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Technical Report

SOME BOUNDARY VALUE PROBLEMS OF MATHEMATICAL PHYSICS

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ABSTRACT

In this thesis the following problems are solved: the homogeneous Laplace equation, with boundary conditions given on a radially finite wedge; the Helmholtz equation, both homogeneous and inhomogeneous, satisfied inside a radially infinite wedge; a half-space problem for the elastic wave equation; several infinite space, half-space, and slab problems, with point sources, in linear transport theory; an integral equation arising from the theory of Mathematical Statistics; and certain generalized wave equations from quantum field theory.

Essential use is made throughout of the Fourier and related transforms, especially in combination with methods based on the theory of Cauchy integrals (including the Wiener-Hopf technique), and certain, usually elementary properties of generalized functions.

I. INTRODUCTION

The elementary theory of the Fourier transformation, depending as it does upon translation invariance, is not obviously applicable to problems involving boundaries. This report, in which a number of boundary value problems from various areas of Mathematical Physics are solved, illustrates several of the means by which the theory can be applied to such problems.

Specifically, we obtain by the Fourier and related transforms the solutions to: wedge problems for the Laplace and Helmholtz equations (Section II); a half-space problem for the elastic wave equation (Section III); several infinite-space, half-space, and slab problems, with point sources, in linear transport theory (Section IV); an integral equation arising from the theory of Mathematical Statistics (Section V); and certain generalized wave equations from quantum field theory (Section VI).

The specific technique used to solve each problem is briefly discussed at the beginning of the appropriate section. Here, it is convenient to summarize the notation and basic mathematical tools which will be generally relevant below.

The Fourier transform of $f(x)$ will be denoted by $\tilde{f}(k)$ and defined by

$$\tilde{f}(k) = \int_{-\infty}^{\infty} e^{ikx}\, f(x)dx \tag{1}$$

$$\Rightarrow f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\, \tilde{f}(k)dk$$

We will often have occasion to use a conventional decomposition of $\tilde{f}(k)$:

$$\tilde{f}(k) = \tilde{f}_+(k) + \tilde{f}_-(k) \tag{2}$$

$$\tilde{f}_+(k) = \int_0^\infty e^{ikx} f(x)dx \; ; \; \tilde{f}_-(k) = \int_{-\infty}^0 e^{ikx} f(x)dx$$

If f(x) has at most polynomial growth at infinity, we have the important fact that

$$\tilde{f}_\pm(k) \text{ is analytic in the } \begin{Bmatrix}\text{upper}\\ \text{lower}\end{Bmatrix} \text{ half k-plane}$$

The $\tilde{f}_\pm(k)$ may have singularities on the real axis; for example, if

$$f(x) = 1 \quad -\infty < x < \infty$$

then

$$\tilde{f}_\pm(k) = \frac{\pm i}{k \pm io} \tag{3}$$

where the infinitesimal imaginary part specifies the interpretation of the pole in the usual way. Alternatively, we may consider equation (3) as defining the generalized function

$$\frac{1}{k \pm io} = P\,\frac{1}{k} \mp i\pi\delta(k) \tag{4}$$

where P denotes "principal value."

The above remarks can easily be made rigorous by requiring f(x) to be a generalized function in the space S' of Gel'fand.(1) Then $\tilde{f}(k)$ is also in S' and the $\tilde{f}_\pm(k)$ always exist. On the occasions when we must assume f(x) to be in a different generalized function space, we will draw attention to the fact, although generally such matters will not be of crucial interest.

Whenever $\tilde{f}(k) = o(k^{-1})$ for $k \sim \infty$ we can easily deduce the useful formulae

$$\tilde{f}_{\pm}(k) = \pm \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\tilde{f}(k')dk'}{k'-k\mp io} \tag{5}$$

In general, subscripts will be used to denote functions which are defined and analytic in appropriate half-planes[the functions need not be Fourier transforms of any f(x)]. Superscripts, on the other hand, will denote the boundary values along the cuts of functions which are sectionally holomorphic in the sense of Muskhelishvili.(2) In particular, if ℓ is any sufficiently smooth arc (or union of arcs) and

$$\Phi(z) = \frac{1}{2\pi i} \int_{\ell} \frac{\varphi(t')dt'}{t'-z}$$

then for each $t \in \ell$, we define(2)

$$\Phi^{\pm}(t) = \lim \Phi(z) \text{ as } z \to t \text{ from the } \begin{cases} \text{left of } \ell \\ \text{right of } \ell \end{cases} \tag{6}$$

If $\varphi(t)$ satisfies a Hölder condition, the $\Phi^{\pm}(t)$ can be shown to exist, and the following useful formulae of Plemelj(3) hold:

$$\left. \begin{aligned} \Phi^{+}(t) + \Phi^{-}(t) &= \frac{1}{\pi i} P \int_{\ell} \frac{\varphi(t')}{t'-t} dt' \\ \Phi^{+}(t) - \Phi^{-}(t) &= \varphi(t) \end{aligned} \right\} \tag{7}$$

As an example, observe that equation (5) could be stated as

$$F^{\pm}(k) = \pm \tilde{f}_{\pm}(k) \quad (\mathrm{Im}(k) = o) \tag{8}$$

where, for $\mathrm{Im}(k) \neq o$,

$$F(k) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\tilde{f}(k')}{k'-k} dk' \tag{9}$$

and $\tilde{f}(k)$ is suitably behaved. The second Plemelj formula, applied to $\tilde{f}(k)$, is now merely the statement of (2).

II. CLASSICAL WEDGE PROBLEMS

Two planes, intersecting along and terminating at the z-axis, constitute what we will call a wedge. In this section we solve some classical two-dimensional differential equations with simple boundary conditions specified on wedges.

In Part 1 we find that solution of Laplace's equation which attains a given, constant value on a wedge of finite "width" (width is measured in the radial direction—see Figure 1), by the Wiener-Hopf technique. It is found that, in a space of generalized functions, an infinite set of solutions exists; of these, only the "least singular" solution is uniquely determined by the classical boundary conditions. The special case of the strip is also discussed in some detail.

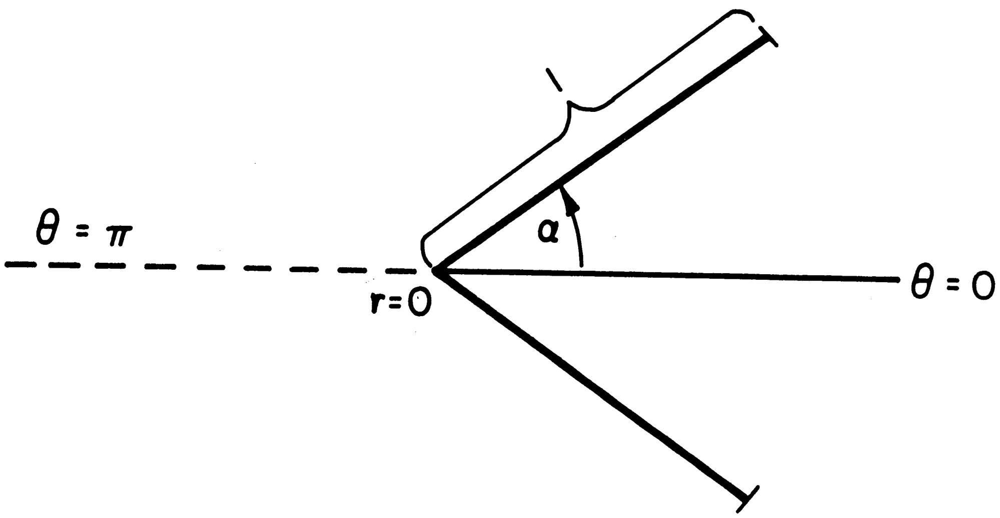

Figure 1. The wedge of Section II.1.

Part 2 is concerned with the Helmholtz equation, satisfied inside a radially infinite wedge. The particular problems considered, which already have known solutions,[4] were chosen so as to clearly illustrate the main features of our method; this[5] involves a modification of the Fourier transformation.

