NONLINEAR DECOUPLING THEORY WITH APPLICATIONS TO ROBOTICS

by
In Joong Ha

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Doctoral Committee :

Professor Elmer G. Gilbert, Chairman
Professor N. Harris McClamroch
Professor Semyon M. Meerkov
Professor William L. Root
Associate Professor Carl P. Simon

ABSTRACT

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In Joong Ha

Chairman : Elmer G. Gilbert

Some theoretical results on nonlinear decoupling theory are presented and their applications to robotic manipulator control are discussed.

First, refinements and extensions of some known results on feedback decoupling of nonlinear systems are given. Precise definitions of decoupling and decomposition are stated. Some conditions under which the two definitions are equivalent for nonlinear systems are found. A previously known condition is shown to be necessary as well as sufficient for a system to be decouplable or locally decomposable.

Second, we obtain new results which characterize the whole class of nonlinear feedback control laws which decouple or decompose. These results are important from both mathematical and engineering viewpoints. For instance, there exist systems where our results allow the stable decoupling of a decouplable system, while former results do not. The class of decoupling control laws is characterized by solutions of certain first order partial differential equations. The class of decomposing control laws is characterized by simple feedback laws applied to a standard decomposed system (SDS). The SDS is similar to the

decomposed system of Isidori, Krener, Gori - Giorgi, and Monaco but has finer structure. These new results are provided by a generalization of ideas used by Gilbert for linear systems.

Third, we discuss a form of approximate decoupling. We neglect fast dynamics of a system to obtain a computationally simple control law. It is shown that when the neglected dynamics are sufficiently fast, the simplified law decouples the actual system "approximately" in a certain sense.

Finally, these results are applied to decoupled control of robotic manipulators. Two cases are considered. In the first case, actuator dynamics are completely neglected. In the second case, the dynamics of a significant class of actuators are taken into account. Our formulas for the complete class of decoupling control laws unify and generalize previous results on the decoupled control of robotic manipulators. For example, it is possible to achieve decoupled control of the end - effector.

To my late friend, **Myung Kyu Kim**

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CHAPTER 1

INTRODUCTION

In this dissertation, we present some theoretical results on nonlinear decoupling theory and discuss their applications to robotics. Let us begin the introduction with a simple discussion of the main ideas.

Suppose we have a nonlinear system :

$$(1.1) \qquad \dot{x}(t) = f(x(t), u(t)), \quad y(t) = h(x(t)),$$

where $x(t) \in R^n$, $u(t) \in R^m$, $y(t) \in R^m$. A new, closed - loop, system is obtained by using a nonlinear feedback control law $u = K(x, u)$:

$$(1.2) \qquad \dot{x}(t) = f(x(t), K(x(t), \hat{u}(t))), \quad y(t) = h(x(t)),$$

where $K : R^n \times R^m \rightarrow R^m$, $\hat{u}(t) \in R^m$. This system, with input $\hat{u}$, has different dynamics and input-output characteristics than (1.1).

Roughly speaking, the system, (1.1) is **decoupled** if for $i = 1, \ldots, m$, the ith component u_i of u effects only the ith component y_i of y. If the system (1.2) is decoupled by a control law K, we say (1.1) is **decouplable**. In particular, if the input - output map of the system is described by $y_i = \phi_i(u_1, \ldots, u_m)$, $i = 1, \ldots, m$, decoupling requires $\phi_i(u_1, \ldots, u_m) = \Psi_i(u_i)$, $i = 1, \ldots, m$. The control law K which decouples (1.1) is called a **decoupling control law**. The concept of decoupling can be easily generalized for the case $y(t) \in R^{\ell}$, where $\ell > m$ and y is partitioned into m subvectors. But, in this dissertation, we consider only the most common case, $\ell = m$.

Some applications of decoupling theory are found in robotics ([Fre.2, Fre.3, Nij.5, Sin.4, Tar.1, Yua.1]). A simple illustrative example is as follows. The rigid body equations of motion for a mechanical manipulator with D.C. motor drives can be described by

$$(1.3) \qquad M(q)\ddot{q} + F(\dot{q}, q) = u, \quad y = q,$$

where $q, u \in R^m$ and $M(q)$ is an $(m \times m)$ nonsingular matrix and we have simplified the notation by not showing the explicit dependence on t. Then, applying the nonlinear feedback control law $u = M(q)\hat{u} + F(\dot{q}, q)$ leads to a simple decoupled linear system :

$$(1.4) \qquad \ddot{q} = \hat{u}, \quad y = q.$$

The system, (1.4) may be decoupled in a stable way by a linear control law $\hat{u} = \gamma_1 \dot{q} + \gamma_2 q + \gamma_3 \tilde{u}$, where $\tilde{u}(t) \in R^m$ is the new closed-loop input and $\gamma_1, \gamma_2, \gamma_3$ are appropriate $(m \times m)$ diagonal constant matrices. If there are additional dynamics representing actuators or structural flexibility, a solution of the decoupling problem may not be so straightforward. This motivates a more general and deeper investigation into nonlinear decoupling theory.

With respect to (1.2), there are four questions of obvious importance :

(a) Under what condition, is decoupling possible ?

(b) What is the class of control laws which decouple ?

(c) What is the class of decoupled closed - loop systems ?

(d) What is the correspondence between elements of the classes mentioned in (b) and (c) ?

If a given system can be decoupled but the decoupled system is not internally stable, decoupling does not make sense. Furthermore, the decoupled system may need to have desirable input - output characteristics. These problems can be fully investigated only by characterizing the whole class of decoupling control laws. Thus, question (b) is important in decoupling theory. The questions (c), (d) are related to the structural aspects of decoupled systems.

Decoupling theory was first developed for linear systems of the form :

$$(1.5) \qquad f(x, u) \triangleq Ax + Bu, \quad h(x) \triangleq Cx$$

$$(1.6) \qquad K(x, u) \triangleq Fx + Gu,$$

where A, B, C, F, and G are constant matrices with appropriate dimensions. For question (a), Morgan [Mor.1] first presented a concrete definition of decoupling with a sufficient condition for decoupling. Then, a complete answer to question (a) was established by Falb and Wolovich ([Fab.1]). The remaining questions (b), (c), and (d) were answered first by Gilbert ([Gil.1, Gil.2]).

Wonham & Morse considered the general case, $\ell = m$, using a novel geometric approach ([Mos.1, Mos.2, Won.1, Won.2]).

The literature on nonlinear decoupling is more recent. The case which has been considered extensively is :

$$(1.7) \qquad f(x, u) \triangleq f_0(x) + \sum_{i=1}^{m} f_i(x) u_i,$$

$$(1.8) \qquad K(x, \hat{u}) \triangleq \alpha(x) + \beta(x) \hat{u},$$

where $f_i : R^n \to R^n$, $i = 0, \ldots, m$, $\alpha : R^n \to R^n$, and $\beta : R^n \to R^{m \times m}$. For this nonlinear system, the theory is still incomplete compared with linear decoupling theory. All of our following discussion applies to systems and feedback control laws of the forms (1.7), (1.8). Clearly, the system (1.3) can be written in the form (1.7) if $M(q)$ is nonsingular, $q \in R^m$.

For question (a), the earliest works are [Naz.1, Maj.1, Por.1, Sin.1] with [Cla.1, Fre.1, Sih.1] appearing later. These papers present nonlinear versions of Falb and Wolovich's necessary and sufficient condition for linear decoupling, where the definition of decouping is based on the input - output behaviour of systems. Later, authors consider decomposition of the above class of nonlinear systems ([Isi.1, Nij.2, Res.1]). Decomposition concerns dynamic structure of the systems in state space. The system, (1.1) is **decomposed** if in an appropriate system of coordinates, (1.1) appears as a system having m independent subsystems such that for the ith subsystem, the input and the output are the ith components of u and y, respectively. In other words, the system

(1.1) is decomposed if there exists a mapping $T : R^n \to R^n$ such that through the state transformation $\bar{x} = T(x)$, (1.1) is expressed as

$$(1.9) \quad \dot{\bar{x}}_i(t) = \bar{f}_i(\bar{x}_i(t)\, u_i), \quad y_i(t) = \bar{h}_i(\bar{x}(t)), \; i = 1, \ldots, m,$$
$$\dot{\bar{x}}_{m+1}(t) = \bar{f}_{m+1}(\bar{x}(t), u(t)),$$

where $\bar{x} \triangleq (\bar{x}_1, \ldots, \bar{x}_m, \bar{x}_{m+1})$. If the system (1.2) is decomposed by a control law K, we say (1.1) is **decomposable**. The control law K is called a **decomposing control law**. In [Isi.1], decomposition is called **noninteracting control**. Decomposition is a strong definition for nonlinear decoupling. It is clear that conditions for decomposition are also sufficient conditions for nonlinear decoupling. In most of the papers on decomposition, the philosophic approach is to generalize to nonlinear systems the line of attack introduced by Wonham and Morse.

For question (b), some partial results are found in [Cla.1, Fre.1, Maj.1, Sin.1, Sin.2, Sin.3, Sih.1]. The class of decoupling control laws in these papers is given by linear or nonlinear functions of outputs and their time derivatives. As will be shown later, this is not the most general form of decoupling control law. For question (c), (d), no results have been presented.

In this dissertation, we give precise definitions of decoupling and decomposition. Then, we present various detailed results concerning questions (a), (b), (c), and (d). This is our main contribution. Next, we consider "approximate decoupling" for systems with "fast" dynamics. Finally, we apply these results to decoupled control of robotic manipulators.

Now, we describe in greater detail the major contributions and organization of the dissertation. Chapter 2 contains the general mathematical background on which the development in later chapters is based. In particular, some elements of differential geometry are reviewed. For example, Lie algebraic tools are introduced because they play the same role in the treatment of nonlinear systems that linear algebra plays in the treatment of linear systems.

Chapter 3 is the main part of this dissertation. We begin by defining decoupling (Definition 3. 1. 3) and decomposition (Definition 3. 1. 5). Our definition of decoupling is based on the concept of input-output map. It is an extension of Hirshorn's definition of disturbance decoupling ([Hir.2]). An earlier origin may be found in the work of Silverman and Payne ([Sil.1]).

We give algebraic conditions for decoupling (Theorem 3. 2. 1) and decomposition (Theorem 3. 2. 2). Theorem 3. 2. 1 is a minor extension of the results on disturbance decoupling in [Hir.2, Isi.1]. But, we believe our proof is clearer and simpler. With additional steps, Theorem 3. 2. 2 is implied by arguments contained in [Isi.1]. The conditions for decomposition are more complex than those for decoupling. But, in Thorem 3. 2. 3, we present some conditions under which two concepts are equivalent.

For question (a), we prove rigorously that a nonlinear version of Falb & Wolovich's condition is both necessary and sufficient for a system to be decouplable (Theorem 3. 4. 1). This has been considered in [Maj.1, Sin.1] but with unclear proofs. The nonlinear version of Falb & Wolovich's condition is also a necessary and sufficient condition for a system to be "locally decomposable" (Theorem 3. 4. 2). The proof is a refinement of an argument contained in [Isi.1]. An important implication of Theorem 3. 3. 1 and Theorem 3. 3. 2 is that decouplability and decomposability are, under the hypotheses which they share, equivalent.

For question (b), we characterize the whole class of decoupling control laws (Theorem 3. 4. 1) and decomposing control laws (Theorem 3. 4. 2). In the case of linear systems, (1.5), (1.6), the characterizations reduce to a single result contained in [Gil.1]. Through an example (Example 3. 5. 1), we illustrate that while previous work on the class of decoupling control laws may not allow a system to be decoupled in a stable way, our characterization of the whole class of decoupling control laws may. We show that the class found by previous authors can be the whole class of decoupling control laws only under very restrictive assumptions (Remark 3. 4. 7).

For questions (c), (d), first, we introduce a standard form of decoupled systems (Definition 3. 3. 1). It is a nonlinear version of the form proposed by Gilbert ([Gil.1]) in the case of linear systems. Then, we show that a class of decoupled systems has the standard form in an appropriate state representation (Theorem 3. 3. 3 and Theorem 3. 3. 4). For this class of systems, we obtain answers to questions, (c), (d) (Theorem 3. 4. 5). The underlying idea is to characterize the whole class of the control laws which decouple or decompose the standard form.

Chapter 4 concerns approximate decoupling. We neglect fast dynamics of a system to obtain computationally simple control laws.

They decouple the simplified model but do not decouple the actual system. It is shown that when the neglected dynamics are sufficiently fast, the simplified law decouples the actual system "approximately" in a certain sense (Theorem 4. 2. 1).

In Chapter 5, the results of earlier chapters are applied to decoupled control of robotic manipulators. Two cases are considered. In Section 5. 1, actuator dynamics are completely neglected. In Section 5. 2, the dynamics of a significant class of actuators are taken into account. Our general formulas give a unified and generalized framework for previous results on the decoupled control of robotic manipulators.

Finally, Chapter 6 contains a brief summary of the results presented in the previous chapters and discuss some of their possible extensions.

CHAPTER 2

MATHEMATICAL BACKGROUND

In this chapter, we present the general mathematical background on which our development in later chapters is based. In Section 2.1, general notation and definitions of differential calculus are introduced. In Section 2.2, some basic concepts of differential geometry are introduced. Section 2.3 contains some theorems from differential geometry. Readers who are familiar with differential geometry can use this chapter as a reference for notation and proceed directly to the following chapters. For full details, see [Boo.1, Die.1, Mun.1, Wag.1, War.1].

2.1. General Notation and Definitions

Let $\mathbf{N} \triangleq \{0, 1, 2, \cdots\}$. Let $p, q \in \mathbf{N}$. Then, $\mathbf{M}_{p,q}$ denotes the set $\{ j \in \mathbf{N} : p \le j \le q \}$. For $i \in \mathbf{M}_{1,m}$, $\bar{\mathbf{M}}_i$ denotes the set $\{ j : j \in \mathbf{M}_{1,m}$ but $j \neq i \}$. The real line, its upper half line $[0, \infty)$ are denoted by R, R^+, respectively. The $(p \times p)$ identity matrix is denoted by $\mathbf{I}_p$. The transpose of a $(p \times q)$ matrix Q is Q^T.

In this paragraph, $\mathbf{X}$, $\hat{\mathbf{X}}$, $\hat{\hat{\mathbf{X}}}$ are topological spaces. Let F be a **mapping** from $\mathbf{X}$ into $\hat{\mathbf{X}}$ (if $\hat{\mathbf{X}} \triangleq R$, F is a **function**). The

image of a subset U of $\mathcal{X}$ by F, denoted by F(U), is defined by

$$F(U) \triangleq \{\hat{x} \in \hat{\mathcal{X}} : \hat{x} = F(x),\ x \in U\}. \tag{2.1.1}$$

The **inverse image** of a subset V of $\mathcal{X}$ by F, denoted by $F^{-1}(V)$, is defined by

$$F^{-1}(V) \triangleq \{x \in \mathcal{X} : \hat{x} = F(x),\ \hat{x} \in V\}. \tag{2.1.2}$$

Let H be a mapping from $\hat{\mathcal{X}}$ into $\tilde{\mathcal{X}}$. The composition of H and F, denoted by $H \circ F$, is defined by $H \circ F(x) = H(F(x))$, $x \in \mathcal{X}$. A mapping $F : \mathcal{X} \rightarrow \hat{\mathcal{X}}$ is **invertible** if there exists a mapping $G : \hat{\mathcal{X}} \rightarrow \mathcal{X}$ such that $F \circ G$ and $G \circ F$ are the identity mappings on the sets $\hat{\mathcal{X}}$, $\mathcal{X}$, respectively. Since G is unique, it is called the **inverse mapping** of F and denoted by F^{-1}. Let U be an open subset of $\mathcal{X}$. If for each open subset V of $\hat{\mathcal{X}}$, the intersection $U \cap F^{-1}(V)$ is open in U, F is **continuous** on U. If F is continuous on $\mathcal{X}$ and has a continuous inverse mapping F^{-1}, F is a **homeomorphism** on $\mathcal{X}$.

A topological space $\mathcal{X}$ is **Hausdorff** if for each pair x, z of distinct points of $\mathcal{X}$, there exist open neighborhoods U, V of x, z, respectively, that are disjoint. A topological space $\mathcal{X}$ is **connected** if the only subsets of $\mathcal{X}$ that are open and closed in $\mathcal{X}$ are the empty set and $\mathcal{X}$ itself.

The vector space of n - tuples of real numbers with componentwise addition and multiplication is denoted by R^n. The element $x \in R^n$ is written as a column vector. The transpose of x, denoted by x^T, stands for its expression as a row vector. Note that R^n with a norm $|\cdot|$ is a Banach space.

Finally, we introduce some basic definitions of differential calculus on Banach spaces, which are found in [Die.1, Wag.1]. From now on until the end of this section, $\mathcal{X}, \hat{\mathcal{X}}$ are Banach spaces. The set of all continuous linear mappings from $\mathcal{X}$ into $\hat{\mathcal{X}}$ is denoted by $\mathcal{B}(\mathcal{X} ; \hat{\mathcal{X}})$. Then, it can be shown that $\mathcal{B}(\mathcal{X} ; \hat{\mathcal{X}})$ with its induced norm, $\|\xi\| \triangleq \sup\{|\xi x| ; |x| \leq 1\}$ is a Banach space. For simplity, in the rest of this section, the norms of Banach spaces are identically denoted by $|\cdot|$. Let F be a continuous mapping from an open subset S of $\mathcal{X}$ into $\hat{\mathcal{X}}$. Let $x_0 \in S \subset \mathcal{X}$. If there exists a $v \in \mathcal{B}(\mathcal{X} ; \hat{\mathcal{X}})$ such that

$$(2.1.3) \qquad \lim_{\substack{x \to x_0 \\ x \in S - \{x_0\}}} |F(x) - F(x_0) - v(x - x_0)| / |x - x_0| = 0,$$

then, F is **differentiable** at x_0. The mapping v is usually denoted by $DF(x_0)$ and is called the **first derivative** of F at x_0 since it is actually unique. When $\mathcal{X} \triangleq R^n$ and $\hat{\mathcal{X}} \triangleq R^q$, $DF(x_0)$ is a $(q \times n)$ matrix and is called the **Jacobian matrix** of F at x_0. Then, the **rank** of F at $x_0 \in \mathcal{X}$ is the rank of its Jacobian matrix at x_0. If F is differentiable at each $x_0 \in S$, F is **differentiable on** S. Then, DF is a mapping from S into $\mathcal{B}(\mathcal{X} ; \hat{\mathcal{X}})$. If it is continuous on S, F is **continuously differentiable on** S.

Suppose that F is continuously differentiable on S and DF is differentiable at x_0. Then, F is **twice differentiable** at x_0. The derivative of DF at x_0 is called the **second derivative** of f at x_0 and denoted by $D^2F(x_0)$. If F is twice differentiable at each $x_0 \in S$, it is **twice differentiable on** S. Then, D^2F is a mapping from S into $\mathbf{B}(\mathcal{X};\mathbf{B}(\mathcal{X};\hat{\mathcal{X}}))$. If it is a continuous mapping from S into $\mathbf{B}(\mathcal{X};\mathbf{B}(\mathcal{X};\hat{\mathcal{X}}))$, F is **twice continuously differentiable on** S. Inductively, we can define higher order derivatives. The details are omitted. If F is p times continuously differentiable on S, we write $F \in C^p$ on S. Particularly, when F is continuous (infinitely continuously differentiable or smooth), we write $F \in C^0$ (C^∞) on S. If $F \in C^\infty$ on S and at each $x_0 \in S$, there exists a neighborhood U of x_0 such that U is open in S and F can be expanded on U as an infinite Taylor series, F is **real analytic** (C^ω) on S.

Now, suppose that $\mathcal{X}$ is the product space of two Banach spaces $\mathcal{X}_1$, $\mathcal{X}_2$; $\mathcal{X} \triangleq \mathcal{X}_1 \times \mathcal{X}_2$. For each $x_0 \triangleq (a_1, a_2) \in \mathcal{X}$, we can consider the partial mappings $x_1 \mapsto F(x_1, a_2)$ and $x_2 \mapsto F(a_1, x_2)$ of open subsets of $\mathcal{X}_1$ and $\mathcal{X}_2$, respectively, into $\hat{\mathcal{X}}$. If the partial mapping $x_1 \mapsto F(x_1, a_2)$ ($x_2 \mapsto F(a_1, x_2)$) is differentiable at a_1 (a_2), F is **differentiable with respect to the first (second) argument** at x_0. The derivative of that mapping, which is an element of $\mathbf{B}(\mathcal{X}_1 ; \hat{\mathcal{X}})$ ($\mathbf{B}(\mathcal{X}_2 ; \hat{\mathcal{X}})$) is called the

first partial derivative of F at x_0 with respect to the first(second) argument and written as $D_1F(a_1, a_2)$ or $(\partial F/\partial x_1)_{X=X_0}$ ($D_2F(a_1,a_2)$ or $(\partial F/\partial x_2)_{X=X_0}$). Inductively, we can define the second and higher order partial derivatives. Details are omitted. Note that $D_iD_jF(x_1, x_2) \in \mathcal{B}(\mathcal{X}_i ; \mathcal{B}(\mathcal{X}_j ; \hat{\mathcal{X}}))$, i, j $\in \mathcal{M}_{1,2}$. Note that any bilinear mapping in $\mathcal{B}(\mathcal{X}_i ; \mathcal{B}(\mathcal{X}_j ; \hat{\mathcal{X}}))$ can be identified with a bilinear mapping in $\mathcal{B}(\mathcal{X}_i \times \mathcal{X}_j ; \hat{\mathcal{X}})$, i, j $\in \mathcal{M}_{1,2}$. Therefore, $D_iD_jF(x_1, x_2)$ is written $(v, w) \mapsto D_iD_jF(x_1, x_2)\,[v]\,[w]$. In particular when n_1, $n_2 \in \mathcal{N}$, $\mathcal{X}_1 \triangleq R^{n_1}$, $\mathcal{X}_2 \triangleq R^{n_2}$, and $\hat{\mathcal{X}} \triangleq R$, we have for i, j $\in \mathcal{M}_{1,2}$,

(2.1.4) $$D_iD_jF(x_1, x_2)\,[v]\,[w] = w^T D_i(D_jF(x_1, x_2))^T v,$$

where $D_i(D_jF(x_1, x_2))^T$ is an $(n_j \times n_i)$ matrix.

Now, let [0, L) be an interval of the real line R. Suppose that there is a partition of [0, L) such that $0 = t_0 < t_1 < \cdots < t_q = L$. Let F be a mapping from [0, L) into $\mathcal{X}$. Let i $\in \mathcal{M}_{1,q}$. The mapping F has an **extension** (F_i, U_i) on the interval $[\, t_{(i-1)}, t_i)$ if there exists a mapping F_i from an open interval U_i into $\mathcal{X}$ such that $[\, t_{(i-1)}, t_i) \subset U_i$ and $F_i(t) = F(t)$, $t \in [\, t_{(i-1)}, t_i)$. The mapping F is **piecewise C^0** on [0, L) if on each interval $[\, t_{(i-1)}, t_i)$, i $\in \mathcal{M}_{1,q}$, it has an extension (F_i, U_i) such that F_i is bounded and C^0 on U_i. More generally, when on each interval $[\, t_{(i-1)}, t_i)$, i $\in \mathcal{M}_{1,q}$, F has an extension (F_i, U_i) such that F_i is bounded and C^p (C^∞) [C^ω] instead of C^0 on U_i, F is **piecewise C^p (C^∞) [C^ω]** on [0, L). Note that piecewise C^∞ and C^ω mappings can actually be discontinuous at the points t_i. Thus, derivatives of F in usual sense are not defined at the points t_i. However, we will find it convenient to define the kth derivative of F, D^kF in the following way : for t ε [$t_{(i-1)}$, t_i), $D^kF(t) = D^kF_i(t)$.

2. 2. Basic Concepts of Differential Geometry

A **manifold $\mathcal{X}$ of dimension n , or n - dimensional manifold** is a topological space with the following properties :

(1) $\mathcal{X}$ is Hausdorff,

(2) At each p ε $\mathcal{X}$, there is a pair (U, ϕ) such that U is an open neighborhood of p and ϕ is a homeomorphism from U onto an open subset of R^n ,

(3) $\mathcal{X}$ has a countable basis of open sets.

The pair (U, ϕ) is called a **coordinate neighborhood** or **chart** . Charts (U, ϕ), (V, ψ) are **C^∞- compatible** if $U \cap V$ nonempty implies that composite functions $\phi \circ \psi^{-1}$, $\psi \circ \phi^{-1}$ are C^∞ - diffeomorphisms of the open subsets $\phi(U \cap V)$ and $\psi(U \cap V)$ of R^n. A **smooth structure** or C^∞- **atlas** on a manifold $\mathcal{X}$ is a family $\mathcal{A}$ = { U_α, ϕ_α } of charts such that

(1)' The U_α cover $\mathcal{X}$, (2)' For any α, β, the charts (U_α, ϕ_α) and (V_β, ψ_β) are C^∞- compatible,

(3)' The collection $\mathcal{A}$ is maximal : any chart (V, ψ) C^∞ - compatible with every (U, ϕ) $\in \mathcal{A}$ is itself in $\mathcal{A}$.

A **smooth manifold** is a manifold with a C^∞- atlas.

If in the previous paragraph we replace "C^∞ and smooth" by C^ω and real analytic", we obtain the definitions of **C^ω - compatibility, C^ω - atlas,** and **real analytic manifold** instead of C^∞ - compatibility, C^∞- atlas, and smooth manifold. Clearly, any open subset of R^n is a real analytic manifold.

Let $\mathcal{X}$, $\hat{\mathcal{X}}$ be smooth manifolds of dimension n, m, respectively. A mapping T from $\mathcal{X}$ into $\hat{\mathcal{X}}$ is C^∞ if for each p $\in \mathcal{X}$, there exist charts (U, ϕ) of p and (V, ψ) of T(p) with T(U) $\subset$ V such that the mapping $\tilde{T} \triangleq \psi \circ T \circ \phi^{-1}$ from ϕ(U) into ψ(V) is C^∞ in the sense defined in Section 2. 1. The **rank** of T at p is the rank of $\tilde{T}$ at ϕ(p) (see Section 2. 1). Note that the rank is independent of the choice of charts. If T : $\mathcal{X} \to \hat{\mathcal{X}}$ is a homeomorphism and $T^{-1} : \hat{\mathcal{X}} \to \mathcal{X}$ is C^∞, it is a **C^∞- diffeomorphism** on $\mathcal{X}$. Suppose that $n \le m$. If T : $\mathcal{X} \to \hat{\mathcal{X}}$ has a rank n at all points of $\mathcal{X}$, it is an **immersion** of $\mathcal{X}$ in $\hat{\mathcal{X}}$. If T : $\mathcal{X} \to \hat{\mathcal{X}}$ is an immersion and one - to - one on $\mathcal{X}$, it is an **one - to - one immersion**.

Let $\mathcal{S}$ be a set. An n - ary operation on $\mathcal{S}$ is a mapping from $\mathcal{S}^n$ into $\mathcal{S}$. A system consisting of a set and one or more n-ary operations on the set is an **algebraic system** or simply **algebra**. An algebraic system is usually denoted by $\langle \mathfrak{S}, f_1, \cdots, f_k \rangle$ where $\mathfrak{S}$ is a nonempty set and $f_1, \cdots, f_k$ are operations on $\mathfrak{S}$.

Given any point $p \in \mathfrak{X}$, we define $\langle \mathfrak{C}^\infty(p), +, \cdot \rangle$ as the algebra of C^∞-functions whose domain of definition includes some open neighborhood of p. Here, the binary operations $+$, $\cdot$ are the usual addition, multiplication of two functions, respectively. Any two functions are considered equal if they agree on any open neighborhood of p.

We define the **tangent space** $\mathfrak{T}_p(\mathfrak{X})$ to $\mathfrak{X}$ at $p \in \mathfrak{X}$ to be the set of all mappings $Y_p : \mathfrak{C}^\infty(p) \to R$ satisfying, for any $\phi, \psi \in \mathfrak{C}^\infty(p)$, the following three conditions:

(i) $Y_p(\phi + \psi) = Y_p\phi + Y_p\psi$,

(ii) $Y_p\phi = 0$ if ϕ is a constant mapping,

(iii) $Y_p(\phi \cdot \psi) = (Y_p\phi) \cdot \psi(p) + (Y_p\psi) \cdot \phi(p)$,

with the vector space operations in $\mathfrak{T}_p(\mathfrak{X})$ over R defined by $(Y_p + Z_p)\phi \triangleq Y_p\phi + Z_p\phi$, $(aY_p)\phi \triangleq a(Y_p\phi)$ for Y_p, $Z_p \in \mathfrak{T}_p(\mathfrak{X})$ and for $a \in R$.

A **tangent vector** to $\mathfrak{X}$ at $p \in \mathfrak{X}$ is any $Y_p \in \mathfrak{T}_p(\mathfrak{X})$.

A **cotangent space** $\mathfrak{T}_p^*(\mathfrak{X})$ to $\mathfrak{X}$ at $p \in \mathfrak{X}$ is the dual space to $\mathfrak{T}_p(\mathfrak{X})$ at $p \in \mathfrak{X}$, defined by the set of all linear mappings σ_p from $\mathfrak{T}_p(\mathfrak{X})$ into R with the vector space operations in $\mathfrak{T}_p^*(\mathfrak{X})$

such that for $Y_p \in \mathcal{T}_p(\mathcal{X})$, $a \in R$, and σ_p, $\zeta_p \in \mathcal{T}_p{}^*(\mathcal{X})$,

(i)' $(\sigma_p + \zeta_p) Y_p = \sigma_p Y_p + \zeta_p Y_p$,

(ii)' $(a\sigma_p) Y_p = a\, \sigma_p Y_p$.

A **cotangent vector** to $\mathcal{X}$ at $p \in \mathcal{X}$ is any $\sigma_p \in \mathcal{T}_p{}^*(\mathcal{X})$.

A **vector field** Y on $\mathcal{X}$ is a mapping assigning to each point $p \in \mathcal{X}$ a tangent vector $Y_p \in \mathcal{T}_p(\mathcal{X})$. A **covector field** or **one form** σ on $\mathcal{X}$ is a function assigning to each point $p \in \mathcal{X}$ a cotangent vector $\sigma_p \in \mathcal{T}_p{}^*(\mathcal{X})$.

Any function ϕ from $\mathcal{X}$ into R defines a covector field, denoted by $d\phi$, on $\mathcal{X}$ by the formula :

(2.2.1) $\quad d\phi_p Y_p = Y_p \phi, \quad p \in \mathcal{X}$ for any vector field Y on $\mathcal{X}$.

This covector field $d\phi$ is called the **differential** of ϕ and $d\phi_p$, its value at p, the differential of ϕ at p. We may often write as Y(p) (σ(p)) the tangent vector Y_p (the cotangent vector σ_p) assigned to a point $p \in \mathcal{X}$ by a vector field Y (a covector field σ). Similarly, we often write $d\phi(p)$, $Y\phi(p)$, $\sigma Y(p)$ instead of $d\phi_p$, $Y_p\phi$, $\sigma_p Y_p$.

The vector fields Y_i, $i \in \mathcal{N}_{1,k}$ on $\mathcal{X}$ are **linearly independent** on an open subset W of $\mathcal{X}$ if at each $p \in W$, the tangent vectors $(Y_i)_p$, $i \in \mathcal{M}_{1,k}$ are linearly independent. The covector fields σ_i, $i \in \mathcal{M}_{1,k}$ on $\mathcal{X}$ are **linearly independent on** an open subset W of $\mathcal{X}$ if at each $p \in W$, the cotangent vectors $(\sigma_i)_p$, $i \in \mathcal{M}_{1,k}$ are linearly independent. Note that if the vector fields are not linearly independent at a point $p \in \mathcal{X}$, they are not linearly independent on any open neighborhood of p. Let ϕ_i, $i \in \mathcal{M}_{1,k}$ be C^1-functions from $\mathcal{X}$ into R. Let $\phi \triangleq (\phi_1, \cdots, \phi_k)$. The functions ϕ_i, $i \in \mathcal{M}_{1,k}$ are **functionally independent** ([Gou.1, Hil.1]) on an open set W of $\mathcal{X}$ if there does not exist any C^1-function $\psi : \phi(W) \to R$ such that $\psi \circ \phi(x) = 0$, $x \in W$ but ψ is not identically zero on $\phi(W)$. It is easy to show that if $d\phi_i$, $i \in \mathcal{M}_{1,k}$ are linearly independent on W, then, ϕ_i, $i \in \mathcal{M}_{1,k}$ are functionally independent on W. But, the converse statement is not necessarily true. A simple example to show this is $\phi_1(x_1,x_2) = x_1 \sin x_2$, $\phi_2(x_1,x_2) = x_1 \cos x_2$, $(x_1, x_2) \in R^2$.

Let T be a C^∞-mapping from an n-dimensional smooth manifold $\mathcal{X}$ into an m-dimensional smooth manifold $\hat{\mathcal{X}}$. For each $p \in \mathcal{X}$, it induces a linear mapping $T_{*p} : \mathcal{T}_p(\mathcal{X}) \to \mathcal{T}_{T(p)}(\hat{\mathcal{X}})$, defined by

$$(2.2.2) \qquad T_{*p}(Y_p)\hat{\phi} = Y_p(\hat{\phi} \circ T) \quad \text{for } \hat{\phi} \in \mathbf{C}^\infty(T(p)), \quad Y_p \in \mathcal{T}_p(\mathcal{X}).$$

The mapping T_{*p} is often called the **differential** of T at p and denoted by dT_p.

The dimensions of the tangent space $\mathcal{T}_p(\mathcal{X})$ and the cotangent space $\mathcal{T}_p^*(\mathcal{X})$ to $\mathcal{X}$ at $p \in \mathcal{X}$ are the same as that of $\mathcal{X}$. Therefore, at each $p \in \mathcal{X}$, there is a set of n linearly independent tangent vectors which span $\mathcal{T}_p(\mathcal{X})$. It is called a **basis** of $\mathcal{T}_p(\mathcal{X})$. Since $\mathcal{T}_p^*(\mathcal{X})$ is the dual space to $\mathcal{T}_p(\mathcal{X})$, the basis of $\mathcal{T}_p^*(\mathcal{X})$ is uniquely determined by a basis of $\mathcal{T}_p(\mathcal{X})$. So, it is called the **dual basis** of $\mathcal{T}_p^*(\mathcal{X})$. Suppose that $\mathcal{X}$ is an open subset of R^n. Let $(x_1, \cdots, x_n)$ be the coordinate vector. Then, the n vector fields $\partial/\partial x_i$, $i \in \mathcal{M}_{1,n}$ are linearly independent on $\mathcal{X}$ and at each point $p \in \mathcal{X}$, $\{ (\partial/\partial x_i)_p,\ i \in \mathcal{M}_{1,n} \}$ is a basis of $\mathcal{T}_p(\mathcal{X})$. We call this basis a **canonical basis.** The **canonical dual basis** $\{ dx_i,\ i \in \mathcal{M}_{1,n} \}$ is determined by

(2.2.3) $$(dx_i)_p (\partial/\partial x_j)_p \triangleq (\partial x_i/\partial x_j)_p = \delta_{i,j},\ p \in \mathcal{X},\ i, j \in \mathcal{M}_{1,n},$$

where $\delta_{i,j} \triangleq 1$ if $i = j$, $\delta_{i,j} \triangleq 0$, otherwise.

Now, consider an n - dimensional manifold $\mathcal{X}$ which is not necessarily an open subset of R^n. Let (ϕ, U) be a chart at $p \in \mathcal{X}$. Then, by the definition of smooth manifold and the observations in the previous paragraph, it follows that there exist vector fields E_i, $i \in \mathcal{M}_{1,n}$ on U such that

(a) E_i, $i \in \mathcal{M}_{1,n}$ are linearly independent on U,

(b) At each $q \in U$, $\{(E_i)_q,\ i \in \mathcal{M}_{1,n}\}$ is a basis of $\mathcal{T}_q(\mathcal{X})$. One possible choice is

$$(2.2.4)\qquad (E_i)_q \triangleq \phi^{-1}_{*\phi(q)}((\partial/\partial x_i)_{\phi(q)}),\quad i \in \mathcal{M}_{1,n},\quad q \in U.$$

Correspondingly, there exist n covector fields ω_i, $i \in \mathcal{M}_{1,n}$, on U such that

(a)' $(\omega_i)_q (E_j)_q = \delta_{i,j}$, $\quad q \in U$, $\quad i,j \in \mathcal{M}_{1,n}$,

(b)' At each $q \in U$, $\{(\omega_i)_q,\ i \in \mathcal{M}_{1,n}\}$ is a basis of $\mathcal{T}^*_q(\mathcal{X})$.

Using the above notations, any vector field Y and covector field σ on $\mathcal{X}$ can be locally represented, respectively, by

$$(2.2.5)\qquad Y_q \triangleq \sum_{i=1}^{n} a_i(q)\,(E_i)_q,\quad q \in U,$$

$$(2.2.6)\qquad \sigma_q \triangleq \sum_{i=1}^{n} b_i(q)\,(\omega_i)_q,\quad q \in U,$$

where a_i, b_i are functions from U into R. When $\mathcal{X}$ is an open subset of R^n, any vector field Y and covector field σ on $\mathcal{X}$ are globally identified by (2.2.5) and (2.2.6) with $E_i = \partial/\partial x_i$, $\omega_i = dx_i$, $i \in \mathcal{M}_{1,n}$. When the a_i, b_i are C^∞ on U, Y and σ are, respectively, a **C^∞- vector field** on U and a **C^∞- covector field** on U. If at each $p \in \mathcal{X}$, there exists a chart (U, ϕ) such that Y, σ are C^∞ on U, they are respectively a C^∞- vector field on $\mathcal{X}$ and a C^∞- covector field on $\mathcal{X}$.

If in the previous paragraph, C^∞ is replaced by C^ω and $\mathcal{X}$

is a real analytic manifold, then Y, σ are a C^ω - **vector field**, a C^ω - **covector field**, respectively.

A **Lie algebra** is a vector space $\mathbf{L}$ over R which, in addition to its vector space structure, possesses a product [,] satisfying the following properties:

(1) $\mathbf{L}$ is closed under the product : $[Y, Z] \in \mathbf{L}$ if $Y, Z \in \mathbf{L}$,

(2) The product is bilinear over R : for a, b $\in$ R and for X, Y, Z $\in \mathbf{L}$,

$$[aX + bY, Z] = a[X, Z] + b[Y, Z], \tag{2.2.7}$$

$$[X, aY + bZ] = a[X, Y] + b[X, Z], \tag{2.2.8}$$

(3) The product is skew commutative :

$$[Z, Y] = -[Y, Z] \text{ for } Y, Z \in \mathbf{L}, \tag{2.2.9}$$

(4) The product satisfies the Jacobi identity:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0. \tag{2.2.10}$$

Let $\mathbf{C}^\infty(\mathcal{X})$ ($\mathbf{C}^\omega(\mathcal{X})$) be the set of all C^∞ (C^ω) - functions from $\mathcal{X}$ into R. Let $V^\infty(\mathcal{X})$ ($V^\omega(\mathcal{X})$) be the set of all C^∞ (C^ω) - vector fields on $\mathcal{X}$. Then, $V^\infty(\mathcal{X})$ is a vector space over R and a module over $C^\infty(\mathcal{X})$. We define a product [Y, Z] of Y, Z $\in V^\infty(\mathcal{X})$ by

$$[Y, Z]_p \phi = Y_p(Z\phi) - Z_p(Y\phi), \quad \phi \in \mathbf{C}^\infty(p), \quad p \in \mathcal{X} \tag{2.2.11}$$

The product defined by (2.2.11) is called the **Lie bracket** of Y, Z. Clearly, it satisfies the properties (1) - (4) in the previous paragraph and so $V^{\infty}(\mathcal{X})$ with the Lie bracket is a **Lie algebra**. Let (U, ϕ) be a chart at $p \in \mathcal{X}$. Then, for $Y \triangleq \sum_{i=1}^{n} a_i(\cdot) E_i$, $Z \triangleq \sum_{i=1}^{n} b_i(\cdot) E_i$, $[Y, Z]$ can be expressed locally on U as

$$(2.2.12) \quad [Y, Z]_q = \sum_{i=1}^{n} \Big[\sum_{j=1}^{n} a_j(q) (E_j)_q b_i - \sum_{j=1}^{n} b_j(q) (E_j)_q a_i \Big] (E_i)_q, \quad q \in U.$$

Of course, when $\mathcal{X}$ is an open subset of R^n, this expression holds globally on $\mathcal{X}$ with $E_i = \partial/\partial x_i$, $i \in \mathbf{n}_{1,n}$.

If a subset E of $V^{\infty}(\mathcal{X})$ is closed under the Lie bracket, it is **involutive**. A **subalgebra** of $V^{\infty}(\mathcal{X})$ is an involutive linear subspace of $V^{\infty}(\mathcal{X})$ over R. A **distribution** Δ on $\mathcal{X}$ is a mapping which assigns to each $p \in \mathcal{X}$, a subspace Δ_p of $T_p(\mathcal{X})$. If Y is a vector field such that $Y_p \in \Delta_p$, $p \in \mathcal{X}$, we write $Y \in \Delta$ on $\mathcal{X}$. A distribution Δ on $\mathcal{X}$ is **involutive** if for all vector fields Y, Z such that $Y, Z \in \Delta$ on $\mathcal{X}$, $[Y, Z] \in \Delta$ on $\mathcal{X}$. If dimension of Δ_p is k, $p \in \mathcal{X}$, Δ has a **dimension** k on $\mathcal{X}$. A distribution Δ on $\mathcal{X}$ is C^{∞} (C^{ω}) if at each $p \in \mathcal{X}$, there exist an open neighborhood U and k(p) linearly independent C^{∞} (C^{ω}) - vector fields Y_i, $i \in \mathbf{n}_{1,k}$ on U such that at each point $q \in U$, $\{(Y_i)_q, i \in \mathbf{n}_{1,k}\}$ spans Δ_q. Note that a subspace of $V^{\infty}(\mathcal{X})$ (e.g. a subalgebra of $V^{\infty}(\mathcal{X})$) generates a C^{∞} - distribution on $\mathcal{X}$. Thus, the subspace may have a dimension on $\mathcal{X}$. Note this dimension is a pointwise concept, not a function space concept. Note that in general, the dimension of a C^∞-distribution Δ may not be defined on $\mathcal{X}$ (the number of basis vectors for Δ_q may depend on q). Let $V_*^\infty(\mathcal{X})$ be the set of all C^∞-covector fields on $\mathcal{X}$. The C^∞-codistribution $\Delta^\perp$ of a C^∞-distribution Δ on $\mathcal{X}$ is defined by

$$(2.2.13) \quad \Delta_p^\perp \triangleq \{ \sigma_p \in T_p^*(\mathcal{X}) : \sigma_p Y_p = 0,\ Y_p \in \Delta_p \},\ p \in \mathcal{X}.$$

Let Z be a vector field on $\mathcal{X}$. A distribution Δ on $\mathcal{X}$ is Z - **invariant** if $[Z, Y] \in \Delta$ whenever $Y \in \Delta$. The codistribution $\Delta^\perp$ of a distribution Δ on $\mathcal{X}$ is Z - **invariant** if for any $\phi \in \mathbf{C}^\infty(\mathcal{X})$, $d\phi \in \Delta^\perp$ always implies that $dZ\phi \in \Delta^\perp$. Simple calculations show that a distribution Δ on $\mathcal{X}$ is Z - invariant if and only if the codistribution $\Delta^\perp$ of Δ is Z - invariant.

