E-03
2	5	9.3949	25.2091	.21478E-06	.97311E-07	.44180E-06
3	1	3.7691	10.1136	.54657E-02	.14137E 02	.12983E 02
3	2	5.5099	14.7846	-.12259E-02	.16205E 00	-.96376E 00
3	3	6.9865	18.7467	.25530E-03	.14700E-01	.98438E-01
3	4	8.6224	23.1363	-.15952E-04	.22696E-03	-.12421E-02
4	1	4.7167	12.6563	.32667E-02	.78602E 02	.32250E 02
4	2	6.5877	17.6766	-.97264E-03	.40879E 00	-.17830E 01
4	3	8.0320	21.5521	.30833E-03	.38995E-01	.29389E 00
4	4	9.6000	25.7594	-.33371E-04	.13856E-02	-.84608E-02
4	5	11.2617	30.2183	.20087E-05	.14365E-04	.76772E-04
4	6	12.9565	34.7659	-.10214E-06	.88790E-07	-.43934E-06
5	1	5.6538	15.1707	.21502E-02	.46216E 03	.81718E 02
5	2	7.6418	20.5051	-.76749E-03	.12621E 01	-.33426E 01
5	3	9.1097	24.4438	.30522E-03	.77710E-01	.59942E 00

REFERENCES

1. Abramson, H. N., Ransleben, G., "Simulation of Fuel Sloshing Characteristics in Missile Tanks by Use of Small Models," 20 March 1959, TR No. 3, Contract: DA-23-072-ORD-1251, Southwest Research Institute, San Antonio, Texas.

2. Bauer, H. F., "Fluid Oscillations in a Circular Cylindrical Tank," 18 April 1958, Rep. No. DA-TR-1-58, Army Ballistic Missile Agency, Redstone Arsenal, Ala.

3. Bauer, H. F., "The Moment of Inertia of a Liquid in a Circular Cylindrical Tank," 23 April 1958, Rep. No. DA-TR-5-58, Army Ballistic Missile Agency, Redstone Arsenal, Ala.

4. Bogoriad, I. B., "On the Solution by a Variational Method of the Problem of the Oscillations of a Liquid Partially Filling a Cavity," Journal of Applied Mathematics and Mechanics, Vol. 26, No. 6, 1962, pp. 1122-1127.

5. Campbell, R., "Théorie générale de l'équation de Mathieu," Masson et Cie Editeurs, Paris, 1955.

6. Courant, R., Hilbert, D., "Methods of Mathematical Physics, Vol. I," Interscience Publishers, First English Ed., 1953, New York.

7. Dodge, F. T., Kana, D. D., Abramson, H. N., "Liquid Surface Oscillations in Longitudinally Excited Rigid Cylindrical Containers," AIAA Journal, Vol. 3, No. 4, April 1965.

8. Housner, G. W., "Dynamic Pressures on Accelerated Fluid Containers," Bulletin of the Seismological Society of America, Vol. 47, No. 1, pp. 15-35, January 1951.

9. Housner, G. W., "The Dynamic Behavior of Water Tanks," Bulletin of the Seismological Society of America, Vol. 53, No. 2, pp. 381-387, February 1963.

10. Jacobson, L. S., Ayre, R. S., "Hydrodynamic Experiments with Rigid Cylindrical Tanks Subjected to Transient Motions", Bulletin of the Seismological Society of America, Vol. 41, 1951.

11. Jahnke, E., Emde, F., "Tables of Functions", Dover Publications, New York, 1945.

12. Krasnoshchekov, P. S., "On Oscillations of a Physical Pendulum having Cavities Filled with a Viscous Liquid," Journal of Applied Mathematics and Mechanics, Vol. 27, No. 2, 1963, pp. 193-202.

13. Lamb, H., "Hydrodynamics," Dover Publishers, Sixth Edition, 1945.

14. McLachlan, N. W., "Theory and Application of Mathieu Functions," Dover Publications, Inc., New York, 1964.

15. Meixner, J., Schäfke, F. W., "Mathieusche Funktionen und Sphäroidfunktionen," Springer-Verlag, 1954.

16. Moiseev, N. N., "Introduction to the Theory of Oscillations of Liquid-Containing Bodies," Advances in Applied Mechanics, Vol. 8, pp. 183-232, 1964.

17. Pozharitskii, G. K., "The Minimum Problem of the Stability of Equilibrium of a Solid Body Partially Filled with a Liquid," Journal of Applied Mathematics and Mechanics, Vol. 26, No. 4, 1962, pp. 593-605.

18. Rumyantsev, V., "Stability of Motion of Solid Bodies with Liquid-Filled Cavities by Lyapunov's Methods," Advances in Applied Mechanics, Vol. 8, pp. 183-232, 1964.

19. "Tables Relating to Mathieu Functions", The Computation Laboratory of the National Applied Mathematics Laboratories, National Bureau of Standards, Columbia University Press, New York, 1951.

20. Tolstov, G. P., "Fourier Series," Prentice-Hall, Inc., 1962.

21. Veubeke de, B. F., "The Inertia Tensor of an Incompressible Fluid Bounded by Walls in Rigid Body Motion," International Journal of Engineering Science, January-March 1963, pp. 23.

22. Zurmühl, R., "Praktische Mathematik für Ingenieure und Physiker," Springer-Verlag, Third Edition, 1961.