1. THE LAPLACE EQUATION

We want to find $\varphi(r,\Theta)$ where, for each (r,Θ) not on the wedge,

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\Theta^2} = 0 \tag{1}$$

and

$$\varphi(r,\alpha) = \varphi(r,-\alpha) = \varphi_o \text{ for } r \leq 1 \tag{2}$$

Here the wedge subtends an angle 2α, and we have conveniently assumed it to have unit width. With the orientation indicated in Figure 1, it is clear from symmetry that we can restrict our attention to the region $0 \leq \Theta \leq \pi$, and that (assuming $\alpha \neq 0,\pi$),

$$\varphi'(r,\Theta) \equiv \frac{\partial\varphi(r,\Theta)}{\partial\Theta} = 0 \text{ on } \Theta = 0 \text{ and } \Theta = \pi \tag{3}$$

For a unique solution, an additional boundary condition must be specified: the total charge per unit length on the wedge. (Since the wedge looks asymptotically like a line of charge, we could equivalently specify the behavior of φ for large r. The fact that both φ_o and the charge must be given is essentially due to the fact that the wedge extends to infinity in the z-direction.) If we define

$$D[f]_\Theta = \lim_{\epsilon\to 0}[f(\Theta+\epsilon) - f(\Theta-\epsilon)] = f(\Theta+) - f(\Theta-)$$

then the charge density, $q(r)$, on the wedge is clearly given by

$$q(r) = -\frac{1}{r} D[\varphi']_\alpha \quad 0 \leq r \leq 1 \tag{4}$$

and the specified quantity is

$$Q = 2 \int_0^1 q(r)dr \tag{5}$$

Finally, we remark that with the exception of φ' on $\Theta = \alpha$, $0 \leq r \leq 1$, φ and φ' are continuous for all $\Theta \in (0,\pi)$. In particular,

$$\varphi'(r,\alpha+) = \varphi'(r,\alpha-) \quad \text{for} \quad r \geq 1 \tag{6}$$

Equations (2) and (6) together constitute "mixed boundary conditions" and suggest use of the method of Wiener and Hopf.[(6)]

The Least Singular Solution

Our procedure for finding the $\varphi(r,\Theta)$ satisfying (1) — (6) may be summarized in two steps:

(i) a change of variable $r \to u = -\ln r$, followed by Fourier transforming with respect to u*; solution of the resulting ordinary differential equation to obtain the Θ-dependence.

(ii) use of the mixed boundary conditions to obtain a Wiener-Hopf equation for the transform variable dependence; solution of the Wiener-Hopf equation and inversion of the transformation of (i).

Step (i): the angular dependence

In terms of the variable $u = -\ln r$, our differential equation (1) takes the form [we write $\varphi(e^{-u},\Theta) = \varphi(u,\Theta)$ for convenience]

*Equivalently, we could have taken the Mellin transform with respect to r.

$$\frac{\partial^2\varphi}{\partial u^2} + \frac{\partial^2\varphi}{\partial\Theta^2} = 0 \qquad (7)$$

with boundary conditions

$$\varphi(u,\alpha) = \varphi_o \quad 0 \leq u \leq \infty \qquad (8)$$

$$D[\varphi']_\alpha = 0 \quad -\infty \leq u \leq 0 \qquad (9)$$

$$\varphi'(u,0) = \varphi'(u,\pi) = 0 \quad -\infty \leq u \leq \infty \qquad (10)$$

$$-2\int_o^\infty D[\varphi']_\alpha du = Q \qquad (11)$$

Using the convention specified in Section I, we take the Fourier transform of (7) to obtain

$$\frac{\partial^2\tilde{\varphi}(k,\Theta)}{\partial\Theta^2} - k^2\,\tilde{\varphi}(k,\Theta) = 0 \qquad (12)$$

Recalling the continuity conditions discussed above, and using equation (10), it is a simple matter to solve (12):

$$\tilde{\varphi}(k,\Theta) = A(k)\begin{cases} \dfrac{\cosh k\Theta}{\cosh k\alpha} & 0 \leq \Theta \leq \alpha \\ \dfrac{\cosh k(\pi-\Theta)}{\cosh k(\pi-\alpha)} & \alpha \leq \Theta \leq \pi \end{cases} \qquad (13)$$

where A(k) is to be determined.

Step (ii): the Wiener-Hopf equation

With the definitions of $\tilde{\varphi}_\pm(k,\Theta)$ from Section I, and equation (8), we observe that

$$\tilde{\varphi}_+(k,\alpha) = \frac{i\varphi_o}{k+io} \qquad (14)$$

while from (9) we have

$$\tilde{\varphi}_-'(k,\alpha+) = \tilde{\varphi}_-'(k,\alpha-) \qquad (15)$$

Now note from equation (4) that

$$\tilde{q}_+(k) = \tilde{\varphi}_+'(k,\alpha-) - \tilde{\varphi}_+'(k,\alpha+)$$

is the transform of the charge density. Using (13) and (15) we find

$$\tilde{q}_+(k) = kA(k)\left\{\frac{\sinh k(\pi-\alpha)}{\cosh k(\pi-\alpha)} + \frac{\sinh k\alpha}{\cosh k\alpha}\right\} = kA(k)\left[\frac{\sinh k\pi}{\cosh k(\pi-\alpha)\cosh k\alpha}\right] \qquad (16)$$

On the other hand, equation (13) implies that

$$A(k) = \tilde{\varphi}(k,\alpha) = \tilde{\varphi}_+(k,\alpha) + \tilde{\varphi}_-(k,\alpha)$$

Substituting this into (16), and using (14), we finally obtain the desired Wiener-Hopf equation:

$$\tilde{q}_+(k) = H(k)[\tilde{\varphi}_-(k,\alpha) + \frac{i\varphi_o}{k+io}] \qquad (17)$$

where

$$H(k) = \frac{k \sinh k\pi}{\cosh k\alpha \cosh k(\pi-\alpha)} \qquad (18)$$

Because of their known analyticity properties (cf., Section I), both of the unknown functions $\tilde{q}_+$ and $\tilde{\varphi}_-$ can be determined from equation (17)* by the Wiener-Hopf technique, as follows:

Suppose we can find functions $h_+(k)$ and $h_-(k)$ such that

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*In deriving (17) from (7)-(11) we followed what is called, in Reference 6, Jones Method.

(a) $H(k) = k^2 h_+(k)\, h_-(k)$

(b) $h_\pm(k)$ is analytic and non-zero for $\mathrm{Im}(k) \gtrless 0$

(c) $h_\pm(k)$ has at most polynomial growth at ∞

Then equation (17) implies that

$$\frac{q_+(k)}{h_+(k)} = k\, h_-(k)[i\varphi_o + k\tilde{\varphi}_-(k,\alpha)] \qquad (19)$$

The function F(k) defined by

$$F(k) = \begin{cases} \dfrac{\tilde{q}_+(k)}{h_+(k)} & \mathrm{Im}(k) > 0 \qquad (20) \\ kh_-(k)[i\varphi_o + k\tilde{\varphi}_-(k,\alpha)] & \mathrm{Im}(k) < 0 \qquad (21) \end{cases}$$

is easily seen to be entire: it is analytic for $\mathrm{Im}(k) \neq 0$ and, by (19), continuous across the real axis. Now assuming $\tilde{q}_+$ and $\tilde{\varphi}_-$ to be generalized functions in the space S', and noting condition (c) above on the $h_+(k)$, we see that F(k) is bounded by a polynomial at infinity. Hence, by the "extended" Liouville theorem,[7] F(k) must itself be a polynomial:

$$F(k) = B_o + B_1 k + \ldots + B_n k^n \qquad (22)$$

Assuming, for the present, the B_i to be known, equations (20)—(22) provide the desired solution to (19).