Let T be a C^∞-mapping from an n - dimensional smooth manifold $\mathcal{X}$ into an m - dimensional smooth manifold $\mathcal{X}$. A vector field $\hat{Y} \in V^\infty(\hat{\mathcal{X}})$ is T - **related on** $\mathcal{X}$ to a vector field $Y \in V^\infty(\mathcal{X})$ if

$$(2.2.14) \quad \hat{Y}_{T(p)}\hat{\phi} = Y_p(\hat{\phi} \circ T), \quad \hat{\phi} \in \mathbf{C}^\infty(T(p)), \quad p \in \mathcal{X}.$$

A function $\hat{\phi} \in \mathbf{C}^\infty(\hat{\mathcal{X}})$ is T - **related** on $\mathcal{X}$ to a function $\phi \in \mathbf{C}^\infty(\mathcal{X})$ if

$$(2.2.15) \quad \phi(p) = \hat{\phi} \circ T(p), \quad p \in \mathcal{X}.$$

Other definitions such as integral curve and Lie derivative will be introduced in the next section.

2. 3. Some Fundamental Results

We state some well-known results without proofs. As in Section 2. 2, $\mathcal{X}$, $\hat{\mathcal{X}}$ are smooth manifolds of dimensions n, m, respectively. Using the definitions in Section 2. 2, the following facts may be easily verified.

Fact 2. 3. 1 ([Boo.1], p. 155). For $\phi, \psi \in \mathbf{C}^{\infty}(\mathcal{X})$ and Y, Z $\in$ $V^{\infty}(\mathcal{X})$, the following equality holds for all p $\in$ $\mathcal{X}$.

$$(2.3.1) \quad [\,\phi Y, \psi Z\,]_p = \phi(p)\,\psi(p)\,[\,Y, Z\,]_p + \phi(p)\,(Y_p\psi)\,Z_p - \psi(p)\,(Z_p\phi)\,Y_p. \quad \square$$

Fact 2. 3. 2 ([Boo.1], p. 154). If $\hat{Y}, \hat{Z} \in V^{\infty}(\hat{\mathcal{X}})$ are T-related on $\mathcal{X}$ to Y, Z $\in$ $V^{\infty}(\mathcal{X})$, respectively, then $[\,\hat{Y}, \hat{Z}\,]$ is T-related on $\mathcal{X}$ to $[\,Y, Z\,]$. $\square$

Fact 2. 3. 3. If $\hat{\phi} \in \mathbf{C}^{\infty}(\hat{\mathcal{X}})$, $\hat{Y} \in V^{\infty}(\hat{\mathcal{X}})$ are T-related on $\mathcal{X}$ to $\phi \in \mathbf{C}^{\infty}(\mathcal{X})$, Y $\in$ $V^{\infty}(\mathcal{X})$, respectively, then $\hat{Y}\hat{\phi}$ is T-related on $\mathcal{X}$ to $Y\phi$. $\square$

Let X $\in$ $V^{\infty}(\mathcal{X})$. If a C^{∞}-mapping F from an open interval J of R into $\mathcal{X}$ satisfies

(2.3.2) $\quad F_{*_t}((\partial/\partial\tau)_t) = X_{F(t)}, \quad t \in J,$

the mapping F is an **integral curve** of X. Customarily, we write $\dot{F}(t)$ instead of $F_{*_t}((\partial/\partial\tau)_t)$. The following theorem is concerned about the existence of integral curves for a given vector field X. It is essentially a restatement of the existence theorem for ordinary differential equations.

Theorem 2.3.1 ([Boo.1], p. 132). Let $X \in V^{\infty}(\mathcal{X})$. Then, for each $p \in \mathcal{X}$, there exist an open neighborhood U of p, a real number $\delta(p) > 0$, and a C^{∞}-mapping $\theta^X : (-\delta, \delta) \times U \to \mathcal{X}$ satisfying

(2.3.3) $\quad \dot{\theta}^X(t,q) = X_{\theta^X(t,q)}, \quad \theta^X(o, q) = q, \; q \in U.$ □

When we emphasize that $\theta^X(t, q)$ is a function of q for a fixed time $t \in R$, we may write $\theta^X_t(q)$.

Theorem 2.3.2 ([Boo.1], p. 133). Let $X \in V^{\infty}(\mathcal{X})$. Then, for each $p \in \mathcal{X}$, there is a maximal open interval $I_p \triangleq \{ a(p) < t < b(p) \}$ containing $t = 0$, on which the integral curve $\theta^X(\cdot, p)$ of X passing through p at $t = 0$ is defined. Moreover, the integral curve $\theta^X(\cdot, p)$ of X passing through p at $t = 0$ is unique on I_p. □

By Theorem 2.3.2, for each $t \in R$, we can define a subset $\mathfrak{D}_t^X$ of $\mathcal{X}$ by

(2.3.4) $\quad \mathfrak{D}_t^X \triangleq \{ p \in \mathcal{X} : t \in I_p \}.$

A vector field X on $\mathcal{X}$ is **complete** if $\mathfrak{D}_t^X = \mathcal{X}$ for all $t \in R$.

Theorem 2.3.3 ([War.1], p. 37). Let $X \in V^{\infty}(\mathcal{X})$. Then, the following properties hold.

(i) $\mathfrak{D}_t^X$ is open for each $t \in R$,

(ii) $\bigcup_{t>0} \mathfrak{D}_t = \mathcal{X}$,

(iii) For each $t \in R$, θ_t^X is a C^{∞}- diffeomorphism from $\mathfrak{D}_t^X$ onto $\mathfrak{D}_{-t}^X$ with inverse θ_{-t}^X,

(iv) On the domain of $\theta_s^X \circ \theta_t^X$,

(2.3.5) $\quad \theta_s^X \circ \theta_t^X = \theta_{t+s}^X.$ □

Vector fields can be differentiated with respect to a vector field. The vector field $L_Y Z$, called the **Lie derivative** of Z with respect to Y at $p \in \mathcal{X}$ is defined by

(2.3.6) $\quad (L_Y Z)_p = \lim_{t \to 0} [(\theta_{-t}^Y)_{*\theta_t^Y(p)} (Z_{\theta_t^Y(p)}) - Z_p] / t.$

The following result connects the Lie bracket with the Lie derivative we defined just above.

Theorem 2. 3. 4 ([Boo.1], p. 153). Let $Y, Z \in V^{\infty}(\mathcal{X})$. Then,

$$(2.3.7) \qquad (L_Y Z)_p = [\,Y, Z\,]_p\,, \quad p \in \mathcal{X}. \qquad \square$$

By Theorem 2. 3. 4, we shall confuse $L_Y Z$ with $[\,Y, Z\,]$ and define the successive Lie brackets of $Y, Z \in V^{\infty}(\mathcal{X})$ by

$$(2.3.8) \qquad L_Y^k Z \triangleq [\,Y, L^{(k-1)} Z\,], \quad k \in \mathcal{N}_{1,\infty},$$

where

$$(2.3.9) \qquad L_Y^0 Z \triangleq Z.$$

Next, consider

Theorem 2. 3. 5 (Cambell - Baker - Hausdorff Formula). Let Y, Z be C^{ω}- vector fields on $\mathcal{X}$. Then, at each $p \in \mathcal{X}$, there exists a real number $\delta > 0$ such that

$$(2.3.10) \qquad (\theta_t^Y)_{*_p} Z_p = \sum_{k=0}^{\infty} \{\,(-t)^k / k\,!\,\}\,(L_Y^k Z)_{\theta_t^Y(p)}, \quad t \in (\,-\delta\,,\,\delta\,). \qquad \square$$

Although this results appears in many places, it is remarkable that no proof is given in the standard references. The proof follows from Theorem 2. 3. 4 and, while not exactly obvious, is not too difficult. Note that if Y, Z are C^∞, this formula does not necessarily hold. Most of all the results which we derive in future, where real analyticity is required, come from Theorem 2.3.5.

Now, we state two Inverse Function Theorems, the Constant Mapping Theorem, and the Frobenius Theorem.

Theorem 2.3.6 (**Local Inverse Function Theorem**, [War.1], p. 30). Let T be C^∞ (C^ω)- mapping from $\mathfrak{X}$ into $\hat{\mathfrak{X}}$. Suppose that at a point p $\in \mathfrak{X}$, T_{*_p} is an isomorphism from $\mathfrak{T}_p(\hat{\mathfrak{X}})$ onto $\mathfrak{T}_{T(p)}(\mathfrak{X})$. Then, there is an open neighborhood U of p such that T is a C^∞ (C^ω) - diffeomorphism from U onto the open subset T(U) of $\mathfrak{X}$. □

Theorem 2.3.7 ([Gui.1], p. 18). Let T be a C^∞ (C^ω) - mapping from an open subset $W \subset \mathfrak{X}$ into $\hat{\mathfrak{X}}$. Then, T is a C^∞ (C^ω) - diffeomorphism from W onto $T(W) \subset \hat{\mathfrak{X}}$ if and only if

(1) At each point p $\in$ W, T_{*_p} is an isomorphism from $\mathfrak{T}_p(\mathfrak{X})$ onto $\mathfrak{T}_{T(p)}(\hat{\mathfrak{X}})$,

(2) T is one - to - one on W. □

Theorem 2.3.8 (**The Constant Mapping Theorem**, [War.1], p.18). Let T be a C^∞- mapping from $\mathfrak{X}$ into $\hat{\mathfrak{X}}$. Suppose that $\mathfrak{X}$ is connected and $T_{*_p} = 0$, p $\in \mathfrak{X}$. Then, there exists a constant c $\in$

R such that

$$(2.3.11) \quad T(x) = c, \; x \in \mathcal{X}. \qquad \square$$

Before we state the Frobenius Theorem, we define an (immersed) submanifold and integral manifold. Let W be a subset of $\mathcal{X}$. W is an **(immersed) submanifold** of $\mathcal{X}$ if there exist an r - dimensional smooth manifold $\mathcal{N}$ and an one - to - one immersion $T : \mathcal{N} \to \mathcal{X}$ such that $r \leq n$ and $W = T(\mathcal{N})$. An **integral manifold** of a C^∞- distribution Δ is a connected submanifold $\mathcal{E}$ of $\mathcal{X}$ with the property that $\Delta_p = T_p(\mathcal{E})$, $p \in \mathcal{E}$. For a more general definition of integral manifold, see [Boo.1]. A C^∞- distribution Δ on $\mathcal{X}$ of dimension k is **completely integrable** on $\mathcal{X}$ if each point $p \in \mathcal{X}$ has a chart (U, ϕ) such that the k vector fields $E_i \triangleq d\phi^{-1}(\partial/\partial x_i)$, $i \in \mathbf{n}_{1,k}$ are a local basis on U for Δ, where $x_1, \cdots, x_n$ are the local coordinates. In this case, an integral manifold $\mathcal{E}$ of Δ through $q \in U$ is

$$(2.3.12) \quad \mathcal{E} \triangleq \phi^{-1}(\{ x \in \phi(U) : x_{k+1} = a_{k+1}, \cdots, x_n = a_n \}),$$

where $(a_1, \cdots, a_n) \triangleq \phi(q)$.

Theorem 2. 3. 9 (Local Frobenius Theorem, [Boo.1], p. 159). Let Δ be a C^∞- distribution on $\mathcal{X}$ with dimension k. Then, Δ is involutive on $\mathcal{X}$ if and only if it is completely integrable on $\mathcal{X}$. $\square$

CHAPTER 3

NONLINEAR DECOUPLING THEORY

This chapter contains results on decoupling and decomposition. In Section 3.1, further notation and definitions on systems are introduced on the basis of the general notation and definitions in Chapter 2. In particular, the precise definitions of decoupling (Definition 3. 1. 3) and decomposition (Definition 3. 1. 5) are proposed. In Section 3. 2, we present the results on decoupling (Theorem 3. 2. 1) and decomposition (Theorem 3. 2. 2 and Theorem 3. 2. 3). In Section 3. 3, the results on decomposability (Theorem 3. 3. 1), decouplability (Theorem 3. 3. 2), and the standard decomposed system (Theorem 3. 3. 3 and Theorem 3. 3. 4) are presented. In Section 3. 4, we characterize the whole class of control laws which decouple or decompose nonlinear systems (Theorem 3. 4. 1 - Theorem 3. 4. 4). Then, for a class of nonlinear systems, we discuss the class of closed - loop decoupled systems generated by the whole class of decoupling control laws (Theorem 3. 4. 5). In Section 3. 5, three examples are considered which illustrate the significance of the results developed in the previous sections. Section 3. 6 makes comments on the results discussed in this chapter.

3. 1. Definitions

Recall the system (1.1), (1.7) of Chapter 1. We now give it an alternative abstract formulation. For each $i \in \mathbf{M}_{0,m}$, we may view f_i as the coordinate representation of a vector field X_i on R^n in the canonical basis $\{\partial/\partial x_j,\ j \in \mathbf{M}_{1,n}\}$ such that

$$(3.1.1) \qquad X_i \triangleq \sum_{j=1}^{n} f_{i,j}(\cdot)\ \partial/\partial x_j,$$

where $f_{i,j}$ is the jth component of f_i, $j \in \mathbf{M}_{1,n}$. Then, we can write (1.1), (1.7) as the vector field representation :

$$(3.1.2) \qquad \dot{x} = F(x, u) \triangleq X_0(x) + \sum_{i=1}^{m} X_i(x)\, u_i, \quad y = H(x).$$

Here, $x(t) \in R^n$, $\dot{x}$ is interpreted as $\dot{x}(t) = x_{*t}((\partial/\partial\tau)_t)$, $u_i(t) \in R$ is the ith component of u, and $H = h$. Conversely, suppose that in (3.1.2), X_i, $i \in \mathbf{M}_{0,m}$ are vector fields defined, more generally, on an n-dimensional manifold $\mathcal{X}$. Then, at each $p \in \mathcal{X}$, there exists a chart (U, ϕ) such that in the coordinates ϕ, the system (3.1.2) has the form of (1.1), (1.7). We shall denote by $\{F, H, \mathcal{X}\}$ the system (3.1.2) defined on an n-dimensional manifold $\mathcal{X}$. Its local representation (1.1) defined on U is denoted by $\{f, h, U\}$. Note that if $\mathcal{X}$ is an open subset of R^n, then $h = H$ and $U = \mathcal{X}$. Through the vector field representation, we can tackle abstract systems defined on manifolds which are not necessarily an open subsets of R^n. Moreover, as will be seen later, the vector field representation of the system gives an efficient notation for handling the complex differentiations involved in our developments. Also, it is easier to compare our results with results in the prior literature. We denote by y_i, h_i, H_i the ith scalar components of y, h, H, respectively.

Let $\mathcal{U}^\infty$ be the set of all piecewise C^∞ - mappings from R^+ into R^m. We say the system $\{F, H, \mathcal{X}\}$ is **smooth** if

(i) $X_i \in V^\infty(\mathcal{X})$, $i \in \mathcal{M}_{0,m}$,

(ii) $u \in \mathcal{U}^\infty$,

(iii) $H : \mathcal{X} \to R^m$ is C^∞,

(iv) $\mathcal{X}$ is a n - dimensional smooth manifold.

To simplify our definitions, all systems considered in this section are assumed to be smooth. At the end of the section, we will indicate the appropriate extensions to real analytic systems.

Consider the local representation (1.1), (1.7) of (3.1.2). For $u \in \mathcal{U}^\infty$ and $t \in R^+$, $f(\cdot, u(t))$ is C^∞. On the other hand, for $x \in \mathcal{X}$, $f(x, u(\cdot))$ is piecewise C^∞. These observations with well known results ([Hal.1, War.1, Var.1]) on the existence of solutions of differential equations imply the following. For each $x(0) \triangleq x_0 \in \mathcal{X}$ and each $u \in \mathcal{U}^\infty$, there exists a maximal interval $[0, L) \in R^+$, $L = L(x_0, u)$, such that (3.1.2) has a unique solution $x : [0, L) \to \mathcal{X}$ which is continuous but piecewise C^∞. Both x, y are not differentiable in the usual sense but they are differentiable in the peculiar sense discussed in Section 2. 1. As will be seen later, some proofs of our results utilize piecewise constant inputs. This is the main reason for the introduction of piecewise differentiability in Section 2. 1 and the set $\mathcal{U}^\infty$ in this section.

Define the set $\mathcal{Y}^\infty$ by

(3.1.3) $\mathcal{Y}^{\infty} \triangleq \{ (y, L) : L > 0$ and $y : [0, L) \to R^m$ is continuous and piecewise $C^{\infty} \}$.

Then, we can view the input - output behavior of smooth system $\{F, H, \mathcal{X}\}$ as a mapping Φ from $\mathcal{U}^{\infty} \times \mathcal{X}$ into $\mathcal{Y}^{\infty}$. Similar concepts are found in [Gil.3, Sus.2]. Explicitly, we write $(y, L) = (\Phi(u, x_0), L(u, x_0))$ for an input u and an initial state $x(0) \triangleq x_0$.

Let $x_0 \in \mathcal{X}$ and $\bar{u}, \tilde{u} \in \mathcal{U}^{\infty}$. Then, since $L(x_0, \bar{u})$, $L(x_0, \tilde{u})$ are not necessarily equal, the comparisons of the outputs $\bar{y} = \Phi(\bar{u}, x_0)$, $\tilde{y} = \Phi(\tilde{u}, x_0)$ are restricted to their common interval $[0, L)$, where $L \triangleq \min. \{ L(x_0, \bar{u}), L(x_0, \tilde{u}) \}$. For instance, we write $\Phi(\bar{u}, x_0) = \Phi(\tilde{u}, x_0)$ if they are equal on $[0, L)$. Similarly, we write $\bar{u} = \tilde{u}$ if they are same on $[0, L)$.

The following definition concerns state transformations between systems.

Definition 3. 1. 1. Suppose for two systems $\{F, H, \mathcal{X}\}$, $\{\hat{F}, \hat{H}, \hat{\mathcal{X}}\}$, there exists a C^{∞}- diffeomorphism $T : \mathcal{X} \to \hat{\mathcal{X}}$ such that

(i) $\hat{X}_i$ is T - related on $\mathcal{X}$ to X_i, $i \in \mathbf{M}_{0,m}$,

(ii) $\hat{H}_i$ is T - related on $\mathcal{X}$ to H_i, $i \in \mathbf{M}_{1,m}$.

Then, $\{\hat{F}, \hat{H}, \hat{\mathcal{X}}\}$ is T - **related** on $\mathcal{X}$ to $\{F, H, \mathcal{X}\}$. □

The intuitive idea of this definition is that we obtain $F(x, u)$, $H(x)$ from $\hat{F}(\hat{x}, \hat{u})$, $\hat{H}(\hat{x})$ by the "substitution" of variables $\hat{x}$

$= T(x)$, $\hat{u} = u$. See Fig. 3.1.1 for a schematic representation. The definition yields the following obvious conclusion. If $\{\hat{F}, \hat{H}, \hat{\mathcal{X}}\}$ is T-related on $\mathcal{X}$ to $\{F, H, \mathcal{X}\}$, then for any input $u \in \mathcal{U}^{\infty}$ and any initial state $x(0) \triangleq x_0 \in \mathcal{X}$,

$$(3.1.4) \qquad \Phi(u, x_0) = \hat{\Phi}(u, T(x_0)).$$

A definition similar to Definition 3.1.1 is found in [Sus.2].

Next, we introduce a general relation between systems, which takes into account both state and input-feedback transformations. Let T, α, β be mappings from $\mathcal{X}$ into $\hat{\mathcal{X}}$, R^m, and $R^{m \times m}$, respectively, such that $\beta(x)$ is nonsingular, $x \in \mathcal{X}$. Define a mapping $J : \mathcal{X} \times R^m \to \hat{\mathcal{X}} \times R^m$ by

$$(3.1.5) \qquad J(x, u) \triangleq \begin{bmatrix} T(x) \\ -[\beta(x)]^{-1}\alpha(x) + [\beta(x)]^{-1} u \end{bmatrix}, \qquad (x, u) \in \mathcal{X} \times R^m.$$

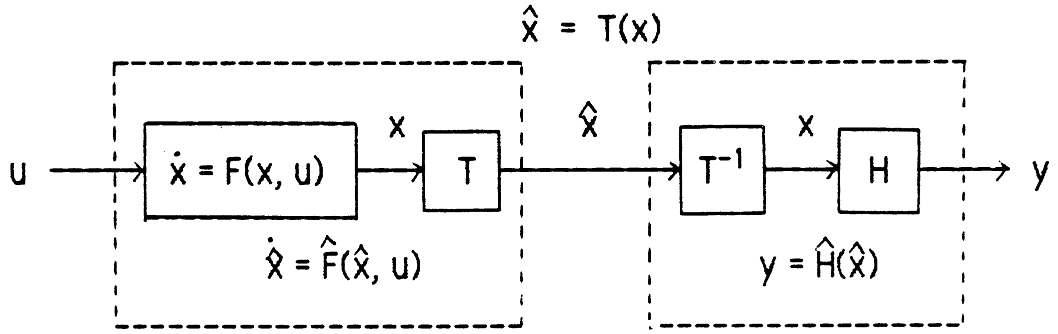

Figure 3.1.1. $\{\hat{F}, \hat{H}, \hat{\mathcal{X}}\}$ is T-related on $\mathcal{X}$ to $\{F, H, \mathcal{X}\}$

We often write $J = \{\alpha, \beta, T\}$. We denote by $\{F, H, \mathcal{X}\}^{\alpha,\beta}$ the feedback system of $\{F, H, \mathcal{X}\}$ corresponding to a control law $u = \alpha(x) + \beta(x)\hat{u}$. In other words, $\{F, H, \mathcal{X}\}^{\alpha,\beta}$ stands for the system $\dot{x} = \hat{F}(x, \hat{u}) \triangleq F(x, \alpha(x) + \beta(x)\hat{u})$, $y = \hat{H}(x) \triangleq H(x)$. Or, $\{F, H, \mathcal{X}\}^{\alpha,\beta} = \{\hat{F}, \hat{H}, \mathcal{X}\}$. A control law $u = \alpha(x) + \beta(x)\hat{u}$ is **smooth** if $\alpha : \mathcal{X} \to R^m$ and $\beta : \mathcal{X} \to R^{m \times m}$ are C^∞. All control laws considered in this section are assumed to be smooth.

Definition 3.1.2. Suppose there exists a C^∞-diffeomorphism $J : \mathcal{X} \times R^m \to \hat{\mathcal{X}} \times R^m$ defined by (3.1.5) such that $\{\hat{F}, \hat{H}, \hat{\mathcal{X}}\}$ is T-related on $\mathcal{X}$ to the system $\{F, H, \mathcal{X}\}^{\alpha,\beta}$. Then, $\{\hat{F}, \hat{H}, \hat{\mathcal{X}}\}$ is J-**feedback related on** $\mathcal{X}$ to $\{F, H, \mathcal{X}\}$. □

The intuitive idea of this definition is that we obtain $F(x, u)$, $H(x)$ from $\hat{F}(\hat{x}, \hat{u})$, $\hat{H}(\hat{x})$ by the "substitution" of variables $\hat{x} = T(x)$, $\hat{u} = [\beta(x)]^{-1}(u - \alpha(x))$. See Fig. 3.1.2 for a schematic representation. The definition is a nonlinear version of **the Control law equivalence** used for linear systems in [Gil.1]. Similar definitions are found in [Bro.1, Hun.1, Hun.2, Hun.3, Isi.2, Mey.1, Jak.1, Sur.1], where the systems do not have outputs, i.e, they are a pair $\{F, \mathcal{X}\}$.

The J-feedback relation is actually an equivalence relation on the set of all smooth systems defined on n-dimensional smooth manifolds. Consider three systems $\{F, H, \mathfrak{X}\}$, $\{\hat{F}, \hat{H}, \hat{\mathfrak{X}}\}$, $\{\bar{F}, \bar{H}, \bar{\mathfrak{X}}\}$. Suppose $\{\hat{F}, \hat{H}, \hat{\mathfrak{X}}\}$ is J - feedback related on $\mathfrak{X}$ to $\{F, H, \mathfrak{X}\}$ and $\{\bar{F}, \bar{H}, \bar{\mathfrak{X}}\}$ is $\hat{J}$ - feedback related on $\mathfrak{X}$ to $\{\hat{F}, \hat{H}, \hat{\mathfrak{X}}\}$. Then, it is easy to see that $\{\bar{F}, \bar{H}, \bar{\mathfrak{X}}\}$ is $\bar{J}$ - feedback related on $\mathfrak{X}$ to $\{F, H, \mathfrak{X}\}$, where $\bar{J} \triangleq \hat{J} \circ J$. Thus, the J - feedback relation is transitive. It is obvious that the J - feedback relation is symmetric and reflexive. Two systems belonging to the same equivalence class are the same with respect to what can be accomplished by feedback. This fact motivates much of our later work.

In order to make precise definitions of decoupling and decomposition, the following technical details are needed. Let $x_0 \in \mathfrak{X}$ and $i \in \mathbf{M}_{1,m}$. Let Φ_i be the ith component of Φ.

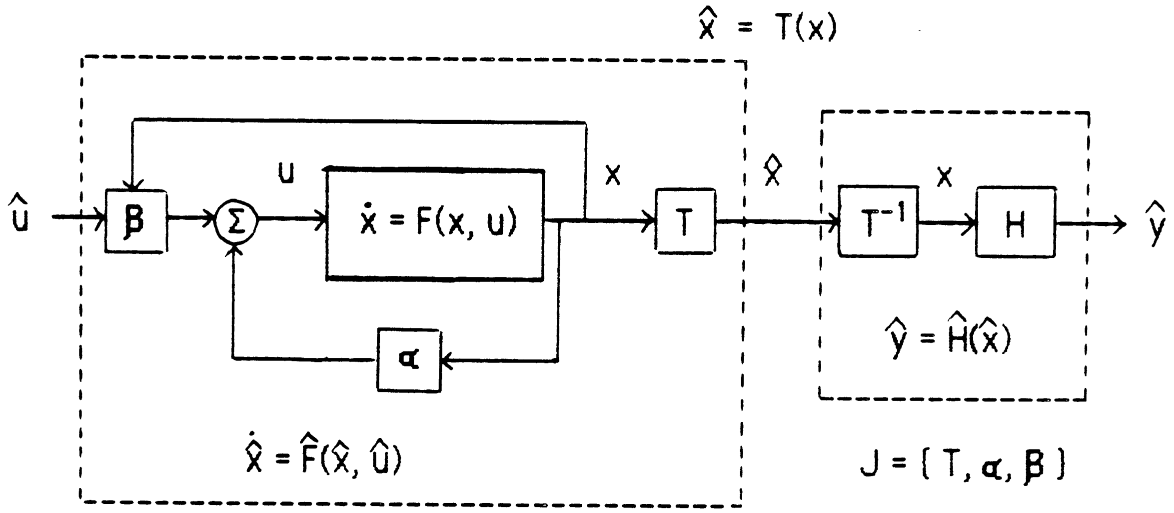

Figure 3. 1. 2. $\{\hat{F}, \hat{H}, \hat{\mathfrak{X}}\}$ is J - feedback related on $\mathfrak{X}$ to $\{F, H, \mathfrak{X}\}$

Suppose $\Phi_i(\bar{u}, x_0) = \Phi_i(\tilde{u}, x_0)$ for all inputs $\bar{u}$, $\tilde{u}$ ε $\mathcal{U}^\infty$ such that $\bar{u}_i = \tilde{u}_i$. Then, y_i is **decoupled at** x_0. If y_i is decoupled at every x_0 ε $\mathcal{X}$, y_i is **decoupled on** $\mathcal{X}$. A similar definition for disturbance decoupling is found in [Hir.2]. The intuitive idea of decoupling for y_i is that y_i is not "connected" to u_j, j ε $\bar{\mathcal{M}}_i$. If y_i is decoupled for i ε $\mathcal{M}_{1,m}$, the **system** is decoupled. The following definition makes this notion precise.

Definition 3. 1. 3.

(1) $[F, H, \mathcal{X}]$ is **decoupled at** x_0 ε $\mathcal{X}$ if y_i is decoupled at x_0, i ε $\mathcal{M}_{1,m}$. If $[F, H, \mathcal{X}]$ is decoupled at each x_0 ε $\mathcal{X}$, $[F, H, \mathcal{X}]$ is **decoupled on** $\mathcal{X}$.

(2) $[F, H, \mathcal{X}]$ is **decouplable at** x_0 ε $\mathcal{X}$ if there exists a control law $u = \alpha(x) + \beta(x)\hat{u}$ such that $[F, H, \mathcal{X}]^{\alpha,\beta}$ is decoupled at x_0. $[F, H, \mathcal{X}]$ is **decouplable on** $\mathcal{X}$ if there exists a control law $u = \alpha(x) + \beta(x)\hat{u}$ such that $[F, H, \mathcal{X}]^{\alpha,\beta}$ is decoupled on $\mathcal{X}$. □

For some applications, we may need a stronger definition of decoupling. Let i ε $\mathcal{M}_{1,m}$ and x_0 ε $\mathcal{X}$. Let $\Phi_i(\bar{u}, x_0) \neq \Phi_i(\tilde{u}, x_0)$ for all inputs $\bar{u}$, $\tilde{u}$ ε $\mathcal{U}^\infty$ such that $\bar{u}_i \neq \tilde{u}_i$ but $\bar{u}_j = \tilde{u}_j$, j ε $\bar{\mathcal{M}}_i$. Then, y_i is **connected** at x_0 to u_i.

Definition 3. 1. 4.

(1) If $\{F, H, \mathcal{X}\}$ is decoupled at $x_0 \in \mathcal{X}$ and y_i is connected at x_0 to u_i, $i \in \mathcal{M}_{1,m}$, $\{F, H, \mathcal{X}\}$ is **input - output decoupled** at x_0. If $\{F, H, \mathcal{X}\}$ is input - output decoupled at each $x_0 \in \mathcal{X}$, it is **input - output decoupled on $\mathcal{X}$**.

(2) $\{F, H, \mathcal{X}\}$ is **input - output decouplable at** $x_0 \in \mathcal{X}$ if there exists a control law $u = \alpha(x) + \beta(x)\hat{u}$ such that $\{F, H, \mathcal{X}\}^{\alpha,\beta}$ is input - output decoupled at x_0. $\{F, H, \mathcal{X}\}$ is **input - output decouplable on $\mathcal{X}$** if there exists a control law $u = \alpha(x) + \beta(x)\hat{u}$ such that $\{F, H, \mathcal{X}\}^{\alpha,\beta}$ is input - output decoupled on $\mathcal{X}$. □

These definitions of decoupling and decouplability are based entirely on the input - output maps for the systems. There is a different concept of decoupling, which is based on the structural forms of state equations. For this idea of decoupling, we use the term **decomposition**.

Definition 3. 1. 5.

(1) $\{F, H, \mathcal{X}\}$ is **decomposed at** $x_0 \in \mathcal{X}$ if there exist : (a) an open neighborhood $\mathcal{E}$ of x_0 ; (b) an open subset $\bar{\mathcal{X}}$ of R^n ; (c) a C^∞- diffeomorphism $T : \mathcal{E} \rightarrow \bar{\mathcal{X}}$; (d) integers $s_i \geq 1$, $i \in \mathcal{M}_{1,m}$ and $s_{m+1} \geq 0$ satisfying $n = \sum_{i=1}^{m+1} s_i$; and (e) a system $\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\}$

which is T-related on $\mathcal{E}$ to $\{F, H, \mathcal{E}\}$ such that its coordinate representation $\{\bar{f}, \bar{h}, \bar{\mathcal{X}}\}$ has the form :

$$(3.1.6) \qquad \dot{\bar{x}}_i = \bar{f}_i(\bar{x}_i) + \bar{g}_i(\bar{x}_i)\bar{u}_i, \quad \bar{y}_i = \bar{h}_i(\bar{x}_i), \ i \in \mathcal{M}_{1,m},$$
$$\dot{\bar{x}}_{m+1} = \bar{f}_{m+1}(\bar{x}) + \sum_{j=1}^{m} \bar{b}_j(\bar{x})\bar{u}_j,$$

where $\bar{x}_i(t) \in R^{s_i}$, $i \in \mathcal{M}_{1,m+1}$, and $\bar{x} \triangleq (\bar{x}_1, \cdots, \bar{x}_m, \bar{x}_{m+1})$. If $\mathcal{E} = \mathcal{X}$ in the above statement, $\{F, H, \mathcal{X}\}$ is **decomposed on** $\mathcal{X}$.

(2) $\{F, H, \mathcal{X}\}$ is **decomposable at** $x_0 \in \mathcal{X}$ if there exists a control law $u = \alpha(x) + \beta(x)\hat{u}$ such that $\{F, H, \mathcal{X}\}^{\alpha,\beta}$ is decomposed at x_0. $\{F, H, \mathcal{X}\}$ is **decomposable on** $\mathcal{X}$ if there exists a control law $u = \alpha(x) + \beta(x)\hat{u}$ such that $\{F, H, \mathcal{X}\}^{\alpha,\beta}$ is decomposed on $\mathcal{X}$. □

Note from these definitions that if $\{F, H, \mathcal{X}\}$ is decomposed at $x_0 \in \mathcal{X}$, then there exists an open neighborhood $\mathcal{E}$ of x_0 such that $\{F, H, \mathcal{E}\}$ is decoupled on $\mathcal{E}$. The converse statement is not necessarily true. It is obvious from [Gil.1] that a linear system is decoupled on R^n if and only if it is decomposed on R^n.

In Section 3.2, we show that if $\{F, H, \mathcal{X}\}$ is real analytic and is decomposed at each $x_0 \in \mathcal{X}$, it is decoupled on $\mathcal{X}$. Unfortunately, it is not clear that the same result holds for smooth systems.

Definitions similar to Definition 3.1.5 are found in many papers including [Isi.1, Nij.2, Res.1]. Some of these papers use the terminology **noninteracting control** for decomposed control.

Thus, papers which concern noninteracting control apply to decomposition (not to decoupling).

Next, we define some distributions based on the smooth system $\{F, H, \mathcal{X}\}$. Let $\mathcal{L}^0(\{F, H, \mathcal{X}\}) \triangleq \{L^k_{X_0} X_i,\ k \in \mathcal{M}_{0,\infty},\ i \in \mathcal{M}_{1,m}\}$, where $L^0_{X_0} X_i \triangleq X_i$. Define $\mathcal{L}(\{F, H, \mathcal{X}\})$ as the smallest subalgebra which contains $\mathcal{L}^0(\{F, H, \mathcal{X}\})$. We say $\{F, H, \mathcal{X}\}$ satisfies the **controllability rank condition** if

(3.1.7) dimension of $\mathcal{L}_p(\{F, H, \mathcal{X}\}) = n, \quad p \in \mathcal{X}$

For each $i \in \mathcal{M}_{1,m}$, let

(3.1.8) $\Delta^0_i(\{F, H, \mathcal{X}\}) \triangleq \{ L_{X_{i_1}} L_{X_{i_2}} \cdots L_{X_{i_k}} X_j : i_r \in \{0, i\},$

$r \in \mathcal{M}_{1,k},\ k \in \mathcal{M}_{0,\infty}, \text{ and } j \in \bar{\mathcal{M}}_i \},$

where $L_{X_{i_1}} L_{X_{i_2}} \cdots L_{X_{i_k}} X_j \triangleq X_j$ if $k = 0$. Define $\Delta_i(\{F, H, \mathcal{X}\})$ as the smallest subalgebra containing $\Delta^0_i(\{F, H, \mathcal{X}\})$.

Some insight about these definitions may be gained by considering the linear system $\{f, h, R^n\}$ in (1.5). Let $[\partial/\partial x_1 \cdots \partial/\partial x_n]$ be the n-row vector of the vector fields $\partial/\partial x_i$, $i \in \mathcal{M}_{1,n}$. Recall that for $w \in R^n$, $W \triangleq [\partial/\partial x_1 \cdots \partial/\partial x_n] w = \sum_{i=1}^{n} w_i\, \partial/\partial x_i$ is a vector field in $V^\infty(R^n)$, where w_i is the ith component of w. Let B_i be the ith column of B. Then, the distributions $\mathcal{L}$, Δ_i take on familiar forms :

$$(3.1.9) \qquad \mathcal{L}(\{F, H, \mathcal{X}\}) = \mathrm{span}\{[\partial/\partial x_1 \cdots \partial/\partial x_n] A^k B_i, \; i \in \mathcal{M}_{1,m}, \; k \in \mathcal{M}_{0,\infty}\},$$

$$(3.1.10) \qquad \Delta_i(\{F, H, \mathcal{X}\}) = \mathrm{span}\{[\partial/\partial x_1 \cdots \partial/\partial x_n] A^k B_j, \; j \in \bar{\mathcal{M}}_i, \; k \in \mathcal{M}_{0,\infty}\}.$$

We conclude this section with definitions of invertibility, reachability, and the precise concept of real analytic systems. Let $x_0 \in \mathcal{X}$. $\{F, H, \mathcal{X}\}$ is **invertible** at x_0 if $\Phi(\bar{u}, x_0) \neq \Phi(\tilde{u}, x_0)$ for all distinct inputs $\bar{u}, \tilde{u} \in \mathcal{U}^\infty$. If $\{F, H, \mathcal{X}\}$ is **invertible** at each $x_0 \in \mathcal{X}$, it is invertible on $\mathcal{X}$. Similar definitions appear in the literature on invertibility of systems (see for instance, [Sil.1, Hir.1]).

We say ([Her.1, Sus.1]) that $x_1 \in \mathcal{X}$ is **reachable** from $x_0 \in \mathcal{X}$ at time $t_1 > 0$ if there exists $u \in \mathcal{U}^\infty$ such that the solution of (3.1.2), $x(t) \in \mathcal{X}$, $t \in [0, t_1]$, $x(0) \triangleq x_0$, satisfies $x(t_1) \triangleq x_1$. We denote by $\mathcal{R}(x_0, t_1)$ the set of all points in $\mathcal{X}$ which are reachable from $x_0 \in \mathcal{X}$ at $t = t_1$.

If C^∞ in the definitions of the sets $\mathcal{U}^\infty$, $\mathcal{Y}^\infty$ is replaced by C^ω, we obtain the definitions for the sets $\mathcal{U}^\omega$, $\mathcal{Y}^\omega$. Similarly, if $V^\infty(\mathcal{X})$, $\mathcal{U}^\infty$, C^∞, "smooth manifold" in the definitions of smooth system and smooth control law are replaced, respectively, by $V^\omega(\mathcal{X})$, $\mathcal{U}^\omega$, C^ω, "real analytic manifold", we obtain the definitions of **real analytic system** and **real analytic control law**.

3. 2. Decoupling and Decomposition

By the definitions of decoupling and decomposition, it is clear that a decomposed system is always decoupled. In this section, we show that the conditions for decomposition are more complex than those for decoupling. But, we present some conditions under which the two concepts are at least locally equivalent.

To state our results a variety of assumptions are needed. To simplify the presentation we list them together here.

(A. 1) The system $\{F, H, \mathcal{X}\}$ is smooth,

(A. 1)' The system $\{F, H, \mathcal{X}\}$ is real analytic,

(A. 2) For each constant input $u(t) \in R^m$, the vector field $F(\cdot, u)$ is complete,

(A. 3) The system $\{F, H, \mathcal{X}\}$ satisfies the controllability rank condition on $\mathcal{X}$,

(A. 4) The codistribution $\Delta_i^{\perp}(\{F, H, \mathcal{X}\})$ has constant dimension $p_i \geq 1$ on $\mathcal{X}$, $i \in \mathcal{M}_{1,m}$.

We begin by giving a necessary and sufficient condition for decoupling.

Theorem 3. 2. 1. Suppose that (A.1)', (A.2) are satisfied. Then, $\{F, H, \mathcal{X}\}$ is decoupled on $\mathcal{X}$ if and only if

(3.2.1) $\quad dH_i \in \Delta_i^{\perp}(\{F, H, \mathcal{X}\})$ on $\mathcal{X}$, $i \in \mathcal{M}_{1,m}$. □

Theorem 3. 2. 1 is important because it gives an algebraic condition for nonlinear decoupling. Before presenting its proof, some discussion of the Theorem may be useful. The condition in (3.2.1) simply requires that $YH_i = 0$ on $\mathcal{X}$ for all $Y \in \Delta_i([F, H, \mathcal{X}])$. As will be seen in the examples of Section 3. 5, the distribution Δ_i is, in most cases, spanned by a finite number of vector fields. Thus, (3.2.1) does not necessarily require an infinite number of calculations. Suppose that $[F, H, \mathcal{X}]$ is a linear system, (1.5) such that $\mathcal{X} = R^n$. Then, by (3.1.10), the condition in (3.2.1) becomes

$$(3.2.2) \qquad C_i A^k B_j = 0, \quad k \in \mathbf{M}_{0,\infty}, \quad i, j \in \mathbf{M}_{1,m}, \text{ if } i \neq j,$$

where C_i is the ith row of C. It is not difficult to show, by using the well - known expansion for $(s I_n - A)^{-1}$ ([Gan.1]), that (3.2.2) is a necessary and sufficient condition for the Laplace - transform transfer function matrix of the linear system to be diagonal. This is the definition of linear decoupling given in [Gil.1] and it is equivalent to saying that the (linear) input - output map is diagonal. For other, essentially equivalent, definitions of linear decoupling, see [Fal.1, Sil.1, Won.2].

We begin our proof of Theorem 3. 2. 1 with the following two lemmas. The first lemma is a well known result on the reachable set, $\mathbf{R}(x_0, t)$.

Lemma 3.2.1. (Theorem 4.5 in [Sus.1]) Suppose that (A.1)', (A.2) are satisfied. Let $x_0 \in \mathcal{X}$ and $t > 0$. Let $I(\mathcal{L}(\{F, H, \mathcal{X}\}), x_1)$ be the largest integral manifold of $\mathcal{L}(\{F, H, \mathcal{X}\})$ passing through a point $x_1 \in \mathcal{R}(x_0, t)$. Then, $\mathcal{R}(x_0, t) \subset I(\mathcal{L}(\{F, H, \mathcal{X}\}), x_1)$. Moreover, the interior of $\mathcal{R}(x_0, t)$ relative to $I(\mathcal{L}(\{F, H, \mathcal{X}\}), x_1)$ is dense in $\mathcal{R}(x_0, t)$ and not empty. □

To state the second lemma, we need the following notation. Let $i \in \mathcal{M}_{1,m}$. Define a multiindex I_i by any finite sequence of integers taken from $\mathcal{M}_{0,m}$ such that at least one of its elements must belong to the set $\bar{\mathcal{M}}_i$. For such a multiindex $I_i \triangleq \{i_1, i_2, \cdots, i_k\}$, let X_{I_i} be defined by

(3.2.3) $$X_{I_i} \triangleq X_{i_1} X_{i_2} \cdots X_{i_k}$$

Then, for each $i \in \mathcal{M}_{1,m}$, define

(3.2.4) $$\mathcal{D}_i \triangleq \{ \sum_{\gamma \in \ell} a_\gamma X_\gamma : a_\gamma \in R \text{ and } \ell \text{ is any finite collection of multiindices } I_i \}$$

Using these notation, we can state the following result.

Lemma 3.2.2.

(i) Let $\phi \in C^\infty(\mathcal{X})$. Then, (3.2.5) $\quad d\phi \in \Delta_i^{\perp}(\{F, H, \mathcal{X}\})$ on $\mathcal{X}$

if and only if

(3.2.6) $\quad \sigma\phi = 0$ on $\mathcal{X}$, $\quad \sigma \in \mathcal{D}_i(\{F, H, \mathcal{X}\})$.