Thus, the problem reduces to finding functions $h_\pm(k)$ which satisfy conditions (a), (b), and (c). For the H(k) of equation (18), this is not difficult. Using known[7] representations of the hyperbolic functions, we have

$$H(k) = \frac{k^2\Gamma(\frac{1}{2} - ik\,\frac{\alpha}{\pi})\Gamma(\frac{1}{2} - ik\,\frac{\pi-\alpha}{\pi})\Gamma(\frac{1}{2} + ik\,\frac{\alpha}{\pi})\Gamma(\frac{1}{2} + ik\,\frac{\pi-\alpha}{\pi})}{\pi\;\Gamma(1-ik)\Gamma(1+ik)}$$

Noting that $\Gamma(z)$ (which is nowhere zero), is analytic except for simple poles at $z = o, -1, -2, \ldots$, we can immediately exhibit functions satisfying conditions (a) and (b):

$$h_+(k) = \frac{1}{\pi} \frac{\Gamma(\frac{1}{2} - ik\frac{\alpha}{\pi})\Gamma(\frac{1}{2} - ik\frac{\pi-\alpha}{\pi})}{\Gamma(1-ik)} \chi(k,\alpha) \qquad (23)$$

$$h_-(k) = \frac{\Gamma(\frac{1}{2} + ik\frac{\alpha}{\pi})\Gamma(\frac{1}{2} + ik\frac{\pi-\alpha}{\pi})}{\Gamma(1+ik)} \frac{1}{\chi(k,\alpha)} \qquad (24)$$

where the $\chi(k,\alpha)$ factors are to be chosen according to condition (c). Using Stirling's formula we find

$$\ln h_+(k) \xrightarrow[k\to\infty]{} \ln\chi - ik[\frac{\alpha}{\pi} \ln \frac{\alpha}{\pi-\alpha} + \ln \frac{\pi-\alpha}{\pi}] - \frac{1}{2} \ln k$$

so that $h_+(k)$ will have exponential growth at infinity unless

$$\chi(k,\alpha) = (\frac{\alpha}{\pi-\alpha})^{ik\frac{\alpha}{\pi}} (\frac{\pi-\alpha}{\pi})^{ik} = \chi(k,\pi-\alpha) \qquad (25)$$

This choice of χ insures the proper asympotic behavior of $h_-(k)$ also. In fact, it follows from (23)—(25) that

$$h_\pm(k) \sim k^{-\frac{1}{2}} \text{ for } k \sim \infty \text{ in the } \begin{Bmatrix}\text{upper}\\ \text{lower}\end{Bmatrix} \text{ half plane} \qquad (26)$$

We can determine the B_i, which fix the behavior of $\tilde{\varphi}_-$ and $\tilde{q}_+$ near $k = \infty$, only by appropriately restricting the class of functions which will be considered acceptable solutions. For the present, let us require, in the "classical" manner, that $\tilde{\varphi}_-(k,\alpha)$ be square-integrable, i.e.,

$$\tilde{\varphi}(k,\alpha) = o(k^{-1/2}) \quad k \to \infty \qquad (27)$$

The $\varphi(u,\Theta)$ which results from this requirement will be square-integrable except for the trivial singularity unavoidably associated with equation (8) and will be the "least singular" solution in the sense of Case.[8] Postponing discussion of the more singular solutions, we observe that (27) implies

$$B_i = 0 \text{ for } i > 0 \tag{28}$$

since, from (21) we have

$$\tilde{\varphi}_-(k,\alpha) = \frac{F(k)}{(k-io)^2 h_-(k)} - \frac{i\varphi_o}{k-io} \tag{29}$$

which, with (26) and (27), gives (28). The remaining constant, B_o, is determined from the final boundary condition, equation (11):

$$\frac{Q}{2} = - \int_o^\infty [\varphi'(u,\alpha_+) - \varphi'(u,\alpha_-)]du = \tilde{q}_+(o) \tag{30}$$

From (20), (22), and (23) we have

$$\tilde{q}_+(o) = \frac{B_o}{h_+(o)} = B_o$$

so that

$$B_o = \frac{Q}{2} \tag{31}$$

Now it follows from (14) and (28)—(31) that

$$A(k) = \tilde{\varphi}(k,\alpha) = \frac{Q}{2(k-io)^2\ h_-(k)} + i\varphi_o\left(\frac{1}{k+io} - \frac{1}{k-io}\right)$$

which equation, combined with (13), determines $\tilde{\varphi}(k,\Theta)$:

$$\tilde{\varphi}(k,\Theta) = \left[\frac{Q}{2(k-io)^2 h_-(k)} + i\varphi_o\left(\frac{1}{k+io} - \frac{1}{k-io}\right)\right]\begin{cases}\dfrac{\cosh k\Theta}{\cosh k\alpha} & 0 \le \Theta \le \alpha \\ \dfrac{\cosh k(\pi-\Theta)}{\cosh k(\pi-\alpha)} & \alpha \le \Theta \le \pi\end{cases} \tag{32}$$

Similarly [from (20)],

$$\tilde{q}_+(k) = \frac{Q}{2} h_+(k) \tag{33}$$

where the $h_\pm(k)$ are given by equations (23)—(25). Our problem in k-space is solved.

As is clear from (23)—(25), the right-hand sides of (32) and (33) are meromorphic and well-behaved at infinity. Thus the inverse transform integral,

$$\varphi(u,\Theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iku}\,\tilde{\varphi}(k,\Theta)dk$$

involves only an elementary computation of residues (i.e., the contour can be closed by a semi-circle in the half-plane appropriate to the sign of u). In terms of the physical variable $r = e^{-u}$, the results of this computation can be given in the following form:

For $0 < \alpha < \pi$,

$$\varphi(r,\Theta) = \varphi_o - \frac{Q}{2} r^{\frac{\pi}{2\alpha}} \sum_{n=o}^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma[(n+\frac{1}{2})\frac{\pi}{\alpha}]\chi(k_n,\alpha)}{\Gamma[\frac{1}{2} + (n+\frac{1}{2})\frac{\pi-\alpha}{\alpha}](n+\frac{1}{2})\pi} \cos[(n+\frac{1}{2})\frac{\pi\Theta}{\alpha}] r^{n\frac{\pi}{\alpha}}$$

$$\text{for } 0 \le \Theta \le \alpha,\ 0 \le r \le 1 \tag{34}$$

$$= \varphi_o - \frac{Q}{2}\left\{\frac{\gamma + 2\ln 2 + \ln r}{\pi}\right.$$

$$+ \frac{1}{r} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)^2} \frac{\cos[(n+1)\theta] r^{-n}}{\cos[(n+1)\alpha]\Gamma[\frac{1}{2} - \frac{\alpha}{\pi}(n+1)]\Gamma[\frac{1}{2} - \frac{\pi-\alpha}{\pi}(n+1)]\chi(t_n,\alpha)} \Bigg\}$$

$$\text{for } 0 \leq \theta \leq \pi, \ 1 \leq r \leq \infty \qquad (35)$$

Here, $k_n = -i(n+\frac{1}{2}) \frac{\pi}{\alpha}$, $t_n \equiv i(n+1)$, χ is given by (25) and $\gamma \equiv \frac{\alpha}{\pi} \ln \frac{\alpha}{\pi-\alpha} + \ln \frac{\pi-\alpha}{\pi}$.