(ii) $\Delta_i^{\perp}(\{F, H, \mathcal{X}\})$ is X_0 - invariant and X_i - invariant on $\mathcal{X}$.

Proof. Consider (i). It is clear that (3.2.6) implies (3.2.5). We prove (3.2.5) implies (3.2.6) by induction. By (3.2.5), we have

(3.2.7) $\quad X_j\phi = X_jX_0\phi = X_jX_i\phi = 0$ on $\mathcal{X}$ if $j \in \bar{\mathcal{M}}_i$.

Suppose that for $k = 0, 1, \cdots, \ell$,

(3.2.8) $\quad X_jX_{i_1}X_{i_2}\cdots X_{i_k}\phi = 0$ on $\mathcal{X}$, $j \in \bar{\mathcal{M}}_i$, $i_r \in \{0, i\}$.

Then, by (3.2.5) and (3.2.8),

(3.2.9) $\quad X_jX_{i_1}X_{i_2}\cdots X_{i_\ell}X_{i_{\ell+1}}\phi = (-1)^{\ell+1}(L_{X_{i_{\ell+1}}}L_{X_{i_\ell}}\cdots L_{X_{i_1}}X_j)\phi = 0,$

if $j \in \bar{\mathcal{M}}_i$ and $i_r \in \{0, i\}$.

Thus, we have for any $k \in \mathcal{M}_{0,\infty}$,

(3.2.10) $\quad X_jX_{i_1}X_{i_2}\cdots X_{i_k}\phi = 0$ if $j \in \bar{\mathcal{M}}_i$ and $i_r \in \{0, i\}$.

From this, (3.2.6) follows immediately. Part (ii) is an easy consequence of (i). □

Note that $\Delta_i \neq \mathfrak{D}_i$ because $\mathfrak{D}_i$ is not a set of vector fields. Now, using these Lemmas, we prove Theorem 3. 2. 1.

Proof of Theorem 3. 2. 1. First, assume (3.2.1) holds. Fix $i \in \mathcal{M}_{1,m}$. Define a multiindex J_i by any finite sequence of integers taken from $\{o, i\}$. For such a multiindex $J_i \triangleq \{i_1, i_2, \cdots, i_k\}$, define

$$(3.2.11) \qquad X_{Ji} \triangleq X_{i} X_{i} \cdots X_{i},$$

$$(3.2.12) \qquad y_{Ji} \triangleq X_{Ji} H_i(x),$$

where $y_{Ji} \triangleq H_i(x)$ if $J_i = \emptyset$. Let $\mathcal{Y}_{Ji}$ be the set of all y_{Ji} defined by (3.2.12). Differentiating $y_{Ji}(t)$ with respect to t in the sense described at the end of Section 2. 1, we obtain, by (3.2.1) and Lemma 3. 2. 2,

$$
\begin{aligned}
(3.2.13) \qquad y_{Ji}(t) &= X_{Ji} H_i(x(t)), \\
y^{(1)}_{Ji}(t) &= X_0 X_{Ji} H_i(x(t)) + \Sigma u_j(t) X_j X_{Ji} H_i(x(t)) \\
&= X_0 X_{Ji} H_i(x(t)) + u_i(t) X_i X_{Ji} H_i(x(t)) \\
y^{(2)}_{Ji}(t) &= X_0^2 X_{Ji} H_i(x(t)) + u_i(t) X_i X_0 X_{Ji} H_i(x(t)) + \\
&\quad u_i^{(1)}(t) X_i X_{Ji} H_i(x(t)) + u_i(t) X_0 X_i X_0 X_{Ji} H_i(x(t)) + \\
&\quad (u_i(t))^2 X_i^2 X_0 X_{Ji} H_i(x(t)),
\end{aligned}
$$

. . . . .

Note that

(3.2.14) $y_{Ji}^{(k)}(t)$ is a finite linear combination of some $y_\gamma \in \mathcal{Y}_{Ji}$ such that each coefficient is 1 or some monomial in u_i and the time derivatives of u_i at t.

Let $\bar{u}$, $\tilde{u}$ be any two inputs belonging to $\mathcal{U}^\omega$ such that

(3.2.15) $\bar{u}_i = \tilde{u}_i$.

Let $x(0) \triangleq x_0 \in \mathcal{X}$. We shall denote by $\bar{x}$, $\tilde{x}$, the solutions of $\{F, H, \mathcal{X}\}$ corresponding to $\bar{u}$, $\tilde{u}$, respectively. Similarly, y_{Ji} in (3.2.12) corresponding to $\bar{x}$, $\tilde{x}$ are denoted by $\bar{y}_{Ji}$, $\tilde{y}_{Ji}$, respectively. Let $L \triangleq \min\{L(x_0, \bar{u}), L(x_0, \tilde{u})\}$. Since $\bar{u}$, $\tilde{u}$ are piecewise real analytic, there exists a partition of $[0, L)$ such that $0 = t_0 < t_1 < t_2 < \ldots < t_r = L$ and on each interval $[t_j, t_{(j+1)})$, $j \in \mathcal{M}_{0,(r-1)}$, both $\bar{u}$ and $\tilde{u}$ are real analytic. Then, by (A.1)',

(3.2.16) $\bar{y}_{Ji}$, $\tilde{y}_{Ji}$ are continuous and piecewise C^ω on $[0, L)$ such that on each interval $[t_j, t_{(j+1)})$, $j \in \mathcal{M}_{0,(r-1)}$, they are C^ω.

Now, using the above facts, we prove by induction on the intervals $[0, t_j)$, $j \in \mathcal{M}_{1,r}$ that

(3.2.17) $\bar{y}_{Ji}(t) = \tilde{y}_{Ji}(t)$, $t \in [0, L)$ for all $y_{Ji} \in \mathcal{Y}_{Ji}$.

This with $J_i = \emptyset$ in (3.2.12) implies

$$(3.2.18) \quad \Phi_i(\bar{u}, x_0) = \Phi_i(\tilde{u}, x_0) \text{ on } [0, L),$$

which is what we need to complete the first part of our proof.

First, consider the time interval $[0, t_1)$. By (3.2.14) and (3.2.15),

$$(3.2.19) \quad \bar{y}_{J_i}^{(k)}(0) = \tilde{y}_{J_i}^{(k)}(0), \quad k \in \mathbb{N}_{0,\infty}.$$

Thus, by (3.2.16) and **analytic continuation** [Die.1], we obtain

$$(3.2.20) \quad \bar{y}_{J_i}(t) = \tilde{y}_{J_i}(t), \quad t \in [0, t_1).$$

Next, suppose that for $1 < j < r$,

$$(3.2.21) \quad \bar{y}_{J_i}(t) = \tilde{y}_{J_i}(t), \; t \in [0, t_j).$$

Then, by (3.2.16), (3.2.21) holds on $[0, t_j]$. By (3.2.14) and (3.2.15), this implies

$$(3.2.22) \quad \bar{y}_{J_i}^{(k)}(t_j) = \tilde{y}_{J_i}^{(k)}(t_j), \; k \in \mathbb{N}_{0,\infty}.$$

Then, by (3.2.16) and analytic continuation, we obtain

$$(3.2.23) \quad \bar{y}_{J_i}(t) = \tilde{y}_{J_i}(t), \; t \in [t_j, t_{j+1}).$$

(3.2.24) $\bar{y}_{Ji}(t) = \tilde{y}_{Ji}(t),\ t \in [0, t_{j+1})$.

Thus, we have shown (3.2.17).

Now, assume that $\{F, H, \mathcal{X}\}$ is decoupled on $\mathcal{X}$. We denote by $x(t, u, x_0)$ the state response of $\{F, H, \mathcal{X}\}$ to an input u and an initial state $x(0) = x_0$. By (A.2), given $\bar{x} \in \mathcal{X}$, $q \in \mathbf{M}_{1,\infty}$, $c_k \in R$, $k \in \mathbf{M}_{1,q}$, we can choose real numbers $\tau_k > 0$, $k \in \mathbf{M}_{0,\infty}$ and $x_0 \in \mathcal{X}$ such that $x(t, u^*, x_0) \in \mathcal{X}$ for all $t \in [0, t_q]$ and $x(t_q, u^*, x_0) = \bar{x}$, where $t_r \triangleq \sum_{i=1}^{r} \tau_i$, $r \in \mathbf{M}_{1,q}$, $u_j^* \triangleq 0$, $j \in \bar{\mathbf{M}}_i$, and u_i^* is given by

(3.2.25) $u_i^*(t) \triangleq c_i,\quad t_{r-1} \le t \le t_r,\ r = 1, \cdots, q.$

Let $\{F, H, \mathcal{X}\}_i$ be the system obtained from $\{F, H, \mathcal{X}\}$ by letting $u_i = 0$. Clearly, for any given $x_0 \in \mathcal{X}$, there exist $t_0 > 0$, $x_0 \in \mathcal{X}$, $u^o \in \mathbf{U}^\omega$ with $u_i^o = 0$ such that $x_0 = x(t_0, u^o, \hat{x}_0)$. Then, by Lemma 3. 2. 1, $\mathbf{R}(\hat{x}_0, t_0) \subset I(\mathbf{L}(\{F, H, \mathcal{X}\}_i), x_0)$. Let $\mathring{\mathbf{R}}(\hat{x}_0, t_0)$ be the interior of $\mathbf{R}(\hat{x}_0, t_0)$ relative to $I(\mathbf{L}(\{F, H, \mathcal{X}\}_i), x_0)$. Then, there are two cases : (i) $x_0 \in \mathring{\mathbf{R}}(\hat{x}_0, t_0)$ and (ii) $x_0 \in \mathbf{R}(\hat{x}_0, t_0) - \mathring{\mathbf{R}}(\hat{x}_0, t_0)$.

First, we consider the case (i). Since (i) holds and $X_j \in \mathbf{L}(\{F, H, \mathcal{X}\}_i)$, $j \in \mathbf{M}_i$, we can choose $\gamma > 0$ such that

(3.2.26) $\theta_\delta^{X_j}(x_0) \in \mathring{\mathbf{R}}(\hat{x}_0, t_0),\ \delta \in (-\gamma, \gamma)$.

Then, for any $\delta \in (-\gamma, \gamma)$ and $j \in \bar{\mathcal{M}}_i$, there exists an input u^s with $u_i^s = 0$ such that

$$(3.2.27) \quad x(t_0, u^s, \hat{x}_0) = \theta_\delta^{X_j}(x_0).$$

Now, construct two piecewise real analytic inputs $\bar{u}$, $\tilde{u}$ as follows:

$$(3.2.28) \quad \bar{u}_i(t) = \tilde{u}_i(t) \triangleq \begin{cases} 0, & 0 \le t \le t_0 \\ u_i^*(t - t_0), & t_0 \le t \le t_0 + t_q, \end{cases}$$

$$(3.2.29) \quad \bar{u}_j(t) \triangleq \begin{cases} u_j^o(t), & 0 \le t \le t_0, \\ 0, & 0 \le t \le t_0 + t_q, \end{cases} \qquad j \in \bar{\mathcal{M}}_i$$

$$(3.2.30) \quad \tilde{u}_j(t) \triangleq \begin{cases} u_j^s(t), & 0 \le t \le t_0, \\ 0, & t_0 \le t \le t_0 + t_q. \end{cases} \qquad j \in \bar{\mathcal{M}}_i$$

Let $\bar{X}_k \triangleq X_0 + c_k X_i$, $k \in \mathcal{M}_{1,q}$. Consider the response of the original system $\{F, H, \mathcal{X}\}$. By the above construction of $\bar{u}$, $\bar{x}$ = $x(t_0 + t_q, \bar{u}, \hat{x}_0)$. Since $\{F, H, \mathcal{X}\}$ is decoupled on $\mathcal{X}$, we have

$$(3.2.31) \quad H_i(\bar{x}) = H_i(x(t_0 + t_q, \tilde{u}, \hat{x}_0))$$
$$= H_i(\theta_{\tau_q}^{\bar{X}_q} \circ \ldots \circ \theta_{\tau_1}^{\bar{X}_1} \circ \theta_\delta^{X_j} \circ \theta_{-\tau_1}^{\bar{X}_1} \circ \ldots \circ \theta_{-\tau_q}^{\bar{X}_q}(\bar{x})).$$

Note that (3.2.31) holds for all $\delta \in (-\gamma, \gamma)$. Therefore, differentiating (3.2.31) with respect to δ and letting $\delta = 0$ yields

$$(3.2.32) \quad 0 = dH_i(\bar{x})\,(\theta_{\tau_q}^{\bar{X}_q})_* \ldots (\theta_{\tau_1}^{\bar{X}_1})_* X_j(\theta_{-\tau_1}^{\bar{X}_1} \circ \ldots \circ \theta_{-\tau_q}^{\bar{X}_q}(\bar{x})).$$

Applying the **Cambell - Baker - Hausdorff formula** (Theorem 2. 3. 5) to (3.2.32) successively q times leads to the following fact : there exists $\eta > 0$ such that

$$(3.2.33) \quad 0 = \sum_{\ell_1=0}^{\infty} \cdots \sum_{\ell_q=0}^{\infty} \frac{(-\tau_1)^{\ell_1} \cdots (-\tau_q)^{\ell_q}}{\ell_1! \cdots \ell_q!} dH(\bar{x}) L_{\bar{X}_q}^{\ell_q} \cdots L_{\bar{X}_1}^{\ell_1} X_j(\bar{x}),$$

for all $\tau_k < \eta$, $k \in \mathbf{M}_{1,q}$.

Note that η depends only on $\bar{x}$, X_j, and $\bar{X}_k \in \mathbf{M}_{1,q}$. Thus, when we construct u* in (3.2.25), τ_k, $k \in \mathbf{M}_{1,q}$ can be assumed to be chosen smaller than η. Small variations of τ_k, $k \in \mathbf{M}_{1,q}$ in (3.2.33) yields

$$(3.2.34) \quad dH_i(\bar{x}) L_{\bar{X}_q}^{\ell_q} \cdots L_{\bar{X}_1}^{\ell_1} X_j(\bar{x}) = 0, \ \ell_k \in \mathbf{M}_{0,\infty}, \ k \in \mathbf{M}_{1,q}, \ j \in \mathbf{M}_i.$$

Now, we go back to the case (ii). Note that (ii) does not necessarily imply (3.2.26). To show (3.2.34) is still true for the case (ii), we need a slight modification of the above arguments. By Theorem 2. 3. 3, there exist open neighborhoods U, V of x_0, $\bar{x}$, respectively such that $x(t_q, u^*, \cdot)$ is a C^{∞} - diffeomorphism from U onto V. Let $\bar{U}$ be the intersection of U and $\mathring{\mathbf{R}}(\hat{x}_0, t_0)$. Then, since x_0 is in the closure of $\bar{U}$, there exists a sequence $\{x_0(p)\}$ converging to x_0 such that

$$(3.2.35) \quad x_0(p) \in \bar{U} \subset \mathring{\mathbf{R}}(\hat{x}_0, t_0), \ p \in \mathbf{M}_{1,\infty}.$$

Let $\bar{x}(p) \triangleq x(t_q, u^*, x_0(p))$, $p \in \mathcal{N}_{1,\infty}$. Fix $p \in \mathcal{N}_{1,\infty}$. By (3.2.35), all arguments and equations following (3.2.26) do not change if x_0, $\bar{x}$ are replaced by $x_0(p)$, $x(p)$, respectively. In particular, we have

$$\text{(3.2.36)} \quad dH_i(\bar{x}(p))\, L_{\bar{X}_q}^{\ell_q} \cdots L_{\bar{X}_1}^{\ell_1} X_j(\bar{x}(p)) = 0, \text{ for all } \ell_k \in \mathcal{N}_{0,\infty} \text{ and } j \in \mathcal{N}_l.$$

But since dH_i, $L_{\bar{X}_q}^{\ell_q} \cdots L_{\bar{X}_1}^{\ell_1} X_j$ are continuous on $\mathcal{X}$ and $\bar{x}(p) \to \bar{x}$ as $p \to \infty$, (3.2.36) implies (3.2.34). Thus, we have shown that (3.2.34) holds for both of the cases (i), (ii). Finally, since q, x, and c_k, $k \in \mathcal{N}_{1,q}$ are chosen arbitrarily, (3.2.34) implies

$$\text{(3.2.37)} \quad \sigma H_i = 0 \text{ on } \mathcal{X}, \quad \sigma \in \mathcal{D}_l(\{F, H, \mathcal{X}\}).$$

Then, Lemma 3. 2. 2 completes the proof. □

Remark 3. 2. 1. Theorem 3. 2. 1 is a minor generalization of results for disturbance decoupling of real analytic nonlinear systems, which are stated in [Hir.2, Isi.1]. The first (sufficiency) part of our proof is entirely different from those in [Hir.2, Isi.1]. Our second (necessity) part adopts its main idea from [Hir.2]. We feel that some of the arguments in the proofs by these authors are incomplete. For instance, the details for the case when input is piecewise real analytic are not given ([Hir.2, Isi.1]) and the fact that countably infinite intersections of open sets are not necessarily open is not taken into account ([Hir.2]). We believe our proof is simpler and clearer. Finally, it is interesting to note that while in [Hir.2], $\mathcal{X}$ is required to be connected, it need not to be connected in [Isi.1] and here. □

Theorem 3. 2. 1 concerns decoupling but not input - output decoupling. We can easily show under the hypotheses of Theorem 3. 2. 1 that $\{F, H, \mathcal{X}\}$ is input - output decoupled on $\mathcal{X}$ if and only if (3.2.1) and the following condition are satisfied :

(C) For each $i \in \mathcal{M}_{1,m}$, the single input - single output system $\{F, H_i, \mathcal{X}\}_i$, obtained from the original system $\{F, H_i, \mathcal{X}\}$ by setting $u_j = 0$, $j \in \overline{\mathcal{M}}_i$, is invertible on $\mathcal{X}$.

Algebraic conditions which are either necessary or sufficient for (C) have been obtained. But, those which are both necessary and sufficient have not yet been presented in the literature. A special case of invertibility of nonlinear systems is considered in [Hir.1, Nij.1].

Some results for input-output decoupling of smooth nonlinear systems are stated without proof in [Nij.3, Nij.4]. Their validity is in doubt for the following reasons. The first of two necessary and sufficient conditions for input - output decoupling is similar to (3.2.1), although the assumption made is (A.1). But recall that (A.1)' is crucial in the derivation of (3.2.1). The second condition is for (C). But, it is not clear it is both necessary and sufficient for (C).

Another condition corresponding to (3.2.1) appears in [Tar.1] and is used there as a definition of decoupling for smooth systems (assumption (A.1)). The connection between it and our definition of decoupling would require (A.1)' instead of (A.1).

Note that (A.2) is used only in the necessity part of the proof, where it is required to apply Lemma 3. 2. 1. But, (A.2) can be greatly relaxed. For instance, in Lemma 3. 2. 1 and, hence Theorem 3. 2. 1, (A.2) can be replaced by

(A.2)' There exists a locally path - connected subset Ω of R^m such that for each constant input $u(t) \in \Omega$, the vector field $F(\cdot, u)$ is complete ([Sus.1]).

Next, we consider necessary and sufficient conditions for local decomposition.

Theorem 3. 2. 2. Suppose that (A.1) is satisfied. Then, $[F, H, \mathcal{X}]$ is decomposed at $x_0 \in \mathcal{X}$ if and only if there exist an open neighborhood $\mathcal{E}$ of x_0 and m involutive distributions Δ_i^* on $\mathcal{E}$ which have dimension $r_i < n$ such that on $\mathcal{E}$,

(3.2.38) (i) $dH_i \in (\Delta_i^*)^\perp \subset \Delta_i^\perp$, $i \in \mathcal{M}_{1,m}$,

(ii) $(\Delta_i^*)^\perp$ is X_0, X_i - invariant, $i \in \mathcal{M}_{1,m}$,

(iii) $(\Delta_i^*)^\perp$, $i \in \mathcal{M}_{1,m}$ are mutually disjoint at each $x \in \mathcal{E}$. □

Note that although Theorem 3. 2. 2 requires only the assumption of smoothness, the conditions for decomposition are more complex than those for decoupling. In Section 3.1, we pointed out that if a system $\{F, H, \mathcal{X}\}$ is decomposed at x_0, then there exits an open neighborhood $\mathcal{E}$ of x_0 such that $\{F, H, \mathcal{E}\}$ is decoupled on $\mathcal{E}$. A comparison of the conditions in Theorem 3.2.1 and Theorem 3.2.2 suggests that the converse is not necessarily true.

Theorem 3.2.2 is implied by Theorem 5.1 in [Isi.1], where conditions similar to those in (3.2.38) are stated as being necessary and sufficient for $\{F, H, \mathcal{X}\}$ to be decomposable on $\mathcal{X}$. However, the conditions in [Isi.1] do not necessarily imply the existence of T which is a C^∞-diffeomorphism on $\mathcal{X}$. Thus, they are necessary and sufficient conditions for $\{F, H, \mathcal{X}\}$ to be decomposable at each $x_0 \in \mathcal{X}$ rather than on $\mathcal{X}$. In this sense, Theorem 3.2.2 may be viewed as a corrected version of the result in [Isi.1].

We omit the proof of Theorem 3.2.2 since it can be obtained from [Isi.1] and utilizes some ideas contained in the proof of Theorem 3.2.3. Note that it is not easy to check for the existence of Δ_i^*, $i \in \mathbf{M}_{1,m}$ satisfying conditions specified in Theorem 3.2.2. This is in contrast with the ease of applying the decoupling conditions in (3.2.1). In this respect, the following Corollary is valuable.

Corollary 3.2.1. If $\{F, H, \mathcal{X}\}$ satisfies (A.1)' and is decomposed at each $x_0 \in \mathcal{X}$, it is decoupled on $\mathcal{X}$.

Proof. The given hypotheses imply (3.2.38) - (i) and thus (3.2.1) holds. Since the sufficiency part of the proof for

Theorem 3.2.1 does not require (A.2), (A.1)' and (3.2.1) imply that $\{F, H, \mathcal{X}\}$ is decoupled on $\mathcal{X}$. □

It is uncertain that this Corollary is true for smooth systems. This motivates the following Theorem.

Theorem 3.2.3. Suppose that (A.1), (A.3), (A.4) are satisfied. Then, $\{F, H, \mathcal{X}\}$ is decomposed at each $x_0 \in \mathcal{X}$ if and only if (3.2.1) holds. □

Apart from giving an easily verified condition for decomposition, this result has other important implications. It shows that under assumptions (A.1), (A.3), (A.4), the condition for decomposition of smooth systems is reduced to that of decoupling of real analytic systems. Consequently, we see from Theorem 3.2.1 and Theorem 3.2.3 that under assumptions (A.1)', (A.2), (A.3), (A.4), the concepts of decomposition and decoupling are equivalent.

There are several circumstances where the assumptions of Theorem 3.2.3. hold. The most obvious is the case of controllable linear systems. For real analytic systems that satisfy (A.3), Theorem 3.2.3 holds on a submanifold. In particular, when $\mathcal{X}$ is connected, we can show by analytic continuation that there exists a submanifold $\mathcal{X}_0$ of $\mathcal{X}$ such that $\mathcal{X}_0$ is open and dense in $\mathcal{X}$ and the assumption (A.4) is satisfied on $\mathcal{X}_0$.

We believe Theorem 3.2.3 and the equivalence of decoupling and decomposition is new. The only similar result we know of is in [Nij.4], where the structure of a decoupled system with $m = 2$ was investigated. Although assumptions similar to ours were made, the structure in (3.1.4) was not obtained.

In order to prove Theorem 3. 2. 3, we need the following three Lemmas. Particularly, Lemma 3. 2. 5 is the key to Theorem 3. 2. 3.

Lemma 3. 2. 3. Let Δ be an involutive C^∞- distribution on $\mathcal{X}$ with dimension $r < n$. Then, at each point $p \in \mathcal{X}$, there exist an open neighborhood U of p and $(n-r)$ C^∞- functions θ_j, $j \in \mathcal{M}_{1,(n-r)}$, from U into R such that $d\theta_j$, $j \in \mathcal{M}_{1,(n-r)}$, are linearly independent on U and at each $q \in U$, $\{ d\theta_j(q),\ j \in \mathcal{M}_{1,(n-r)} \}$ is a basis for $\Delta_q^\perp$.

Proof. Since Δ is involutive and has dimension r on $\mathcal{X}$, Theorem 2. 3. 9 (**Frobenius Theorem**) applies and at each $p \in U$, there exists a chart (U, ϕ) such that if $x_1, \cdots, x_n$ denote the local coordinates, then $\{ (E_i)_q \triangleq \phi^{-1}_{*\phi(q)}((\partial/\partial x_i)_{\phi(q)}),\ i \in \mathcal{M}_{1,r} \}$ is a local basis of Δ_q, $q \in U$. Let $\theta_j \triangleq \phi_{(r+j)}$, $j \in \mathcal{M}_{1,(n-r)}$, where ϕ_k is the kth component of ϕ. Then, since

$$(3.2.39)\qquad E_i\,\theta_j = \phi_*^{-1}(\partial/\partial x_i)\,\phi_{(r+j)} = (\partial/\partial x_i)\,(\phi_{(r+j)} \circ \phi^{-1})$$

$$= \partial x_{(r+j)}/\partial x_i = 0, \qquad \text{on } U,\ i \in \mathcal{M}_{1,r},\ j \in \mathcal{M}_{1,(n-r)},$$

the desired result follows immediately. □

Lemma 3.2.4. Let Δ be an involutive C^∞ - distribution on $\mathcal{X}$ with dimension $r < n$. Let θ_j, $j \in \mathcal{M}_{1,(n-r)}$ be any C^∞- functions from an open subset U of $\mathcal{X}$ into R such that $\{ d\theta_j(q),\ j \in \mathcal{M}_{1,(n-r)} \}$ is a basis of $\Delta_q^\perp$, $q \in U$. These functions exist by Lemma 3.2.3. Then, for any C^∞- function η from U into R satisfying

(3.2.40) $\quad d\eta \in \Delta^\perp$ on U,

there exist an open subset $\mathcal{E}$ of U and a C^∞- function g, defined on an open connected subset of $R^{(n-r)}$, such that

(3.2.41) $\quad \eta(x) = g(\theta_1(x), \cdots, \theta_{(n-r)}(x)), \quad x \in \mathcal{E}$.

Conversely, if (3.2.41) holds on U, then (3.2.40) holds.

Proof. Fix $p \in U$. Since $\{ d\theta_j(q),\ j \in \mathcal{M}_{1,(n-r)} \}$ is a basis of Δ_q, $q \in U$, we can choose r C^∞- functions $\theta_{n-r+j} : U \to R$, $j \in \mathcal{M}_{1,r}$ so that

(3.2.42) $\quad$ rank of $T \triangleq (\theta_1, \cdots, \theta_n)$ at $p = n$.

By Theorem 2.3.6, there exist an open neighborhood $\hat{U} \subset U$ of p such that

(3.2.43) $\quad$ T is a C^∞ - diffeomorphism on $\hat{U}$.

Choose an open neighborhood $\mathcal{E} \subset U$ of p so that

(3.2.44) $T(\mathcal{E})$ is open and connected.

Define $g : T(\mathcal{E}) \to R$ by

(3.2.45) $g \triangleq \eta \circ T^{-1}$.

Then, $\eta = g \circ T$. Since $\{ d\theta_j(q),\ j \in \mathcal{M}_{1,(n-r)} \}$ is a basis of Δ_q, $q \in U$, it follows that

(3.2.46) $D_j g(\theta_1(q), \cdots, \theta_n(q)) = 0, \quad q \in \mathcal{E}$ if $j > n-r$.

By Theorem 2.3.8 and (3.2.44), this implies $g(y_1, \ldots, y_n) = c(y_1, \ldots, y_{n-r})$ on $T(\mathcal{E})$ and (3.2.41) follows. Next, suppose that (3.2.41) holds on U. Then, $d\eta(x)$ is a linear combination of $d\theta_j(x)$, $j \in \mathcal{M}_{1,(n-r)}$ at each point $x \in U$. Thus, (3.2.40) holds. □

Lemma 3.2.5. Suppose that (A.1), (A.3), (A.4) are satisfied. Then, at each point $x_0 \in \mathcal{X}$, there exist $(\sum_{i=1}^{m} p_i)$ C^∞- functions $\xi_{i,j}$, $j \in \mathcal{M}_{1,pi}$, $i \in \mathcal{M}_{1,m}$ from an open neighborhood $\mathcal{V}$ of x_0 into R such that

(i) $d\xi_{i,j}$, $j \in \mathcal{M}_{1,pi}$, $i \in \mathcal{M}_{1,m}$ are linearly independent on $\mathcal{V}$,

(ii) $d\xi_{i,j} \in \Delta_i^{\perp}(\{F, H, \mathcal{X}\})$ on $\mathcal{V}$, $j \in \mathcal{M}_{1,pi}$, $i \in \mathcal{M}_{1,m}$.

Proof. Fix $x_0 \in \mathcal{X}$. By Lemma 3.2.3, (A.1) and (A.4) imply that for each $i \in \mathbf{M}_{1,m}$, there exists an open neighborhood $\mathcal{E}_i$ of x_0 and C^∞-functions $\xi_{i,j}$, $j \in \mathbf{M}_{1,pi}$ from $\mathcal{E}_i$ into R such that

(3.2.47) $d\xi_{i,j}$, $j \in \mathbf{M}_{1,pi}$ are linearly independent on $\mathcal{E}_i$,

(3.2.48) $d\xi_{i,j} \in \Delta_i^\perp(\,[F, H, \mathcal{X}]\,)$ on $\mathcal{E}_i$, $j \in \mathbf{M}_{1,pi}$.

Let $\mathcal{V} \triangleq \mathcal{E}_1 \cap \cdots \cap \mathcal{E}_m$. Then, (3.2.48) implies (ii). Now, we show that (A.3) implies (i). Suppose that at a point $\hat{x}_0 \in \mathcal{V}$, there exist constants $\alpha_{i,j}(\hat{x}_0)$, $j \in \mathbf{M}_{1,pi}$, $i \in \mathbf{M}_{1,m}$ such that

$$(3.2.49) \qquad \sum_{i=1}^{m} \sum_{j=1}^{p_i} \alpha_{i,j}(\hat{x}_0)\, d\xi_{i,j}(\hat{x}_0) = 0.$$

Let $\eta \triangleq \sum_{i=1}^{m} \sum_{j=1}^{p_i} \alpha_{i,j}(\hat{x}_0)\, \xi_{i,j}$ on $\mathcal{V}$. Fix $\ell \in \mathbf{M}_{1,m}$. Let $\eta_\ell \triangleq \sum_{j=1}^{p_\ell} \alpha_{\ell,j}(\hat{x}_0)\, \xi_{\ell,j}$ on $\mathcal{V}$. Define a multiindex I_ℓ by any finite sequence of integers taken $\mathbf{M}_{0,m}$ such that at least one of its elements is ℓ. For such a multiindex $I_\ell \triangleq \{\, i_1, \cdots, i_k \,\}$, define a vector field $Y_{I_\ell} \triangleq L_{X_{i_1}} \cdots L_{X_{i_{k-1}}} X_{i_k}$. Let $\bar{\Delta}_\ell$ be the set of all such vector fields Y_{I_ℓ}. Then, from (3.2.49), (ii), and Lemma 3.2.2, it follows that

$$(3.2.50) \qquad Y_{I_\ell}\eta(\hat{x}_0) = Y_{I_\ell}\eta_\ell(\hat{x}_0) = 0, \qquad Y_{I_\ell} \in \bar{\Delta}_\ell.$$

On the other hand, by (ii) and the definition of η, (3.2.51) $\quad Z\eta_{\ell}(\hat{x}_0) = 0, \quad Z \in \Delta_{\ell}(\{F, H, \mathcal{X}\})$.

By the definition of $\mathcal{L}^0(\{F, H, \mathcal{X}\})$, (3.2.50) and (3.2.51) imply

(3.2.52) $\quad Y\eta_{\ell}(\hat{x}_0) = 0, \quad Y \in \mathcal{L}^0(\{F, H, \mathcal{X}\})$

By the definition of $\mathcal{L}(\{F, H, \mathcal{X}\})$, (3.2.52) holds for all $Y \in \mathcal{L}(\{F, H, \mathcal{X}\})$. By (A.3), this implies

(3.2.53) $\quad 0 = d\eta_{\ell}(\hat{x}_0) = \sum_{j=1}^{p_{\ell}} \alpha_{\ell,j}(\hat{x}_0)\, d\xi_{\ell,j}(\hat{x}_0)$,

and from (3.2.47),

(3.2.54) $\quad \alpha_{\ell,j}(\hat{x}_0) = 0, \quad j \in \mathbf{M}_{1,p_{\ell}}$.

Since ℓ was chosen arbitrarily, we conclude that

(3.2.55) $\quad \alpha_{i,j}(\hat{x}_0) = 0, \quad j \in \mathbf{M}_{1,p_i}, \quad i \in \mathbf{M}_{1,m}$,

and our proof of (i) is complete. □

It is interesting to note that although the distributions Δ_i, $i \in \mathbf{M}_{1,m}$ do not satisfy the conditions of Respondek, Tarn and others ([Res.1, Tar.1]), this lemma shows that these distributions are still simultaneously integrable in their terminology. Now, we present the proof of Theorem 3.2.3.

Proof of Theorem 3.2.3. Assume $\{F, H, \mathcal{X}\}$ is decomposed at each $x_0 \in \mathcal{X}$. Fix $x_0 \in \mathcal{X}$. Then, by Definition 3.1.5, there exist: (a) an open neighborhood $\mathcal{E}$ of x_0; (b) an open subset $\bar{\mathcal{X}}$ of R^n; (c) a C^∞-diffeomorphism $T : \mathcal{E} \to \bar{\mathcal{X}}$; (d) integers $s_i \geq 1$, $i \in \mathcal{M}_{1,m}$ and $s_{m+1} \geq 0$ satisfying $n = \sum_{i=1}^{m+1} s_i$; and (e) a system $\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\}$ which is T-related on $\mathcal{E}$ to $\{F, H, \mathcal{E}\}$ such that its coordinate representation $\{\bar{f}, \bar{h}, \bar{\mathcal{X}}\}$ has the form (3.1.6). Let $T \triangleq (T_1, \cdots, T_m, T_{m+1})$ and $T_i \triangleq (T_{i,1}, \cdots, T_{i,si})$, $i \in \mathcal{M}_{1,m+1}$.

Let $\bar{X}_i$, $i \in \mathcal{M}_{0,m}$ be the vector fields corresponding to $\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\}$.

Fix $i \in \mathcal{M}_{1,m}$. Let ϕ_i be any C^∞-function of x_i only. Then, by the special structure of $\{\bar{f}, \bar{h}, \bar{\mathcal{X}}\}$, (3.1.1), and Definition 3.1.1, we obtain

$$(3.2.56) \quad (L_{\bar{X}i_1} \cdots L_{\bar{X}i_k} X_j)\, \phi_i = 0 \quad \text{on } T(\mathcal{E}),$$

$$i_\ell \in \{0, i\},\ j \in \bar{\mathcal{M}}_i,\ k \in \mathcal{M}_{0,\infty}.$$

Since $\bar{X}_i$ is T-related on $\mathcal{E}$ to X_i, $i \in \mathcal{M}_{0,m}$, Fact 2.3.2 and (3.2.56) imply

$$(3.2.57) \quad (L_{Xi_1} \cdots L_{Xi_k} X_j)\, (\bar{\phi}_i \circ T_i) = 0 \quad \text{on } \mathcal{E}$$

$$\text{if } i_\ell \in \{0, i\},\ j \in \bar{\mathcal{M}}_i,\ k \in \mathcal{M}_{0,\infty}.$$

On the other hand,

(3.2.58) $H_i(x) = \bar{H}_i(T_i(x))$, $x \in \mathcal{E}$, $i \in \mathcal{M}_{1,m}$.

By the definition of $\Delta_i(\{F, H, \mathcal{X}\})$, (3.2.57) and (3.2.58) imply

(3.2.59) $Y H_i = 0$ on $\mathcal{E}$, $Y \in \Delta_i(\{F, H, \mathcal{X}\})$, $i \in \mathcal{M}_{1,m}$.

Since x_0 can be arbitrarily chosen, (3.2.59) implies (3.2.1).

Now assume (3.2.1) holds. By (A.1), (A.3), (A.4), Lemma 3. 2. 5 may be applied. Fix $x_0 \in \mathcal{X}$. We use the same notation in Lemma 3. 2. 5 except that $\xi_{i,j}$ is denoted by $T_{i,j}$, $j \in \mathcal{M}_{1,p_i}$, $i \in \mathcal{M}_{1,m}$. Let $T_i \triangleq (T_{i,1}, \ldots, T_{i,p_i})$, $i \in \mathcal{M}_{1,m}$. Let $p_{m+1} \triangleq n - \sum_{i=1}^{m} p_i$. If $p_{m+1} > 0$, it is possible to choose other C^∞-functions $T_{m+1,j}$, $j \in \mathcal{M}_{1,p_{m+1}}$ from $\mathcal{V}$ into $R^{p_{m+1}}$ so that $T \triangleq (T_1, \cdots, T_m, T_{m+1})$ has rank n at x_0 where $T_{m+1} \triangleq (T_{m+1,1}, \cdots, T_{m+1,p_{m+1}})$. Then, by Theorem 2. 3. 6, there exists an open neighborhood $W \subset \mathcal{V}$ of x_0 such that T is a C^∞-diffeomorphism from W into R^n. It then follows that there exist C^∞-functions $\bar{f}_{m+1,j}$, $\bar{b}_{i,j}$, $i \in \mathcal{M}_{1,m}$, $j \in \mathcal{M}_{1,p_{m+1}}$ such that

(3.2.60) $X_0 T_{m+1,j}(x) = \bar{f}_{m+1,j}(T(x))$, $X_i T_{m+1,j}(x) = \bar{b}_{i,j}(T(x))$, $x \in W$.

On the other hand, for each $i \in \mathcal{M}_{1,m}$, Lemma 3. 2. 4 holds with $r \triangleq n - p_i$ and $\theta_j \triangleq T_{i,j}$, $j \in \mathcal{M}_{1,p_i}$. Thus, by (3.2.1) and Lemma

3.2.4, there exist an open neighborhood $U \subset W$ of x_0 and C^∞-functions h_i, defined on an appropriate subset of R^{pi} such that

(3.2.61) $H_i(x) = \bar{h}_i(T_i(x)), \quad x \in U, \quad i \in \mathbf{M}_{1,m}$.

Next, by Lemma 3.2.2-(ii) and Lemma 3.2.5-(ii),

(3.2.62) $dX_0 T_{i,j},\ dX_i T_{i,j} \in \Delta_i^\perp(\{F, H, \mathbf{X}\})$ on U, $j \in \mathbf{M}_{1,pi}$, $i \in \mathbf{M}_{1,m}$.

Consequently, applying Lemma 3.2.4, it follows that there exist an open neighborhood $\mathbf{E} \subset U$ of x_0 and C^∞-functions $\bar{f}_{i,j}$, $\bar{g}_{i,j}$, $j \in \mathbf{M}_{1,pi}$, $i \in \mathbf{M}_{1,m}$ such that

(3.2.63) $X_0 T_{i,j}(x) = \bar{f}_{i,j}(T_i(x)), \quad X_i T_{i,j}(x) = \bar{g}_{i,j}(T_i(x)), \quad x \in \mathbf{E}$

Let $\bar{\mathbf{X}} \triangleq T(\mathbf{E})$. Let $\bar{x} \triangleq (\bar{x}_1, \cdots, \bar{x}_m, \bar{x}_{m+1}) \triangleq (T_1(x), \cdots, T_{m+1}(x))$. Let $\bar{f}_i \triangleq (\bar{f}_{i,1}, \ldots, \bar{f}_{i,pi})$, $\bar{g}_i \triangleq (\bar{g}_{i,1}, \cdots, \bar{g}_{i,pi})$, $i \in \mathbf{M}_{1,m}$. Let $\bar{f}_{m+1} \triangleq (\bar{f}_{m+1,1}, \ldots, \bar{f}_{m+1,pm+1})$, $\bar{b}_i \triangleq (\bar{b}_{i,1}, \cdots, \bar{b}_{i,pm+1})$, $i \in \mathbf{M}_{1,m}$. Let $\bar{H} \triangleq \bar{h} = (\bar{h}_1, \cdots, \bar{h}_m)$. Define vector fields $\bar{X}_i$, $i \in \mathbf{M}_{0,m}$ by

(3.2.64) $\bar{X}_0(\bar{x}) \triangleq \sum_{i=1}^{m} \sum_{j=1}^{p_i} \bar{f}_{i,j}(\bar{x}_i)\,(\partial/\partial\bar{x}_{i,j}) + \sum_{j=1}^{p_{m+1}} \bar{f}_{m+1,j}(\bar{x})\,(\partial/\partial\bar{x}_{m+1,j})$,

(3.2.65) $\bar{X}_i(\bar{x}) \triangleq \sum_{j=1}^{p_i} \bar{g}_{i,j}(\bar{x}_i)\,(\partial/\partial\bar{x}_{i,j}) + \sum_{j=1}^{p_{m+1}} \bar{b}_{i,j}(\bar{x})\,(\partial/\partial\bar{x}_{m+1,j}), \quad i \in \mathbf{M}_{1,m}$.

Note from (3.2.60), (3.2.62), (3.2.63), that $\bar{X}_i$, $i \in \mathbf{M}_{0,m}$, and $\bar{H}_i$, $i \in \mathbf{M}_{1,m}$ are T-related on $\mathbf{E}$ to X_i, $i \in \mathbf{M}_{0,m}$, and H_i, $i \in \mathbf{M}_{1,m}$, respectively. Let $(\bar{F}, \bar{H}, \bar{\mathcal{X}})$ be the system constructed as above. Then, it is an easy consequence that the above $\mathcal{E}$, $(\bar{F}, \bar{H}, \bar{\mathcal{X}})$ with $\bar{s}_i \triangleq p_i$, $i \in \mathcal{N}_{1,(m+1)}$ meet the requirements of Definition 3. 1. 5. □

In this section, we have elaborated on the difference between decoupling and decomposition and have presented algebraic conditions related to them. Examples of systems which are decoupled and decomposed are easy to construct. See for example, the standard decomposed system of Definition 3. 1. 5. It is not easy to give an example of a system which is decoupled but not decomposed. The various results are connected to one another in the way described in the Figure 3. 2. 1.