For $\alpha \leq \theta \leq \pi$, $0 \leq r \leq 1$, $\varphi(r,\theta)$ is clearly given by (34) with θ,α replaced by $\pi-\theta$, $\pi-\alpha$, respectively. The charge density can be computed either from (33), or (34) with (4). By either method, the result is, for $0 < \alpha < \pi$ and $\alpha \neq \frac{\pi}{2}$,

$$q(r) = \frac{Q}{2r} \sum_{n=0}^{\infty} \frac{1}{n!} \left\{ \frac{\Gamma[(n+\frac{1}{2})\frac{\pi}{\alpha}]\chi(p_n,\alpha) r^{(n+\frac{1}{2})\frac{\pi}{\alpha}}}{\alpha\Gamma[\frac{1}{2} + (n+\frac{1}{2})(\frac{\pi-\alpha}{\alpha})]} + \frac{\Gamma[(n+\frac{1}{2})\frac{\pi}{\pi-\alpha}]\chi(q_n,\alpha) r^{(n+\frac{1}{2})\frac{\pi}{\pi-\alpha}}}{(\pi-\alpha)\Gamma[\frac{1}{2} + (n+\frac{1}{2})(\frac{\alpha}{\pi-\alpha})]} \right\} \qquad (36)$$

where $p_n \equiv -i(n+\frac{1}{2})\frac{\pi}{\alpha}$, $q_n \equiv -i(n+\frac{1}{2})\frac{\pi}{\pi-\alpha}$.

The Strip

As we have implied, the cases $\alpha = 0$, $\alpha = \pi$, and, for $q(r)$, $\alpha = \frac{\pi}{2}$, are somewhat exceptional [this is clear from equation (32); note that for all three of these angles, the wedge reduces to a strip]. Nonetheless, the inverse transform can be found in essentially the same way as for the unexceptional α. We only give the result for $\alpha = 0$, which is typical.

$$\varphi(r,\theta) = \varphi_o - Qr^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n!} \frac{\Gamma(n+3/2)}{(n+\frac{1}{2})^2 \pi^{3/2}} \cos[(n+\tfrac{1}{2})(\pi-\theta)] r^n$$

$$\text{for } \alpha = 0 \leq \Theta \leq \pi,\ 0 \leq r \leq 1 \tag{37}$$

$$= \varphi_o - Q\,[\frac{1}{\pi}(2 \ln 2 + \ln r) + \frac{1}{r}\sum_{n=o}^{\infty} \frac{(-1)^n}{n!} \frac{\cos[(n+1)\Theta]r^{-n}}{\pi^{1/2}\,\Gamma(-\frac{1}{2} - n)} \tag{38}$$

$$\text{for } \alpha = 0 \leq \Theta \leq \pi,\ 1 \leq r \leq \infty$$

$$q(r) = \frac{Q}{\pi} \frac{1}{r^{1/2}(1-r)^{1/2}} \quad 0 \leq r \leq 1 \tag{39}$$

where we have used the fact that

$$\sum_{n=o}^{\infty} \frac{(-1)^n}{n!} \frac{r^n \sqrt{\pi}}{\Gamma[1-(n + \frac{1}{2})]} = (1-r)^{-1/2} \tag{40}$$

Note that the factor of 2 in equation (5) is erroneous in the case $\alpha = 0$; since the wedge is "closed," the integral of (5) automatically includes the charge on both wedge-planes. We have accounted for this in (37)-(38) and it is easily checked that (39) indeed satisfies

$$\int_o^1 q(r)dr = Q \quad (\alpha = 0) \tag{41}$$

The closed form of (39) suggests that we might have solved the strip problem in a much simpler way. That this is indeed the case can be seen in the following digression.

The equations

$$\varphi(r,\Theta) = (\text{constant}) \int_o^1 q(r')\ln|\vec{r}-\vec{r}'|dr' \tag{42}$$

and

$$\frac{\partial\varphi}{\partial r}\Big|_{\Theta = \alpha = 0} = 0 \tag{43}$$

imply

$$P \int_0^1 \frac{q(r')}{r'-r} dr' = 0 \quad 0 < r < 1 \tag{44}$$

Now assume that $q(r)$ satisfies a Hölder condition and let

$$\Phi(z) = \frac{1}{2\pi i} \int_0^1 \frac{q(r')}{r'-z} dr' \tag{45}$$

We note that (3)

(i) $\Phi(z)$ is analytic everywhere except on the line $(0,1)$.

(ii) $\Phi(z) \sim \frac{1}{z}$ for $z \sim \infty$

(iii) $\Phi^{\pm}(r)$ exists [cf. equation (I.6)] for $0 < r < 1$. Near the endpoints,

$$\Phi(z) \leq \frac{\text{constant}}{(z-c)^{\beta}}$$

where $0 < \beta < 1$ and $c = 0$ or 1.

(iv) By the Plemelj formulae and equation (44),

$$\Phi^{+}(r) + \Phi^{-}(r) = 0 \quad 0 < r < 1 .$$

The task of finding a $\Phi(z)$ which satisfies (i)—(iv) constitutes a "Hilbert Problem."(3) Since the equation of statement (iv) is in this case particularly simple, the general method for solving such problems (which method we will use in later sections) is not needed here; it suffices to observe that the function $P(z)\sqrt{z(1-z)}$, where the branch cut extends along the real axis from 0 to 1 and $P(z)$ is any function continuous across this cut, satisfies (iv). By (i) and (iii), $P(z)$ can have no singularities in the finite plane except possibly simple poles at 0 and 1, i.e.,

$$P(z) = \frac{Q(z)}{z(1-z)}$$

where Q(z) is an entire function. By (ii), Q(z) = constant, so we have

$$\Phi(z) = \frac{\text{constant}}{z^{1/2}(1-z)^{1/2}} \tag{46}$$

and, again using the Plemelj formulae,

$$q(r) = \frac{\text{constant}}{r^{1/2}(1-r)^{1/2}} \tag{47}$$

as in equation (39).

We remark that a generalization of this method can be effectively applied to the case of several strips aligned, say, along the real axis, with varying, given potentials on each strip.

Behavior of q(r) Near the Endpoints

When the wedge is not a strip, we cannot express q(r) in closed form. However, it is easy to obtain such expressions for q(r) valid near the extremities of the wedge.

From equation (36) it is clear that near the vertex,

$$q(r) \sim (\text{constant}) \ \text{Max}\left[r^{\frac{\pi}{2\alpha}-1}, \ r^{\frac{\pi}{2(\pi-\alpha)}-1} \right]$$

$$\text{for } r \sim 0, \ 0 < \alpha < \pi \tag{48}$$

For $r \sim 1$ (near the edge), we use a well known[9] Tauberian theorem: for $-1 < \eta < 0$,

$$[\tilde{f}_+(k) \sim k^{-\eta-1} \text{ for } k \sim \infty] \Rightarrow [f(u) \sim u^{\eta} \text{ for } u \sim 0] \tag{49}$$

Applying this to $\tilde{q}_+(k)$, by (33) and (26) we have

$$q(u) \sim u^{-1/2} \quad \text{for } u \sim 0$$

i.e.,

$$q(r) \sim (-\ln r)^{-1/2}$$

$$\sim \frac{(\text{constant})}{\sqrt{1-r}} \quad \text{for } r \sim 1 \tag{50}$$

So that near the edge, the charge distribution of the wedge is asymptotic to that of the strip, as we would expect.

More Singular Solutions

If we relax the requirement of equation (27), and thus allow $F(k)$ to be a polynomial of degree greater than zero, the resulting $\tilde{\varphi}(k,\Theta)$ no longer possesses an inverse Fourier transform in the ordinary, classical sense. However, we may still obtain useful solutions, provided we allow them to be generalized functions in the sense of Gel'fand.[1]

Let us restrict our attention to the charge distribution. From (20) and (22) we have

$$q(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iku}[B_o + B_1 k + \ldots + B_n k^n] h_+(k)dk \tag{51}$$

Of course the integral does not exist, but in the generalized function space S', which we now explicitly use, the inverse Fourier transform always exists, and we have in general

$$F^{-1}[k\tilde{f}] = i\frac{d}{du}F^{-1}[\tilde{f}]$$

where $F^{-1}[\tilde{f}]$ denotes the inverse Fourier transform of $\tilde{f}(k)$. Thus

$$q(u) = F^{-1}[(B_o + B_1 k + \ldots + B_n k^n)h_+(k)]$$

$$= [B_o + B_1 i\frac{d}{du} + \ldots + B_n(i\frac{d}{du})^n]q_o(u) \tag{52}$$

where $q_o(u)$ is the least singular charge distribution obtained above (except that the constant $\frac{Q}{2}$ is no longer significant); $q_o(u)$ is a <u>regular</u> generalized function in that all its singularities are integrable. Equation (52) is typical:[8] the general solution is a linear combination of the least singular solution and its derivatives.