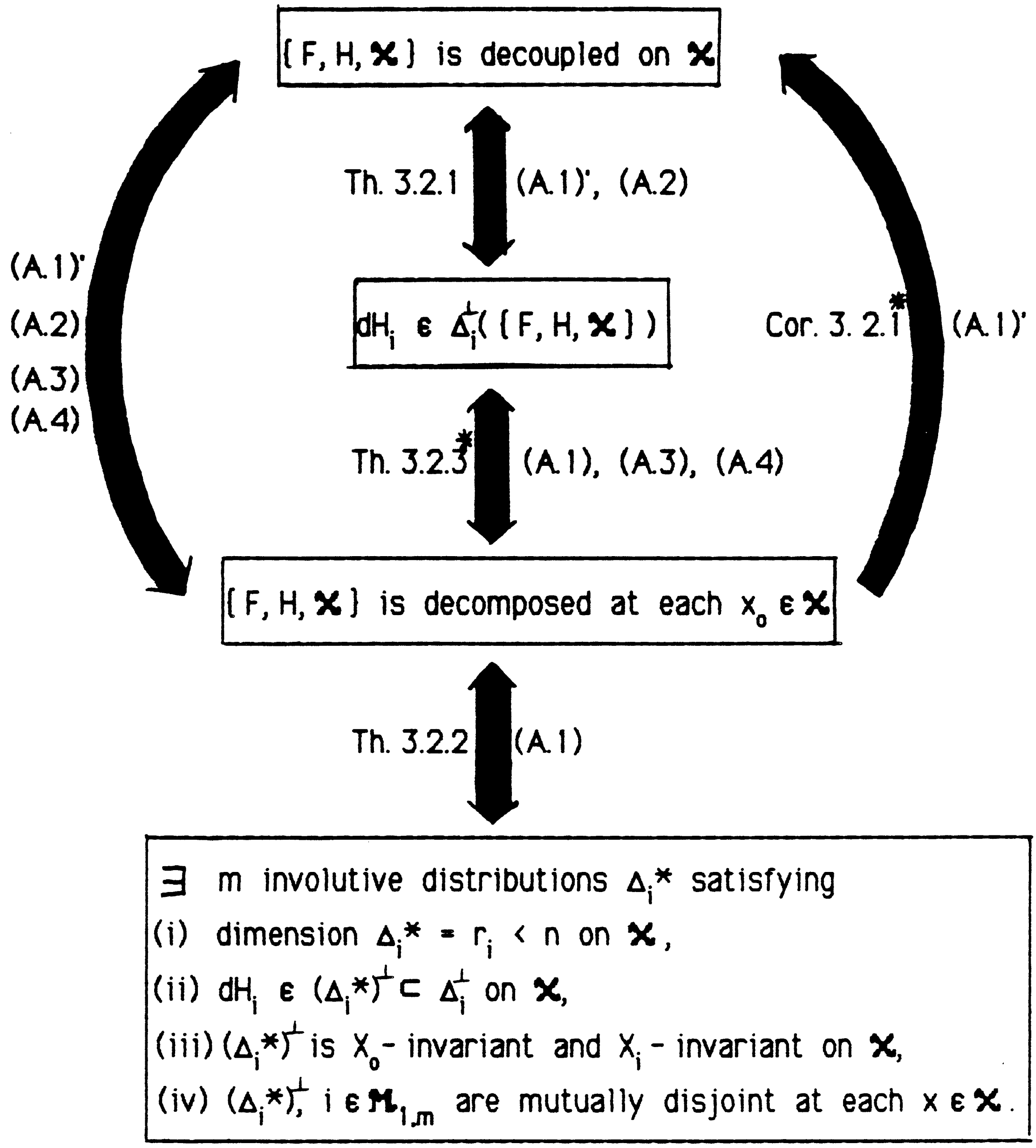

Figure 3.2.1. Summary of main results in Section 3.2 showing assumptions required for each implication.

Section 3. 3. Decouplability and Decomposability

We now turn to the question of when a system is decouplable by a control law. Results concerning local decomposability (when a system is decomposable at a point $x_0 \in \mathcal{X}$) are found in [Isi.1, Nij.2] , where Wonham and Morse's geometric approach ([Mos.1, Mos.2, Won.1, Won.2]) is generalized to nonlinear systems. They are sufficient conditions for local decouplability since a locally decomposable system is always locally decouplable. Little has been done on the question of global decomposability (when a system is decomposable on $\mathcal{X}$). Results concerning global decouplability (the system is decouplable on $\mathcal{X}$) are found in [Cla.1, Fre.1, Maj.1, Por.1, Sin.1, Sih.1]. These results extend Falb and Wolovich's result on linear decoupling to nonlinear systems. The earliest of all the above papers is by Singh and Rugh ([Sin.1]). Subsequent papers add relatively little to their results.

In this section, we discuss global decouplablity based on our precise definition of decoupling (Definition 3. 1. 3) and investigate the connections between decouplability and decomposability. We add the following assumptions to the list we made in Section 3. 2.

(A. 5) The control law $u = \alpha(x) + \beta(x)\hat{u}$ is smooth,

(A. 5)' The control law $u = \alpha(x) + \beta(x)\hat{u}$ is real analytic,

(A. 6) $\beta(x)$ is nonsingular on $\mathcal{X}$,

(A. 7) There exist $d_i \in \mathbf{N}$, $i \in \mathcal{M}_{1,m}$ such that the following m - row vector conditions are satisfied :

$$(3.3.1) \qquad [\, X_1 X_0^k H_i(x) \cdots X_m X_0^k H_i(x)\,] = 0, \; x \in \mathcal{X}, \; k \in \mathbf{M}_{0,(d_i-1)},$$

$$(3.3.2) \qquad D_i^*(x) \triangleq [\, X_1 X_0^{d_i} H_i(x) \cdots X_m X_0^{d_i} H_i(x)\,] \neq 0, \; x \in \mathcal{X}.$$

We begin this section by giving a necessary and sufficient condition for global decouplability. When (A.7) is satisfied, let $D^*(x)$ and $A^*(x)$ denote, respectively the $(m \times m)$ and $(m \times 1)$ matrices of functions defined by

$$(3.3.3) \qquad D^*(x) \triangleq \begin{bmatrix} D_1^*(x) \\ \cdots \\ D_m^*(x) \end{bmatrix}, \qquad A^*(x) \triangleq \begin{bmatrix} X_0^{(d_1+1)} H_1(x) \\ \cdots \\ X_0^{(d_m+1)} H_m(x) \end{bmatrix}.$$

Theorem 3. 3. 1. Suppose $\{F, H, \mathcal{X}\}$ satisfies (A.1), (A.7) and the class of control laws satisfies (A.5), (A.6). Then, $\{F, H, \mathcal{X}\}$ is decouplable on $\mathcal{X}$ if and only if

$$(3.3.4) \qquad D^*(x) \text{ is nonsingular at each } x \in \mathcal{X}.$$

Furthermore, $u = [D^*(x)]^{-1}(\hat{u} - A^*(x))$ decouples $\{F, H, \mathcal{X}\}$ on $\mathcal{X}$. That is, for $\alpha(x) \triangleq -[D^*(x)]^{-1}A^*(x)$ and $\beta(x) \triangleq [D^*(x)]^{-1}$, the system $\{F, H, \mathcal{X}\}^{\alpha,\beta}$ is decoupled on $\mathcal{X}$. □

This is actually a nonlinear version of the well known result by Falb and Wolovich on linear decoupling ([Fal.1]). The sufficiency part of Theorem 3. 3. 1 is proved in [Cla.1, Fre.1, Maj.1, Por.1, Sin.1, Sih.1]. Its necessity part is stated in [Maj.1, Sin.1], but the arguments are not entirely clear. We shall prove the necessity part rigorously, based on our precise definition of decoupling.

We need the following lemma for the proof of Theorem 3.3.1.

Lemma 3.3.1. Suppose that $\{F, H, \mathcal{X}\}$ satisfies (A.1) and (A.7). Consider control laws $u = \alpha(x) + \beta(x)\hat{u}$ which satisfy (A.5) and (A.6). Let $\{\hat{F}, \hat{H}, \hat{\mathcal{X}}\}$ be J- feedback related on $\mathcal{X}$ to $\{F, H, \mathcal{X}\}$ by $J = \{T, \alpha, \beta\}$. Then :

(i) (A.7) is satisfied on $T(\mathcal{X})$ with $\hat{d}_i = d_i$, $i \in \mathcal{M}_{1,m}$,

(ii) $\hat{D}^*(T(x)) = D^*(x)\beta(x)$, $\hat{A}^*(T(x)) = A^*(x) + D^*(x)\alpha(x)$, $x \in \mathcal{X}$,

(iii) $\hat{X}_0{}^k \hat{H}_i(T(x)) = X_0{}^k H_i(x)$, $x \in \mathcal{X}$, $k \in \mathcal{M}_{0,di}$.

Proof. Let I_0 be the identity mapping from $\mathcal{X}$ onto $\mathcal{X}$. We show that for two special cases, $J = \{I_0, \alpha, \beta\}$ and $J = \{T, 0, I_m\}$, (i) and (ii) hold. Then, by the transitivity of J - feedback relations, (i) and (ii) hold for the general case, $J = \{T, \alpha, \beta\}$.

First, we consider the case of $J = \{I_0, \alpha, \beta\}$. Then, $\{\hat{F}, \hat{H}, \hat{\mathcal{X}}\} = \{F, H, \mathcal{X}\}^{\alpha,\beta}$ with $\hat{\mathcal{X}} = \mathcal{X}$. Let $\hat{X}_i$, $i \in \mathcal{M}_{0,m}$ be the vector fields corresponding to $\{F, H, \mathcal{X}\}$. Then, for all $x \in \mathcal{X}$,

$$(3.3.5) \qquad \hat{X}_0 = X_0 + \sum_{i=1}^{m} \alpha_i(\cdot) X_i,$$

$$(3.3.6) \qquad \hat{X}_j = \sum_{i=1}^{m} \beta_{i,j}(\cdot) X_i, \quad \hat{H}_i = H_j, \ j \in \mathcal{M}_{1,m}.$$

where α_i is the ith component of α, $\beta_{i,j}$ is the (i, j) th component of β. From this and the definition of $\{d_i, i \in \mathcal{M}_{1,m}\}$, it follows that for all $i \in \mathcal{M}_{1,m}$ and $x \in \mathcal{X}$,

$$(3.3.7) \qquad \hat{X}_0^k \hat{H}_i = X_0^k H_i, \quad k \in \mathcal{M}_{0,di},$$

$$(3.3.8) \qquad \hat{X}_j \hat{X}_0^k \hat{H}_i = \hat{X}_j X_0^k H_i = 0, \quad j \in \mathcal{M}_{1,m}, \; k \in \mathcal{M}_{0,(di-1)},$$

$$\hat{X}_j \hat{X}_0^{d_i} \hat{H}_i = \sum_{k=1}^{m} \beta_{k,j}(\cdot) X_k X_0^{d_i} H_i, \quad j \in \mathcal{M}_{1,m}.$$

By (3.3.8),

$$(3.3.9) \qquad [\hat{X}_1 \hat{X}_0^{d_i} \hat{H}_i(x) \ \ldots \ \hat{X}_m \hat{X}_0^{d_i} \hat{H}_i(x)] = D_i^*(x)\, \beta(x) \quad \text{on } \mathcal{X}.$$

By (A.6) and (3.3.2), this implies

$$(3.3.10) \qquad [\hat{X}_1 \hat{X}_0^{d_i} \hat{H}_i(x) \ \ldots \ \hat{X}_m \hat{X}_0^{d_i} \hat{H}_i(x)] \neq 0, \quad x \in \mathcal{X}.$$

The definition of $\{\hat{d}_i,\ i \in \mathcal{M}_{1,m}\}$ for $\{\hat{F}, \hat{H}, \hat{\mathcal{X}}\}$, (3.3.7), (3.3.8), and (3.3.10) imply (i), (ii), and (iii).

Next, we consider the case of $J = \{T, 0, I_m\}$. Then, $\{\hat{F}, \hat{H}, \hat{\mathcal{X}}\}$ is T-related on $\mathcal{X}$ to $\{F, H, \mathcal{X}\}$. Therefore, by Definition 3.1.1 and Fact 2.3.3, we have

$$(3.3.11) \qquad \hat{X}_0^k \hat{H}_i(T(x)) = X_0^k H_i(x), \quad x \in \mathcal{X}, \; k \in \mathcal{M}_{0,\infty}, \; i \in \mathcal{M}_{1,m},$$

$$(3.3.12) \qquad \hat{X}_j \hat{X}_0^k \hat{H}_i(T(x)) = X_j X_0^k H_i(x), \quad x \in \mathcal{X}, \; k \in \mathcal{M}_{0,\infty}, \; i, j \in \mathcal{M}_{1,m}.$$

This implies (i), (ii), and (iii). □

Lemma 3.3.1 is a nonlinear version of the invariant property of the integers, d_i, $i \in \mathcal{M}_{1,m}$ on linear systems, which is shown in [Gil.1]. The case of $J \triangleq \{ I_0, \alpha, \beta \}$ and $\mathcal{X} \triangleq R^n$ is proved in [Por.1]. Now, we prove Theorem 3. 3. 1.

Proof of Theorem 3. 3. 1 First, assume (3.3.4) holds. Let $\{ \hat{F}, \hat{H}, \mathcal{X} \} \triangleq \{ F, H, \mathcal{X} \}^{\alpha,\beta}$ with $\alpha(x) \triangleq -[D^*(x)]^{-1}A^*(x)$ and $\beta(x) \triangleq [D^*(x)]^{-1}$. Then, $\{ F, H, \mathcal{X} \}$ is J - feedback related on $\mathcal{X}$ to $\{ \hat{F}, \hat{H}, \mathcal{X} \}$ by $J = \{ I_0, A^*, D^* \}$. The vector fields associated with $\{ \hat{F}, \hat{H}, \mathcal{X} \}$ are given by (3.3.5) and (3.3.6). By these observations and Lemma 3. 3. 1, direct computation shows that

$$(3.3.13) \quad \hat{X}_0^{\,k} \hat{H}_i(x) = X_0^{\,k} H_i(x), \qquad k \in \mathcal{M}_{0,di},$$

$$\hat{X}_0^{(d_i+1)} \hat{H}_i(x) = 0,$$

$$(3.3.14) \quad \hat{X}_j \hat{X}_0^{\,k} \hat{H}_i(x) = \begin{cases} 1 & \text{if } j = i \text{ and } k = d_i, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\hat{y}_i$ be the ith output of $\{ \hat{F}, \hat{H}, \mathcal{X} \}$. Then, by (3.3.13) and (3.3.14), differentiating $\hat{y}_i$ (d_i+1) times with respect to t leads to

$$(3.3.15) \quad \hat{y}_i(t) = \hat{H}_i(x(t)),$$

$$\hat{y}_i^{(1)}(t) = \hat{X}_0 \hat{H}_i(x(t)) + \sum_{j=1}^{m} \hat{u}_j(t) \hat{X}_j \hat{H}_i(x(t))$$

$$= \begin{cases} \hat{u}_i(t), & d_i = 0, \\ \hat{X}_0 \hat{H}_i(x(t)), & d_i > 0, \end{cases}$$

. . . . .

(3.3.16) $\hat{y}_i^{(d_i+1)}(t) = \hat{u}_i(t),$

where initial conditions are given by

(3.3.17) $\hat{y}_i^{(k)}(0) = \hat{X}_0^k \hat{H}_i(x(0)), \quad k \in \mathbf{M}_{0,di}.$

By Definition 3. 1. 3, this implies $\{\hat{F}, \hat{H}, \mathbf{X}\}$ is decoupled on $\mathbf{X}$.

Next, assume that there exists a control law $u = \alpha(x) + \beta(x)\hat{u}$ such that $\{F, H, \mathbf{X}\}^{\alpha,\beta}$ is decoupled on $\mathbf{X}$. Let $\{\hat{F}, \hat{H}, \mathbf{X}\} = \{F, H, \mathbf{X}\}^{\alpha,\beta}$. Let $\hat{X}_i$, $i \in \mathbf{M}_{1,m}$ be the vector fields corresponding to $\{\hat{F}, \hat{H}, \mathbf{X}\}$. Let $\hat{y}_i$ be the ith output of $\{\hat{F}, \hat{H}, \mathbf{X}\}$. Since $\{\hat{F}, \hat{H}, \mathbf{X}\}$ is J - feedback related on $\mathbf{X}$ to $\{F, H, \mathbf{X}\}$ by J = $\{I_0, \alpha, \beta\}$, we have by Lemma 3. 3. 1,

(3.3.18) $\hat{y}_i^{(k)}(t) = \hat{X}_0^k \hat{H}_i(x(t)), \quad k \in \mathbf{M}_{0,di},$

$$\hat{y}_i^{(d_i+1)}(t) = \hat{X}_0^{(di+1)}\hat{H}_i(x(t)) + \sum_{j=1}^{m} \hat{u}_j(t)\,\hat{X}_j\,\hat{X}_0^{d_i}\hat{H}_i(x(t)).$$

But, by Definition 3. 1. 3, it follows that for any initial state $\hat{x}(0) \triangleq \hat{X}_0 \in \mathbf{X}$ and for any two inputs $\bar{u}$, $\tilde{u} \in \mathbf{U}^\infty$ with $\bar{u}_i = \tilde{u}_i$,

(3.3.19) $\Delta\hat{y}_i \triangleq \hat{\Phi}_i(\bar{u}, \hat{x}_0) - \hat{\Phi}_i(\tilde{u}, \hat{x}_0) = 0.$

This implies

(3.3.20) $\Delta\hat{y}_i^{(k)}(0) = 0, \quad k \in \mathbf{M}_{0,\infty}.$

Then, by this and (3.3.18), we must have

$$(3.3.21)\qquad \Delta\hat{y}_i^{(d_i+1)}(0) = \sum_{j=1}^{m} \{ \bar{u}_j(0) - \tilde{u}_j(0) \} \hat{X}_j \hat{X}_0^{d_i} \hat{H}_i(\hat{x}_0) = 0.$$

Since we can choose $\bar{u}_j(0)$, $\tilde{u}_j(0)$, $j \in \bar{\mathcal{M}}_1$, and $\hat{x}_0$ arbitrarily, (3.3.21) implies

$$(3.3.22)\qquad \hat{X}_j \hat{X}_0^{d_i} \hat{H}_i = 0 \text{ on } \mathcal{X}, \quad j \in \bar{\mathcal{M}}_1.$$

By Lemma 3. 3. 1 - (i), and the definition of $\{ d_i,\ i \in \mathcal{M}_{1,m} \}$, (3.3.22) implies

$$(3.3.23)\qquad \lambda_i(x) \triangleq \hat{X}_i \hat{X}_0^{d_i} \hat{H}_i(x) \neq 0,\ x \in \mathcal{X},\ i \in \mathcal{M}_{1,m}.$$

On the other hand, by Lemma 3. 3. 1 - (ii), (3.3.22), and (3.3.23),

$$(3.3.24)\qquad D^*(x)\,\beta(x) = \operatorname{diag} \lambda_i(x), \quad x \in \mathcal{X}.$$

Then, (3.3.4) is a direct consequence of (A.6), (3.3.23), and (3.3.24).

□

Because of its importance in our subsequent developments, we henceforth reserve the notation $\{ F^*, H^*, \mathcal{X} \}$ for the system $\{ F, H, \mathcal{X} \}^{\alpha,\beta}$, where $\alpha(x) \triangleq -[D^*(x)]^{-1} A^*(x)$, $\beta(x) \triangleq [D^*(x)]^{-1}$.

Remark 3. 3. 1: The input - output map for $\{ F^*, H^*, \mathcal{X} \}$ is determined by equations of the form (3.3.16). By Definition 3. 1. 4, this implies that $\{F^*, H^*, \mathcal{X}\}$ is also input-output decoupled on $\mathcal{X}$. Thus, under the assumptions (A.1), (A.5), (A.6), and (A.7), (3.3.4) is a necessary and sufficient condition for both decouplablity and input - output decouplability. Our result on input - output decouplability of smooth systems is stronger than the one which appears in [Nij.3, Nij.4]. The result there is local. Moreover, it is derived on the basis of algebraic conditions for decoupling whose validity is not clear, as was discussed in Section 3. 2. □

Next, we consider decomposability.

Theorem 3. 3. 2. Suppose that the hypotheses in Theorem 3. 3. 1 are satisfied. Then, $\{F, H, \mathcal{X}\}$ is decomposable at each $x_0 \in \mathcal{X}$ if and only if (3.3.4) holds. □

Theorem 3. 3. 1 and Theorem 3. 3. 2 have the important implication that under the assumptions (A.1), (A.5), (A.6), and (A.7), decouplability and decomposability are equivalent. In [Isi.1], the sufficiency of Theorem 3. 3. 2 follows under additional assumptions, which are basically equivalent to assuming that $dX_0^k H_i(x)$, $k \in \mathbf{N}_{1,di}$, $i \in \mathbf{N}_{1,m}$ are linearly independent on $\mathcal{X}$. But, as will be shown in Lemma 3. 3. 3, this is automatically implied by (A.7) and (3.3.4). We believe that the necessity of Theorem 3. 3. 2 is new. The following Lemmas are needed for the proof of Theorem 3. 3. 2.

Lemma 3. 3. 2. Let $\ell \in \mathbf{N}$, $\phi \in C^{\infty}(\mathcal{X})$. Let Y, Z $\in V^{\infty}(\mathcal{X})$. If

$$(3.3.25) \quad dZY^{k}\phi = 0 \text{ on } \mathcal{X},\ k \in \mathcal{M}_{0,\ell},$$

then,

$$(3.3.26) \quad (L^{i}_{Y}Z)\,Y^{(j-i)}\phi = (-1)^{i}\,Z\,Y^{j}\phi \text{ on } \mathcal{X},\ j \in \mathcal{M}_{0,(\ell+1)},\ i \in \mathcal{M}_{0,j}.$$

Proof. From [Var.1], we have

$$(3.3.27) \quad L^{i}_{Y}Z = (-1)^{i}\sum_{k=0}^{i}(-1)^{k}\,[\,i!\,/\,(\,k!\,(i-k)!\,)\,]\,Y^{k}\,Z\,Y^{(i-k)},\ i \in \mathcal{M}_{0,\infty}.$$

Postmultiplying $Y^{(j-i)}\phi$ on the both sides of (3.3.27) and using (3.3.25) yield (3.3.26). □

Lemma 3. 3. 3. Suppose that a system $\{F, H, \mathcal{X}\}$ satisfies (A.7) and (3.3.4). Then, $dX_0^{\,k}H_i$, $k \in \mathcal{M}_{0,di}$, $i \in \mathcal{M}_{1,m}$ are linearly independent on $\mathcal{X}$.

Proof. Let $\{\hat{F}, \hat{H}, \mathcal{X}\} \triangleq \{F^{*}, H^{*}, \mathcal{X}\}$. Then, (3.3.14) holds. By Lemma 3. 3. 2, this implies

$$(3.3.28) \quad (L^{k}_{\hat{X}_0}\hat{X}_j)\,\hat{X}_0^{\,r}\,\hat{H}_i = \begin{cases} (-1)^{k} & \text{if } j = i,\ k = d_i - r,\ r \in \mathcal{M}_{0,di}, \\ 0 & \text{otherwise.} \end{cases}$$

By using (3.3.28), we now show that $dX_0^k H_i(x)$, $k \in \mathbf{M}_{0,di}$, $i \in \mathbf{M}_{1,m}$ are linearly independent at each point $x \in \mathbf{X}$.

Suppose that at a point $x_0 \in \mathbf{X}$, there exist constants $\gamma_{i,j}(x_0)$, $j \in \mathbf{M}_{0,di}$, $i \in \mathbf{M}_{1,m}$ such that

$$(3.3.29) \quad \sum_{i=1}^{m} \sum_{j=0}^{d_i} \gamma_{i,j}(x_0)\, d\hat{X}_0^j \hat{H}_i(x_0) = 0.$$

Define a C^∞- function η from $\mathbf{X}$ into R by

$$(3.3.30) \quad \eta \triangleq \sum_{i=1}^{m} \sum_{j=0}^{d_i} \gamma_{i,j}(x_0)\, \hat{X}_0^j \hat{H}_i .$$

Then, by (3.3.29),

$$(3.3.31) \quad (L^k_{\hat{X}_0}\hat{X}_j)\,\eta(x_0) = 0, \quad k \in \mathbf{M}_{0,\infty},\ j \in \mathbf{M}_{1,m}.$$

Applying (3.3.28) to (3.3.31) and choosing k, j appropriately lead to

$$(3.3.32) \quad \gamma_{i,j}(x_0) = 0, \quad j \in \mathbf{M}_{0,di}, \quad i \in \mathbf{M}_{1,m}.$$

This with (3.3.13) completes the proof. □

Before presenting the proof of Theorem 3. 3. 1, we give some comments on Lemma 3. 3. 3. In [Sin.1, Fre.1], it is shown that under the same assumptions, $X_0^k H_i$, $k \in \mathbf{M}_{0,di}$, $i \in \mathbf{M}_{1,m}$ are functionally independent on $\mathbf{X}$. But as mentioned in Section 2. 2, this does not necessarily imply that $dX_0^k H_i$, $k \in \mathbf{M}_{0,di}$, $i \in \mathbf{M}_{1,m}$ are linearly independent on $\mathcal{X}$. The converse is, however, always true. In [Isi.1], linear independence of $dX_0^k H_i$, $k \in \mathbf{M}_{0,di}$, $i \in \mathbf{M}_{1,m}$ is assumed in addition to (A.7) and (3.3.4). Now, we prove Theorem 3. 3. 2.

Proof of Theorem 3. 3. 2 Suppose there exists α, β such that $\{F, H, \mathcal{X}\}^{\alpha,\beta}$ is decomposed at each $x_0 \in \mathcal{X}$. Let $\{\hat{F}, \hat{H}, \mathcal{X}\} \triangleq \{F, H, \mathcal{X}\}^{\alpha,\beta}$. Then, by Theorem 3. 2. 2,

(3.3.33) $\quad d\hat{H}_i \in \Delta_i^{\perp}(\{\hat{F}, \hat{H}, \mathcal{X}\})$ on $\mathcal{X}$, $i \in \mathbf{M}_{1,m}$.

By Lemma 3. 3. 1 - (i), $\hat{d}_i = d_i$. This with (3.3.33) and Lemma 3. 2. 2 shows that (3.3.22) holds for each $i \in \mathbf{M}_{1,m}$. The remaining arguments are exactly the same as those following (3.3.22).

Next, assume (3.3.4). Let $\{\hat{F}, \hat{H}, \mathcal{X}\} \triangleq \{F^*, H^*, \mathcal{X}\}$. Then, (3.3.13) and (3.3.14) hold. Let $T_{i,j} \triangleq \hat{X}_0^{(j-1)} \hat{H}_i$, $j \in \mathbf{M}_{1,(di+1)}$, $i \in \mathbf{M}_{1,m}$. Then, by (3.3.13) and (3.3.14), the following equations hold on $\mathcal{X}$:

$$\text{(3.3.34)} \quad \hat{X}_0 T_{i,k} = \begin{cases} T_{i,(k+1)}, & k \in \mathbf{M}_{1,di},\ i \in \mathbf{M}_{1,m}, \\ 0, & k = d_i + 1,\ i \in \mathbf{M}_{1,m}, \end{cases}$$

$$\text{(3.3.35)} \quad \hat{X}_j T_{i,k} = \begin{cases} 1, & j = i,\ k = d_i + 1,\ i \in \mathbf{M}_{1,m}. \\ 0, & j \in \mathbf{M}_i,\ k \in \mathbf{M}_{1,(di+1)},\ i \in \mathbf{M}_{1,m}. \end{cases}$$

$$\text{(3.3.36)} \quad \hat{H}_i = T_{i,1}, \quad i \in \mathbf{M}_{1,m}.$$

On the other hand, by Lemma 3. 3. 3 and (3.3.13), $d\hat{X}_0^k \hat{H}_i$, $k \in$

$\mathcal{M}_{0,di}$, $i \in \mathcal{M}_{1,m}$ are linearly independent on $\mathcal{X}$. Let $T_i \triangleq (T_{i,1}, \cdots, T_{i,(di+1)})$, $i \in \mathcal{M}_{1,m}$. Let $p \triangleq \sum_{i=1}^{m} (d_i+1)$ and $p_{m+1} \triangleq n - p$. Fix $x_0 \in \mathcal{X}$. Because $d\hat{X}_0{}^k\hat{H}_i$, $k \in \mathcal{M}_{0,di}$, $i \in \mathcal{M}_{1,m}$ are linearly independent on $\mathcal{X}$, it is possible to choose a C^∞-mapping $T_{m+1} : \mathcal{X} \to R^{pm+1}$ such that $T \triangleq (T_1, \cdots, T_m, T_{m+1})$ has rank n at x_0. Then, by Theorem 2. 3. 6, there exists an open neighborhood $\mathcal{E}$ of x_0 such that T is a C^∞-diffeomorphism from $\mathcal{E}$ into R^n. Consequently, there exist C^∞-functions $\bar{f}_{m+1,j}$, $\bar{b}_{i,j}$, $i \in \mathcal{M}_{1,m}$, $j \in \mathcal{M}_{1,pm+1}$ defined on appropriate subsets of R^n such that

$$(3.3.37)\quad \hat{X}_0 T_{m+1,j}(x) = \bar{f}_{m+1,j}(T(x)),\quad \hat{X}_i T_{m+1,j}(x) = \bar{b}_{i,j}(T(x)),\quad x \in \mathcal{E}$$

Now, Let $\bar{x} \triangleq (\bar{x}_1, \cdots, \bar{x}_m, \bar{x}_{m+1}) \triangleq (T_1(x), \cdots, T_m(x), T_{m+1}(x))$. Let $\bar{\mathcal{X}} \triangleq T(\mathcal{E})$. Define vector fields $\bar{X}_i$, $i \in \mathcal{M}_{0,m}$ and functions $\bar{H}_i$, $i \in \mathcal{M}_{1,m}$ by

$$(3.3.38)\quad \bar{X}_0(\bar{x}) \triangleq \sum_{i=1}^{m}\sum_{j=1}^{d_i} \bar{x}_{i,(j+1)} (\partial/\partial\bar{x}_{i,j}) + \sum_{j=1}^{p_{m+1}} \bar{f}_{m+1,j}(\bar{x}) (\partial/\partial\bar{x}_{m+1,j}),$$

$$(3.3.39)\quad \bar{X}_i(\bar{x}) \triangleq \partial/\partial\bar{x}_{i,(di+1)},\quad i \in \mathcal{M}_{1,m},$$

$$(3.3.40)\quad \bar{H}_i(\bar{x}) \triangleq \bar{x}_{i,1},\quad i \in \mathcal{M}_{1,m},$$

where, $\bar{x}_{i,j}$ is the jth component of $\bar{x}_i$. Let $\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\}$ be the system constructed as above. Then, the above $\mathcal{E}$, $\bar{\mathcal{X}}$, and $\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\}$ with $s_i \triangleq d_i + 1$, $i \in \mathcal{M}_{1,m}$, $s_{m+1} \triangleq p_{m+1}$ meet the requirement of Definition 3. 1. 5.

In particular, its coordinate representation $\{\bar{f}, \bar{h}, \bar{\mathcal{X}}\}$ has the form (3.1.6) such that for each $i \in \mathbf{M}_{1,m}$,

$$(3.3.41) \quad \bar{f}_i(\bar{x}_i) \triangleq \bar{A}_i \bar{x}_i, \quad \bar{g}_i(\bar{x}_i) \triangleq \bar{B}_i, \quad \bar{h}_i(\bar{x}_i) \triangleq \bar{C}_i \bar{x}_i,$$

where $\bar{A}_i \triangleq \begin{bmatrix} 0 & I_{di} \\ 0 & 0 \end{bmatrix}$, $\bar{B}_i \triangleq \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\bar{C}_i \triangleq [\,1\ 0 \cdots 0\,]$.

Since $\{F^*, H^*, \mathcal{X}\}$ is decomposed at each $x_0 \in \mathcal{X}$, $\{F, H, \mathcal{X}\}$ is decomposable at each $x_0 \in \mathcal{X}$ by the control law $u = [D^*(x)]^{-1}(u - A^*(x))$. □

Remark 3.3.2. We applied Theorem 3.2.2 to prove the necessity part of Theorem 3.3.2. Theorem 3.2.2 also yields an alternative proof of the sufficiency part, which is basically the one given in [Isi.1]. The argument goes as follows. Under the hypotheses of Theorem 3.2.2, it is not difficult to find the distributions Δ_i^*, $i \in \mathbf{M}_{1,m}$ required in Theorem 3.2.2. For $\{F^*, H^*, \mathcal{X}\}$, choose Δ_i^*, $i \in \mathbf{M}_{1,m}$ by

$$(3.3.42) \quad \Delta_i^* \triangleq \{ Y \in V^\infty(\mathcal{X}) : dX_0^k H_i Y = 0 \text{ on } \mathcal{X}, \ k \in \mathbf{M}_{0,di} \}.$$

Then, by (3.3.13), (3.3.14), and Lemma 3.3.3, Δ_i^* has a constant dimension $(n-d_i-1)$ on $\mathcal{X}$ and $(\Delta_i^*)^\perp$, $i \in \mathbf{M}_{1,m}$ are mutually disjoint at each $x \in \mathcal{X}$. Moreover, Δ_i^* is involutive on $\mathcal{X}$. By

(3.3.13), (3.3.14), and Lemma 3. 2. 2, (3.2.1) holds for $\{F^*, H^*, \mathcal{X}\}$. Thus, (3.2.38) - (i) is implied. Since by (3.3.13) and (3.3.14), Δ_i^* is $\hat{X}_0$ - invariant and $\hat{X}_i$ - invariant on $\mathcal{X}$, it automatically follows that $(\Delta_i^*)^{\perp}$ is $\hat{X}_0$ - invariant and $\hat{X}_i$ - invariant on $\mathcal{X}$. Thus, the distributions Δ_i^*, $i \in \mathcal{M}_{1,m}$ meet all requirements for Theorem 3. 2. 2 to hold. Consequently, $\{F^*, H^*, \mathcal{X}\}$ is decomposed at each $x_0 \in \mathcal{X}$. □

By adding further assumptions to those in Theorem 3. 3. 2, we can obtain a more detailed structure for $\{\bar{f}, \bar{h}, \bar{\mathcal{X}}\}$ than the one in (3.1.6) and (3.3.41). First, we define a standard decomposed system.

Definition 3. 3. 1. Let $\bar{\mathcal{X}}$ be an open connected subset of R^n. A system $\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\}$ is a **standard decomposed system** if its coordinate representation $\{\bar{f}, \bar{h}, \bar{\mathcal{X}}\}$ has the following properties :

(1) There exist nonegative integers $\bar{d}_i$, $i \in \mathcal{M}_{1,m}$ and $\bar{p}_i$, $i \in \mathcal{M}_{1,m+1}$ satisfying $n = \sum_{i=1}^{m+1} \bar{p}_i$ and $\bar{p}_i \geq \bar{d}_i + 1$, $i \in \mathcal{M}_{1,m}$ so that $\{\bar{f}, \bar{h}, \bar{\mathcal{X}}\}$ has a form :

$$\dot{\bar{x}}_i = \bar{f}_i(\bar{x}_i, \bar{u}_i) \triangleq \begin{bmatrix} \bar{A}_i \bar{x}_i \\ \bar{\theta}_i(\bar{x}_i) \end{bmatrix} + \begin{bmatrix} \bar{B}_i \\ \bar{\gamma}_i(\bar{x}_i) \end{bmatrix} \bar{u}_i, \quad \bar{y}_i = \bar{h}_i(\bar{x}_i) = \bar{C}_i \bar{x}_i, \quad i \in \mathcal{M}_{1,m}, \tag{3.3.43}$$

$$\dot{\bar{x}}_{m+1} = \bar{f}_{m+1}(\bar{x}) + \sum_{i=1}^{m} \bar{b}_i(\bar{x}) \bar{u}_i, \tag{3.3.44}$$

where: $\bar{x}_i(t) \in R^{p_i}$, $i \in \mathcal{M}_{1,m+1}$, $\bar{x} \triangleq (\bar{x}_1, \ldots, \bar{x}_{m+1}) \in R^n$; $\bar{A}_i$, $\bar{B}_i$, $\bar{C}_i$ are respectively $(\bar{d}_i+1) \times \bar{p}_i$, $(\bar{d}_i+1) \times 1$, $1 \times \bar{p}_i$ matrices such that

$$\bar{A}_i \triangleq \begin{bmatrix} 0 & I_{di} & : \\ 0 & 0 & : \end{bmatrix}, \quad \bar{B}_i \triangleq \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \bar{C}_i \triangleq [\,1\, 0 \cdots 0\,],$$

(2) Let $\bar{\mathcal{X}}_i \triangleq \{\bar{x}_i : \bar{x} \triangleq (\bar{x}_1, \ldots, \bar{x}_{m+1}) \in \bar{\mathcal{X}}\}$. Each subsystem $\{\bar{f}_i, \bar{h}_i, \bar{\mathcal{X}}_i\}$, $i \in \mathcal{M}_{1,m}$, in (3.3.43) satisfies the controllability rank condition on $\bar{\mathcal{X}}_i$,

(3) $\dim. \Delta_i^{+}(\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\}) = \bar{p}_i$ on $\bar{\mathcal{X}}$, $i \in \mathcal{M}_{1,m}$. □

Remark 3.3.3. The standard decomposed system in Definition 3.3.1 is a nonlinear version of the system obtained by Gilbert ([Gil.1]). It is worth noting that properties (2), (3) together imply the standard decomposed system $\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\}$ satisfies (A.3). When $\{F, H, \mathcal{X}\}$ is a linear system, it can be shown that property (3) is equivalent to condition (iv) in Definition 6 of [Gil.1]. □

Now, we are ready to state the following result.

Theorem 3.3.3. Suppose that the hypotheses in Theorem 3.3.1, (3.3.4), and (A.3) are satisfied. Further, assume that $\{F^*, H^*, \mathcal{X}\}$ satisfies (A.4). Then, at each $x_0 \in \mathcal{X}$, there exist : (a) an open neighborhood $\mathcal{E}$ of x_0 ; (b) an open connected subset $\bar{\mathcal{X}}$ of R^n ; (c) a C^∞ - diffeomorphism $T : \mathcal{E} \rightarrow \bar{\mathcal{X}}$; and (d) the system $\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\}$, which is T-related on $\mathcal{E}$ to $\{F^*, H^*, \mathcal{E}\}$, is a standard decomposed system with $\bar{d}_i = d_i$, $\bar{p}_i = p_i$, $i \in \mathcal{M}_{1,m}$, and $\bar{p}_{m+1} = p_{m+1} \triangleq n - \sum_{i=1}^{m} p_i$, where the p_i and d_i appear in (A.4) and (A.7). □

See Figure 3.3.1 for a schematic description of the result of Theorem 3.3.3. Since $\{F, H, \mathcal{E}\}$ and $\{\bar{F}, \bar{H}, T(\mathcal{E})\}$ are J-feedback related, they are equivalent with respect to what can be accomplished by feedback (recall Section 3.1). Thus, the value of Theorem 3.3.3 lies in that the class of decoupling control laws can be characterized by looking at the standard decomposed system instead of the general system. This motivates some results in Section 3.4. For the proof of Theorem 3.3.3, we need the following Lemma.

Lemma 3.3.4. Suppose that $\{F, H, \mathcal{X}\}$ satisfies (A.1) and $\{\hat{F}, \hat{H}, \hat{\mathcal{X}}\}$ is J-feedback related on $\mathcal{X}$ to $\{F, H, \mathcal{X}\}$ by $J = \{T, \alpha, \beta\}$, where J satisfies (A.5) and (A.6). Then, if $\{F, H, \mathcal{X}\}$ satisfies (A.3) on $\mathcal{X}$, $\{\hat{F}, \hat{H}, \hat{\mathcal{X}}\}$ satisfies (A.3) on $T(\mathcal{X})$.

Proof. First, we consider the case of $J = \{I_0, \alpha, \beta\}$. Then, we have

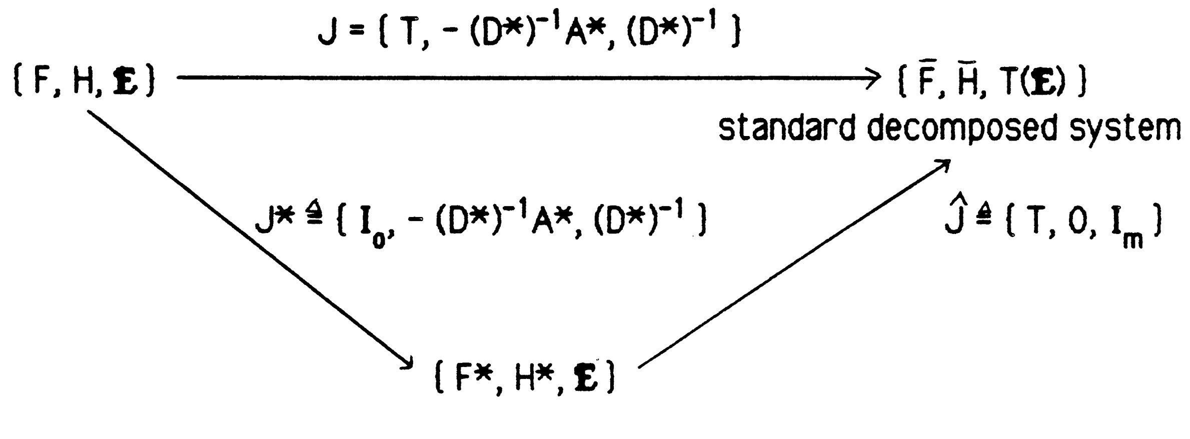

Figure 3.3.1. A standard decomposed system $\{\bar{F}, \bar{H}, T(\mathcal{E})\}$ is J-feedback related on $\mathcal{E}$ to the system $\{F, H, \mathcal{E}\}$.

(3.3.45) $X_0 = \hat{X}_0 + \sum_{i=1}^{m} \alpha_i(\cdot)\hat{X}_i,$

(3.3.46) $X_j = \sum_{i=1}^{m} \beta_{i,j}(\cdot)\hat{X}_i, \quad j \in \mathcal{M}_{1,m},$

where α_i is the ith component of α and $\beta_{i,j}$ is the (i,j)th component of β. Then, by Fact 2. 3. 1, these imply

(3.3.47) $\mathcal{L}_p(\{F, H, \mathcal{X}\}) \subset \mathcal{L}_p(\{\hat{F}, \hat{H}, \hat{\mathcal{X}}\}), \quad p \in \mathcal{X}.$

By (A.3), this implies that dim. $\mathcal{L}_p(\{F, H, \mathcal{X}\}) = n, \ p \in \mathcal{X}$.

Next, we consider the case of $J = \{T, 0, I_m\}$. Then, by Definition 3. 1. 1, $\hat{X}_i$ is T - related on $\mathcal{X}$ to X_i, $i \in \mathcal{M}_{0,m}$. By Fact 2. 3. 2, each vector field $\hat{Y} \in \mathcal{L}(\{\hat{F}, \hat{H}, \hat{\mathcal{X}}\})$ is T - related on $\mathcal{X}$ to a vector field $Y \in \mathcal{L}(\{F, H, \mathcal{X}\})$. Since T is a C^∞- diffeomorphism on $\mathcal{X}$, this implies that at each $p \in \mathcal{X}$, $\mathcal{L}_{T(p)}(\{\hat{F}, \hat{H}, \hat{\mathcal{X}}\})$ is isomorphic to $\mathcal{L}_p(\{F, H, \mathcal{X}\})$. Thus, dim. $\mathcal{L}_q(\{\hat{F}, \hat{H}, \hat{\mathcal{X}}\}) = n$, $q \in T(\mathcal{X})$. Our assertion follows easily from the two cases of J and the transitivity of J - feedback relations. □

Now, we prove Theorem 3. 3. 3.