For clarity we specialize to the case of the strip ($\alpha = 0$) and rewrite (52) in terms of r:

$$q(r) = \frac{A_1}{\sqrt{r(1-r)}} + A_2 r\frac{d}{dr}\frac{1}{\sqrt{r(1-r)}} + A_3(r\frac{d}{dr} + r^2\frac{d^2}{dr^2})\frac{1}{\sqrt{r(1-r)}}$$

$$+ \ldots \tag{53}$$

Two remarks are in order:

(i) The above q(r) is not a regular generalized function, since the singularity at r = 1 is not integrable. Thus the derivatives with respect to r must be interpreted in the usual generalized function sense; e.g.,

$$\int r\frac{dq_o}{dr}\psi(r)dr = (q_o'(r), r\psi(r)) = -(q_o,[r\psi]') \tag{54}$$

where $\psi(r)$ is one of an appropriately restricted class of test functions.

(ii) The boundary conditions used to determine the least singular solution are not sufficient—nor even applicable, since the integral of (5) no longer exists—to determine the more general solution of equation (53). Instead, some sort of "edge conditions" would seem to be called for.

This concludes our discussion of Laplace's equation.

2. THE HELMHOLTZ EQUATION

The Fourier transform, of course, owes its usefulness to the relation

$$\frac{d}{dx}\,[e^{ikx}] = ik\,[e^{ikx}]$$

The observation that

$$r^2\{\frac{1}{r}\frac{\partial}{\partial r}\,r\,\frac{\partial}{\partial r} - \kappa^2\}\,[e^{i\kappa r \sinh\tau}] = \frac{\partial^2}{\partial\tau^2}\,[e^{i\kappa r \sinh\tau}] \tag{1}$$

suggests[5] a new transform; this will simplify the Helmholtz operator in polar coordinates, much as the Fourier transform simplifies the differential operator in Cartesian coordinates.

For $\varphi(r,\theta)$, defined for all $r \in [-\infty, \infty]$ (and sufficiently well-behaved), we define the "τ-transform," $\psi(\tau,\theta)$, of $\varphi(r,\theta)$, by

$$\psi(\tau,\theta) = \cosh\,\tau \int_{-\infty}^{\infty} dr\;\varphi(r,\theta)e^{i\kappa r \sinh\tau} \tag{2}$$

The inverse transform is given by

$$\varphi(r,\theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau\;\psi(\tau,\theta)e^{-i\kappa r \sinh\tau} \tag{3}$$

as can easily be seen by making the substitution $k = \kappa\,\sinh\,\tau$: we see that $\psi(\tau,\theta)$ is simply $\cosh\,\tau \cdot\ [\tilde{\varphi}(\kappa\sinh\tau,\theta)]$

Our purpose here is to examine the properties and usefulness of this transformation by means of a few examples.

The Homogeneous Helmholtz Equation Inside a Wedge

We find $\varphi(r,\theta)$ such that

$$\{\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}-\kappa^2\}\varphi(r,\theta)=0 \tag{4}$$

$$\text{for } 0<\theta<\alpha,\ 0<r<\infty$$

and

$$\varphi(r,0)=\varphi(r,\alpha)=0\ (0<\alpha<2\pi)\ \text{for } 0<r<\infty \tag{5}$$

(Note that here the wedge is radially infinite and has a different angle and orientation from that used in the previous problem.) We first assume (4) and (5) to hold for <u>all</u> r, and formally apply the transformation (2). Using (1) we find that

$$\{\frac{\partial^2}{\partial\tau^2}+\frac{\partial^2}{\partial\theta^2}\}\ \psi(\tau,\theta)=0 \tag{6}$$

$$\psi(\tau,0)=\psi(\tau,\alpha)=0 \tag{7}$$

Clearly

$$\psi(\tau,\theta)=\left[Ae^{\frac{n\pi}{\alpha}\tau}+Be^{-\frac{n\pi}{\alpha}\tau}\right]\sin\frac{n\pi}{\alpha}\theta\quad n=1,2,\ldots \tag{8}$$

We will see [cf., equation (11)] that both terms in (8) give the same result. Hence we choose B = 0 and apply the inverse transform (3) to obtain

$$\varphi(r,\theta)=A\sin\frac{n\pi}{\alpha}\theta\int_{-\infty}^{\infty}d\tau\ e^{\frac{n\pi}{\alpha}\tau}e^{-i\kappa r\sinh\tau} \tag{9}$$

With the change of variable $\tau\longrightarrow\tau+i\frac{\pi}{2}$,

$$\varphi(r,\theta) = A \sin \frac{n\pi}{\alpha}\theta \int_{-\infty - i\frac{\pi}{2}}^{\infty - i\frac{\pi}{2}} e^{-\frac{n\pi}{\alpha}\left(\tau + i\frac{\pi}{2}\right)} e^{-\kappa r \cosh\tau}\, dt \tag{10}$$

In view of the rapidly decreasing behavior of $e^{-\kappa r \cosh\tau}$ for large $|\tau|$, it is clear from Cauchy's theorem that the integration path in (10) is equivalent to one along the real axis. That is

$$\varphi(r,\theta) = (\text{constant}) \sin \frac{n\pi}{\alpha}\theta \int_{-\infty}^{\infty} e^{-\frac{n\pi}{\alpha}\tau} e^{-\kappa r \cosh\tau}\, d\tau \tag{11}$$

In this form, we recognize(10) the integral and conclude

$$\varphi(r,\theta) = (\text{constant}) \sin \frac{n\pi}{\alpha}\theta\, K_{\frac{n\pi}{\alpha}}(\kappa r) \quad n=1,2,\ldots \tag{12}$$

which is the familiar solution obtained by separation of variables.

This simple example demonstrates the essential features of the τ-transform:

(i) It transforms the Helmholtz operator in polar coordinates into the Laplace operator in Cartesian coordinates [equation (6)].

(ii) It transforms a wedge into a strip [equation (7)].

(iii) It is not one-one; this is clear from equation (11). In general we can say that whenever the transformed function ψ is such that the manipulations of equations (10) and (11) are permissible, then the odd part of ψ will have vanishing inverse transform (since $\cosh\tau$ is even).

The Inhomogeneous Equation

In the above example no difficulties associated with the unphysical domain of r occurred. The necessity of allowing r to be negative does, however, require us to exercise some care in dealing with inhomogeneous problems. To see this, consider the equation

$$(\nabla^2_{r,\theta} - \kappa^2)\varphi(r,\theta) = q(\vec{r}) \quad r > 0 \tag{13}$$

Multiplying by r^2 and taking the transform we find

$$\left[\frac{\partial^2}{\partial\tau^2} + \frac{\partial^2}{\partial\theta^2}\right] \psi(\tau,\theta) = \cosh \tau \int_{-\infty}^{\infty} dr\, r^2\, q(\vec{r}) e^{i\kappa r \sinh\tau} \tag{14}$$

where we have implicitly assumed (13) to hold for all $r \in [-\infty,\infty]$. The significance of this assumption can be seen in the case of a point source. Placing this source at ($x = r_o$, $y = 0$) for convenience, we have

$$q(\vec{r}) = q\, \delta(x-r_o)\, \delta(y)$$

The point is that

$$q(\vec{r}) \neq \frac{q}{r_o}\, \delta(r-r_o)\delta(\theta) \tag{15}$$

In fact

$$q(\vec{r}) = q\, \delta(r \cos \theta - r_o)\delta(r \sin \theta) \tag{16}$$

$$= q\, \delta(r \cos \theta - r_o)\left[\frac{1}{|r|}\, \delta(\sin \theta) + \frac{1}{|\sin \theta|}\, \delta(r)\right] \tag{17}$$

The $\delta(r)$ term evidently gives no contribution to the integral in equation (14), while[11]

$$\delta(\sin \theta) = \sum_{n=-\infty}^{\infty} \delta(\theta - n\pi) \tag{18}$$

so that (since $r_o > 0$)

$$q(\vec{r}) = q\,\delta(r\cos\Theta - r_o)\frac{1}{|r|}\sum_{n=-\infty}^{\infty}\delta(\Theta - n\pi) = \frac{q}{r_o}\left\{\sum_{n=-\infty}^{\infty}\delta(\Theta - 2n\pi)\delta(r - r_o) + \sum_{n=-\infty}^{\infty}\delta[\Theta-(2n+1)\pi]\delta(r+r_o)\right\} \quad (19)$$

which provides the correct source function to be inserted into equation (14). More generally, for a point source at $(x_o, y_o) = (r_o \cos \Theta_o, r_o \sin \Theta_o)$, we have

$$\delta(\vec{r}-\vec{r}_o) = \frac{1}{r_o}\left\{\sum_{n=-\infty}^{\infty}\delta(\Theta-\Theta_o-2n\pi)\ \delta(r-r_o) + \sum_{n=-\infty}^{\infty}\delta[\Theta-\Theta_o-(2n+1)\pi]\ \delta(r+r_o)\right\} \quad (20)$$

If we restrict out attention to the region $0 < \Theta < 2\pi$, this reduces to

$$\delta(\vec{r}-\vec{r}_o) = \frac{1}{r_o}\{\delta(\Theta-\Theta_o)\ \delta(r-r_o) + \delta(\Theta-\Theta_o \pm \pi)\ \delta(r+r_o)\}$$

$$\text{for } \Theta_o \gtrless \pi,\ 0 < \Theta < 2\pi \quad (21)$$

With these remarks we can find the <u>Green's function for an infinite wedge</u>[4]. This satisfies

$$(\nabla^2_{r,\Theta} - \kappa^2)\varphi_G(r,\Theta) = q\,\delta(\vec{r}-\vec{r}_o) \quad 0 < \Theta < \alpha \quad (22)$$

with the boundary condition

$$\varphi_G(r,0) = \varphi_G(r,\alpha) = 0 \quad (23)$$

Transforming these equations according to (2), and using equations (14) and (21), we obtain

$$\nabla^2\psi_G(\tau,\theta) = f(\tau)\ \delta(\theta-\theta_o) + f(-\tau)\ \delta(\theta-\theta_o-\pi) \quad 0 < \theta < \alpha \tag{24}$$

$$\psi_G(\tau,0) = \psi_G(\tau,\alpha) = 0 \quad 0 < \alpha < 2\pi$$

where (choosing $q = \pi\ \kappa$ for convenience)

$$f(\tau) = \pi\ \kappa\ r_o\ \cosh\ \tau\ e^{i\kappa r_o \sinh\ \tau}$$

$$= -i\pi \frac{d}{d\tau}\ e^{i\kappa r_o \sinh\ \tau} \tag{25}$$

and we have assumed $\theta_o < \pi$, the modification for $\theta_o > \pi$ being trivial [equation (21)].

Our task is to solve the equations (24); the solution to (22)—(23) will then be given by (3). The solution to (24) can clearly be written in the form

$$\psi_G(\tau,\theta;\theta_o) = \psi_1(\tau,\theta;\theta_o) + \psi_1(-\tau,\theta;\theta_o+\pi) \tag{26}$$

where ψ_1 satisfies

$$\left.\begin{aligned} \nabla^2\psi_1(\tau,\theta) &= f(\tau)\ \delta(\theta-\theta_o) \quad 0 < \theta < \alpha \\ \psi_1(\tau,0) &= \psi_1(\tau,\alpha) = 0 \end{aligned}\right\} \tag{27}$$

We find ψ_1 by first determining the Green's function G satisfying

$$\nabla^2 G(\tau,\theta;\tau',\theta') = \delta(\tau-\tau')\ \delta(\theta-\theta') \tag{28}$$

$$G(\tau,\theta;\tau',\theta') = G(\tau,\alpha;\tau',\theta') = 0 \tag{29}$$

By Green's identity we have

$$\int_V Gf\, \delta(\Theta-\Theta_o) d\tau d\Theta - \psi_1(\tau', \Theta') = \int_\Gamma \left(G \frac{\partial\psi_1}{\partial n} - \psi_1 \frac{\partial G}{\partial n}\right) d\Gamma = 0 \tag{30}$$

where V is the strip $0 < \Theta < \alpha$ and Γ its boundary (G and ψ_1 must of course be suitably behaved for large τ). Renaming variables we have [since $G(\tau_o, \Theta_o; \tau, \Theta) = G(\tau, \Theta; \tau_o \Theta_o)$]

$$\psi_1(\tau, \Theta) = \int_{-\infty}^{\infty} G(\tau, \Theta; \tau_o, \Theta_o)\, f(\tau_o) d\tau_o \tag{31}$$

so that the problem is solved if we can find a G satisfying (28) and (29). But this is elementary: G is clearly the electrostatic potential due to a point charge between parallel conducting plates, and can be constructed in a well-known way from an infinite sequence of "image" charges:

$$2\pi\, G(\tau, \Theta; \tau_o, \Theta_o) = \text{Re}\, \{F(z; z_o)\} \tag{32}$$

where

$$z = \tau + i\Theta, \quad z_o = \tau_o + i\Theta_o$$

and

$$F(z, z_o) = \sum_{n=-\infty}^{\infty} \ln \left[\frac{z - z_o + i2n\alpha}{z - \bar{z}_o + i2n\alpha}\right] \tag{33}$$

Each term in (33) represents of course the field due to a point charge; the charges are positioned so as to satisfy (29). It is not hard to show[7] that

$$F(z; z_o) = \ln \left[\frac{z - z_o}{z - \bar{z}_o} \prod_{n \neq o} \frac{1 + \frac{z - z_o}{i2n\alpha}}{1 + \frac{z - \bar{z}_o}{i2n\alpha}}\right] \tag{34}$$

$$= \ln \frac{\sinh \frac{\pi}{2\alpha}(z-z_o)}{\sinh \frac{\pi}{2\alpha}(z-\bar{z}_o)} \tag{35}$$

Hence

$$G = \frac{1}{2\pi} \ln \left| \frac{\sinh \frac{\pi}{2\alpha}(z-z_o)}{\sinh \frac{\pi}{2\alpha}(z-\bar{z}_o)} \right| \tag{36}$$

$$= \frac{1}{2\pi} \ln \left[\frac{\cosh \frac{\pi}{\alpha}(\tau-\tau_o)-\cos \frac{\pi}{\alpha}(\theta-\theta_o)}{\cosh \frac{\pi}{\alpha}(\tau-\tau_o)-\cos \frac{\pi}{\alpha}(\theta+\theta_o)} \right] \tag{37}$$

Substituting this expression into (31), and, as suggested by equation (25), integrating by parts, we obtain

$$\psi_1(\tau,\theta) = \frac{1}{2i} \int_{-\infty}^{\infty} e^{i\kappa r_o \sinh \tau_o} \left\{ \frac{\sinh \frac{\pi}{\alpha}(\tau-\tau_o)}{\cosh \frac{\pi}{\alpha}(\tau-\tau_o)-\cos \frac{\pi}{\alpha}(\theta-\theta_o)} - \frac{\sinh \frac{\pi}{\alpha}(\tau-\tau_o)}{\cosh \frac{\pi}{\alpha}(\tau-\tau_o)-\cos \frac{\pi}{\alpha}(\theta+\theta_o)} \right\} d\tau_o \tag{38}$$

or, with the change of variable $\xi = i(\tau-\tau_o)$,

$$\psi_1(\tau,\theta) = \frac{1}{2i} \int_{i\infty}^{-i\infty} e^{i\kappa r_o \sinh(\tau+i\xi)} \left\{ \frac{\sin \frac{\pi}{\alpha}\xi}{\cos \frac{\pi}{\alpha}\xi-\cos \frac{\pi}{\alpha}(\theta-\theta_o)} - \frac{\sin \frac{\pi}{\alpha}\xi}{\cos \frac{\pi}{\alpha}\xi-\cos \frac{\pi}{\alpha}(\theta+\theta_o)} \right\} d\xi \tag{39}$$

where the integration path must be understood as infinitesimally displaced to the right of the imaginary axis. In the corresponding expression for $\psi_1(-\tau,\theta;$