Proof of Theorem 3. 3. 3. Let $\{\hat{F}, \hat{H}, \mathcal{X}\} = \{F^*, H^*, \mathcal{X}\}$. By given hypotheses and Lemma 3. 3. 4, $\{\hat{F}, \hat{H}, \mathcal{X}\}$ satisfies (A.3). Fix $x_0 \in \mathcal{X}$. Then, by Lemma 3. 2. 5, there exist an open neighborhood

$\mathcal{V}$ of x_0 and $(\sum_{i=1}^{m} p_i)$ C^∞- functions $\phi_{i,j} : \mathcal{V} \rightarrow R$, $j \in \mathbf{M}_{1,pi}$, $i \in \mathbf{M}_{1,m}$ such that on $\mathcal{V}$,

(3.3.48) $d\phi_{i,j}$, $j \in \mathbf{M}_{1,pi}$, $i \in \mathbf{M}_{1,m}$ are linearly independent,

(3.3.49) $d\phi_{i,j} \in \Delta_i^{\perp}(\{\hat{F}, \hat{H}, \mathbf{X}\})$, $j \in \mathbf{M}_{1,pi}$, $i \in \mathbf{M}_{1,m}$.

As was shown in the proof of Theorem 3. 3. 2, $\{\hat{F}, \hat{H}, \mathbf{X}\}$ is decomposed at x_0. Therefore (see the proof of Theorem 3. 2. 3 and (3.2.59)), there exists an open neighborhood $\hat{\mathcal{V}} \subset \mathcal{V}$ of x_0 such that

(3.3.50) $d\hat{H}_i \in \Delta_i^{\perp}(\{\hat{F}, \hat{H}, \mathbf{X}\})$ on $\hat{\mathcal{V}}$, $i \in \mathbf{M}_{1,m}$.

This and Lemma 3. 2. 2 - (ii) implies

(3.3.51) $d\hat{X}_0{}^k\hat{H}_i \in \Delta_i^{\perp}(\{\hat{F}, \hat{H}, \mathbf{X}\})$ on $\hat{\mathcal{V}}$, $k \in \mathbf{M}_{0,di}$, $i \in \mathbf{M}_{1,m}$.

This, (3.3.12), and Lemma 3. 3. 3 shows

(3.3.52) $p_i \geq d_i + 1$, $i \in \mathbf{M}_{1,m}$.

Next, we show that there exists an open neighborhood $W \subset \hat{\mathcal{V}}$ of x_0 and a basis of $\Delta_i^{\perp}(\{\hat{F}, \hat{H}, \mathbf{X}\})$ on W which contains $d\hat{X}_0{}^k\hat{H}_i$, $k \in \mathbf{M}_{0,di}$. By (3.3.48), (3.3.49), (3.3.51), and Lemma 3. 2. 4, for each $i \in \mathbf{M}_{1,m}$, there exist an open neighborhood $V_i \subset \hat{\mathcal{V}}$ of x_0

and C^∞- functions $\psi_{i,j}$ from an appropriate subset of R^{pi} into R, $j \in \mathcal{M}_{1,(di+1)}$ such that

$$(3.3.53) \quad T_{i,j}(x) \triangleq \hat{X}_0^{(j-1)}\hat{H}_i(x) = \psi_{i,j}(\phi_{i,1}(x), \cdots, \phi_{i,pi}(x)),$$
$$x \in V_i, \quad j \in \mathcal{M}_{1,(di+1)}.$$

Then by Lemma 3. 3. 3, (3.3.12), and (3.3.48), $D\psi_{i,j}(\phi_{i,1}(x_0), \cdots, \phi_{i,pi}(x_0))$, $j \in \mathcal{M}_{1,(di+1)}$ are linearly independent $(1 \times p_i)$ row vectors. Now, for each $i \in \mathcal{M}_{1,m}$, let $r_i \triangleq p_i - d_i - 1$ and choose r_i $(1 \times p_i)$ row vectors $\eta_{i,j}$ such that

$$(3.3.54) \quad Q_i \triangleq \begin{bmatrix} D\psi_{i,1}(\phi_{i,1}(x_0), \cdots, \phi_{i,pi}(x_0)) \\ \cdots \\ D\psi_{i,(di+1)}(\phi_{i,1}(x_0), \cdots, \phi_{i,pi}(x_0)) \\ \eta_{i,1} \\ \cdots \\ \eta_{i,ri} \end{bmatrix}.$$

is a nonsingular $(p_i \times p_i)$ matrix. Let

$$(3.3.55) \quad T_i \triangleq (\phi_{i,1}, \cdots, \phi_{i,pi}), \quad T_{i,(di+1+j)} \triangleq \eta_{i,j} T_i, \quad j \in \mathcal{M}_{1,ri}, \; i \in \mathcal{M}_{1,m}.$$

Then, by the construction of $T_{i,j}$, $j \in \mathcal{M}_{1,pi}$, $i \in \mathcal{M}_{1,m}$,

(3.3.56) $dT_{i,j}(x_0)$, $j \in \mathcal{M}_{1,pi}$, $i \in \mathcal{M}_{1,m}$ are linearly independent, (3.3.57) $\quad dT_{i,j} \in \Delta_i^{\perp}(\{\hat{F}, \hat{H}, \mathcal{X}\})$ on V_i, $j \in \mathbf{M}_{1,pi}$, $i \in \mathbf{M}_{1,m}$.

Let $V \triangleq V_1 \cap \cdots \cap V_m$ and $p_{m+1} \triangleq n - \sum_{i=1}^{m} p_i$. If $p_{m+1} \geq 1$, choose a C^{∞}- mapping T_{m+1} from V into R^{pm+1} such that T has rank n at x_0, where

(3.3.58) $\quad T \triangleq (T_1, \cdots, T_m, T_{m+1})$, $\quad T_i \triangleq (T_{i,1}, \cdots, T_{i,pi})$.

Then, by Theorem 2. 3. 6, there exists an open neighborhood $W \subset V$ of x_0 such that

(3.3.59) $\quad$ T is a C^{∞}- diffeomorphism on W,

(3.3.60) $\quad \{dT_{i,j}(p),\ j \in \mathbf{M}_{1,pi}\}$ is a basis of $(\Delta_i^{\perp})_p(\{\hat{F}, \hat{H}, \mathcal{X}\})$, $p \in W$.

Now, using (3.3.59) and (3.3.60), we show property (I) of Definition 3. 3. 1. By Lemma 3. 2. 2 - (ii), (3.3.60) implies

(3.3.61) $\quad d\hat{X}_0 T_{i,j},\ d\hat{X}_i T_{i,j} \in \Delta_i^{\perp}(\{\hat{F}, \hat{H}, \mathcal{X}\})$ on W, $j \in \mathbf{M}_{1,pi}$, $i \in \mathbf{M}_{1,m}$.

Then, by (3.3.60), (3.3.61), and Lemma 3. 2. 4, there exist an open neighborhood $\mathcal{E} \subset W$ of x_0 and C^{∞} - functions $\bar{\theta}_{i,j}$, $\bar{\gamma}_{i,j}$ from appropriate open connected subsets of R^{pi} into R, $j \in \mathbf{M}_{1,pi}$, $i \in \mathbf{M}_{1,m}$ such that

(3.3.62) $\quad X_0 T_{i,(di+1+j)}(x) = \bar{\theta}_{i,j}(T_i(x))$, $\quad X_i T_{i,(di+1+j)}(x) = \bar{\gamma}_{i,j}(T_i(x))$, $\quad x \in \mathcal{E}$.

On the other hand, by (3.3.59), there exist C^{∞} - functions $\bar{f}_{m+1,j}$, $\bar{b}_{i,j}$, $i \in \mathcal{M}_{1,m}$, $j \in \mathcal{M}_{1,pm+1}$ defined on appropriate subsets of R^n such that

$$(3.3.63) \quad \hat{X}_0 T_{m+1,j}(x) = \bar{f}_{m+1,j}(T(x)), \quad \hat{X}_i T_{m+1,j}(x) = \bar{b}_{i,j}(T(x)), \quad x \in \mathcal{E}$$

Let $\bar{\mathcal{X}} \triangleq T(\mathcal{E})$. Let $\bar{x} \triangleq (\bar{x}_1, \cdots, \bar{x}_m, \bar{x}_{m+1}) \triangleq (T_1(x), \cdots, T_{m+1}(x))$. Let $\bar{\theta}_i \triangleq (\bar{\theta}_{i,1}, \cdots, \bar{\theta}_{i,ri})$, $\bar{\gamma}_i \triangleq (\bar{\gamma}_{i,1}, \cdots, \bar{\gamma}_{i,ri})$, $i \in \mathcal{M}_{1,m}$. Let $\bar{f}_{m+1} \triangleq (\bar{f}_{m+1,1}, \cdots, \bar{f}_{m+1,pm+1})$, $\bar{b}_i \triangleq (\bar{b}_{i,1}, \cdots, \bar{b}_{i,p_{m+1}})$, $i \in \mathcal{M}_{1,m}$. Define vector fields $\bar{X}_i$, $i \in \mathcal{M}_{0,m}$ by

$$(3.3.64) \quad \bar{X}_0(\bar{x}) \triangleq \sum_{i=1}^{m} \Big(\sum_{j=1}^{d_i} \bar{x}_{i,(j+1)} \, \partial/\partial\bar{x}_{i,j} + \sum_{j=d_i+2}^{p_i} \bar{\theta}_{i,(j-di-1)}(\bar{x}_i) \, \partial/\partial\bar{x}_{i,j} \Big) + \sum_{j=1}^{p_{m+1}} \bar{f}_{m+1,j}(\bar{x}) \, \partial/\partial\bar{x}_{m+1,j},$$

$$(3.3.65) \quad \bar{X}_i(\bar{x}) \triangleq \partial/\partial\bar{x}_{i,(di+1)} + \sum_{j=d_i+2}^{p_i} \bar{\gamma}_{i,(j-di-1)}(\bar{x}_i) \, \partial/\partial\bar{x}_{i,j} \Big) + \sum_{j=1}^{p_{m+1}} \bar{b}_{i,j}(\bar{x}) \, \partial/\partial\bar{x}_{m+1,j}, \quad i \in \mathcal{M}_{1,m},$$

$$(3.3.66) \quad \bar{H}_i(\bar{x}) = \bar{x}_{i,1}, \quad i \in \mathcal{M}_{1,m}.$$

where, $\bar{x}_{i,j}$ is the jth component of $\bar{x}_i$. Let $\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\}$ be the system constructed as above. Then, the coordinate representation $\{\bar{f}, \bar{h}, \bar{\mathcal{X}}\}$ of $\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\}$ has the form indicated in (1), where $\bar{d}_i = d_i$, $i \in \mathcal{M}_{1,m}$ and $\bar{p}_i = p_i$, $i \in \mathcal{M}_{1,m+1}$.

Let $\bar{Y}_i$ be a C^{∞}- vector field in $\Delta_i(\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\})$. Then, using (3.3.64) and (3.3.65), we can show that if $\bar{Y}_i = \sum_{j=1}^{m+1} \sum_{k=1}^{p_j} \bar{\gamma}_{j,k}(\,.\,) \, \partial/\partial\bar{x}_{j,k}$

is a local representation of $\bar{Y}_i$ on $\bar{\mathcal{X}}$,

(3.3.67) $\bar{Y}_{i,k}(\bar{x}) = 0, \quad \bar{x} \in \bar{\mathcal{X}}, \quad k \in \mathcal{M}_{1,\rho i}$.

By Lemma 3. 3. 4, $\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\}$ must satisfy (A.3). Thus, (3.3.67) implies property (2) of Definition 3. 3. 1. Property (3) follows from the fact that by (3.3.59), $(\Delta_i)_p(\{\hat{F}, \hat{H}, \mathcal{E}\})$ and $(\Delta_i)_{T(p)}(\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\})$ are isomorphic at each $p \in \mathcal{E}$. □

Remark 3. 3. 4. The system $\{F, H, \mathcal{X}\}$ is locally J - feedback related to a standard decomposed system, where $\tilde{J} \triangleq J^{-1} = \{T^{-1}, A^* \circ T^{-1}, D^* \circ T^{-1}\}$. As is shown in the proof of Theorem 3. 3. 3, the choice of T is not unique. Thus, there are infinitely many standard decoupled systems which can be J - feedback related. to $\{F, H, \mathcal{X}\}$. □

Finally, we state a converse result of Theorem 3. 3. 3.

Theorem 3. 3. 4. Suppose that $\{F, H, \mathcal{X}\}$ satisfies (A.1) and the class of control laws satisfies (A.5), (A.6). Suppose further that at each $x_0 \in \mathcal{X}$, there exist : (a) an open neighborhood $\mathcal{E}$ of x_0 ; (b) an open connected subset $\bar{\mathcal{X}}$ of R^n ; (c) mappings $T : \mathcal{E} \to \mathcal{X}$, $\alpha : \mathcal{E} \to R^m$, $\beta : \mathcal{E} \to R^{m \times m}$; and (d) the system $\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\}$, which is J - feedback related on $\mathcal{E}$ to $\{F, H, \mathcal{E}\}$ by $J \triangleq \{T, \alpha, \beta\}$, is a standard decomposed system. Then, the following properties hold :

(i) $\{F, H, \mathcal{X}\}$ satisfies (3.3.4), (A.3), and (A.7) with $d_i = \bar{d}_i$, $i \in \mathbf{M}_{1,m}$,

(ii) $\{F^*, H^*, \mathcal{X}\}$ satisfies (A.4) with $p_i = \bar{p}_i$, $i \in \mathbf{M}_{1,m}$,

(iii) $\alpha(x) = -[D^*(x)]^{-1}A^*(x)$ and $\beta(x) = [D^*(x)]^{-1}$.

Proof. By Remark 3. 3. 3, $\{\bar{F}, \bar{H}, \mathcal{X}\}$ satisfies (A.3). By Lemma 3. 3. 4, this implies that $\{F, H, \mathcal{X}\}$ satisfies (A.3). Direct computation shows that $\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\}$ satisfies (A.7) with $\bar{D}^*(\bar{x}) = I_m$ and $\bar{A}^*(\bar{x}) = 0$. By this, Lemma 3. 3. 1, and (A.5), we see that $\{F, H, \mathcal{X}\}$ satisfies (3.3.4), (A.7) with $d_i = \bar{d}_i$, $i \in \mathbf{M}_{1,m}$, and, furthermore (iii). Since $\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\}$ is J-feedback related on $\mathbf{E}$ to $\{F, H, \mathbf{E}\}$ by $J \triangleq \{T, \alpha, \beta\}$, (iii) implies that $\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\}$ is T-related on $\mathbf{E}$ to $\{F^*, H^*, \mathcal{X}\}$. Consequently, $(\Delta_i)_q(\{F^*, H^*, \mathcal{X}\})$ and $(\Delta_i)_{T(q)}(\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\})$ are isomorphic at each $q \in \mathbf{E}$. This implies (ii). □

In this section, we have shown that (3.3.4) is a necessary and sufficient condition for both decouplability and decomposability. We have also specified a class of nonlinear systems which are J-feedback related to standard decomposed systems. See Figure 3. 3. 1 for a schematic description of the results obtained in this section.

Finally, we remark that (A.7) can be weakened by

(A.7)' There exist $d_i \in \mathbf{N}$, $i \in \mathbf{M}_{1,m}$ satisfying (3.3.1) and

(3.3.2)' There is at least a point $x_0 \in \mathcal{X}$ such that

$$[\, X_1 X_0^{d_1} H_1(x_0) \cdots X_m X_0^{d_1} H_1(x_0) \,] \neq 0.$$

If $\{F, H, \mathcal{X}\}$ is smooth, (A.7)' implies that there exists an open neighborhood $\mathcal{X}' \subset \mathcal{X}$ of x_0 such that (A.7) is satisfied on $\mathcal{X}'$ instead of $\mathcal{X}$. Thus, when (A.7) is replaced by (A.7)', all results in this section hold with $\mathcal{X} = \mathcal{X}'$. In other words, they are locally valid.

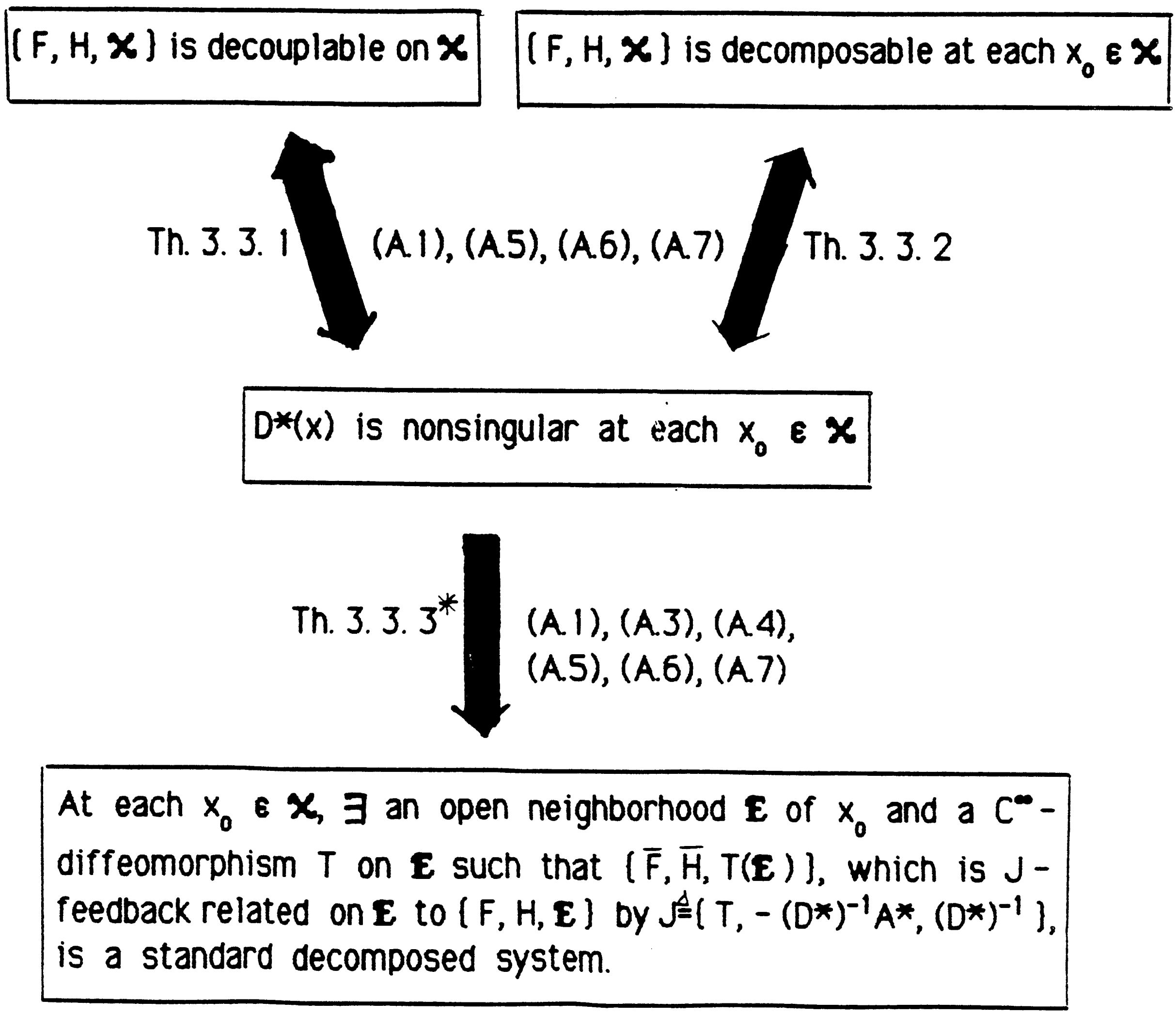

Figure 3. 3. 2. Summary of main results in Section 3. 3 showing assumptions required for each implication.

Section 3. 4. The Whole Class of Decoupling and Decomposing Control Laws

In this section, we consider the class of control laws which decouple or decompose a nonlinear system and thus obtain some answers to questions (b), (c), (d) in Chapter 1. We believe our results are new and are important contributions. We begin by discussing at some length the significance of characterizing the whole class of decoupling control laws.

Let $\{F, H, \mathcal{X}\}$ be a system which satisfies the hypotheses in Theorem 3. 3. 1 and (3.3.4). Let $\{\hat{F}, \hat{H}, \mathcal{X}\} \triangleq \{F^*, H^*, \mathcal{X}\}$. Then, the input - output map for $\{\hat{F}, \hat{H}, \mathcal{X}\}$ is determined by (3.3.16) and (3.3.17). Now, suppose we choose the following control law for $\{\hat{F}, \hat{H}, \mathcal{X}\}$:

$$(3.4.1) \qquad \hat{u}_i = \sum_{k=1}^{d_i+1} F_{i,k}\ \hat{y}_i^{(k-1)} + c_i\ \bar{u}_i,$$
$$= \sum_{k=1}^{d_i+1} F_{i,k}\ X_0^{(k-1)}H_i(x) + c_i\ \bar{u}_i, \quad i \in \mathcal{M}_{1,m},$$

where the $F_{i,k}$ are constants, the c_i are nonzero constants, and the last equality comes from (3.3.13) and (3.3.18). Note that this procedure corresponds to choosing for the original system $\{F, H, \mathcal{X}\}$ a control law of the form $u = \alpha(x) + \beta(x)\,\bar{u}$ where

$$(3.4.2) \qquad \alpha(x) \triangleq [D^*(x)]^{-1} \left\{ \begin{bmatrix} \sum_{k=1}^{d_1+1} F_{1,k}\ X_0^{(k-1)}H_1(x) \\ \cdots \\ \sum_{k=1}^{d_m+1} F_{m,k}\ X_0^{(k-1)}H_m(x) \end{bmatrix} - A^*(x) \right\},$$

$$(3.4.3) \qquad \beta(x) \triangleq [D^*(x)]^{-1}\ \mathrm{diag}\ c_i\,.$$

By (3.3.16) and (3.4.1), the input - output map for $\{F, H, \mathcal{X}\}^{\alpha,\beta}$ is given by (3.3.17) and

$$(3.4.4) \qquad \hat{y}_i^{(d_i+1)}(t) = \sum_{k=1}^{d_i+1} F_{i,k}\, \hat{y}_i^{(k-1)} + c_i\, \bar{u}_i, \quad i \in \mathbf{M}_{1,m}.$$

Therefore, $\{F, H, \mathcal{X}\}^{\alpha,\beta}$ is decoupled on $\mathcal{X}$ and the control law (3.4.2), (3.4.3) is a decoupling control law for $\{F, H, \mathcal{X}\}$. Moreover, appropriate selection of the constants $F_{i,k}$ and c_i gives good input - output dynamic characteristics. The class of decoupling control laws (3.4.2), (3.4.3) was considered in [Fre.1, Sin.1, Sih.1].

A nonlinear control law more general than the one in (3.4.1) is :

$$(3.4.1)' \qquad \hat{u}_i = \phi_i(\hat{y}_i^{(d_i)}, \ldots, \hat{y}_i) + \psi_i(\hat{y}_i^{(d_i)}, \ldots, \hat{y}_i)\, \bar{u}_i,$$

where ϕ_i, ψ_i are arbitrary C^∞ - functions of their arguments. The corresponding control law $u = \alpha(x) + \beta(x)\bar{u}$ for $\{F, H, \mathcal{X}\}$ is

$$(3.4.5) \qquad \alpha(x) = [D^*(x)]^{-1} \left\{ \begin{bmatrix} \eta_1(x) \\ \cdots \\ \eta_m(x) \end{bmatrix} - A^*(x) \right\},$$

$$(3.4.6) \qquad \beta(x) = [D^*(x)]^{-1} \operatorname{diag} \lambda_i(x),$$

where

(3.4.7) $\quad \eta_i(x) \triangleq \phi_i(H_i(x),\ X_0 H_i(x), \ldots, X_0^{d_i} H_i(x)),$

(3.4.8) $\quad \lambda_i(x) \triangleq \psi_i(H_i(x),\ X_0 H_i(x), \ldots, X_0^{d_i} H_i(x)).$

Now, by (3.3.16) and (3.4.1)', the input-output map for $[F, H, \mathcal{X}]^{\alpha,\beta}$ is given by (3.3.17) and

(3.4.2)' $\quad \hat{y}_i^{(d_i+1)}(t) = \phi_i(\hat{y}_i^{(d_i)}, \ldots, \hat{y}_i) + \psi_i(\hat{y}_i^{(d_i)}, \ldots, \hat{y}_i)\ \bar{u}_i, \quad i \in \mathbf{M}_{1,m}.$

We see that the new feedback system $[F, H, \mathcal{X}]^{\alpha,\beta}$ is still decoupled on $\mathcal{X}$. Thus, the control law (3.4.5) - (3.4.8) is a decoupling control law and is more general than the control law (3.4.2), (3.4.3). The class of decoupling control laws (3.4.5) - (3.4.8) was suggested in some examples which appear in [Cla.1, Sin.2].

Can we find a still more general class of decoupling control laws ?. Knowledge of a more general class of decoupling control laws allows more flexibility in choosing a decoupling control law. For an instance, as will be shown later by an example (Example 3.5.1 in Section 3.5), a decoupling control law (3.4.2), (3.4.3) may not decouple a system in a "stable" way but it may be possible by finding a more general decoupling control law. Thus, characterizing the whole class of decoupling control laws is a significant question both from engineering and mathematical viewpoints. For future purposes, we define several classes of control laws.

Definition 3.4.1. $\mathcal{S}_o^\infty(\{F,H,\mathcal{X}\})(\mathcal{S}_o^\omega(\{F,H,\mathcal{X}\}))$ is the class of control laws $u = \alpha(x) + \beta(x)\hat{u}$ satisfying (A.5) ((A.5)'), (A.6), and (3.4.5) - (3.4.8). □

Definition 3.4.2. $\mathcal{S}^\infty(\{F,H,\mathcal{X}\})(\mathcal{S}^\omega(\{F,H,\mathcal{X}\}))$ is the class of control laws $u = \alpha(x) + \beta(x)\hat{u}$ satisfying (A.5) ((A.5)'), (A.6), (3.4.5), (3.4.6), and

(3.4.9) $\quad d\eta_i,\ d\lambda_i \in \Delta_i^\perp(\{F^*,H^*,\mathcal{X}\})$ on $\mathcal{X}$. □

Remark 3.4.1. By (3.3.13) and (3.3.51), the smooth functions η_i, λ_i in (3.4.7), (3.4.8) satisfy (3.4.9). Thus,

(3.4.10) $\quad \mathcal{S}_o^\infty \subset \mathcal{S}^\infty$ and $\mathcal{S}_o^\omega \subset \mathcal{S}^\omega$.

In general, $\mathcal{S}_o^\infty$ ($\mathcal{S}_o^\omega$) is a very limited subset of $\mathcal{S}^\infty$ ($\mathcal{S}^\omega$). When $n = \sum_{i=1}^{m}(d_i+1)$, it is usually true that $\mathcal{S}^\infty(\{F,H,\mathcal{X}\}) = \mathcal{S}_o^\infty(\{F,H,\mathcal{X}\})$. But, as will be seen in Example 3.5.2 of Section 3.5, this is not always so. Further discusssion will be given in Remark 3.4.7. □

The following theorem shows that $\mathcal{S}^\omega$ is actually the whole class of real analytic decoupling control laws for a real analytic nonlinear system.

Theorem 3.4.1. Suppose that $\{F,H,\mathcal{X}\}$ satisfies (A.1)', (A.7).

Suppose that class of control laws satisfies the following assumptions : (A.5)', (A.6), and for $u = \alpha(x) + \beta(x)\bar{u}$ in the class, $\{F, H, \mathcal{X}\}^{\alpha,\beta}$ satisfies (A.2). Then, the control law $u = \alpha(x) + \beta(x)\bar{u}$ decouples $\{F, H, \mathcal{X}\}$ on $\mathcal{X}$ if and only if it belongs to $\mathcal{S}^{\omega}(\{F, H, \mathcal{X}\})$ and (3.3.4) holds. □

Remark 3. 4. 2. The condition (3.4.9) is equivalent to

$$(3.4.11) \quad Y\eta_i = Y\lambda_i = 0 \text{ on } \mathcal{X} \text{ for all } Y \in \Delta_i\{F^*, H^*, \mathcal{X}\}.$$

Thus, Theorem 3. 4. 1 reduces the problem of characterizing the whole class of decoupling control laws to that of finding all solutions of the set of the first order linear partial differential equations specified by (3.4.11). When $\{F, H, \mathcal{X}\}$ is a linear system and the class of control laws is restricted to be linear (e. g., (1.5) and (1.6)), (3.4.11) is reduced to a set of linear algebraic equations. Moreover, Theorem 3. 4.1 is reduced to a result contained in [Gil.1]. □

We need the following Lemma for the proof of Theorem 3. 4. 1.

Lemma 3. 4. 1. Let ξ_i, ψ_i, $\phi_{i,j}$, $j \in \mathbf{M}_{1,m}$, $i \in \mathbf{M}_{1,m}$ be any C^{∞}- functions from $\mathcal{X}$ into R such that

$$(3.4.12) \quad d\xi_i,\ d\psi_i,\ d\phi_{i,j} \in \Delta_i^{\perp}(\{F, H, \mathcal{X}\}) \text{ on } \mathcal{X},\ i \in \mathbf{M}_{1,m}.$$

Define C^∞ - vector fields $\tilde{X}_i$, $i \in \mathcal{M}_{0,m}$ by

$$(3.4.13) \quad \tilde{X}_0 \triangleq X_0 + \sum_{j=1}^{m} \psi_j(\cdot) X_j,$$

$$(3.4.14) \quad \tilde{X}_i \triangleq \sum_{j=1}^{m} \phi_{i,j}(\cdot) X_j, \quad i \in \mathcal{M}_{1,m}.$$

Let $i \in \mathcal{M}_{1,m}$. Let k be any finite nonnegative integer. Then, if $i_j \in \{o, i\}$, $j \in \mathcal{M}_{1,k}$,

$$(3.4.15) \quad d\tilde{X}_{i1}\tilde{X}_{i2} \cdots \tilde{X}_{ik} \xi_i \in \Delta_i^\perp(\{F, H, \mathcal{X}\}) \text{ on } \mathcal{X}.$$

Proof. By Lemma 3. 2. 2 - (ii),

$$(3.4.16) \quad dX_0\xi_i, \ dX_i\xi_i \in \Delta_i^\perp(\{F, H, \mathcal{X}\}) \text{ on } \mathcal{X}.$$

Recall that $X_j \in \Delta_i$, $j \in \bar{\mathcal{M}}_i$ if $j \neq i$. This with (3.4.12) - (3.4.16) implies

$$(3.4.17) \quad d\tilde{X}_0\xi_i, \ d\tilde{X}_i\xi_i \in \Delta_i^\perp(\{F, H, \mathcal{X}\}) \text{ on } \mathcal{X}.$$

Successive application of this result yields (3.4.15), immediately. □

Now, we prove Theorem 3. 4. 1.

Proof of Theorem 3. 4. 1. Let $u = \alpha(x) + \beta(x)\bar{u}$ be a control law which satisfies (3.4.5), (3.4.6), (3.4.9), and (3.4.10). Let $\bar{X}_i$, i

$\varepsilon\ \mathbf{M}_{0,m}$ be vector fields corresponding to $[F, H, \mathcal{X}]^{\alpha,\beta}$. Let $\hat{X}_i$, $i \varepsilon$ $\mathbf{M}_{0,m}$ be vector fields corresponding to $[F^*, H^*, \mathcal{X}]$. Then, we have

$$(3.4.18) \quad \bar{X}_0 = \hat{X}_0 + \sum_{j=1}^{m} \eta_j(\cdot)\hat{X}_j,$$

$$(3.4.19) \quad \bar{X}_j = \lambda_j(\cdot)\hat{X}_j, \quad j \ \varepsilon\ \mathbf{M}_{1,m}.$$

Let $i \ \varepsilon\ \mathbf{M}_{1,m}$. Let k be any finite nonnegative integer. Then, by (3.3.33), (3.4.9), (3.4.18), (3.4.19), and Lemma 3. 4. 1, we see that if $i_q \varepsilon [0, i]$, $q \ \varepsilon\ \mathbf{M}_{1,k}$,

$$(3.4.20) \quad d\bar{X}_{i1}\bar{X}_{i2}\cdots\bar{X}_{ik}\bar{H}_i \ \varepsilon\ \Delta_i^{\perp}([F^*, H^*, \mathcal{X}]) \text{ on } \mathcal{X}.$$

But note that $\hat{X}_j \ \varepsilon\ \Delta_i([F^*, H^*, \mathcal{X}])$ on $\mathcal{X}$, $j \ \varepsilon \bar{\mathbf{M}}_i$. Hence, (3.4.19) and (3.4.20) imply that if $i_q \varepsilon [0, i]$, $q \ \varepsilon\ \mathbf{M}_{1,k}$ and $j \ \varepsilon\ \bar{\mathbf{M}}_i$,

$$(3.4.21) \quad \bar{X}_j\bar{X}_{i1}\bar{X}_{i2}\cdots\bar{X}_{ik}\bar{H}_i = 0 \text{ on } \mathcal{X}.$$

This with Lemma 3. 2. 2 - (i) and Theorem 3. 2. 1 implies that $[F, H, \mathcal{X}]^{\alpha,\beta}$ is decoupled on $\mathcal{X}$.

Next, suppose that $u = \alpha(x) + \beta(x)\bar{u}$ decouples $[F, H, \mathcal{X}]$ on $\mathcal{X}$. Let $[\bar{F}, \bar{H}, \mathcal{X}] \triangleq [F, H, \mathcal{X}]^{\alpha,\beta}$. Let $\bar{X}_i$, $i \ \varepsilon\ \mathbf{M}_{0,m}$ be vector fields corresponding to $[\bar{F}, \bar{H}, \mathcal{X}]$. Then, by Theorem 3. 2. 1 and Lemma 3. 2. 2 - (i),

$$(3.4.22)\quad \bar{X}_j \bar{X}_{i_1} \bar{X}_{i_2} \cdots \bar{X}_{i_k} \bar{H}_i = 0 \quad \text{on } \mathcal{X},$$

$$i_q \in \{0, i\},\ q \in \mathcal{M}_{1,k},\ k \in \mathcal{M}_{0,\infty},\ j \in \bar{\mathcal{M}}_i,\ i \in \mathcal{M}_{1,m}.$$

On the other hand, by Theorem 3. 3. 1, (3.3.4) holds. Let $\hat{X}_i$, $i \in \mathcal{M}_{0,m}$ be vector fields corresponding to $\{F^*, H^*, \mathcal{X}\}$. Define C^∞-mappings $\eta : \mathcal{X} \to R^m$, $\Gamma : \mathcal{X} \to R^{m \times m}$ by

$$(3.4.23)\quad \eta(x) \triangleq D^*(x)\,\alpha(x) + A^*(x), \quad \Gamma(x) \triangleq D^*(x)\,\beta(x).$$

Then, we see from Fig 3. 4. 1 that

$$(3.4.24)\quad \bar{X}_0 = \hat{X}_0 + \sum_{j=1}^{m} \eta_j(\cdot)\,\hat{X}_j,$$

$$(3.4.25)\quad \bar{X}_j = \sum_{i=1}^{m} \Gamma_{i,j}(\cdot)\,\hat{X}_i, \quad j \in \mathcal{M}_{1,m},$$

where $\Gamma_{i,j}$ is the (i, j)th component of Γ. On the other hand, by Lemma 3. 3. 1 - (i), direct computation with (3.4.24) shows that

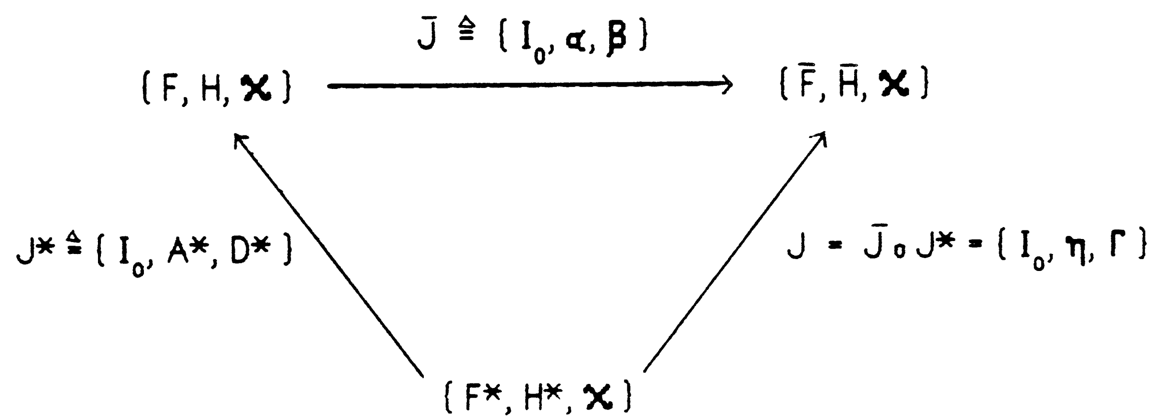

Fig 3. 4. 1. Relationships between $\{F, H, \mathcal{X}\}$, $\{\bar{F}, \bar{H}, \mathcal{X}\}$, and $\{F^*, H^*, \mathcal{X}\}$

(3.4.26) $\bar{X}_0{}^k\bar{H}_i = \hat{X}_0{}^k\hat{H}_i$, $k \in \mathcal{M}_{0,di}$, $i \in \mathcal{M}_{1,m}$.

This with (3.3.13), (3.3.14), (3.4.24), and (3.4.25) yields

(3.4.27) $\bar{X}_0{}^{(d_i+1)}\bar{H}_i = \eta_i$, $\bar{X}_i\bar{X}_0{}^{d_i}\bar{H}_i = \Gamma_{i,i}$ on $\mathcal{X}$, $i \in \mathcal{M}_{1,m}$.

This with (3.4.22) and Lemma 3. 2. 2 - (i) shows that

(3.4.28) $d\eta_i$, $d\Gamma_{i,i}$ $\in \Delta_i^{\perp}(\{\bar{F}, \bar{H}, \mathcal{X}\})$ on $\mathcal{X}$, $i \in \mathcal{M}_{1,m}$.

Note that by (3.3.23), (3.3.24), and (3.4.23), we must have

(3.4.29) $\Gamma_{i,j}(x) = 0$, $x \in \mathcal{X}$, $i \neq j$,

(3.4.30) $\lambda_i(x) \triangleq \Gamma_{i,i}(x) \neq 0$, $x \in \mathcal{X}$, $i \in \mathcal{M}_{1,m}$.

Consequently, we can write (3.4.24), (3.4.25) as

(3.4.31) $\hat{X}_0 = \bar{X}_0 - \sum_{j=1}^{m} (\eta_j(\cdot)/\lambda_i(\cdot))\bar{X}_j$,

(3.4.32) $\hat{X}_j = (1/\lambda_j(\cdot))\bar{X}_j$, $j \in \mathcal{M}_{1,m}$.

Since $d\lambda_i \in \Delta_i^{\perp}(\{\bar{F}, \bar{H}, \mathcal{X}\})$ implies $d(\lambda_i)^{-1} \in \Delta_i^{\perp}(\{\bar{F}, \bar{H}, \mathcal{X}\})$, these equations with (3.4.22), (3.4.28), and Lemma 3. 4. 1 lead to :

(3.4.33) $d\hat{X}_{i1}\hat{X}_{i2}\cdots\hat{X}_{ik}\eta_i$, $d\hat{X}_{i1}\hat{X}_{i2}\cdots\hat{X}_{ik}\lambda_i$ $\in \Delta_i^{\perp}(\{\bar{F}, \bar{H}, \mathcal{X}\})$ on $\mathcal{X}$,

$i_q \in \{0, i\}$, $q \in \mathcal{M}_{1,k}$, $k \in \mathcal{M}_{0,\infty}$, $j \in \bar{\mathcal{M}}_i$, $i \in \mathcal{M}_{1,m}$.

Recall that $\bar{X}_j \in \Delta_i(\{\bar{F}, \bar{H}, \mathcal{X}\})$ on $\mathcal{X}$. $j \in \bar{\mathcal{M}}_i$. Therefore, (3.4.32) and (3.4.33) imply

$$(3.4.34)\quad \hat{X}_j\hat{X}_{i1}\hat{X}_{i2}\cdots\hat{X}_{ik}\eta_i = \hat{X}_j\hat{X}_{i1}\hat{X}_{i2}\cdots\hat{X}_{ik}\lambda_i = 0 \text{ on } \mathcal{X}.$$

$$i_q \in \{0, i\},\ q \in \mathbf{M}_{1,k},\ k \in \mathbf{M}_{0,\infty},\ j \in \bar{\mathbf{M}}_i,\ i \in \mathbf{M}_{1,m}.$$

This and Lemma 3. 2. 2 - (i) complete the proof. □

Remark 3. 4. 3. A result on the characterization of decoupling control laws is found in [Sin.1], where it is shown that if a smooth control law $u = \alpha(x) + \beta(x)\bar{u}$ decouples $\{F, H, \mathcal{X}\}$ on $\mathcal{X}$, then

$$(3.4.35)\quad \bar{X}_j \bar{X}_0^k \bar{H}_i = 0 \text{ on } \mathcal{X},\ k \in \mathbf{M}_{0,\infty} \text{ if } i \neq j,$$

where $\bar{X}_j$, $j \in \mathbf{M}_{0,m}$ are vector fields corresponding to $\{F, H, \mathcal{X}\}^{\alpha,\beta}$ and $\bar{H}_i \triangleq H_i$, $i \in \mathbf{M}_{1,m}$. A more complete result is that under hypotheses of Theorem 3. 2. 1, a control law $u = \alpha(x) + \beta(x)\bar{u}$ decouples $\{F, H, \mathcal{X}\}$ on $\mathcal{X}$ if and only if (3.4.22) holds. But, (3.4.22) is an implicit and complex condition for $u = \alpha(x) + \beta(x)\bar{u}$ to be a decoupling control law. It results in high order partial differential equations. On the other hand, the condition given by Theorem 3. 4. 1 is explicit and involves only the first order partial differential equations. Thus, (3.4.35) is not so useful for characterizing the class of decoupling control laws as our condition. □

Unfortunately, we are not able to prove that Theorem 3.4.1 is valid for smooth systems and smooth control laws. But, we can show that $\mathcal{S}^\infty(\{F, H, \mathcal{X}\})$ is the whole class of smooth decomposing control laws for a special class of smooth systems.

Theorem 3.4.2. Suppose the hypotheses for Theorem 3.3.3 are satisfied. Then, a control law $u = \alpha(x) + \beta(x)\bar{u}$ decomposes $\{F, H, \mathcal{X}\}$ at each $x_0 \in \mathcal{X}$ if and only if it belongs to $\mathcal{S}^\infty(\{F, H, \mathcal{X}\})$ and (3.3.4) holds.

Proof. Suppose a control law $u = \alpha(x) + \beta(x)\bar{u}$ decomposes $\{F, H, \mathcal{X}\}$ at each $x_0 \in \mathcal{X}$. Let $\{\bar{F}, \bar{H}, \mathcal{X}\} \triangleq \{F, H, \mathcal{X}\}^{\alpha,\beta}$. Let $\bar{X}_i$, $i \in \mathcal{M}_{0,m}$ be vector fields corresponding to $\{\bar{F}, \bar{H}, \mathcal{X}\}$. Then, by (3.2.38) - (i) and Lemma 3.2.2 - (i), (3.4.22) holds. Then, the remaining arguments are exactly the same as those following (3.4.22) except that (3.3.4) holds by Theorem 3.2.2 instead of Theorem 3.3.1.

Next, suppose that a control law $u = \alpha(x) + \beta(x)\bar{u}$ belongs to $\mathcal{S}^\infty(\{F, H, \mathcal{X}\})$. Fix $x_0 \in \mathcal{X}$. Then, by Theorem 3.3.3, there exist an open neighborhood $\mathcal{E}$ of x_0 and a mapping $T : \mathcal{E} \to R^n$ such that $\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\}$ which is J - feedback related on $\mathcal{E}$ to $\{F, H, \mathcal{E}\}$ by $J \triangleq \{T, -(D^*)^{-1}A^*, (D^*)^{-1}\}$ is a standard decomposed system with $\bar{\mathcal{X}} = T(\mathcal{E})$, $\bar{d}_i = d_i$, $i \in \mathcal{M}_{1,m}$, and $\bar{p}_i = p_i$, $i \in \mathcal{M}_{1,m+1}$. The mapping T constructed by (3.3.53), (3.3.55), and (3.3.58) satisfies (3.3.59) and (3.3.60) on $\mathcal{E}$. Then, by (3.4.9) and Lemma 3.2.4, there exist an open neighborhood $\mathcal{H} \subset \mathcal{E}$ of x_0 and C^∞-functions $\bar{\eta}_i$, $\bar{\lambda}_i$, defined on appropriate subsets of R^{pi}, $i \in \mathcal{M}_{1,m}$ such that

$$\text{(3.4.36)} \quad \eta_i(x) = \bar{\eta}_i(T_i(x)), \;\; \lambda_i(x) = \bar{\lambda}_i(T_i(x)), \; x \in \mathcal{H}.$$

Let $\bar{\eta} \triangleq (\bar{\eta}_1, \ldots, \bar{\eta}_m)$ and $\bar{\Gamma} \triangleq \text{diag}\, \bar{\lambda}_i$. Let $J_1 \triangleq \{T, 0, I_m\}$ and $J_2 \triangleq \{I_0, \eta, \text{diag}\,\lambda_i\}$. Then, as can be seen from Figure 3.4.2, $\{F, H, \mathcal{H}\}^{\alpha,\beta}$ is J_3 - feedback related to $\{F, H, T(\mathcal{H})\}$ on $T(\mathcal{H})$ by $J_3 = J_2 \circ J_1^{-1}$. Direct computation shows that $J_2 \circ J_1^{-1} = \{T^{-1}, \bar{\eta}, \bar{\Gamma}\}$. The form of the standard decomposed system, the form of η, Γ, and Definition 3.1.5 imply $\{F, H, \mathcal{X}\}^{\alpha,\beta}$ is decomposed at x_0. □

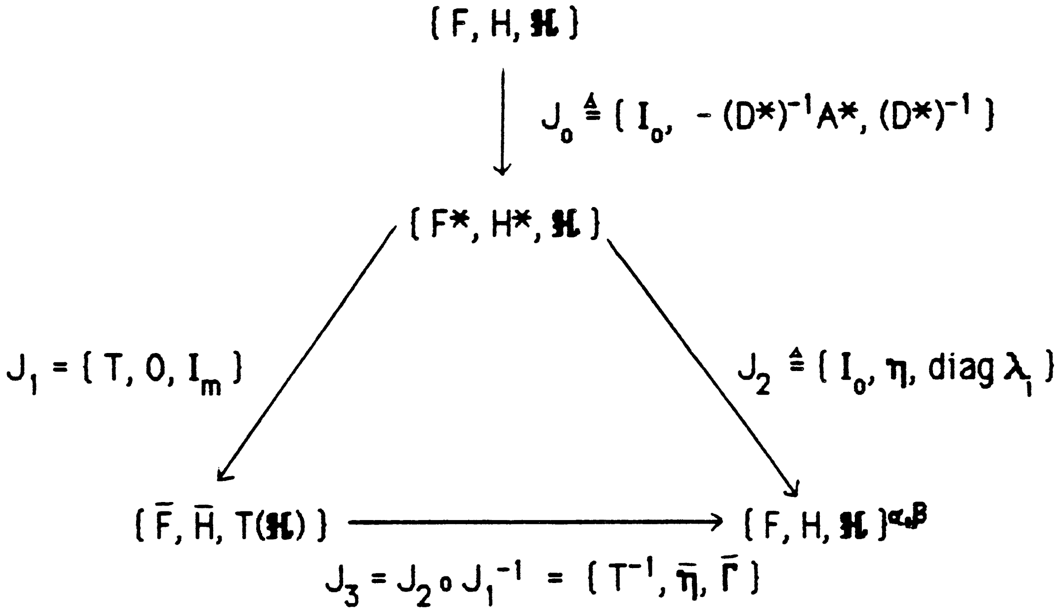

Fig. 3.4.2. Relationships between $\{F, H, \mathcal{H}\}$, $\{F^*, H^*, \mathcal{H}\}$, $\{F, H, \mathcal{H}\}^{\alpha,\beta}$, and $\{\bar{F}, \bar{H}, T(\mathcal{H})\}$.

This result has other implications. Recall that if $\{F, H, \mathcal{X}\}$ is decomposed at $x_0 \in \mathcal{X}$, then, there exists an open neighborhood $\mathcal{E}$ of x_0 such that $\{F, H, \mathcal{E}\}$ is decoupled on $\mathcal{E}$. Therefore, Theorem 3.4.2 shows that under its hypotheses, $\mathcal{S}^\infty(\{F, H, \mathcal{X}\}$ is a class of smooth control laws which decouple $\{F, H, \mathcal{X}\}$ at least locally around each point $x_0 \in \mathcal{X}$.

If $\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\}$ is a standard decomposed system, we might expect intuitively from its special structure that its decoupling control laws are of the form $\bar{u}_i = \bar{\eta}_i(\bar{x}_i) + \bar{\lambda}_i(\bar{x}_i)\tilde{u}_i$, $i \in \mathcal{M}_{1,m}$. In the next Theorem, we show that this is really the case. Before doing so, we formalize the class of control laws.

Definition 3.4.3. Let $\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\}$ be a standard decomposed system. $\bar{\mathcal{S}}^\infty(\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\})$ $(\bar{\mathcal{S}}^\omega(\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\}))$ is the class of control laws $\bar{u} = \bar{\alpha}(\bar{x}) + \bar{\beta}(\bar{x})\tilde{u}$ satisfying (A.5) ((A.5)'), (A.6), and

$$(3.4.37) \quad \bar{\alpha}(\bar{x}) = \begin{bmatrix} \bar{\eta}_1(\bar{x}_1) \\ \cdots \\ \bar{\eta}_m(\bar{x}_m) \end{bmatrix}, \quad \bar{\beta}(\bar{x}) = \operatorname{diag} \bar{\lambda}_i(\bar{x}_i),$$

where $\bar{\eta}_i$, $\bar{\lambda}_i$ are functions from $\bar{\mathcal{X}}_i$ into R, $i \in \mathcal{M}_{1,m}$. □

Theorem 3.4.3. Let $\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\}$ be the standard decomposed system in Definition 3.3.1. Suppose that $\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\}$ satisfies (A.1)'. Suppose that class of control laws satisfies the following assumptions : (A.5)', (A.6), and for $u = \bar{\alpha}(\bar{x}) + \bar{\beta}(\bar{x})\tilde{u}$ in the class, $\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\}^{\bar{\alpha},\bar{\beta}}$ satisfies (A.2). Then, the control law $\bar{u} = \bar{\alpha}(\bar{x}) + \bar{\beta}(\bar{x})\tilde{u}$ decouples $\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\}$ on $\bar{\mathcal{X}}$ if and only if it belongs to $\bar{\mathcal{S}}^{\omega}(\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\})$.

Proof. Suppose that a control law $\bar{u} = \bar{\alpha}(\bar{x}) + \bar{\beta}(\bar{x})\tilde{u}$ belongs to $\bar{\mathcal{S}}^{\omega}(\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\})$. Then, since $\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\}^{\bar{\alpha},\bar{\beta}}$ is decomposed on $\bar{\mathcal{X}}$, it is decoupled on $\bar{\mathcal{X}}$.

Next, suppose $\bar{u} = \bar{\alpha}(\bar{x}) + \bar{\beta}(\bar{x})\tilde{u}$ decouples $\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\}$ on $\bar{\mathcal{X}}$. Direct computation shows

(3.4.38) $\bar{D}^*(\bar{x}) = I_m, \quad \bar{A}^*(\bar{x}) = 0 \quad \text{on } \bar{\mathcal{X}}$

By Theorem 3. 4. 1, this implies that $\bar{\alpha}$, $\bar{\beta}$ must have the following properties :

(3.4.39) $\bar{\alpha}(\bar{x}) = \begin{bmatrix} \bar{\eta}_1(\bar{x}_1) \\ \cdots \\ \bar{\eta}_m(\bar{x}_m) \end{bmatrix}, \quad \bar{\beta}(\bar{x}) = \text{diag}\, \bar{\lambda}_i(\bar{x}), \quad \text{on } \bar{\mathcal{X}},$

(3.4.40) $d\bar{\eta}_i,\ d\bar{\lambda}_i \in \Delta_i^{\perp}(\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\}) \text{ on } \bar{\mathcal{X}}, \quad i \in \mathcal{M}_{1,m}\,,$

(3.4.41) $\bar{\lambda}_i(\bar{x}_i) \neq 0, \quad \bar{x}_i \in \bar{\mathcal{X}}_i, \quad i \in \mathcal{M}_{1,m}.$

Direct computation using the property (1) of $\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\}$ in Definition 3. 3. 1 shows that if $\bar{Y}_i \triangleq \sum_{j=1}^{m+1} \sum_{k=1}^{p_j} \bar{\gamma}_{j,k}(\cdot)\, \partial/\partial\bar{x}_{j,k}$ belongs to $\Delta_i(\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\})$, (3.3.67) must hold. This and the property (3) of $\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\}$ imply that if a covector field $\bar{\sigma}_i \triangleq \sum_{j=1}^{m+1} \sum_{k=1}^{p_j} \bar{\delta}_{j,k}(\cdot)\, d\bar{x}_{j,k}$

belongs to $\Delta_i^{\perp}(\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\})$ on $\bar{\mathcal{X}}$,

$$(3.4.42) \quad \bar{\delta}_{j,k}(\bar{x}) = 0, \ \bar{x} \in \bar{\mathcal{X}}, \quad k \in \mathbf{M}_{1,p_j} \ \text{ if } j \neq i.$$

Since $\bar{\mathcal{X}}$ is connected, this and Theorem 2.3.8 imply that any C^{ω}-function ψ from $\bar{\mathcal{X}}$ into R satisfying (3.4.40) must be the function of $\bar{x}_i$ only. □

Note that the property (3) of the standard decomposed system is essential in obtaining Theorem 3.4.2. Just as with Theorem 3.4.1, we are not able to prove that Theorem 3.4.3 extends to smooth systems and smooth control laws. But, we can show that $\bar{\mathbf{S}}^{\infty}(\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\})$ is the whole class of decomposing control laws for the smooth standard decomposed system $\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\}$.

Theorem 3.4.4. Let $\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\}$ be a standard decomposed system in Definition 3.4.1. Suppose that $\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\}$ satisfies (A.1) and class of control laws satisfies (A.5) and (A.6). Then, the control law $\bar{u} = \bar{\alpha}(\bar{x}) + \bar{\beta}(\bar{x})\tilde{u}$ decomposes $\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\}$ on $\bar{\mathcal{X}}$ if and only if it belongs to $\bar{\mathbf{S}}^{\infty}(\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\})$.

Proof. Let $\bar{u} = \bar{\alpha}(\bar{x}) + \bar{\beta}(\bar{x})\tilde{u}$ be a smooth control law in $\bar{\mathbf{S}}^{\infty}(\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\})$. Obviously, $\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\}^{\bar{\alpha},\bar{\beta}}$ is decomposed on $\bar{\mathcal{X}}$.

Next, suppose that a control law $\bar{u} = \bar{\alpha}(\bar{x}) + \bar{\beta}(\bar{x})\tilde{u}$ decomposes

$\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\}$ on $\bar{\mathcal{X}}$. This implies that $\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\}^{\bar{\alpha},\bar{\beta}}$ is decomposed at each $\bar{x}_0 \in \bar{\mathcal{X}}$. Let $\{\tilde{F}, \tilde{H}, \bar{\mathcal{X}}\} \triangleq \{\bar{F}, \bar{H}, \bar{\mathcal{X}}\}^{\bar{\alpha},\bar{\beta}}$. Let $\tilde{X}_i$, $i \in \mathbf{M}_{0,m}$ be vector fields corresponding to $\{\tilde{F}, \tilde{H}, \bar{\mathcal{X}}\}$. Then, by Theorem 3. 2. 2 and Lemma 3. 2. 2 - (i),

$$\tilde{X}_j \tilde{X}_{i_1} \tilde{X}_{i_2} \cdots \tilde{X}_{i_k} \tilde{H}_i = 0 \quad \text{on } \bar{\mathcal{X}}, \tag{3.4.43}$$

$$i_q \in \{o, i\},\ q \in \mathbf{M}_{1,k},\ k \in \mathbf{M}_{0,\infty},\ j \in \tilde{\mathbf{M}}_i,\ i \in \mathbf{M}_{1,m}.$$

Using this and (3.4.38), we can show that (3.4.39) - (3.4.41) hold. The arguments are very similar to those following (3.4.22) except for minor differences in notation. Once (3.4.39) - (3.4.41) hold, the remaining arguments are exactly the same as those following (3.4.41). □

The control laws in the sets $\mathcal{S}^{\omega}(\{F, H, \mathcal{X}\})$, $\mathcal{S}^{\infty}(\{F, H, \mathcal{X}\})$ are closely related to those in the sets $\bar{\mathcal{S}}^{\omega}(\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\})$, $\bar{\mathcal{S}}^{\infty}(\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\})$. We show that for a class of nonlinear systems, there is a one - to - one correspondence between them.

Theorem 3. 4. 5. Suppose that the hypotheses of Theorem 3. 3. 3 are satisfied. Let $x_0 \in \mathcal{X}$. Let $\mathcal{E}$, T, $\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\}$ be the open neighborhood of x_0, the mapping, and standard decomposed system given by Theorem 3. 3. 3. Then, there exist an open neighborhood $\mathcal{H} \subset \mathcal{E}$ of x_0 such that

(i) For every $u = \alpha(x) + \beta(x)\hat{u}$ in $\mathcal{S}^{\infty}(\{F, H, \mathcal{H}\})$, there exists a unique control law $\bar{u} = \bar{\alpha}(\bar{x}) + \bar{\beta}(\bar{x})\tilde{u}$ in $\bar{\mathcal{S}}^{\infty}(\{\bar{F}, \bar{H}, T(\mathcal{H})\})$ such that $\{\bar{F}, \bar{H}, T(\mathcal{H})\}^{\bar{\alpha},\bar{\beta}}$ is T-related on $\mathcal{H}$ to $\{F, H, \mathcal{H}\}^{\alpha,\beta}$. Conversely, for every $\bar{u} = \bar{\alpha}(\bar{x}) + \bar{\beta}(\bar{x})\tilde{u}$ in $\mathcal{S}^{\infty}(\{\bar{F}, \bar{H}, T(\mathcal{H})\})$, there exists a unique control law $u = \alpha(x) + \beta(x)\hat{u}$ in $\mathcal{S}^{\infty}(\{F, H, \mathcal{H}\})$ such that $\{F, H, \mathcal{H}\}^{\alpha,\beta}$ is T^{-1}-related on $T(\mathcal{H})$ to $\{\bar{F}, \bar{H}, T(\mathcal{H})\}^{\bar{\alpha},\bar{\beta}}$,

(ii) Let $u = \alpha(x) + \beta(x)\hat{u}$, $\bar{u} = \bar{\alpha}(\bar{x}) + \bar{\beta}(\bar{x})\tilde{u}$ be control laws in $\mathcal{S}^{\infty}(\{F, H, \mathcal{H}\})$, $\bar{\mathcal{S}}^{\infty}(\{\bar{F}, \bar{H}, T(\mathcal{H})\})$, respectively. Suppose they are in the one-to-one correspondence described in (i). Then,

$$(3.4.44) \quad \alpha(x) = [D^*(x)]^{-1}[\bar{\alpha}(T(x)) - A^*(x)], \quad \beta(x) = [D^*(x)]^{-1}\bar{\beta}(T(x)).$$

(iii) In particular, when T is a C^{∞}-diffeomorphism on $\mathcal{X}$ and $\mathcal{X}$ is connected, the above (i), (ii) hold with $\mathcal{H} = \mathcal{X}$.

Proof. First consider (i). Suppose $u = \alpha(x) + \beta(x)\hat{u}$ belongs to $\mathcal{S}^{\infty}(\{F, H, \mathcal{H}\})$. Then, following the second part of the proof for Theorem 3.4.3 leads to the fact that there exist an open neighborhood $\mathcal{H} \subset \mathcal{E}$ of x_0 and C^{∞}-functions $\bar{\eta}_i$, $\bar{\lambda}_i$, defined on appropriate subsets of R^{p_i}, $i \in \mathcal{M}_{1,m}$ such that (3.4.36) holds. Note that given T and $\mathcal{H}$, the $\bar{\eta}_i$ and $\bar{\lambda}_i$ are unique. Define $\bar{\alpha} \triangleq (\bar{\eta}_1, \ldots, \bar{\eta}_m)$ and $\beta \triangleq \text{diag}\, \bar{\lambda}_i$. Then, $\bar{u} = \bar{\alpha}(\bar{x}) + \bar{\beta}(\bar{x})\tilde{u}$ belongs to

$\mathcal{S}^\infty(\{\bar{F}, \bar{H}, T(\mathcal{H})\})$. Furthermore, $\{\bar{F}, \bar{H}, T(\mathcal{H})\}^{\bar{\alpha},\bar{\beta}}$ is T-related on $\mathcal{H}$ to $\{F, H, \mathcal{H}\}^{\alpha,\beta}$. Next, consider the converse statement. Suppose $\bar{u} = \bar{\alpha}(\bar{x}) + \bar{\beta}(\bar{x})\tilde{u}$ belongs to $\mathcal{S}^\infty(\{\bar{F}, \bar{H}, T(\mathcal{H})\})$. Define α, β by (3.4.44). Then, by (3.3.60), it follows that $u = \alpha(x) + \beta(x)\hat{u}$ belongs to $\mathcal{S}^\infty(\{F, H, T(\mathcal{H})\})$. Clearly, $\{F, H, \mathcal{H}\}^{\alpha,\beta}$ is T^{-1}-related on $\mathcal{H}$ to $\{\bar{F}, \bar{H}, T(\mathcal{H})\}^{\bar{\alpha},\bar{\beta}}$.

Part (ii) has been shown implicitly above. Part (iii) follows from that given hypotheses imply that (3.3.59), (3.3.60) hold on $\mathcal{X}$ and $T(\mathcal{X})$ is connected. By the arguments similar to those following (3.2.44), (3.4.36) holds globally on $\mathcal{X}$. □

Remark 3. 4. 4. See Figure 3. 4. 3 for a schematic description of Theorem 3. 4. 5. Systems $\{\bar{F}, \bar{H}, T(\mathcal{H})\}^{\bar{\alpha},\bar{\beta}}$ described in (i) of Theorem 3. 4. 5 have the forms :

$$(3.4.45) \qquad \dot{\tilde{x}}_i = \tilde{f}_i(\tilde{x}_i, \tilde{u}_i) \triangleq \begin{bmatrix} \bar{A}_i \tilde{x}_i \\ \bar{\theta}_i(\tilde{x}_i) \end{bmatrix} + \bar{\eta}_i(\tilde{x}_i)\begin{bmatrix} \bar{B}_i \\ \bar{\gamma}_i(\tilde{x}_i) \end{bmatrix} + \bar{\lambda}_i(\tilde{x}_i)\begin{bmatrix} \bar{B}_i \\ \bar{\gamma}_i(\tilde{x}_i) \end{bmatrix}\tilde{u}_i,$$

$$\tilde{y}_i = \tilde{h}_i(\tilde{x}_i) \triangleq \bar{C}_i \tilde{x}_i, \qquad i \in \mathcal{M}_{1,m},$$

$$(3.4.46) \qquad \dot{\tilde{x}}_{m+1} = \bar{f}_{m+1}(\tilde{x}) + \sum_{i=1}^{m} \bar{b}_i(\tilde{x})\bar{\eta}_i(\tilde{x}_i) + \sum_{i=1}^{m} \bar{\lambda}_i(\tilde{x}_i)\bar{b}_i(\tilde{x})\tilde{u}_i .$$

Thus, part (i) characterizes the class of closed-loop locally decomposed or decoupled systems. Part (ii) shows connection between a given closed-loop system and a feedback control law. □

Remark 3.4.5. Since $\{\bar{F}, \bar{H}, T(\mathcal{H})\}^{\bar{\alpha},\bar{\beta}}$ is T-related on $\mathcal{H}$ to $\{F, H, \mathcal{H}\}^{\alpha,\beta}$, the solutions of the differential equations for the two syatems are related by T (i. e., $\bar{x}(t) = T(x(t))$). Also the two systems have the same input - output maps. When $\mathcal{H} = \mathcal{X}$, these results are valid globally on $\mathcal{X}$. □

Remark 3.4.6. For a standard decomposed system $\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\}$ in Definition 3.3.1, let

$$(3.4.47)\quad \bar{\mathcal{X}}_i^* \triangleq \{ (\bar{x}_{i,1}, \ldots, \bar{x}_{i,\bar{d}i+1}) \in R^{\bar{d}i+1} : \bar{x} \triangleq (\bar{x}_1, \ldots, \bar{x}_m, \bar{x}_{m+1}) \in \bar{\mathcal{X}} \},\ i \in \mathcal{N}_{1,m}.$$

Define $\bar{\mathcal{S}}_o^{\infty}(\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\})$ ($\bar{\mathcal{S}}_o^{\omega}(\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\})$) by a set of all control laws

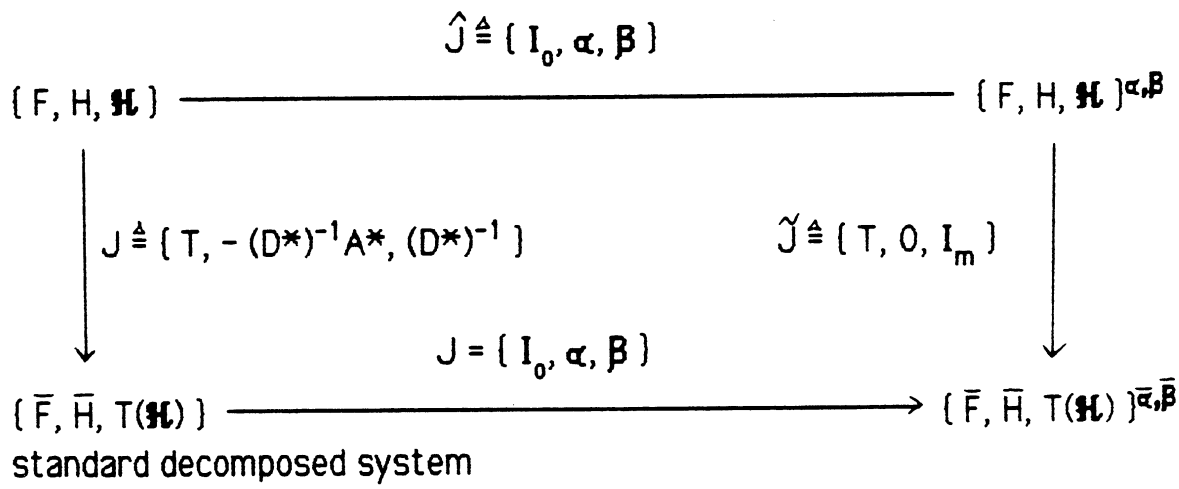

Fig. 3.4.3. A schematic description of Theorem 3.4.5, where $u = \alpha(x) + \beta(x)\hat{u}$, $\bar{u} = \bar{\alpha}(\bar{x}) + \bar{\beta}(\bar{x})\tilde{u}$ are control laws in $\mathcal{S}^{\infty}(\{F, H, \mathcal{H}\})$, $\bar{\mathcal{S}}^{\infty}(\{\bar{F}, \bar{H}, T(\mathcal{H})\})$, respectively.

$\bar{u} = \bar{\alpha}(\bar{x}) + \bar{\beta}(\bar{x})\tilde{v}$ satisfying (A.5) ((A.5)'), (A.6), and

$$(3.4.48) \quad \bar{\alpha}(\bar{x}) = \begin{bmatrix} \tilde{\eta}_1(\bar{x}_{1,1}, \ldots, \bar{x}_{1,\bar{d}1+1}) \\ \cdots \\ \tilde{\eta}_m(\bar{x}_{m,1}, \ldots, \bar{x}_{m,\bar{d}m+1}) \end{bmatrix},$$

$$(3.4.49) \quad \bar{\beta}(\bar{x}) = \mathrm{diag}\ \tilde{\lambda}_i(x_{i,1}, \ldots, x_{i,\bar{d}i+1}),$$

where $\tilde{\eta}_i$, $\tilde{\lambda}_i$ are arbitrary functions from $\bar{\mathcal{X}}_i^*$ into R. Clearly, $\bar{\mathcal{S}}_o^\infty(\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\})$ is a subset of $\bar{\mathcal{S}}^\infty(\{\bar{F}, \bar{H}, \bar{\mathcal{X}}\})$. Let $\{F, H, \mathcal{X}\}$ be a system which satisfies the hypotheses of Theorem 3. 4. 5. All statements in Theorem 3. 4. 5 still hold with $\mathcal{S}^\infty(\{F, H, \mathcal{X}\})$, $\bar{\mathcal{S}}^\infty(\{\bar{F}, \bar{H}, T(\mathcal{X})\})$ replaced by $\mathcal{S}_o^\infty(\{F, H, \mathcal{X}\})$, $\bar{\mathcal{S}}_o^\infty(\{\bar{F}, \bar{H}, T(\mathcal{X})\})$, respectively. □

Remark 3. 4. 7. Suppose the hypotheses of Theorem 3. 3. 1, (3.3.4), and $n = \sum_{i=1}^{m}(d_i+1)$ are satisfied. Then, the hypotheses of Theorem 3. 4. 5 are satisfied trivially. In particular, $p_i = d_i + 1$, $i \in \mathbf{M}_{1,m}$ and T is given by $T \triangleq (T_1, \ldots, T_m)$, where $T_i \triangleq (T_{i,1}, \ldots, T_{i,di+1})$, and $T_{i,j} \triangleq X_0^{(j-1)} H_i$. Then, (ii) of Theorem 3. 4. 5 shows that at least locally, $\mathcal{S}^\infty(\{F, H, \mathcal{X}\}) = \mathcal{S}_o^\infty(\{F, H, \mathcal{X}\})$. When T is a C^∞- diffeomorphism on $\mathcal{X}$ and $\mathcal{X}$ is connected, (iii) confirms that $\mathcal{S}^\infty(\{F, H, \mathcal{X}\}) = \mathcal{S}_o^\infty(\{F, H, \mathcal{X}\})$. Note that for this case, we do not need to solve the partial differential equations (3.4.11) to characterize $\mathcal{S}^\infty(\{F, H, \mathcal{X}\})$. But, if T is not a C^∞-diffeomorphism, $\mathcal{S}^\infty(\{F, H, \mathcal{X}\}) = \mathcal{S}_o^\infty(\{F, H, \mathcal{X}\})$ is not necessarily true. This will be shown through Example 3. 5. 2 in Section 3. 5. □

In this section, we have presented results concerning questions (b), (c), (d) in Chapter 1. They are described in Figure 3. 4. 4 in a schematic way. The simplicity of the results for standard decomposed systems, together with Remark 3. 4. 4 and 3. 4. 5, suggests that in system design it may be easier to deal with the standard decomposed system than with the original system. But, it should be noted that in order to transform the original system into the standard decomposed system, we have to compute a mapping T (see Theorem 3. 3. 3). Computing the mapping T is usually a difficult job since it is basically equivalent to solving a set of the first order linear partial differential equations.

3. 5. Examples

In this section, we present three examples which illustrate the significance of the results developed in the previous sections.

Example 3. 5. 1 is a real analytic system $\{F, H, R^3\}$ which is decouplable and decomposable on R^3. For this example, $\mathfrak{F}_o^{\omega}(\{F, H, \mathcal{X}\})$ is a proper subset of $\mathfrak{F}^{\omega}(\{F, H, \mathcal{X}\})$. While there is no control law in $\mathfrak{F}_o^{\omega}(\{F, H, R^3\})$ which decouples $\{F, H, R^3\}$ on R^3 with **Bounded Input - Bounded State (BIBS) stability**, there are many control laws in $\mathfrak{F}^{\omega}(\{F, H, R^3\})$ which decouples $\{F, H, R^3\}$ on R^3 with BIBS stability.

Example 3. 5. 2 shows that $n = \sum_{i=1}^{m} (d_i + 1)$ does not necessarily imply $\mathfrak{F}^{\omega}(\{F, H, \mathcal{X}\}) = \mathfrak{F}_o^{\omega}(\{F, H, \mathcal{X}\})$. For this example, T defined in Remark 3.4.7 is a C^{ω}-diffeomorphism locally at each point of R^3 but not globally

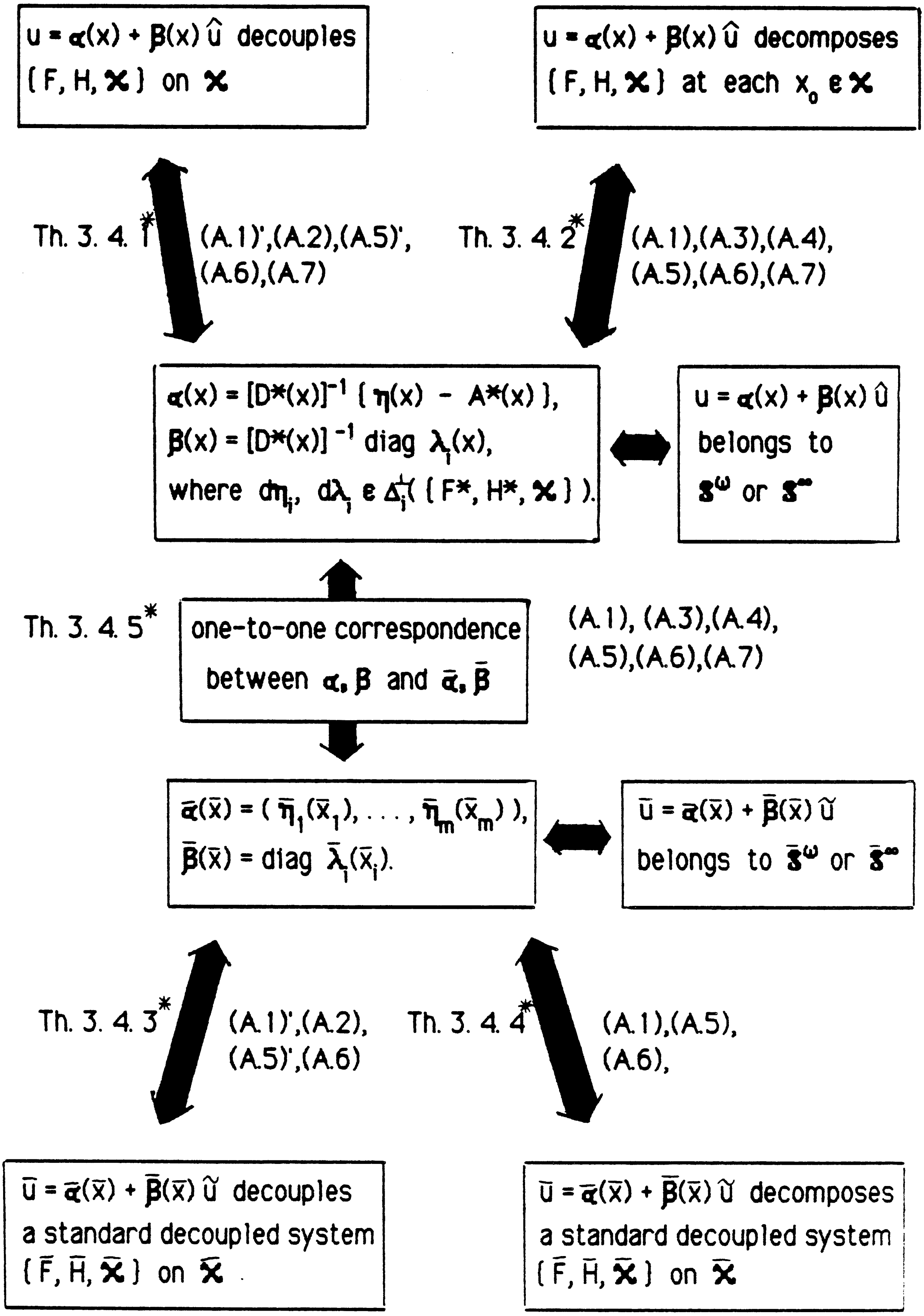

Figure 3. 4. 4. Summary of main results in Section 3. 4 showing assumptions required for each implication.

on R^3. Thus, this example shows that if T is not a C^ω- diffeomorphism, $\mathfrak{s}^\omega(\{F, H, R^3\}) = \mathfrak{s}_o^\omega(\{F, H, R^3\})$ is not necessarily true.

Example 3. 5. 3 was considered in [Sin.1]. We show that for this example, T defined in Remark 3. 4. 7 is a C^∞ - diffeomorphism on R^3 and hence $\mathfrak{s}^\omega(\{F, H, R^3\}) = \mathfrak{s}_o^\omega(\{F, H, R^3\})$. In [Sin.1], a necessary condition for a control law to decouple $\{F, H, R^3\}$ is given in a form of partial differential equations and a class of decoupling control laws is specified. We give a more complete solution for this example.

Example 3. 5. 1. Let us consider a real analytic system $\{F, H, R^3\}$ with $m = 2$ and

(3.5.1) $\quad X_0(x) \triangleq (x_2 + x_1 x_3)\,\partial/\partial x_2,$

(3.5.2) $\quad X_1(x) \triangleq \partial/\partial x_1 + (1 + x_1 - x_3)\,\partial/\partial x_2 - \partial/\partial x_3,$

(3.5.3) $\quad X_2(x) \triangleq \partial/\partial x_1 + (1 - x_3)\,\partial/\partial x_2,$

(3.5.4) $\quad H_1(x) \triangleq x_1, \quad H_2(x) \triangleq x_1 + x_3.$

Direct computation shows that all hypotheses in Theorem 3. 3. 1 and (3.3.4) are satisfied with

(3.5.5) $$d_1 = d_2 = 0, \quad D^*(x) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A^*(x) = 0.$$

Thus, by Theorem 3. 3. 1 and Theorem 3. 3. 2, $\{F, H, R^3\}$ is decouplable on R^3 and decomposable at each $x_0 \in R^3$.

To characterize the whole class of real analytic decoupling control laws, we have to compute $\Delta_i(\{F^*, H^*, R^3\})$, $i \in \mathfrak{M}_{1,2}$. Let $\hat{X}_i$, $i \in \mathfrak{M}_{0,2}$ be the vector fields corresponding to the decoupled system $\{F^*, H^*, R^3\}$. Then, by (3.5.5), we have

(3.5.6) $\hat{X}_0(x) = (x_2 + x_1 x_3)\,\partial/\partial x_2$,

(3.5.7) $\hat{X}_1(x) = \partial/\partial x_1 + (1 + x_1 - x_3)\,\partial/\partial x_2 - \partial/\partial x_3$,

(3.5.8) $\hat{X}_2(x) = -x_1\,\partial/\partial x_2 + \partial/\partial x_3$.

From these, we can compute

(3.5.9) $L_{\hat{X}_0}\hat{X}_2(x) = L_{\hat{X}_1}\hat{X}_2(x) = L_{\hat{X}_2}\hat{X}_1(x) = 0, \quad L_{\hat{X}_2}L_{\hat{X}_0}\hat{X}_1(x) = 0,$

(3.5.10) $L_{\hat{X}_0}\hat{X}_1(x) = -\,\partial/\partial x_2, \quad L^2_{\hat{X}_0}\hat{X}_1(x) = \partial/\partial x_2$.

From these, it is easy to see that on R^3,

(3.5.11) $\Delta_1(\{F^*, H^*, R^3\}) = \text{span}\{\hat{X}_2\}$,

(3.5.12) $\Delta_2(\{F^*, H^*, R^3\}) = \text{span}\{\hat{X}_1, L_{\hat{X}_0}\hat{X}_1\}$.

These with (3.5.5) determine $\mathfrak{S}^\omega(\{F, H, R^3\})$.

Note (3.5.7) - (3.5.10) imply (A.3). On the other hand, by (3.5.11), (3.5.12),

(3.5.13) $\dim.\,\Delta_1^\perp(\{F^*, H^*, R^3\}) = 2, \quad \dim.\,\Delta_2^\perp(\{F^*, H^*, R^3\}) = 1.$

Thus, (A.4) is satisfied by $p_1 = 2$, $p_2 = 1$. Consequently, all hypotheses of Theorem 3.4.5 are satisfied. Define C^ω - functions

$T_{i,j}$, $j \in \mathbf{M}_{1,pi}$, $i \in \mathbf{M}_{1,2}$ by

(3.5.14) $T_{1,1}(x) \triangleq H_1(x)$, $T_{1,2}(x) \triangleq x_2 + x_1 x_3$, $T_{2,1}(x) \triangleq H_2(x)$.

Let $T \triangleq (T_{1,1}, T_{1,2}, T_{2,1})$. Then, we can easily show that T is a C^{ω}-diffeomorphism from R^3 onto R^3 and $\{ dT_{i,j}(q),\ j \in \mathbf{M}_{1,pi} \}$ is a basis of $(\Delta_i^{\perp})_q(\{F^*, H^*, R^3\})$, $q \in R^3$, $i \in \mathbf{M}_{1,2}$. Let $\{\bar{F}, \bar{H}, R^3\}$ be a standard decoupled system whose coordinate representation is

(3.5.15)
$$\begin{bmatrix} \dot{\bar{x}}_{1,1} \\ \dot{\bar{x}}_{1,2} \end{bmatrix} = \begin{bmatrix} 0 \\ \bar{x}_{1,2} \end{bmatrix} + \bar{u}_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \bar{y}_1 = \bar{x}_{1,1},$$
$$\dot{\bar{x}}_2 = \bar{u}_2, \qquad \bar{y}_2 = \bar{x}_2.$$

Then, we can check that the above T and $\{\bar{F}, \bar{H}, R^3\}$ with $\mathbf{E} = R^3$ are those described in Theorem 3. 3. 3 and Theorem 3. 4. 5.

By (3.4.44) and (3.5.5), $\mathbf{S}^{\omega}(\{F, H, R^3\})$ is given by

(3.5.16)
$$\alpha(x) = \begin{bmatrix} \phi_1(x_1, x_2 + x_1 x_3) - \phi_2(x_1 + x_3) \\ \phi_2(x_1 + x_3) \end{bmatrix},$$

(3.5.17)
$$\beta(x) = \begin{bmatrix} \psi_1(x_1, x_2 + x_1 x_3) & \vdots & -\psi_2(x_1 + x_3) \\ 0 & \vdots & \psi_2(x_1 + x_3) \end{bmatrix},$$

where ϕ_i, ψ_i, $i \in \mathbf{M}_{1,2}$ are arbitrary C^{ω}-functions of their arguments such that $\psi_1(z_1, z_2) \neq 0$, $(z_1, z_2) \in R^2$ and $\psi_2(z_3) \neq 0$, $z_3 \in R$. On the other hand, by Definition 3. 4. 1 and (3.5.5), $\mathfrak{S}^{\omega}_{\circ}(\{F, H, R^3\})$ is given by

$$\text{(3.5.18)} \quad \mathring{\alpha}(x) = \begin{bmatrix} \mathring{\phi}_1(x_1) - \mathring{\phi}_2(x_1 + x_3) \\ \mathring{\phi}_2(x_1 + x_3) \end{bmatrix},$$

$$\text{(3.5.19)} \quad \mathring{\beta}(x) = \begin{bmatrix} \mathring{\psi}_1(x_1) & \vdots & -\mathring{\psi}_2(x_1 + x_3) \\ 0 & & \mathring{\psi}_2(x_1 + x_3) \end{bmatrix},$$

where $\mathring{\phi}_i$, $\mathring{\psi}_i$, $i \in \mathbf{M}_{1,2}$ are arbitrary C^{ω}-functions of their arguments such that $\mathring{\psi}_i(z) \neq 0$, $z \in R$, $i \in \mathbf{M}_{1,2}$. From (3.5.16) - (3.5.19), we see that $\mathfrak{S}^{\omega}_{\circ}(\{F, H, R^3\}) \neq \mathfrak{S}^{\omega}(\{F, H, R^3\})$ but $\mathfrak{S}^{\omega}_{\circ}(\{F, H, R^3\}) \subset \mathfrak{S}^{\omega}(\{F, H, R^3\})$.

By Theorem 3. 4. 3, $\mathfrak{S}^{\omega}(\{\bar{F}, \bar{H}, R^3\})$ is given by

$$\text{(3.5.16)'} \quad \bar{\alpha}(\bar{x}) = \begin{bmatrix} \bar{\phi}_1(\bar{x}_{1,1}, \bar{x}_{1,2}) \\ \bar{\phi}_2(\bar{x}_2) \end{bmatrix},$$

$$\text{(3.5.17)'} \quad \bar{\beta}(\bar{x}) = \begin{bmatrix} \bar{\psi}_1(\bar{x}_{1,1}, \bar{x}_{1,2}) & 0 \\ 0 & \bar{\psi}_2(\bar{x}_2) \end{bmatrix},$$

where $\bar{\phi}_i$, $\bar{\psi}_i$, $i \in \mathbf{M}_{1,2}$ are arbitrary C^{ω}-functions of their arguments such that $\bar{\psi}_1(\bar{x}_1, \bar{x}_2) \neq 0$, $(\bar{x}_1, \bar{x}_2) \in R^2$ and $\bar{\psi}_2(\bar{x}_3) \neq 0$, $\bar{x}_3 \in R$. Note that as is indicated by Theorem 3. 4. 5, there is one - to - one correspondence between the control laws of $\mathfrak{I}^{\omega}(\{F, H, R^3\})$ in (3.5.16), (3.5.17) and those of $\bar{\mathfrak{I}}^{\omega}(\{\bar{F}, \bar{H}, R^3\})$ in (3.5.16)', (3.5.17)'.