$\Theta_o+\pi$), it is convenient to make the further substitution $\xi\to -\xi+\pi$, yielding

$$\psi_1(-\tau,\Theta;\Theta_o+\pi) = \frac{1}{2i}\int_{-i\infty+\pi}^{i\infty+\pi} e^{i\kappa r_o \sinh(\tau+i\xi)}$$

$$\left\{\frac{\sin\frac{\pi}{\alpha}(\xi-\pi)}{\cos\frac{\pi}{\alpha}(\xi-\pi)-\cos\frac{\pi}{\alpha}(\Theta-\Theta_o-\pi)} - \frac{\sin\frac{\pi}{\alpha}(\xi-\pi)}{\cos\frac{\pi}{\alpha}(\xi-\pi)-\cos\frac{\pi}{\alpha}(\Theta+\Theta_o+\pi)}\right\}d\xi \qquad (40)$$

Here, the integration path is such that Re (ξ) is infinitesimally less than π.

Equations (39) and (40) provide, with (26), the solution to the problem (24) in τ-space. Our prescription is to substitute this solution into (3); i.e., the Green's function for the wedge is

$$\varphi_G(r,\Theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[\psi_1(\tau,\Theta;\Theta_o) + \psi_1(-\tau,\Theta;\Theta_o+\pi)]e^{-i\kappa r\sinh\tau}d\tau \qquad (41)$$

Now for $\xi\,\epsilon(0,\pi)$, it can be shown (cf., Appendix) that

$$\int_{-\infty}^{\infty} e^{-i\kappa r\sinh\tau} e^{i\kappa r_o\sinh(\tau+i\xi)}d\tau = 2K_o(\kappa|\vec{r}-\vec{r}_o|_\xi) \qquad (42)$$

where

$$|\vec{r}-\vec{r}_o|_\xi \equiv \sqrt{r^2 + r_o^2 - 2rr_o\cos\xi} \qquad (43)$$

Hence, combining equations (39)—(41), and inverting orders of integration, we finally obtain

$$\varphi_G(r,\Theta) = \frac{1}{2\pi i}\int_{i\infty}^{-i\infty} K_o(\kappa|r-r_o|_\xi)\left\{\frac{\sin\frac{\pi}{\alpha}\xi}{\cos\frac{\pi}{\alpha}\xi-\cos\frac{\pi}{\alpha}(\Theta-\Theta_o)} - \frac{\sin\frac{\pi}{\alpha}\xi}{\cos\frac{\pi}{\alpha}\xi-\cos\frac{\pi}{\alpha}(\Theta+\Theta_o)}\right\}d\xi$$

$$+ \frac{1}{2\pi i} \int_{-i\infty+\pi}^{i\infty+\pi} K_o(\kappa|\vec{r}-\vec{r}_o|_\xi) \left\{ \frac{\sin\frac{\pi}{\alpha}(\xi-\pi)}{\cos\frac{\pi}{\alpha}(\xi-\pi)-\cos\frac{\pi}{\alpha}(\Theta-\Theta_o-\pi)} \right.$$

$$\left. - \frac{\sin\frac{\pi}{\alpha}(\xi-\pi)}{\cos\frac{\pi}{\alpha}(\xi-\pi)-\cos\frac{\pi}{\alpha}(\Theta+\Theta_o+\pi)} \right\} d\xi \qquad (44)$$

Equation (44) provides a solution to the problem for arbitrary wedge angle $\alpha \in [0,2\pi]$. Its form is such that it is easily checked in the particular case $\alpha = \frac{\pi}{n}$, $n = 1,2,\ldots$, since for such α the two expressions in braces in (44) are identical. Thus

$$\varphi_G(r,\Theta) = \frac{1}{2\pi i} \int_\Gamma K_o(\kappa|\vec{r}-\vec{r}_o|_\xi) \left\{ \frac{\sin\frac{\pi}{\alpha}\xi}{\cos\frac{\pi}{\alpha}\xi-\cos\frac{\pi}{\alpha}(\Theta-\Theta_o)} \right.$$

$$\left. - \frac{\sin\frac{\pi}{\alpha}\xi}{\cos\frac{\pi}{\alpha}\xi-\cos\frac{\pi}{\alpha}(\Theta+\Theta_o)} \right\} d\xi$$

$$\text{for } \frac{\pi}{\alpha} = n = 1,2,\ldots \qquad (45)$$

where Γ is the path indicated in Figure 2. It is easily seen that Γ can be deformed into the path Γ' as shown; thus we can evaluate the integral in (45) by residues. For definiteness, say $\alpha = \pi/3$. Then the relevant poles of the first term in braces occur at

$$\xi_o = \Theta-\Theta_o,\ \xi_1 = \Theta-\Theta_o + \frac{2\pi}{3},\ \xi_2 = -(\Theta-\Theta_o) + \frac{2\pi}{3}$$

and the residue at ξ_n ($n=o,1,2$) is $K_o(\kappa|\vec{r}-\vec{r}_o|_{\xi_n})$; similarly for the second term (with its minus sign). Thus equation (45) gives the well-known solution by images.

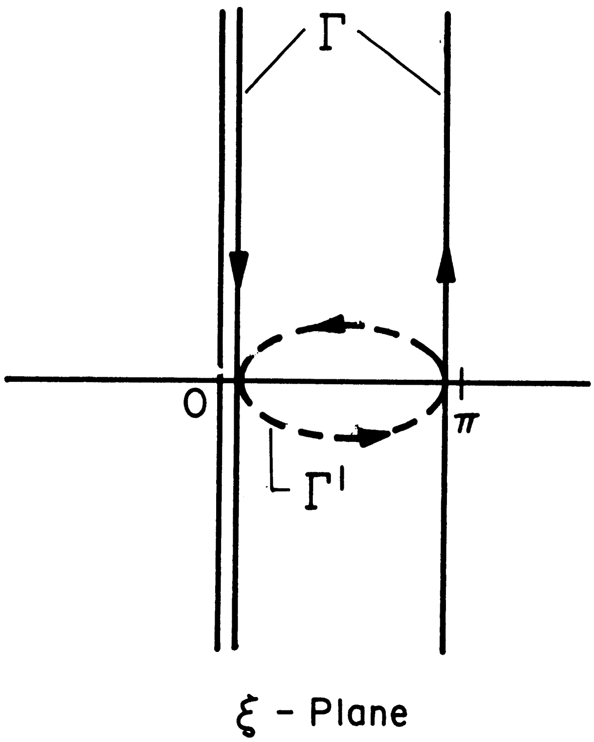

Figure 2. Integration paths for Section II.2.

Note that since the homogeneous equation, (4), possesses (an infinite set of) nontrivial solutions, we cannot expect the solution to (22) obtained above to be unique. In particular, for $\alpha < \pi$, the inverse τ-transform of $\psi_1(\tau,\Theta)$, by itself, is a solution; this is already clear from equation (24). We do not pursue this point, but merely emphasize the fact that for $\alpha > \pi$, and for other problems (such as the infinite space Green's function—see the Appendix), the inclusion of sources at negative r is essential.

III. A HALF-SPACE GREEN'S FUNCTION FOR ELASTIC WAVES

A method for computing the elastic radiation from a small source in the earth's interior has been presented by Case and Colwell.[13] This method, which assumes the earth to be an infinite medium, could be modified to include effects due to the earth's surface simply by replacing the (known) infinite space Green's function which is used, by an appropriate half-space Green's function. It is the purpose of this section to derive a representation for the latter function.