Using the standard decomposed system, it is easy to see how to choose control laws which decouple $\{F, H, R^3\}$ in a stable way. Suppose we want to decouple $\{F, H, R^3\}$ on R^3 with BIBS stability. First, consider $\bar{u} = \bar{\alpha}(\bar{x}) + \bar{\beta}(\bar{x})\tilde{u}$ where $\bar{\alpha}$, $\bar{\beta}$ satisfy (3.5.16)', (3.5.17)'. Let $\{\tilde{f}, \tilde{h}, R^3\}$ be the coordinate representation of $\{\bar{F}, \bar{H}, R^3\}^{\bar{\alpha},\bar{\beta}}$. Then, $\{\tilde{f}, \tilde{h}, R^3\}$ is described by

(3.5.20)
$$\begin{bmatrix} \dot{\tilde{x}}_{1,1} \\ \dot{\tilde{x}}_{1,2} \end{bmatrix} = \begin{bmatrix} \bar{\phi}_1(\tilde{x}_{1,1}, \tilde{x}_{1,2}) \\ \tilde{x}_{1,2} + \bar{\phi}_1(\tilde{x}_{1,1}, \tilde{x}_{1,2}) \end{bmatrix} + \begin{bmatrix} \bar{\psi}_1(\tilde{x}_{1,1}, \tilde{x}_{1,2}) \\ \bar{\psi}_1(\tilde{x}_{1,1}, \tilde{x}_{1,2}) \end{bmatrix} \tilde{u}_1, \quad \tilde{y}_1 = \tilde{x}_{1,1},$$
$$\dot{\tilde{x}}_2 = \bar{\phi}_2(\tilde{x}_2) + \bar{\psi}_2(\tilde{x}_2) u_2, \quad \tilde{y}_2 = \tilde{x}_2.$$

Note from property (2) of Definition 3. 3. 1 and (3.1.9) that $\{\tilde{f}_1, \tilde{h}_1, R^2\}$, $\{\tilde{f}_2, \tilde{h}_2, R\}$ in (3.5.15) are controllable linear systems. Therefore, there are many choice of $\bar{\phi}_1$, $\bar{\phi}_2$ so that $\{\bar{F}, \bar{H}, R^3\}^{\bar{\alpha},\bar{\beta}}$ is decoupled on R^3 with BIBS stability. For such a control law $\bar{u} = \bar{\alpha}(\bar{x}) + \bar{\beta}(\bar{x})\tilde{u}$, choose a control laws $u = \alpha(x) + \beta(x)\hat{u}$ by (3.4.43). Then, $\{\bar{F}, \bar{H}, R^3\}^{\bar{\alpha},\bar{\beta}}$ is T - related on R^3 to $\{F, H, R^3\}^{\alpha,\beta}$. Recall that T is a C^{ω}- diffeomorphism on R^3. Furthermore, by a special form of T in (3.5.14), it follows that for any constant b, $\{x \in R^3 : |T(x)| \leq b\}$ is bounded. These observations imply that $\{F, H, R^3\}^{\alpha,\beta}$ is decoupled on R^3 with BIBS stability. Thus, we have shown that there are many control law $u = \alpha(x) + \beta(x)\hat{u}$ in

$\mathscr{S}^{\omega}(\{F, H, \mathscr{X}\})$ which decouple $\{F, H, R^3\}$ on R^3 in a stable way.

Next, consider $\bar{u} = \bar{\mathring{\alpha}}(\bar{x}) + \bar{\mathring{\beta}}(\bar{x})\,\tilde{u}$ in $\bar{\mathscr{S}}^{\omega}(\{\bar{F}, \bar{H}, R^3\})$. Then, $\bar{\mathring{\alpha}}$, $\bar{\mathring{\beta}}$ have the forms :

$$(3.5.21) \qquad \bar{\mathring{\alpha}}(\bar{x}) = \begin{bmatrix} \bar{\mathring{\phi}}_1(\bar{x}_{1,1}) \\ \bar{\mathring{\phi}}_2(\bar{x}_2) \end{bmatrix}, \qquad \bar{\mathring{\beta}}(x) = \begin{bmatrix} \bar{\mathring{\Psi}}_1(\bar{x}_{1,1}) & 0 \\ 0 & \bar{\mathring{\Psi}}_2(\bar{x}_2) \end{bmatrix},$$

where $\bar{\mathring{\phi}}_i$, $\bar{\mathring{\Psi}}_i$, $i \in \mathbf{n}_{1,2}$ are arbitrary C^{ω}-functions of their arguments such that $\bar{\mathring{\Psi}}_i(z) \neq 0$, $z \in R$, $i \in \mathbf{n}_{1,2}$. Now, let $u = \mathring{\alpha}(x) + \mathring{\beta}(x)\,\hat{u}$ be a control law in $\mathscr{S}^{\omega}_{o}(\{F, H, R^3\})$. By Theorem 3. 4. 5 and Remark 3. 4. 1, we know that for each $u = \mathring{\alpha}(x) + \mathring{\beta}(x)\hat{u}$ in $\mathscr{S}^{\omega}_{o}(\{F, H, R^3\})$, there is a unique control law $\bar{u} = \bar{\mathring{\alpha}}(\bar{x}) + \bar{\mathring{\beta}}(\bar{x})\,\tilde{u}$ in $\bar{\mathscr{S}}^{\omega}_{o}(\{\bar{F}, \bar{H}, R^3\})$ such that $\{\bar{F}, \bar{H}, R^3\}^{\bar{\mathring{\alpha}}, \bar{\mathring{\beta}}}$ is T-related on R^3 to $\{F, H, R^3\}^{\mathring{\alpha}, \mathring{\beta}}$. Let $\{\tilde{f}, \tilde{h}, R^3\}$ be the coordinate representation of $\{\bar{F}, \bar{H}, R^3\}^{\bar{\mathring{\alpha}}, \bar{\mathring{\beta}}}$. Then, $\{\tilde{f}, \tilde{h}, R^3\}$ is described by

$$(3.5.20)' \quad \begin{bmatrix} \dot{\tilde{x}}_{1,1} \\ \dot{\tilde{x}}_{1,2} \end{bmatrix} = \begin{bmatrix} \bar{\mathring{\phi}}_1(\tilde{x}_{1,1}) \\ \tilde{x}_{1,2} + \bar{\mathring{\phi}}_1(\tilde{x}_{1,1}) \end{bmatrix} + \begin{bmatrix} \bar{\mathring{\Psi}}_1(\tilde{x}_{1,1}) \\ \bar{\mathring{\Psi}}_1(\tilde{x}_{1,1}) \end{bmatrix} \tilde{u}_1\,, \qquad \tilde{y}_1 = \tilde{x}_{1,1},$$

$$\dot{\tilde{x}}_2 = \bar{\mathring{\phi}}_2(\tilde{x}_2) + \bar{\mathring{\Psi}}_2(\tilde{x}_2)\,\tilde{u}_2\,, \qquad \tilde{y}_2 = \tilde{x}_2$$

From (3.5.20)', we see that there is no $\bar{\mathring{\phi}}_1$ and $\bar{\mathring{\Psi}}_1$ such that for every bounded $\tilde{u}_1$, $\tilde{x}_{1,2}$ is bounded. By the special structure of T in (3.5.14), this implies that there is no $u = \mathring{\alpha}(x) + \mathring{\beta}(x)\,\hat{u}$ in

$\mathbf{S}^{\omega}_{\circ}(\{F, H, R^3\})$ which decouples $\{F, H, R^3\}$ on R^3 in a stable way. □

Example 3. 5. 2. Let us consider a real analytic system $\{F, H, R^3\}$ with $m = 2$ and

(3.5.22) $X_0(x) \triangleq \partial/\partial x_2,$

(3.5.23) $X_1(x) \triangleq \cos x_2\, \partial/\partial x_1 + \sin x_2\, \partial/\partial x_2, \qquad X_2(x) = \partial/\partial x_3,$

(3.5.24) $H_1(x) \triangleq e^{-x_1} \sin x_2, \quad H_2(x) = x_3 .$

Direct computation shows that all hypotheses of Theorem 3. 3. 1 and (3.3.4) are satisfied and

(3.5.25) $d_1 = 1,\ d_2 = 0,\ D^*(x) = \begin{bmatrix} e^{-x_1} & 0 \\ 0 & 1 \end{bmatrix},\ A^*(x) = \begin{bmatrix} e^{-x_1} \sin x_2 \\ 0 \end{bmatrix},$

Let $\hat{X}_i$, $i \in \mathbf{M}_{0,2}$ be the vector fields corresponding to the decoupled system $\{F^*, H^*, R^3\}$. Then, by (3.5.25), we have

(3.5.26) $\hat{X}_0(x) = (-.5 \sin 2x_2)\, \partial/\partial x_1 + (\cos x_2)^2\, \partial/\partial x_2,$

(3.5.27) $\hat{X}_1(x) = (e^{x_1} \cos x_2)\, \partial/\partial x_1 + (e^{x_1} \sin x_2)\, \partial/\partial x_2 ,$

(3.5.28) $\hat{X}_2(x) = \partial/\partial x_3.$

Note that since $3 = \sum_{i=1}^{2} d_i + 1$, (A.4) is satisfied by $p_1 = 2$, $p_2 = 1$. Define functions $T_{i,j} : R^3 \to R$, $j \in \mathbf{M}_{1,(di+1)}$, $i \in \mathbf{M}_{1,2}$ by

(3.5.29) $\quad T_{1,1}(x) \triangleq x_1, \; T_{1,2}(x) \triangleq x_2, \; T_{2,1}(x) \triangleq x_3.$

We can check easily that at each $q \in R^3$, $\{dT_{i,j}(q),\ j \in \mathbf{M}_{1,pi}\}$ is a basis of $(\Delta_i^{\perp})_q(\{F^*, H^*, R^3\})$, $i \in \mathbf{M}_{1,2}$ and $T \triangleq (T_{1,1}, T_{1,2}, T_{2,1})$ is a C^{ω}- diffeomorphism from R^3 onto R^3. Thus, for each $i \in \mathbf{M}_{1,2}$, Lemma 3. 2. 4 holds with $r = n - p_i$, $\Delta = \Delta_i$, $\theta_j = T_{i,j}$, $\mathbf{X} = U = \mathbf{E} = R^3$. From these observations, Definition 3. 4. 2, and (3.5.25), $\mathbf{S}^{\omega}(\{F, H, R^3\})$ is given by

$$\alpha(x) = \begin{bmatrix} e^{x_1}\eta_1(x_1, x_2) - \sin x_2 \\ \eta_2(x_3) \end{bmatrix}, \quad \beta(x) = \begin{bmatrix} e^{x_1}\lambda_1(x_1, x_2) & 0 \\ 0 & \lambda_2(x_3) \end{bmatrix}, \tag{3.5.30}$$

where η_i, λ_i, $i \in \mathbf{M}_{1,2}$ are arbitrary C^{ω}-functions of their arguments such that $\lambda_1(x_1, x_2) \neq 0$, $(x_1, x_2) \in R^2$ and $\lambda_2(x_3) \neq 0$, $x_3 \in R$. But, (3.5.30) can be more simply described by

$$\alpha(x) = \begin{bmatrix} \phi_1(x_1, x_2) \\ \phi_2(x_3) \end{bmatrix}, \quad \beta(x) = \begin{bmatrix} \psi_1(x_1, x_2) & 0 \\ 0 & \psi_2(x_3) \end{bmatrix}, \tag{3.5.31}$$

where ϕ_i, ψ_i, $i \in \mathbf{M}_{1,2}$ are arbitrary C^{ω}- functions of their arguments such that $\psi_1(x_1, x_2) \neq 0$, $(x_1, x_2) \in R^2$ and $\psi_2(x_3) \neq 0$, $x_3 \in R$. But, by Definition 3. 4. 1 and (3.5.25), $\mathbf{S}_o^{\omega}(\{F, H, R^3\})$ is given by

$$(3.5.32) \quad \mathring{\alpha}(x) = \begin{bmatrix} e^{x_1}\mathring{\eta}_1(e^{-x_1}\sin x_2, \ e^{-x_1}\cos x_2) - \sin x_2 \\ \mathring{\eta}_2(x_3) \end{bmatrix},$$

$$(3.5.33) \quad \mathring{\beta}(x) = \begin{bmatrix} e^{x_1}\mathring{\lambda}_1(e^{-x_1}\sin x_2, \ e^{-x_1}\cos x_2) & 0 \\ 0 & \mathring{\lambda}_2(x_3) \end{bmatrix},$$

where $\mathring{\eta}_i$, $\mathring{\lambda}_i$, $i \in \mathbf{M}_{1,2}$ are arbitrary C^ω-functions of their arguments such that $\mathring{\lambda}_1(x_1, x_2) \neq 0$, $(x_1, x_2) \in R^2$ and $\mathring{\lambda}_2(x_3) \neq 0$, $x_3 \in R$. But, (3.5.32), (3.5.33) can be more simply described by

$$(3.5.34) \quad \mathring{\alpha}(x) = \begin{bmatrix} \mathring{\phi}_1(e^{-x_1}\sin x_2, \ e^{-x_1}\cos x_2) \\ \mathring{\phi}_2(x_3) \end{bmatrix},$$

$$(3.5.35) \quad \mathring{\beta}(x) = \begin{bmatrix} \mathring{\psi}_1(e^{-x_1}\sin x_2, \ e^{-x_1}\cos x_2) & \vdots & 0 \\ 0 & \vdots & \mathring{\psi}_2(x_3) \end{bmatrix},$$

where $\mathring{\phi}_i$, $\mathring{\psi}_i$, $i \in \mathbf{M}_{1,2}$ are arbitrary C^ω-functions of their arguments such that $\mathring{\psi}_1(x_1, x_2) \neq 0$, $(x_1, x_2) \in R^2$ and $\mathring{\psi}_2(x_3) \neq 0$, $x_3 \in R$. Note that $F(x_1, x_2) \triangleq (e^{-x_1}\sin x_2, \ e^{-x_1}\cos x_2)$ is a C^ω-diffeomorphism locally at each point $(x_1, x_2) \in R^2$ but not a C^ω-diffeomorphism globally on R^2. Therefore, there does not exist a C^ω-function $\mathring{\phi}_1$ such that $\mathring{\phi}_1(e^{-x_1}\sin x_2, \ e^{-x_1}\cos x_2) = x_2$, $x_2 \in R$. Thus, although the class of control laws given by (3.5.31) is locally equivalent to the one given by (3.5.34), (3.5.35), they are not globally equivalent. Thus, we have shown that although $\sum_{i=1}^{2}(d_i+1) = 3$, $\mathcal{S}_o^{\omega}(\{F, H, R^3\})$ is only a proper subset of $\mathcal{S}^{\omega}(\{F, H, R^3\})$. □

Example 3.5.3. Consider a real analytic system $\{F, H, \mathcal{X}\}$ with $m = 2$, $\mathcal{X} \triangleq \{x \triangleq (x_1, x_2, x_3) \in R^3 : x_2 > 0\}$, and

(3.5.36) $X_0(x) \triangleq x_1 x_2\ \partial/\partial x_3$,

(3.5.37) $X_1(x) \triangleq \partial/\partial x_1$, $X_2(x) \triangleq \partial/\partial x_2$,

(3.5.38) $H_1(x) \triangleq x_2$, $H_2(x) \triangleq x_3$.

Then, we have

(3.5.39) $d_1 = 0,\ d_2 = 1,\ D^*(x) = \begin{bmatrix} 0 & 1 \\ x_2 & x_1 \end{bmatrix},\ A^*(x) = 0.$

Note that $\mathcal{X}$ is connected and $\sum_{i=1}^{2}(d_i + 1) = 3$.

Define functions $T_{i,j} : R^3 \to R$, $j \in \mathbf{M}_{1,(di+1)}$, $i \in \mathbf{M}_{1,2}$ by

(3.5.40) $T_{1,1}(x) \triangleq x_2$, $T_{2,1}(x) \triangleq x_3$, $T_{2,2}(x) \triangleq x_1 x_2$.

Note that $T_{i,j} = X_0^{(j-1)} H_i$, $j \in \mathbf{M}_{1,pi}$, $i \in \mathbf{M}_{1,2}$. Clearly, $T \triangleq (T_{1,1}, T_{2,1}, T_{2,2})$ is a C^{ω}-diffeomorphism from $\mathcal{X}$ onto $\bar{\mathcal{X}} \triangleq \{\bar{x} \in R^3 :$

$x_1 > 0$}. By Remark 3.4.7 and (3.5.39), $\mathcal{S}^{\omega}(\{F, H, \mathcal{K}\}) = \mathcal{S}_o^{\omega}(\{F, H, \mathcal{K}\})$ and is given by

$$(3.5.41)\quad \alpha(x) = \begin{bmatrix} \{\eta_2(x_1x_2, x_3) - x_1\eta_1(x_2)\}/x_2 \\ \eta_1(x_2) \end{bmatrix},$$

$$(3.5.42)\quad \beta(x) = \begin{bmatrix} -x_1\lambda_1(x_2)/x_2 & \lambda_2(x_1x_2, x_3)/x_2 \\ \lambda_1(x_2) & 0 \end{bmatrix},$$

where η_i, λ_i, $i \in \mathbf{n}_{1,2}$ are arbitrary C^{ω}-functions of their arguments such that $\lambda_1(x_1) \neq 0$, $x_1 \in R^2$ and $\lambda_2(x_2, x_3) \neq 0$, $(x_2, x_3) \in R$.

To compare our solution with the one given by Singh and Rugh ([Sin.1]), we consider the partial differential equations given by (3.4.35). It should be noted that as is pointed out in Remark 3.4.3, (3.4.35) is not a sufficient but a necessary condition for a control law to decouple a system. Through some calculation, we can obtain that α, β solve (3.4.35) if and only if they satisfy (3.5.41), (A.5), and

$$(3.5.43)\quad \beta(x) = \begin{bmatrix} -x_1\tilde{\lambda}_1(x_1, x_2, x_3)/x_2 & \tilde{\lambda}_2(x_1, x_2, x_3)/x_2 \\ \tilde{\lambda}_1(x_1, x_2, x_3) & 0 \end{bmatrix},$$

where $\tilde{\lambda}_i$, $i \in \mathbf{n}_{1,2}$ are arbitrary C^{ω}-functions of their arguments such that $\tilde{\lambda}_i(x_1, x_2, x_3) \neq 0$, $(x_1, x_2, x_3) \in R^3$. A control law u =

$\alpha(x) + \beta(x)\hat{u}$ satisfying (3.5.41), (3.5.43) is not necessarily a decoupling control law. This can be verified by comparing (3.4.42) and (3.4.43). In [Sin.1], the following class of control laws is proposed as a class of decoupling control laws :

$$(3.5.41)' \quad \alpha(x) = \begin{bmatrix} \{\mathring{\eta}_2(x_1 x_2) - x_1 \mathring{\eta}_1(x_2)\} / x_2 \\ \mathring{\eta}_1(x_2) \end{bmatrix},$$

$$(3.5.42)' \quad \beta(x) = \begin{bmatrix} -x_1/x_2 & 1/x_2 \\ 1 & 0 \end{bmatrix},$$

where $\mathring{\eta}_i$, $i \in \mathbf{n}_{1,2}$ are arbitrary C^ω-functions of their arguments. Clearly, this class of decoupling control laws is a proper subset of $\mathcal{S}^\omega(\{F, H, \mathcal{X}\})$ in (3.5.41), (3.5.42). This example shows that the condition on α which (3.4.35) yields is the same as Theorem 3.4.1 does. But, this may not be generally true. □

3.6 Conclusion

In this chapter, we have presented various results on decoupling and decomposition of nonlinear systems. Some of them are refinements or elaborations of previously known results. They are : (a) the definitions of decoupling (Definition 3.1.3) and decomposition (Definition 3.1.5) ; (b) a necessary and sufficient condition for decoupling (Theorem 3.2.1) ; (c) a necessary and sufficient condition for decomposition (Theorem 3. 2. 2, Theorem 3. 2. 3) ; (d) a necessary and sufficient condition for decouplability (Theorem 3. 3.1) ; and (e) a necessary and sufficient condition for decomposability (Theorem 3. 3. 2). We have clarified and / or simplified these known results. This includes the elimination of redundant conditions and proofs for the necessity parts of some of the theorems.

Completely new results are : (1) the characterization of a class of nonlinear systems which are J - related to the standard decomposed systems (Theorem 3. 3. 3 and Theorem 3. 3. 4) ; (2) the characterization of the whole class of decoupling control laws (Theorem 3. 4. 1) and decomposing control laws (Theorem 3. 4. 2) ; (3) the characterization of the class of decoupled closed - loop systems (Theorem 3. 4. 3 - 3. 4. 5). We have distinguished them in the summary Figures 3. 2. 1, 3. 3. 2, and 3. 4. 4 with an asterisk. The new results contribute to the questions (b), (c), (d) in Chapter 1. They provide a deeper and clearer understanding of nonlinear decoupling theory. They provide information about the flexibility we can have in the design of decoupled control systems.

A difficulty exists. In most of these results, it is generally required to solve a set of the first order linear partial differential equations. It is not always possible to find the closed forms of solutions of these partial differential equations. This difficulty is shared with all other literature on the differential geometric approaches.

Finally, we would like to emphasize again the practical

importance of a standard decomposed system. Suppose we have a system and there exists a control law such that through an appropriate input and state transformation (a J - feedback relation, Definition 3. 1. 2), the system with the control law can be described as a standard decomposed system. In this case, the design of decoupled control systems becomes much easier since we can deal with the standard decomposed system instead of the original system. This advantage comes from the simplicity of the results for standard decomposed systems, as is pointed out in the last paragraph of Section 3. 4. Specifically, the class of decoupling control laws for the standard decomposed system is given by (3.437) (see Theorem 3. 4. 3) and for each decoupling control law in this class the decoupling control law for the original system can be obtained through the J - feedback relation (see Theorem 3. 4. 5).
In general, the J - feedback relation which transforms the original system into the standard decomposed system requires the solutions of a set of first order partial differential equations. However, in some applications the J - feedback relation may be found by inspection or rather simple manipulation of the dynamic equations for the original system. This is the case for the robotic manipulators in Chapter 5.

CHAPTER 4

APPROXIMATE DECOUPLING

In practice, some degree of modelling error is unavoidable. Therefore, it may be impossible to achieve "exact" decoupling in the sense of Section 3.1. Even when the exact model is available and decouplable, it may require decoupling control laws which are computationally complex. Thus, it may be more practical to have control laws which require less computation but decouple the system "approximately" in some sense. In this chapter, we neglect fast dynamics of a system to obtain simpler decoupling control laws and investigate the effect of the neglected fast dynamics on the decoupling of the actual system. Section 4.1 contains notation and assumptions, under which we state a result on approximate decoupling in Section 4.2.

4.1. Notation and Assumptions

In the previous chapters, we have considered systems defined on manifolds, which are not necessarily open subsets of R^n. To simplify developments in this chapter, we consider only the class of nonlinear systems defined on open subsets of R^n. Consider the following system, denoted by Σ_λ :

$$(4.1.1) \qquad \dot{x} = b_0(x) + \sum_{j=1}^{r} g_j(x)\, z_j + \sum_{j=1}^{m} b_j(x)\, u_j, \quad y = h(x),$$

$$(4.1.2) \qquad \lambda \dot{z} = A(\lambda)\, z + B_0(x, \lambda) + \sum_{j=1}^{m} B_j(x, \lambda)\, u_j,$$

where : $\mathcal{X}$ is an open subset of R^n containing the origin ; λ_0 is a positive constant scalar and $\lambda \in [0, \lambda_0]$; $g_j : \mathcal{X} \to R^n$, $j \in \mathcal{M}_{1,r}$; $b_j : \mathcal{X} \to R^n$, $j \in \mathcal{M}_{0,m}$; $B_j : \mathcal{X} \times [0, \lambda_0] \to R^r$, $j \in \mathcal{M}_{0,m}$; $A : [0, \lambda_0] \to R^{r \times r}$; $h : \mathcal{X} \to R^m$. We assume:

(B.1) $A(0)$ is a stable matrix.

The **degenerate system** of Σ_λ, denoted by Σ_0, is

$$(4.1.3) \qquad \dot{x} = b_0(x) + \sum_{j=1}^{r} g_j(x)\, z_j + \sum_{j=1}^{m} b_j(x)\, u_j, \quad y = h(x),$$

$$(4.1.4) \qquad 0 = A(0)\, z + B_0(x, 0) + \sum_{j=1}^{m} B_j(x, 0)\, u_j.$$

By (B.1), $A(0)$ is nonsingular. Consequently, Σ_0 can be written as

$$(4.1.5) \qquad \dot{x} = f_0(x) + \sum_{j=1}^{m} f_j(x)\, u_j, \quad y = h(x),$$

where

$$(4.1.6) \qquad f_i(x) \triangleq b_i(x) - [\, g_1(x) \cdots g_q(x)\,]\,[A(0)]^{-1} B_i(x, 0), \quad i \in \mathcal{M}_{0,m}.$$

Note that even when Σ_λ is not decouplable on $\mathcal{X} \times R^r$, Σ_0 may be decouplable on $\mathcal{X}$. Suppose that the degenerate system Σ_0 of Σ_λ

is decouplable on $\mathcal{X}$. Let $u = \alpha(x) + \beta(x)\hat{u}$ be a control law which decouples Σ_0 on $\mathcal{X}$. Let $\Sigma_\lambda^{\alpha,\beta}$ be the feedback system of Σ_λ corresponding to the control law $u = \alpha(x) + \beta(x)\hat{u}$. Then, we can describe $\Sigma_\lambda^{\alpha,\beta}$ by

$$(4.1.7) \qquad \dot{x} = \hat{g}_0(x) + \sum_{j=1}^{r} \hat{g}_j(x) z_j + \sum_{j=1}^{m} \hat{f}_j(x)\hat{u}_j, \quad \hat{y} = \hat{h}(x),$$

$$(4.1.8) \qquad \lambda \dot{z} = A(\lambda) z + \hat{B}_0(x,\lambda) + \sum_{j=1}^{m} \hat{B}_j(x,\lambda)\hat{u}_j,$$

where

$$(4.1.9) \qquad \hat{g}_0(x) \triangleq b_0(x) + [\, b_1(x) \cdots b_m(x)\,]\alpha(x),$$

$$(4.1.10) \qquad \hat{f}_j(x) \triangleq [\, b_1(x) \cdots b_m(x)\,]\beta_j(x), \quad j \in \mathbf{M}_{1,m},$$

$$(4.1.11) \qquad \hat{B}_0(x,\lambda) \triangleq B_0(x,\lambda) + [\, B_1(x,\lambda) \cdots B_m(x,\lambda)\,]\alpha(x),$$

$$(4.1.12) \qquad \hat{B}_j(x,\lambda) \triangleq [\, B_1(x,\lambda) \cdots B^m(x,\lambda)\,]\beta_j(x), \quad j \in \mathbf{M}_{1,m},$$

$$(4.1.13) \qquad \hat{h}(x) \triangleq h(x), \quad \hat{g}_j(x) \triangleq g_j(x), \quad j \in \mathbf{M}_{1,m},$$

and β_j is the jth column of β. Clearly, the degenerate system of $\Sigma_\lambda^{\alpha,\beta}$, denoted by $\Sigma_0^{\alpha,\beta}$, is decoupled on $\mathcal{X}$ but $\Sigma_\lambda^{\alpha,\beta}$ may not be decoupled on $\mathcal{X} \times R^r$. Let L be a positive constant. Let $F \in R^{r\times r}$. Define norms $|\cdot|$, $|\cdot|_L$, $\|\cdot\|$ by

$$(4.1.14) \qquad |x(t)| \triangleq \Big(\sum_{i=1}^{n} |x_i(t)|^2\Big)^{1/2}, \quad |x|_L \triangleq \max.\{\, |x(t)| : t \in [0, L]\,\},$$

$$\|F\| \triangleq \max.\{\, |Fz| : z \in R^r, |z| = 1\,\}.$$

Let $x^* : [0, L] \to \mathcal{X}$ be a **nominal solution** of $\Sigma^{\alpha,\beta}$ for a **nominal initial state** $x^*(0) = x_0^* \in \mathcal{X}$ and a **nominal input** $u^* \in \mathcal{U}^\infty$. For positive real numbers λ_0 and ζ_i, $i \in \mathbf{M}_{1,4}$, define sets $\mathcal{U}_{u^*}$, $\mathcal{R}_{x_0^*}$, $\mathcal{R}_{Z0}$, and $\mathcal{R}_{\lambda_0}$ by

(4.1.15) $\mathcal{U}_{u^*} \triangleq \{ u \in \mathcal{U}^\infty : |u - u^*|_L \le \zeta_1, |u|_L \le \zeta_2 \}$,

(4.1.16) $\mathcal{R}_{x_0^*} \triangleq \{ x_0 \in \mathcal{X} : |x_0 - x_0^*| \le \zeta_3 \}$,

(4.1.17) $\mathcal{R}_{Z0} \triangleq \{ z \in R^r : |z| \le \zeta_4 \}$, $\mathcal{R}_{\lambda_0} \triangleq \{ \lambda \in R : 0 \le \lambda \le \lambda_0 \}$.

We further assume :

(B.2) There exists $\zeta_5 > 0$ such that for all $x(0) \in \mathcal{R}_{x_0^*}$ and $u \in \mathcal{R}_{u^*}$, $\Sigma_0^{\alpha,\beta}$ has a solution $x : [0, L] \to \mathcal{X}$ satisfying $|x|_L \le \zeta_5$.

(B.3) (1) b^j, B^j, $j \in \mathbf{M}_{0,m}$ are C^∞,

(2) g_j, $j \in \mathbf{M}_{1,r}$ are C^∞,

(3) A, h are C^∞.

Note from [Gil.3] that (B.3) implies that (B.2) is true for sufficiently small ζ_1, ζ_3. We denote by $\hat{\Phi}_i(\hat{u}, x_0, z_0, \lambda)$ the ith output, $\hat{y}_i(t)$ of $\Sigma_\lambda^{\alpha,\beta}$ for an input $\hat{u} \in \mathcal{U}_{u^*}$, initial conditions $x(0) \triangleq x_0 \in \mathcal{R}_{x^*}$, $z(0) \triangleq z_0 \in \mathcal{R}_{Z0}$, and $\lambda \in \mathcal{R}_{\lambda_0}$. Now, we are ready to state a result for approximate decoupling.

4.2. Result for Approximate Decoupling

Theorem 4.2.1. Suppose that (B.1), (B.2), and (B.3) are satisfied. Suppose that a control law $u = \alpha(x) + \beta(x)\hat{u}$ satisfies (A.5) and decouples Σ_0. Then, there exist positive real numbers γ, λ_1 such that for every $\lambda \in \mathcal{R}_{\lambda 1}$, every input $u \in \mathcal{U}_{u*}$, and any initial conditions $x(0) \triangleq x_0 \in \mathcal{R}_{x_0^*}$, $z(0) \triangleq z_0 \in \mathcal{R}_{Z0}$, $\Sigma_\lambda^{\alpha,\beta}$ has the following properties :

(i) $\Sigma_\lambda^{\alpha,\beta}$ has a solution $(x, z) : [0, L] \to \mathcal{X} \times R^q$,

(ii) For each $i \in \mathcal{M}_{1,m}$ and for any two inputs $\bar{u}$, $\tilde{u} \in \mathcal{U}_{u*}$ such that $\bar{u}_i = \tilde{u}_i$,

$$(4.2.1) \qquad |\hat{\Phi}_i(\bar{u}, x_0, z_0, \lambda) - \hat{\Phi}_i(\tilde{u}, x_0, z_0, \lambda)|_L < \lambda\gamma. \qquad \square$$

This Theorem shows that if λ is sufficiently small, the control law which decouples the degenerate system still decouples the original system in an approximate way described in (ii). Now, we give the proof of Theorem 4.2.1.

Proof of Theorem 4.2.1. Let $\bar{x}$, $\bar{z}$ be the solutions of the degenerate system $\Sigma_0^{\alpha,\beta}$ in (4.1.7), (4.1.8) for $x(0) \triangleq x_0 \in \mathcal{R}_{x_0^*}$ and $\hat{u} = u \in \mathcal{U}_{u*}$:

$$(4.2.2) \qquad \dot{\bar{x}} = \hat{g}_0(\bar{x}) + \sum_{j=1}^{r} \hat{g}_j(\bar{x})\bar{z}_j + \sum_{j=1}^{m} \hat{f}_j(\bar{x}) u_j, \quad \bar{x}(0) = x_0,$$

$$(4.2.3) \qquad \bar{z} = -[A(0)]^{-1}\{\hat{B}_0(\bar{x}, 0) + \sum_{j=1}^{m} \hat{B}_j(\bar{x}, 0)\, u_j\}.$$

Here, and often in the future, the explicit dependence on t is not shown. From (4.2.3),

$$(4.2.4) \qquad \lambda \dot{\bar{z}} = A(\lambda)\bar{z} + \hat{B}_0(\bar{x}, \lambda) + \sum_{j=1}^{m} \hat{B}_j(\bar{x}, \lambda)\, u_j + \lambda K_0(\bar{x}(t), u(t), \dot{u}(t), \lambda),$$

where

$$(4.2.5) \qquad K_0(\bar{x}(t), u(t), \dot{u}(t), \lambda) \triangleq -[A(0)]^{-1}\{D_1\hat{B}_0(\bar{x},0) + \sum_{j=1}^{m} D_1\hat{B}_j(\bar{x},0)\, u_j\} \cdot$$
$$\{\hat{g}_0(\bar{x}) + \sum_{j=1}^{p} \hat{g}_j(x)\, \bar{z}_j + \sum_{j=1}^{m} \hat{f}_j(\bar{x})\, \hat{u}_j\} - \sum_{j=1}^{m} [A(0)]^{-1}\bar{B}_j(\bar{x}, 0)\, \dot{u}_j +$$
$$\lambda^{-1}\{A(0) - A(\lambda)\}\bar{z} + \lambda^{-1}\{\hat{B}_0(\bar{x}, 0) - \hat{B}_0(\bar{x}, \lambda)\} +$$
$$\lambda^{-1}\sum_{j=1}^{m}\{\hat{B}_j(\bar{x}, 0) - \hat{B}_j(\bar{x}, \lambda)\}\, u_j .$$

For simplicity of notation, we henceforth write $K_0(t, \lambda)$ instead of $K_0(\bar{x}(t), u(t), \dot{u}(t), \lambda)$. This kind of notational abuse will often appear in what follows.

Let η be the solution of the following differential equation :

$$(4.2.6) \qquad \dot{\eta} = \lambda^{-1} A(0)\eta, \quad \eta(0) \triangleq z(0) - \bar{z}(0).$$

The solution is

$$(4.2.7) \qquad \eta(t) = e^{A(0)\, t/\lambda}\{z(0) - \bar{z}(0)\}.$$

But by (B.1), there exist σ', $\zeta_6 > 0$ such that

(4.2.8) $\quad \| e^{A(0)\,t/\lambda} \| \le \zeta_6 \, e^{-\sigma t/\lambda}.$

Let x, z be the solutions of $\Sigma_\lambda^{\alpha,\beta}$ in (4.1.7), (4.1.8) for $x(0) \triangleq x_0$, $z(0) \triangleq z_0$, and $\hat{u} = u$. Let $V \triangleq x - \bar{x}$, $S \triangleq z - \bar{z} - \eta$. It should be clear that the variables V, S are the functions of time t depending implicitly on λ, x_0, z_0, and u. From (4.2.4), (4.2.7), (4.2.8), we obtain the following differential equations. :

(4.2.9) $\quad \dot{V} = W_1(t, V, S, \lambda) + K_1(t, \lambda), \quad V(0) = 0,$

(4.2.10) $\quad \dot{S} = \lambda^{-1} A(0) S + \lambda^{-1} W_2(t, V, S, \lambda) + K_2(t, \lambda), \quad S(0) = 0,$

where

(4.2.11) $$W_1(t, V, S, \lambda) \triangleq [\hat{g}_0(\bar{x} + V) - \hat{g}_0(\bar{x})] + \sum_{j=1}^{r} [\hat{g}_j(\bar{x} + V) - \hat{g}_j(\bar{x})] \cdot (\bar{z}_j + \eta_j) + \sum_{j=1}^{r} \hat{g}_j(\bar{x} + V) S_j + \sum_{j=1}^{m} [\hat{f}_j(\bar{x} + V) - \hat{f}_j(\bar{x})] u_j,$$

(4.2.12) $$K_1(t, \lambda) \triangleq \sum_{j=1}^{r} \hat{g}_j(\bar{x}) \eta_j ,$$

(4.2.13) $$W_2(t, V, S, \lambda) \triangleq [A(\lambda) - A(0)] S + [\hat{B}_0(\bar{x} + V, \lambda) - \hat{B}_j(\bar{x}, \lambda)] + \sum_{j=1}^{m} [\hat{B}_j(\bar{x} + V, \lambda) - \hat{B}_j(\bar{x}, \lambda)] u_j ,$$

(4.2.14) $$K_2(t, \lambda) = \lambda^{-1} [A(\lambda) - A(0)] \eta - K_0(t, \lambda).$$

Choose $\zeta_7 > 0$ so that

(4.2.15) $\quad \bar{x}(t) + V(t) \in \mathcal{X}, \quad t \in [0, L] \;$ if $\; |V|_L \le \zeta_7 \;$ and $\; |\bar{x}|_L \le \zeta_5.$

Define the sets $\mathfrak{R}_V$, $\mathfrak{R}_S$ by

(4.2.16) $\mathfrak{R}_V \triangleq \{\xi \in R^n : |\xi| < \zeta_7\}$, $\mathfrak{R}_S \triangleq \{\xi \in R^r : |\xi| < \zeta_7\}$.

We show that the solutions V, S of (4.2.9), (4.2.10) can be kept within $\mathfrak{R}_V$, $\mathfrak{R}_S$, respectively. Then, as long as the trajectories of V, S stay in the regions $\mathfrak{R}_V$, $\mathfrak{R}_S$, respectively, by (B.2), (B.3), (4.2.3), (4.2.7), (4.2.9), (4.2.11), and (4.2.12), there exists $\zeta_8 > 0$ uniformly with respect to $\mathfrak{R}_{x_o^*}$, $\mathfrak{R}_{Zo}$, $\mathfrak{U}_{u^*}$, $\mathfrak{R}_V$, and $\mathfrak{R}_S$ such that

(4.2.17) $|\dot{V}|_L \le \zeta_8$.

Define $\bar{V}$, $\bar{S}$ by

(4.2.18) $\bar{V}(t, \lambda) \triangleq \int_o^t K_1(\tau, \lambda)\, d\tau$,

(4.2.19) $\bar{S}(t, \lambda) \triangleq \int_o^t e^{A(0)(t-\tau)/\lambda} K_2(\tau, \lambda)\, d\tau$.

Then, by (B.2), (B.3), (4.2.5), (4.2.7), (4.2.8), (4.2.12), and (4.2.14), there exist ζ_9, $\zeta_{10} > 0$ uniformly with respect to $\mathfrak{R}_{x_o^*}$, $\mathfrak{R}_{Zo}$, $\mathfrak{U}_{u^*}$, $\mathfrak{R}_{\lambda o}$, $\mathfrak{R}_V$, and $\mathfrak{R}_S$ such that

(4.2.20) $|\bar{V}|_L \le \lambda \zeta_9$, $|\bar{S}|_L \le \lambda \zeta_{10}$.

On the other hand, by (B.2), (B.3), (4.2.11), (4.2.13), there exist $\zeta_{11}, \zeta_{12} > 0$ uniformly with respect to $\mathfrak{R}_{x_0^*}$, $\mathfrak{R}_{Z_0}$, $\mathfrak{U}_{u^*}$, $\mathfrak{R}_{\lambda_0}$, $\mathfrak{R}_V$, and $\mathfrak{R}_S$ such that

(4.2.21) $\quad |W_1(t, V, S, \lambda)| \leq \zeta_{11} (|V(t, \lambda)| + |S(t, \lambda)|),$

(4.2.22) $\quad |W_2(t, V, S, \lambda)| \leq \zeta_{12} (|V(t, \lambda)| + \lambda |S(t, \lambda)|).$

Finally, we will need some constants related to those we have introduced so far. Take λ_2 such that

(4.2.23) $\quad 0 < \lambda_2 < \min. \{\lambda_0, \sigma / \zeta_{12}\}.$

Define σ_i, $i \in \mathfrak{N}_{1,6}$ by

(4.2.24) $\quad \sigma_1 \triangleq \sigma - \lambda_2 \zeta_{12}, \quad \sigma_2 \triangleq \zeta_{11} (1 + \lambda_2 |\zeta_{12}|^2 / \sigma \sigma_1),$

(4.2.25) $\quad \sigma_3 \triangleq \lambda_2 |\zeta_{12}|^2 \zeta_8 / \sigma \sigma_1 + L \zeta_{12} (\zeta_{12} \zeta_8 / \sigma^2 + \zeta_{10}),$

(4.2.26) $\quad \sigma_4 \triangleq \zeta_{11} \sigma_2 L + \zeta_9, \quad \sigma_5 \triangleq \sigma_4 e^{\sigma_2 L},$

(4.2.27) $\quad \sigma_6 \triangleq \lambda_2 |\zeta_{12}|^2 \sigma_5 / \sigma \sigma_1 + \sigma_3.$

Then, choose λ_1 so that

(4.2.28) $\quad 0 < \lambda_1 < \min. \{\lambda_2, \zeta_7 / \sigma_5, \zeta_7 / \sigma_6\}.$

First, consider part (i). We show that the following statement is true :

(S) If $\lambda \in \mathfrak{R}_{\lambda 1}$, for any input $u \in \mathcal{U}_{u*}$ and any initial states $x(0) \triangleq x_0 \in \mathfrak{R}_{x*}$, $z(0) \triangleq z_0 \in \mathfrak{R}_{Z0}$, the solutions V, S of (4.2.9), (4.2.10) exist on [0, L] and stay in $\mathfrak{R}_V$, $\mathfrak{R}_S$, respectively.

Then, this will imply part (i). We prove (S) by contradiction. Suppose the contrary of (S) :

(S)' There exist $\lambda \in \mathfrak{R}_{\lambda 1}$, $\bar{u} \in \mathcal{U}_{u*}$, $x_0 \in \mathfrak{R}_{x*}$, $z_0 \in \mathfrak{R}_{Z0}$, and $t_0 \in (0, L)$ such that both V, S stay in $\mathfrak{R}_V$, $\mathfrak{R}_S$, respectively, only during the time interval $[0, t_0)$.

By (S)', (4.2.9), (4.2.10), (4.2.18), and (4.2.19), the following Volterra integrals must hold for all $t \in [0, t_0)$:

(4.2.29) $$V(t, \lambda) = \int_0^t W_1(\tau, V(\tau, \lambda), S(\tau, \lambda), \lambda)\, d\tau + \bar{V}(t, \lambda),$$

(4.2.30) $$S(t, \lambda) = \int_0^t \lambda^{-1} e^{A(0)(t-\tau)/\lambda} W_2(\tau, V(\tau, \lambda), S(\tau, \lambda), \lambda)\, d\tau + \bar{S}(t, \lambda).$$

Then, by (4.2.8) and (4.2.20) - (4.2.22), the following inequalities hold for all $t \in [0, t_0)$:

(4.2.31) $$|V(t, \lambda)| \le \zeta_{11} \int_0^t |V(\tau, \lambda)|\, d\tau + \zeta_{11} \int_0^t |S(\tau, \lambda)|\, d\tau + \lambda \zeta_9 .$$

(4.2.32) $$|S(t, \lambda)| \le \zeta_{12} \int_0^t e^{-\sigma(t-\tau)/\lambda} |S(\tau, \lambda)|\, d\tau + \zeta_{12} \int_0^t \lambda^{-1} e^{-\sigma(t-\tau)/\lambda} |V(\tau, \lambda)|\, d\tau + \lambda \zeta_{10}.$$

By (4.2.17), (4.2.32) implies

$$(4.2.33) \quad |S(t,\lambda)| \le \zeta_{12} \int_0^t e^{-\sigma(t-\tau)/\lambda} |S(\tau,\lambda)|\, d\tau + (\zeta_{12}/\sigma)|V(t,\lambda)| + \lambda(\zeta_{12}\zeta_8/\sigma^2 + \zeta_{10}), \quad t \in [0, t_0).$$

Multiplying both sides of (4.2.33) by $e^{\sigma t/\lambda}$, applying Gronwall's Lemma ([Die.1]) and then dividing the result by $e^{\sigma t/\lambda}$,

$$(4.2.34) \quad |S(t,\lambda)| \le \zeta_{12} \int_0^t e^{-(\sigma-\lambda\zeta_{12})(t-\tau)/\lambda} \{ (\zeta_{12}/\sigma)|V(\tau,\lambda)| + \lambda(\zeta_{12}\zeta_8/\sigma^2 + \zeta_{10}) \}\, d\tau, \quad t \in [0, t_0).$$

By (4.2.17), (4.2.23), and (4.2.28), (4.2.34) implies

$$(4.2.35) \quad |S(t,\lambda)| \le \lambda_2(|\zeta_{12}|^2/\sigma\sigma_1)|V(t,\lambda)| + \lambda\sigma_3, \quad t \in [0, t_0).$$

Substituting (4.2.35) into (4.2.31) and applying Gronwall's Lemma leads to

$$(4.2.36) \quad |V(t,\lambda)| \le \lambda\sigma_4(1 + \sigma_2 \int_0^t e^{\sigma_2(t-\tau)}\, d\tau) \le \lambda\sigma_5, \quad t \in [0, t_0).$$

By (4.2.27) and (4.2.36), (4.2.35) implies

$$(4.2.37) \quad |S(t,\lambda)| \le \lambda\sigma_6, \quad t \in [0, t_0).$$

Thus, (4.2.28), (4.2.36), and (4.2.37) show that

$$(4.2.38) \quad |V(t,\lambda)|, \ |S(t,\lambda)| < \zeta_7, \quad t \in [0, t_0).$$

This with (4.2.5), (4.2.7), (4.2.10), (4.2.13), and (4.2.14) shows that there exists $\zeta_{13}(\lambda) > 0$ such that

(4.2.39) $\quad |\dot{S}(t,\lambda)| < \zeta_{13}, \qquad t \in [0, t_0).$

This and (4.2.17) imply that the sequences $\{S(t_r,\lambda)\}$, $\{V(t_r,\lambda)\}$ are convergent sequences in R^r, R^n, respectively, if $\lim_{r\to\infty} t_r = t_0$ and $0 < t_r \leq t_0$, $r \in \mathfrak{N}_{1,\infty}$. Let $S(t_{0-},\lambda) = \lim_{r\to\infty} S(t_r,\lambda)$ and $V(t_{0-},\lambda) = \lim_{r\to\infty} V(t_r,\lambda)$. Then by (4.2.38),

(4.2.40) $\quad |V(t_{0-},\lambda)|, \; |S(t_{0-},\lambda)| \; < \; \zeta_7 .$

This implies that the solutions V, S will continue beyond t_0. This violates the assumption (S)'. Thus, we have shown that $\lambda \leq \lambda_1$ guarantees the existence of solutions $V : [0, L] \to \mathfrak{R}_V$, $S : [0, L] \to \mathfrak{R}_S$.

Next, we prove part (ii). Note that (4.2.36), (4.2.37) hold on [0, L] uniformly with respect to $\mathfrak{R}_{x_0^*}$, $\mathfrak{R}_{Zo}$, $\mathfrak{U}_{u*}$, $\mathfrak{R}_{\lambda 1}$. This fact with (B.3) - (3) shows that there exists $\zeta_{14} > 0$ such that

(4.2.41) $\quad |\hat{h}(x) - \hat{h}(\bar{x})|_L \leq \zeta_{14}\lambda$

holds uniformly with respect to $\mathfrak{R}_{x_0^*}$, $\mathfrak{R}_{Zo}$, $\mathfrak{R}_{u*}$, and $\mathfrak{R}_{\lambda 1}$.

Let $\hat{\Phi}_i^\circ(\hat{u}, x_0)$ denote the ith output of $\Sigma_0^{\alpha,\beta}$ for an input $\hat{u} \in \mathfrak{U}_{u*}$ and an initial state $x(0) \triangleq x_0 \in \mathbf{R}_{X_0^*}$. Let $\bar{u}$, $\tilde{u} \in \mathbf{U}_{u^*}$ be two distinct inputs with $\bar{u}_i = \tilde{u}_i$. Then, since $\Sigma_0^{\alpha,\beta}$ is a decoupled system on $\mathcal{X}$, by (4.2.41), the following inequality holds:

$$(4.2.42) \quad |\hat{\Phi}_i(\bar{u}, x_0, z_0, \lambda) - \hat{\Phi}_i(\tilde{u}, x_0, z_0, \lambda)|_L \le |\hat{\Phi}_i^\circ(\bar{u}, x_0) - \hat{\Phi}_i^\circ(\tilde{u}, x_0)|_L +$$
$$|\hat{\Phi}_i(\bar{u}, x_0, z_0, \lambda) - \hat{\Phi}_i^\circ(\bar{u}, x_0)|_L + |\hat{\Phi}_i^\circ(\tilde{u}, x_0) - \hat{\Phi}_i(\tilde{u}, x_0, z_0, \lambda)|_L$$
$$\le 2\zeta_{14}\lambda,$$

for all $x_0 \in \mathbf{R}_{X_0^*}$, $z_0 \in \mathbf{R}_{Z0}$, and $\lambda \in \mathbf{R}_{\lambda 1}$. □

Remark 4. 2. 1. The proof is a straighforward extension of well-known singular perturbation techniques for systems without inputs and outputs ([Hop.1, Hop.2, Kok.1, Lev.1, Sab.1, Tih.1, Vas.1]). Our proof follows closely the one given in [Lev.1]. But in [Lev.1], part (i) of Theorem 4. 2. 1 was implicitly assumed rather than proven. □

Remark 4. 2. 2. A concept similar to our approximate decoupling appears in [Wil.1, You.1], where **asymptotic** (which corresponds to " approximate ", here) disturbance decoupling of linear systems was considered. □

CHAPTER 5

APPLICATIONS TO ROBOTICS

In this chapter, the results developed in the previous chapters are applied to decoupled control of robotic manipulators. In Section 5. 1, actuator dynamics are completely neglected but in Section 5. 2, the significant part of actuator dynamics are taken into account.

5. 1. Decoupled Control of Robotic Manipulators

Consider the following system :

(5.1.1) $\quad M(q)\ddot{q} + N(\dot{q}, q) = L(\dot{q}, q)\, u, \qquad y = C(q),$

where : $q \in R^m$, $\mathbf{\Sigma}$ is an open connected subset of R^{2m} ; $\mathbf{Q} \triangleq \{q \in R^m : (\dot{q}, q) \in \mathbf{\Sigma}\}$; $M : \mathbf{Q} \to R^{m\times m}$; $N : \mathbf{\Sigma} \to R^m$; $C : \mathbf{Q} \to R^m$; $L : \mathbf{\Sigma} \to R^{m\times m}$. The rigid body dynamics of a robotic manipulator can be described by the above second order differential equation when actuator dynamics are neglected.

We assume

(C.1) M, N, L, C are C^∞,

(C.2) $M(q)$, $DC(q)$ are nonsingular, $q \in \mathbf{Q}$, (C.3) $L(\dot{q}, q)$ is nonsingular, $(\dot{q}, q) \in \mathbf{E}$

We may need the following stronger assumptions :

(C.1)' M, N, L, C are C^{ω},

(C.4) C is one - to - one on $\mathbf{Q}$.

Let $x_1 \triangleq \dot{q}$, $x_2 \triangleq q$, and $x \triangleq (x_1, x_2)$. By (C.2), we can write the system (5.1.1) into the following form :

$$(5.1.2) \qquad \dot{x} = f_0(x) + \sum_{i=1}^{m} f_i(x)\, u_i, \quad y = h(x),$$

where

$$(5.1.3) \qquad f_0(x) \triangleq \begin{bmatrix} -[M(x_2)]^{-1} N(x_1, x_2) \\ x_1 \end{bmatrix}, \quad h(x) \triangleq C(x_2),$$

$$(5.1.4) \qquad f_i(x) \triangleq \begin{bmatrix} [M(x_2)]^{-1} L_i(x_1, x_2) \\ 0 \end{bmatrix}, \quad i \in \mathbf{M}_{1,m}.$$

Here, L_i is the ith column of L. We denote the system (5.1.2) by $[f, h, \mathbf{E}]_0$. In the following theorem, we consider the decoupling of $[f, h, \mathbf{E}]_0$.

Theorem 5.1.1. Suppose that for each of the following parts, (C.1) - (C.3) are satisfied. (i) The system $[f, h, \mathbf{E}]_0$ is decomposable at each $x_0 \in \mathbf{E}$ and decouplable on $\mathbf{E}$ with $d_i = 1$, $i \in$

$\mathcal{M}_{1,m}$. Moreover, the control law $u = \alpha(\dot{q}, q) + \beta(\dot{q}, q)\hat{u}$ decouples $\{f, h, \mathcal{E}\}_0$ on $\mathcal{E}$ if α, β have the following forms on $\mathcal{E}$:

$$(5.1.5)\qquad \alpha(\dot{q}, q) = [L(\dot{q}, q)]^{-1}\{ M(q)[DC(q)]^{-1}(\Pi(\dot{q}, q) - Q_0(\dot{q}, q)\dot{q}) + N(\dot{q},q) \},$$

$$(5.1.6)\qquad \beta(\dot{q}, q) = [L(\dot{q}, q)]^{-1}M(q)\,[DC(q)]^{-1}\Gamma(\dot{q}, q)\,,$$

where

$$(5.1.7)\qquad \Pi(\dot{q}, q) \triangleq \begin{bmatrix} \phi_1(C_1(q), DC_1(q)\,\dot{q}) \\ \cdots \\ \phi_m(C_m(q), DC_m(q)\,\dot{q}) \end{bmatrix}, \quad Q_0(\dot{q}, q) \triangleq \begin{bmatrix} \dot{q}^T D(DC_1(q))^T \\ \cdots \\ \dot{q}^T D(DC_m(q))^T \end{bmatrix},$$

$$(5.1.8)\qquad \Gamma(\dot{q}, q) \triangleq \operatorname{diag} \psi_i(C_i(q), DC_i(q)\,\dot{q})\,,$$

and ϕ_i , ψ_i are arbitrary C^∞- functions of their arguments such that $\Gamma(\dot{q}, q)$ is nonsingular, $(\dot{q}, q) \in \mathcal{E}$

(ii) Suppose that (C.4) is satisfied and the class of control laws satisfies (A.5) and (A.6) of Section 3. 3. Then, $\{f, h, \mathcal{E}\}_0$ is decomposable on $\mathcal{E}$. The class given by (5.1.5) - (5.1.8) is the whole class of smooth decomposing control laws.

(iii) Suppose that (C.1)' is satisfied. Suppose that class of control laws is real analytic and for every control law in the class, $\{f, h, \mathcal{E}\}^{\alpha,\beta}$ satisfies (A.2) of Section 3. 2. Then, the class given by (5.1.5) - (5.1.8) is the whole class of real analytic decoupling control laws.

Proof. First consider part (i). Let X_i, $i \in \mathcal{M}_{1,m}$ be the vector fields corresponding to $\{f, h, \mathcal{E}\}_0$. Fix $i \in \mathcal{M}_{1,m}$. Straightforward computation shows that

$$(5.1.9) \quad X_0 H_i(x) = DC_i(x_2)\, x_1 ,$$

$$(5.1.10) \quad X_0^2 H_i(x) = x_1^T D(DC_i(x_2))^T x_1 - DC_i(x_2)[M(x_2)]^{-1} N(x_1, x_2) ,$$

$$(5.1.11) \quad X_j H_i(x) = 0, \qquad j \in \mathcal{M}_{1,m},$$

for all $x \in \mathcal{E}$. On the other hand, by (C.2) and (C.3),

$$(5.1.12) \quad X_j X_0 H_i(x) = DC_i(x_2)\,[M(x_2)]^{-1} L_j(x_1, x_2) \neq 0,$$

for all $x \in \mathcal{E}$ and $i, j \in \mathcal{M}_{1,m}$. Thus, $d_i = 1$, $i \in \mathcal{M}_{1,m}$ and

$$(5.1.13) \quad D^*(x) = DC(x_2)\,[M(x_2)]^{-1} L(x_1, x_2) ,$$

$$(5.1.14) \quad A^*(x) = Q_0(x_1, x_2) - DC(x_2)\,[M(x_2)]^{-1} L(x_1, x_2).$$

By (C.2), (C.3), (5.1.12), Theorem 3. 3. 1, and Theorem 3. 3. 2, $\{f, h, \mathcal{E}\}_0$ is decouplable on $\mathcal{E}$ and decomposable at each $x_0 \in \mathcal{E}$. $\mathcal{S}^\infty(\{f, h, \mathcal{E}\}_0)$ is given by (5.1.5) and (5.1.6).

Now, consider part (ii). Note that $2m = \sum_{i=1}^{m} (d_i + 1)$. Define a C^∞-mapping T from $\mathcal{E}$ into R^{2m} by

$$(5.1.15) \quad T \triangleq (T_1, \ldots, T_m), \quad T_i \triangleq (T_{i,1}, T_{i,2}), \quad T_{i,1} \triangleq C_i(x_2),$$
$$T_{i,2} \triangleq DC_i(x_2)\, x_1, \quad i \in \mathcal{M}_{1,m} .$$

By (C.2), DT(x) is nonsingular, $x \in \mathbf{\Sigma}$. By Theorem 2.3.7, this with (C.4) implies that T is a C^{∞}-diffeomorphism on $\mathbf{\Sigma}$. From this, Theorem 3.4.4, and Remark 3.4.7, part (ii) follows easily. Part (iii) follows from Theorem 3.4.1, Remark 3.4.7, and the fact that that T is a C^{ω}-diffeomorphism on $\mathbf{\Sigma}$. □

Before making remarks on Theorem 5.1.1, we consider the following system, denoted by Σ_{λ} :

(5.1.16) $\quad M(q)\ddot{q} + F(\dot{q}, q) = \tau, \quad y = C(q),$

(5.1.17) $\quad \lambda \dot{v} = A(\lambda)\, v + B_0(\dot{q}, q, \lambda) + \sum_{j=1}^{m} B_j(\dot{q}, q, \lambda)\, u_j, \quad \tau = G(\dot{q}, q)\, v,$

where : q, M, $\mathbf{\Sigma}$, $\mathbf{Q}$ are defined as in (5.1.1) ; $F : \mathbf{\Sigma} \to R^m$; λ_0 is a positive constant scalar and $\lambda \in [0, \lambda_0]$; $A : [0, \lambda_0] \to R^{r\times r}$; $G : \mathbf{\Sigma} \to R^{m\times r}$; $B_j : \mathbf{\Sigma} \times [0, \lambda_0] \to R^r$, $j \in \mathbf{M}_{0,m}$. The dynamics of a robotic manipulator with D. C. drives ([Asa.1, Erl.1, Daz.1]) or electro-hydraulic actuators ([Mcc.1, Mer.1]) can be described by the above equations. Then, (5.1.16) represents the dynamics of a robotic manipulator, where q is the vector of generalized joint coordinates ; M is a generalized inertia matrix ; F is the vector equivalent forces due to Coriolis and centrifugal effects, friction forces, and gravitation ; and y is the output to be controlled (e. g., the position and orientation of the end-effector). The system (5.1.17) represents additional actuator dynamics, where u is the electrical control input to actuators and τ is the output torque (or force) generated by the actuators.

In the modelling process, when λ is very small (which means that the additional actuator dynamics (5.1.17) are very fast, relatively to the mechanical dynamics (5.1.16)), the additional actuator dynamics are usually neglected. In other words, for simplicity it is assumed that $\lambda = 0$. We denote this system by Σ_0. If (B.1) in Section 4. 1 is assumed, we can write the degenerate system Σ_0 of Σ_λ as (5.1.1), where

(5.1.18) $N(\dot{q}, q) \triangleq F(\dot{q}, q) + G(\dot{q}, q)\,[A(0)]^{-1}B_0(\dot{q}, q, 0),$

(5.1.19) $L(\dot{q}, q) \triangleq -G(\dot{q}, q)\,[A(0)]^{-1}[\,B_1(\dot{q}, q, 0) \cdots B_m(\dot{q}, q, 0)\,].$

Thus, we have shown that when actuator dynamics are neglected, the dynamics of a robotic manipulator can be described by (5.1.1).

Remark 5. 1. 1. Theorem 5. 1. 1 - (i) includes previous results ([Bej.1, Fre.2, Fre.3, Hew.1, Mar.1, Pau.1, Rai.1, Sin.4, Tar.1]) as special cases. For instance, in [Bej.1, Mar.1, Pau.1, Rai.1],

(5.1.20) $m \triangleq 6,\quad C(q) \triangleq q\,,\quad L(\dot{q}, q) \triangleq \mathbf{I}_6,\quad \mathbf{\Sigma} = R^{12}.$

In [Fre.2, Fre.3],

(5.1.21) $m \triangleq 3,\quad L(\dot{q}, q) \triangleq \mathbf{I}_3,\quad C(q) \triangleq (\,q_1\cos q_2\,,\ q_1\sin q_2\,,\ q_3),\quad \mathbf{\Sigma} \triangleq$

$R^3 \times \mathbf{Q},\quad \mathbf{Q} \triangleq \{\,(q_1, q_2, q_3) \in R^3 : 0 < q_1 < \infty,\ 0 < q_2 < 2\pi,\ q_3 \in R\,\}.$

It can be shown that these problems satisfy the assumptions required for Theorem 5. 1. 1. The case of (5.1.20) is called **joint coordinate control**. The case of (5.1.21) is called **hand coordinate control**. The **hand coordinate system** is the Cartesian coordinate system fixed on the gripper or the end - effector. A more general form of the hand coordinate control can be described by

$$(5.1.22) \quad m \triangleq 6, \quad C(q) \triangleq \begin{bmatrix} p(q) \\ \phi(q) \\ \theta(q) \\ \psi(q) \end{bmatrix},$$

where $p(q) \in R^3$ is the position of the origin of the hand coordinate system from the inertial reference coordinate system ; ϕ, θ, ψ are Euler angles of of the hand coordinate system with respect to the inertial reference coordinate system. For the case of (5.1.22), the hypotheses of Theorem 5. 1. 1 hold with $\Sigma = R^6 \times \mathbf{Q}$, where $\mathbf{Q}$ is an open subset of R^6. The details are omitted. □

Remark 5. 1. 2. We believe that Theorem 5. 1. 1 - (ii), (iii) are new. The class of decoupling control laws the above authors consider is, in (5.1.7), (5.1.8),

$$(5.1.23) \quad \phi_i(C_i(q), DC_i(q)\,\dot{q}) = \gamma_{i,1}\, C_i(q) + \gamma_{i,2}\, DC_i(q)\,\dot{q},$$

$$(5.1.24) \quad \psi_i(C_i(q), DC_i(q)\,\dot{q}) = \delta_{i,1}\, C_i(q) + \delta_{i,2}\, DC_i(q)\,\dot{q},$$

where $\gamma_{i,1}$, $\gamma_{i,2}$, $\delta_{i,1}$, $\delta_{i,2}$ are real constants. It is obvious that ours is a more general class of control laws which decouple. It is not so obvious that the class is the most general class. □

Remark 5. 1. 3. In the conventional approaches to control of robotic manipulators ([Luh.2, Luh.3, Mar.1, Pau.2]), the case of (5.1.20) is extensively studied and the design is based on single - input, single - output models for each joint coordinate, treating coupling effects between joint coordinates as disturbance inputs. Though corrections for varying inertias and gravitational loads are sometimes introduced in these approaches, precise and high speed control is difficult to achieve. In the decoupled control investigated in [Fre.2, Fre.3, Hew.1, Rei.1, Pau.1] and here, it is possible. The disadvantage of decoupled control is that it requires a large amount of computation. But methods for reducing the computational complexity and the use of special processors have been investigated by some authors ([Hol.1, Luh.1, Wal.1, Tur.1]). Although these computational methods are proposed originally for the case of (5.1.20), they are also applicable for the general problem considered here. □

Remark 5. 1. 4. An alternative and perhaps more straightforward derivation of Theorem 5. 1. 1 - (i) is as follows. Differentiating y in (5.1.1) twice with respect to t and, in the resulting equation, replacing $\ddot{q}$ by the expression obtained from (5.1.1), we can obtain

$$(5.1.25) \quad \ddot{y} = C(q)[M(q)]^{-1}\{ L(\dot{q}, q)\, u - N(\dot{q}, q) \} + Q_0(\dot{q}, q)\, \dot{q}$$

By (5.1.25), the control law $u = \alpha(\dot{q}, q) + \beta(\dot{q}, q)\hat{u}$ satisfying (5.1.5) - (5.1.8) with $\Pi = 0$ and $\Gamma = I_m$ leads to

(5.1.26) $\ddot{y} = \hat{u}$.

Thus, $\{f, h, \mathcal{E}\}_o$ is decouplable on $\mathcal{E}$. This alternative approach is implied in [Gil.4]. It does not require knowledge of vector fields and is based on the special structure of (5.1.1). The characterization of the entire class of decoupling control laws follows from Remark 3. 4. 4 or Remark 3. 4. 7 (see also the last paragraphs of Section 3. 4, 3. 5). □

Next, let us consider the effect of the neglected fast dynamics (5.1.17) on decoupling of the original system Σ_λ. Let $u = \alpha(\dot{q}, q) + \beta(\dot{q}, q)\hat{u}$ be a control law satisfying (5.1.5) - (5.1.8). We denote by $\Sigma_o^{\alpha,\beta}$, $\Sigma_\lambda^{\alpha,\beta}$, respectively, the feedback systems of Σ_o, Σ_λ corresponding to the control law $u = \alpha(\dot{q}, q) + \beta(\dot{q}, q)\hat{u}$. For the following result, we need

(C.5) F, A, G and B_j, $j \in \mathcal{M}_{o,m}$ are C^∞.

Theorem 5. 1. 2. Suppose that (C.1) - (C.3) and (C.5) are satisfied. Suppose that $\Sigma_o^{\alpha,\beta}$ satisfies (B.1) and (B.2) of Section 4. 1. Then, $\Sigma_\lambda^{\alpha,\beta}$ has the properties (i), (ii) in Theorem 4. 2. 1 with $\mathcal{X} = \mathcal{E}$. □

The theorem shows that although a control law which decouples Σ_0 on $\mathbf{E}$ may not decouples Σ_λ on $\mathbf{E} \times R^r$, it does approximately. Theorem 5. 1. 2 is a direct consequence of Theorem 4. 2. 1 and Theorem 5. 1. 1.

5. 2. Decoupled Control of Robotic Manipulators with Significant Actuator Dynamics

Consider the following system :

(5.2.1) $$M(q)\ddot{q} + N(\dot{q}, q) = g_0(v, \dot{q}, q), \quad y = C(q),$$

(5.2.2) $$\dot{v} = a_0(v, \dot{q}, q) + \sum_{j=1}^{m} a_j(v, \dot{q}, q)\, u_j,$$

where : $\mathbf{X}$ is an open connected subset of R^{3m} ; q, v, $\varepsilon\ R^m$; $\mathbf{E} \triangleq \{(\dot{q}, q) : (v, \dot{q}, q) \in \mathbf{X}\}$; $\mathbf{Q} \triangleq \{q : (\dot{q}, q) \in \mathbf{E}\}$; $M : \mathbf{Q} \to R^{m\times m}$; $N : \mathbf{E} \to R^m$; $C : \mathbf{Q} \to R^m$; $g_0 : \mathbf{X} \to R^m$; $a_i : \mathbf{X} \to R^m$, $i \in \mathbf{M}_{0,m}$. The dynamics of a robotic manipulator can be described as above when significant actuator dynamics are taken into account. Except for the increased complexity, development in this section is quite similar to that in Section 5.1. In addition to (C.1) - (C.3) and (C.4), we assume

(D.1) M, N, C, g_0, a_i, $i \in \mathbf{M}_{0,m}$ are C^∞.

(D.2) $Q_1(v, \dot{q}, q) \triangleq D_1 g_0(v, \dot{q}, q)\,[\, a_1(v, \dot{q}, q) \cdots a_m(v, \dot{q}, q)\,]$ is nonsingular, $(v, \dot{q}, q) \in \mathbf{X}$.

Let $\mathcal{V} \triangleq \{ v \in R^m : (v, \dot{q}, q) \in \mathcal{X} \}$. We may need the following stronger assumptions :

(D.1)' M, N, C, g_0, a_i, $i \in \mathcal{M}_{0,m}$ are C^ω.

(D.3) $g_0(\cdot, \dot{q}, q)$ is one - to - one on $\mathcal{V}$ for each $(\dot{q}, q) \in \mathcal{E}$

Let $x_1 \triangleq v$, $x_2 \triangleq \dot{q}$, $x_3 \triangleq q$, and $x \triangleq (x_1, x_2, x_3)$. By (C.2), we can write the system (5.2.1), (5.2.2) as

(5.2.3) $$\dot{x} = f_0(x) + \sum_{i=1}^{m} f_i(x) u_i, \quad y = h(x),$$

where

(5.2.4) $$f_0(x) \triangleq \begin{bmatrix} a_0(x_1, x_2, x_3) \\ [M(x_3)]^{-1}\{ g_0(x_1, x_2, x_3) - N(x_2, x_3) \} \\ x_2 \end{bmatrix},$$

(5.2.5) $$f_i(x) \triangleq \begin{bmatrix} a_i(x_1, x_2, x_3) \\ 0 \\ 0 \end{bmatrix}, \quad i \in \mathcal{M}_{1,m}.$$

Let $\{ f, h, \mathcal{X} \}_0$ denote the system (5.2.3). Under the above assumptions, we consider decoupling of $\{ f, h, \mathcal{X} \}_0$. We use the following notation (see Section 2. 1 for the definition of the third order derivative) :

(5.2.6) $$Q_{10}(\dot{q}, q) \triangleq \begin{bmatrix} [DM_1^T(q)\, \dot{q}]^T \\ \cdots \\ [DM_m^T(q)\, \dot{q}]^T \end{bmatrix}, \quad Q_{11}(\dot{q}, q) \triangleq \begin{bmatrix} \dot{q}^T D(DC_1(q))^T \\ \cdots \\ \dot{q}^T D(DC_m(q))^T \end{bmatrix},$$

(5.2.7) $Q_9(\dot q, q) \triangleq DC(q)\,[M(q)]^{-1} N(\dot q, q) - Q_{11}(\dot q, q)\,\dot q,$

(5.2.8) $Q_8(\dot q, q) \triangleq DC(q)\,[M(q)]^{-1} D_1N(\dot q, q) - 2\,Q_{11}(\dot q, q),$

(5.2.9) $Q_7(\dot q, q) \triangleq \{\, Q_{11}(\dot q, q) - DC(q)\,[M(q)]^{-1} Q_{10}(\dot q, q)\,\}\,[M(q)]^{-1}.$

(5.2.10) $Q_6(\dot q, q) \triangleq DC(q)\,[M(q)]^{-1} D_2N(\dot q, q)\,\dot q + Q_7(\dot q, q)\,N(\dot q, q) -$

$D^3C(q)\,[\dot q]\,[\dot q]\,[\dot q],$

(5.211) $Q_5(v, \dot q, q) \triangleq [M(q)]^{-1}\{\, g_0(v, \dot q, q) - N(\dot q, q)\,\},$

(5.2.12) $Q_4(v, \dot q, q) \triangleq D_1\, g_0(v, \dot q, q)\, a_0(v, \dot q, q),$

(5.2.13) $Q_3(v, \dot q, q) \triangleq Q_4(v, \dot q, q) + D_2\, g_0(v, \dot q, q)\, Q_5(v, \dot q, q) +$

$D_3\, g_0(v, \dot q, q)\,\dot q,$

(5.2.14) $Q_2(v, \dot q, q) \triangleq Q_8(\dot q, q)\, Q_5(v, \dot q, q) + Q_6(\dot q, q) -$

$Q_7(v, \dot q, q)\, g_0(v, \dot q, q),$

(5.2.15) $\hat Q_1(v, q, q) \triangleq 3\, Q_{11}(\dot q, q)\, Q_5(v, \dot q, q) + D^3C(q)\,[\dot q]\,[\dot q]\,[\dot q],$

(5.2.16) $\hat Q_2(v, q, q) \triangleq D_2N(\dot q, q)\,\dot q - Q_4(v, \dot q, q) - D_3\, g_0(v, \dot q, q)\,\dot q +$

$\{\, D_1N(\dot q, q) + Q_{10}(\dot q, q) - D_2\, g_0(v, \dot q, q)\,\}\, Q_5(v, \dot q, q),$

(5.2.17) $V_i(v, \dot q, q) \triangleq DC_i(q)\, Q_5(v, \dot q, q) + \dot q^T D[DC_i(q)]^T \dot q, \quad i \in \mathcal{M}_{1,m},$

where $M_i^T(q)$ is the transpose of the ith row of $M(q)$ and C_i is the ith component of C. Note that

(5.2.18) $Q_8(\dot q, q) = D_1Q_9(\dot q, q), \quad Q_6(q, q) = D_2Q_9(\dot q, q)\, q.$

Theorem 5. 2. 1. Suppose that for each of the following parts, $\{ f, h, \mathcal{X} \}_0$ satisfies (C.2), (D.1), and (D.2). (i) Then, $\{ f, h, \mathcal{X} \}_0$

is decomposable at each $x_0 \in \mathcal{X}$ and decouplable on $\mathcal{X}$ with $d_i = 2$, $i \in \mathcal{M}_{1,m}$. Moreover, a control law $u = \alpha(v, \dot{q}, q) + \beta(v, \dot{q}, q)\hat{u}$ decouples $[f, h, \mathcal{X}]_0$ on $\mathcal{X}$ if α, β have the forms on $\mathcal{X}$:

$$\text{(5.2.19)} \quad \alpha(v, \dot{q}, q) \triangleq [Q_1(v, \dot{q}, q)]^{-1}\{\hat{Q}_2(v, \dot{q}, q) + M(q)[DC(q)]^{-1}\{\Pi(v, \dot{q}, q) - \hat{Q}_1(v, \dot{q}, q)\},$$

$$\text{(5.2.20)} \quad \beta(v, \dot{q}, q) \triangleq [Q_1(v, \dot{q}, q)]^{-1}M(q)[DC(q)]^{-1}\Gamma(v, \dot{q}, q),$$

where

$$\text{(5.2.21)} \quad \Pi(v, \dot{q}, q) \triangleq \begin{bmatrix} \phi_1(C_1(q), DC_1(q)\dot{q}, V_1(v, \dot{q}, q)) \\ \cdots \\ \phi_m(C_m(q), DC_m(q)\dot{q}, V_m(v, \dot{q}, q)) \end{bmatrix},$$

$$\text{(5.2.22)} \quad \Gamma(v, \dot{q}, q) \triangleq \text{diag}\ \psi_i(C_i(q), DC_i(q)\dot{q}, V_i(v, \dot{q}, q)),$$

and ϕ_i, ψ_i are arbitrary C^∞- functions of their arguments such that $\Gamma(v, \dot{q}, q)$ is nonsingular, $(v, \dot{q}, q) \in \mathcal{X}$.

(ii) Suppose that (C.4), (D.3) are satisfied and the class of control laws satisfies (A.5), (A.6) of Section 3. 3. Then, $[f, h, \mathcal{X}]_0$ is decomposable on $\mathcal{X}$. The class given by (5.2.19) - (5.2.22) is the whole class of smooth decomposing control laws.

(iii) Suppose that (D.1)' is satisfied. Suppose that class of control laws is real analytic and for every control law in the class, $[f, h, \mathcal{X}]^{\alpha,\beta}$ satisfies (A.2) of Section 3. 2. Then, the class given by (5.2.19) - (5.2.22) is the whole class of real analytic decoupling control laws. □

Proof. First consider part (i). Let

$$W_i(x_3) \triangleq DC_i(x_3)\,[M(x_3)]^{-1}, \quad i \in \mathcal{M}_{1,m}. \tag{5.2.23}$$

Then, we can derive

$$[\,D(W_i(x_3))^T x_2\,]^T = (Q_7)_i(x_2, x_3), \quad i \in \mathcal{M}_{1,m}, \tag{5.2.24}$$

where $(Q_7)_i$ is the ith component of Q_7. Let X_j, $j \in \mathcal{M}_{0,m}$ be the vector fields corresponding to $\{f, h, \mathcal{X}\}_0$. Straightforward computation with (5.2.23) and (5.2.24) shows that

$$X_0 H_i(x) = DC_i(x_3)\,x_2\,, \tag{5.2.25}$$

$$X_0^2 H_i(x) = V_i(x_1, x_2, x_3) = W_i(x_3)\,g_0(x_1, x_2, x_3) - (Q_9)_i(x_2, x_3), \tag{5.2.26}$$

$$\begin{aligned} X_0^3 H_i(x) = {} & W_i(x_3)\,Q_4(x_1, x_2, x_3) + \{\,W_i(x_3)\,D_2\,g_0(x_1, x_2, x_3) - \\ & D_1(Q_9)_i(x_2, x_3)\,\}\,Q_5(x_1, x_2, x_3) - D_2(Q_9)_i(x_2, x_3)\,x_2 + \\ & (Q_7)_i(x_2, x_3)\,g_0(x_1, x_2, x_3) + W_i(x_3)\,D_3\,g_0(x_1, x_2, x_3)\,x_2\,. \end{aligned} \tag{5.2.27}$$

where $(Q_9)_i$ is the ith component of Q_9. Note that

$$X_j H_i(x) = X_j X_0 H_i(x) = 0, \quad x \in \mathcal{X}, \quad i, j \in \mathcal{M}_{1,m}. \tag{5.2.28}$$

But (D.2) implies

$$D_i^*(x) = W_i(x_3)\,Q_1(x_1, x_2, x_3) \neq 0, \quad x \in \mathcal{X}, \quad i \in \mathcal{M}_{1,m}. \tag{5.2.29}$$

Thus $d_i = 2$, $i \in \mathcal{M}_{1,m}$. By (C.2), (D.2), Theorem 3. 3. 1, and

Theorem 3. 3. 2, $\{f, h, \mathcal{X}\}_0$ is decouplable on $\mathcal{X}$ and decomposable at each $x_0 \in \mathcal{X}$. $\mathcal{S}^\infty(\{f, h, \mathcal{X}\}_0)$ is given by $\beta(v, \dot{q}, q)$ in (5.2.20) and

$$(5.2.30)\quad \alpha(v,\dot{q},q) = [Q_1(v,\dot{q},q)]^{-1}\{M(q)\,[DC(q)]^{-1}(Q_2(v,\dot{q},q) + \Pi(v,\dot{q},q)) - Q_3(v,\dot{q},q)\}.$$

But, since

$$(5.2.31)\quad M(q)\,[DC(q)]^{-1}Q_2(v,\dot{q},q) = M(q)\,[DC(q)]^{-1}\{Q_8(\dot{q},q)\,Q_5(v,\dot{q},q) - Q_7(\dot{q},q)\,M(q)\,Q_5(v,\dot{q},q) - D_3C(q)\,[\dot{q}]^3\} + D_2N(\dot{q},q)\,\dot{q}$$
$$= -M(q)\,[DC(q)]^{-1}\hat{Q}_1(v,q,q) + D_1N(\dot{q},q)\,Q_5(v,\dot{q},q) + D_2N(\dot{q},q)\,\dot{q} + Q_{10}(\dot{q},q)\,Q_5(v,\dot{q},q),$$

(5.2.30) can be reduced to (5.2.19).

Consider part (ii). Note that $3m = \sum_{i=1}^{m}(d_i + 1)$. Define a mapping T from $\mathcal{X}$ into R^{3m} by

$$(5.2.32)\quad T \triangleq (T_1, \ldots, T_m),\ T_i \triangleq (T_{i,1}, T_{i,2}, T_{i,3}),\ T_{i,1}(x_1, x_2, x_3) \triangleq C_i(x_3),$$
$$T_{i,2}(x_1,x_2,x_3) \triangleq DC_i(x_3)x_2,\ T_{i,3}(x_1,x_2,x_3) \triangleq V_i(x_1,x_2,x_3),\ i \in \mathbf{M}_{1,m}.$$

By (C.2) and (D.2), DT(x) is nonsingular, $x \in \mathcal{X}$. By Theorem 2. 3. 7, this with (C.4), (D.3) implies that T is a C^∞ - diffeomorphism on $\mathcal{X}$. From this, Theorem 3. 4. 4, and Remark 3. 4. 7, part (ii) follows easily.

Part (iii) follows from Theorem 3. 4. 1, Remark 3. 4. 7, and the fact that T is a C^ω - diffeomorphism on $\mathcal{X}$. □

Remark 5. 2. 1. Using the special structure of $\{f, h, \mathcal{X}\}_0$, there is an alternative and perhaps more straightforward way to show that $\{f, h, \mathcal{X}\}_0$ is decouplable on $\mathcal{X}$ and that the control law $u = \alpha(v, \dot{q}, q) + \beta(v, \dot{q}, q)\hat{u}$ satisfying (5.2.19), (5.2.20) decouples $\{f, h, \mathcal{X}\}_0$ on $\mathcal{X}$. Differentiate both sides of the first equation of (5.2.1) with respect to t. Then, in the resulting equation, replace $\ddot{q}$, $\dot{v}$ by expressions obtained from (5.2.1), (5.2.2). Then, we can obtain

$$(5.2.33) \quad M(q)\dddot{q} = Q_1(v, \dot{q}, q)\,u - \hat{Q}_2(v, \dot{q}, q).$$

On the other hand, differentiating the second equation of (5.2.1) three times with respect to t leads to

$$(5.2.34) \quad \dddot{y} = \hat{Q}_1(v, \dot{q}, q) + DC(q)\dddot{q}.$$

From (5.2.33) and (5.2.34),

$$(5.2.35) \quad \dddot{y} = DC(q)\,[M(q)]^{-1}\,\{\,Q_1(v, \dot{q}, q)\,u - \hat{Q}_2(v, \dot{q}, q)\,\} + \hat{Q}_1(v, \dot{q}, q).$$

From (5.2.35), it is clear that the control law $u = \alpha(v, \dot{q}, q) + \beta(v, \dot{q}, q)\hat{u}$ satisfying (5.2.19), (5.2.20) with $\Pi = 0$ and $\Gamma = I_m$ gives

$$(5.2.36) \quad \dddot{y} = u.$$

Thus, $\{f, h, \mathcal{X}\}$ is decouplable. The characterization of the entire class of decoupling control laws follows from Remark 3. 4. 4 or Remark 3. 4. 7 (see also the last paragraphs of Sections 3. 4, 3. 5). □

Remark 5. 2. 2. Nijmeijer ([Nij.4]) considered decoupling of the system in (5.2.1), (5.2.2) with $m = 2$, $g_0(v, \dot{q}, q) = v$, and $C(q) = q$. In [Yua.1], the dynamics of a robotic manipulator with D. C. drives were linearly perturbed around an equilibrium point. Then, the decoupled control of the linearly perturbed system was considered. Thus, the nonlinearity of the system was not fully taken into account. □

Consider the following system, denoted by Σ_λ :

$$(5.2.37) \qquad M(q)\ddot{q} + N(\dot{q}, q) = \tau, \qquad y = C(q),$$

$$(5.2.38) \qquad \dot{v} = b_0(v, \dot{q}, q) + \sum_{j=1}^{r} g_j(v, \dot{q}, q) z_j + \sum_{j=1}^{m} b_j(v, \dot{q}, q) u_j,$$

$$\tau = g_0(v, \dot{q}, q),$$

$$(5.2.39) \qquad \lambda \dot{z} = A(\lambda) z + B_0(v, \dot{q}, q, \lambda) + \sum_{j=1}^{m} B_j(v, \dot{q}, q, \lambda) u_j,$$

where : $\mathcal{X}$, $\mathcal{E}$, $\mathbf{Q}$, q, v, M, N are defined as in (5.2.1), (5.2.2) ; $g_j : \mathcal{X} \to R^m$, $j \in \mathcal{M}_{0,r}$; $b_j : \mathcal{X} \to R^m$, $j \in \mathcal{M}_{0,m}$; λ_0 is a positive constant scalar and $\lambda \in [0, \lambda_0]$; $A : [0, \lambda_0] \to R^{r\times r}$; $B_j : \mathcal{X} \times [0, \lambda_0] \to R^r$, $j \in \mathcal{M}_{0,m}$. As in Section 5. 1, the system (5.2.37) represents the dynamics of a robotic manipulator. Here, the additional actuator dynamics are grouped into two subsystems (5.2.38) , (5.2.39) . The system (5.2.38) ((5.2.39)) represents the slow (fast) part of the additional actuator dynamics. Suppose that we neglect the fast dynamics by letting $\lambda = 0$. Then, the resulting system is the degenerate system Σ_0 of Σ_λ and consists of the systems (5.2.37), (5.2.38), and

(5.2.40) $$0 = A(0)z + B_0(v, \dot{q}, q, 0) + \sum_{j=1}^{m} B_j(v, \dot{q}, q, 0)u_j.$$

If we assume (B.1), Σ_0 can be written as (5.2.1), (5.2.2) with

(5.2.41) $$a_i(v, \dot{q}, q) \triangleq b_i(v, \dot{q}, q) - [g_1(v, \dot{q}, q) \cdots g_q(v, \dot{q}, q)]\,[A(0)]^{-1} B_i(v, \dot{q}, q, 0), \quad i \in \mathcal{M}_{0,m}.$$

Thus, we have shown that the dynamics of a robotic manipulator with the actuator dynamics can be described as (5.2.1), (5.2.2) when the slow part of the actuator dynamics are taken into account.

Finally, we consider the effect of neglected fast part of the actuator dynamics on decoupling of the original system Σ_λ. Let $u = \alpha(v, \dot{q}, q) + \beta(v, \dot{q}, q)\hat{u}$ be a control law satisfying (5.2.19) - (5.2.22). We denote by $\Sigma_0^{\alpha,\beta}$, $\Sigma_\lambda^{\alpha,\beta}$, respectively, the systems Σ_0, Σ_λ with the control law $u = \alpha(v, \dot{q}, q) + \beta(v, \dot{q}, q)\hat{u}$. To apply the theory of Chapter 4 we need :

(D.4) A, B_i, b_i, g_i, $i \in \mathcal{M}_{0,m}$ are C^∞.

Theorem 5. 2. 2. Suppose that (C.2), (D.1), (D.2), and (D.4) are satisfied. Suppose that $\Sigma_0^{\alpha,\beta}$ satisfies (B.1) and (B.2). Then, properties (i), (ii) in Theorem 4. 2. 1 hold. □

CHAPTER 6

CONCLUSION

In the previous chapters, we have addressed various theoretical issues of decoupling and decomposition and their applications to robotics.

In Chapter 3, the major portion of well known results on linear decoupling have been extended to nonlinear systems. Since in Section 3.6, our main contributions have been summarized and some concluding remarks on them have been given, we shall not repeat the same discussion here. Those results contribute to a deeper and clearer understanding of nonlinear decoupling theory. They supply full information about the flexibilities we can have in the design of decoupled systems.

In Chapter 4, a trade - off between the exact decoupling of systems and the computational complexity of decoupling control laws has been considered. We have shown that neglecting the fast dynamics of the systems leads to control laws which require less computation but decouple the systems in an approximate way.

In Chapter 5, these results have been applied to the decoupled control of robotic manipulators. Two cases have been considered. In the first case, actuator dynamics are completely neglected. In the second case, the dynamics of a significant class of actuators are taken into account. We have shown that our formulas for the complete class of decoupling control laws unify and generalize previous results on the decoupled control of robotic manipulators (see comments in Remark 5. 1. 1, 5. 1. 2, and 5. 2. 2). For example, it is possible to achieve decoupled control of the end - effector.

Some of our results may be extended with increased complexity to the general case where the numbers of inputs and outputs are not necessarily equal or the systems do not have the form in (1.7). All our results can be easily extended to time varying nonlinear systems since they can be changed into time invariant nonlinear systems by assigning a new state x_{n+1} to the time variable t.

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