Our method is straightforward. We first formulate the problem in terms of an integral equation (following Case and Colwell), and then solve the integral equation by Fourier transforms. The only complication lies in the fact that, once we choose a definite orientation for our half-space, the matrices which occur are not tensors, i.e., not rotation invariant. Hence tensor theory arguments, with the computational simplifications they often afford, are not available to us.

This section will consistently use the summation convention

$$a_i b_i = \sum_{i=1}^{3} a_i b_i$$

and, augmenting the definitions of Section I, the notations

$$\partial_i = \frac{\partial}{\partial x_i} \quad ; \quad \partial'_i = \frac{\partial}{\partial x'_i}$$

$$\vec{a} = (a_1, a_2) \quad ; \quad \underset{\sim}{a} = (\vec{a}, a_3)$$

We also define two- and three-dimensional Fourier transforms, by trivial extension of (I.1):

$$\hat{f}(\vec{k},x_3) = \int e^{i\vec{k}\cdot\vec{r}} f(\vec{r},x_3)d\vec{r}$$

$$\tilde{f}(\underset{\sim}{k}) = \int e^{ik_3x_3} \hat{f}(\vec{k},x_3)dx_3$$

1. INTEGRAL FORMULATION

We consider first a general region V of $\underset{\sim}{r}$-space, and seek the solution $f_{ij}(\underset{\sim}{r},\underset{\sim}{r}_o)$ to

$$-\omega^2\rho\, f_{ij}(\underset{\sim}{r},\underset{\sim}{r}_o) = \partial_k D_{ikm}(\underset{\sim}{\partial})\, f_{mj}(\underset{\sim}{r},\underset{\sim}{r}_o) + \delta_{ij}\, \delta(\underset{\sim}{r} - \underset{\sim}{r}_o) \qquad (1)$$

$$\underset{\sim}{r},\underset{\sim}{r}_o \in V$$

with the boundary condition

$$n_i\, D_{ikm}(\underset{\sim}{\partial})\, f_{mj}(\underset{\sim}{r}_s,\underset{\sim}{r}_o) = 0 \qquad (2)$$

$$\underset{\sim}{r}_s \in \partial V$$

Here n_i is (the ith component of) the inward normal to the region V and

$$D_{ikm}(\underset{\sim}{\partial}) \equiv \lambda\delta_{ik}\partial_m + \mu(\delta_{im}\partial_k + \delta_{km}\partial_i) \qquad (3)$$

The infinite space Green's function G_{ij} satisfies (1) with V equal to all space:

$$-\omega^2\rho G_{i\ell}(\underset{\sim}{r},\underset{\sim}{r}') = \partial_k D_{ikm}(\underset{\sim}{\partial})\, G_{m\ell}(\underset{\sim}{r},\underset{\sim}{r}') + \delta_{i\ell}\delta(\underset{\sim}{r} - \underset{\sim}{r}') \qquad (4)$$

$$\text{all } \underset{\sim}{r},\underset{\sim}{r}'$$

We now proceed in a standard way to multiply (4) by f_{ij}, (1) by $G_{i\ell}$, and subtract. Using the easily verified identity,

$$G_{i\ell}\partial_k D_{ikm} f_{mj} - f_{ij}\partial_k D_{ikm} G_{m\ell} = \partial_i [G_{k\ell} D_{ikm} f_{mj} - f_{kj} D_{ikm} G_{m\ell}] \qquad (5)$$

we find

$$G_{i\ell}(\underset{\sim}{r},\underset{\sim}{r}')\,\delta_{ij}\delta(\underset{\sim}{r} - \underset{\sim}{r}_o) - f_{ij}(\underset{\sim}{r},\underset{\sim}{r}_o)\,\delta_{i\ell}\delta(\underset{\sim}{r} - \underset{\sim}{r}') + \partial_i [G_{k\ell}(\underset{\sim}{r},\underset{\sim}{r}')\, D_{ikm}(\underset{\sim}{\partial})\, f_{mj}(\underset{\sim}{r},\underset{\sim}{r}_o)$$

$$- f_{kj}(\underset{\sim}{r},\underset{\sim}{r}_o)\, D_{ikm}(\underset{\sim}{\partial})\, G_{m\ell}(\underset{\sim}{r},\underset{\sim}{r}')] = 0 \qquad (6)$$

We now integrate (6) over V, apply the divergence theorem, and note equation (2). Upon renaming variables (and indices), and using the facts that

$$G_{ij}(\underset{\sim}{r},\underset{\sim}{r}_o) = G_{ij}(\underset{\sim}{r}_o,\underset{\sim}{r}) \qquad (7)$$

and

$$D_{ikm}(\underset{\sim}{\partial})\, G_{mj}(\underset{\sim}{r},\underset{\sim}{r}') = - D_{ikm}(\underset{\sim}{\partial}')\, G_{jm}(\underset{\sim}{r}',\underset{\sim}{r}) \qquad (8)$$

we obtain the desired equation

$$f_{ij}(\underset{\sim}{r},\underset{\sim}{r}_o) = G_{ij}(\underset{\sim}{r},\underset{\sim}{r}_o) + S_{ij}(\underset{\sim}{r},\underset{\sim}{r}_o) \qquad (9)$$

$$\underset{\sim}{r} \in V$$

where

$$S_{ij}(\underset{\sim}{r},\underset{\sim}{r}_o) = \int_{\partial V} d^2 r'_s n_k(\underset{\sim}{r}'_s)\, f_{\ell j}(\underset{\sim}{r}'_s,\underset{\sim}{r}_o)\, D_{k\ell m}(\underset{\sim}{\partial}')\, G_{mi}(\underset{\sim}{r}'_s,\underset{\sim}{r}) \qquad (10)$$

Here, of course

$$D_{k\ell m}(\underset{\sim}{\partial}') \; G_{mi}(\underset{\sim}{r}'_s,\underset{\sim}{r}) \equiv [D_{k\ell m}(\underset{\sim}{\partial}') \; G_{mi}(\underset{\sim}{r}',\underset{\sim}{r})]_{\underset{\sim}{r}'=\underset{\sim}{r}_s} \quad \underset{\sim}{r}_s \in \partial V \tag{11}$$

A method for determining $f_{ij}(\underset{\sim}{r},\underset{\sim}{r}_o)$ is clear from equation (9). Indeed, our problem clearly reduces to finding $f_{\ell j}(\underset{\sim}{r}'_s,\underset{\sim}{r}_o)$; and, by taking the limit of (9) as $\underset{\sim}{r} \to \underset{\sim}{r}_s$, we obtain an integral equation which may be solved for $f_{\ell j}(\underset{\sim}{r}'_s,\underset{\sim}{r}_o)$.

Specializing to the case in which V is the half-space $x \geq 0$, we denote the half-space Green's function by g_{ij}:

$$g_{ij}(\underset{\sim}{r},\underset{\sim}{r}_o) = G_{ij}(\underset{\sim}{r},\underset{\sim}{r}_o) + \int d\vec{r}' \; g_{\ell j}(\vec{r}',0;\underset{\sim}{r}_o)[D_{3\ell m}(\underset{\sim}{\partial}') \; G_{mi}(\underset{\sim}{r}',\underset{\sim}{r})]_{x_3'=0} \tag{12}$$

A certain amount of care is required in taking the limit $x_3 \to 0$ of (12), since the integrand is singular on ∂V. In fact, if we define

$$\Sigma_{3\ell i}(\underset{\sim}{r},\underset{\sim}{r}') \equiv D_{3\ell m}(\underset{\sim}{\partial}) \; G_{mi}(\underset{\sim}{r},\underset{\sim}{r}') \tag{13}$$

then it easily follows from the differential equation for G_{ij}, (4), that

$$\Sigma_{3\ell i}(\vec{r},0+;\vec{r}',0) - \Sigma_{3\ell i}(\vec{r},0-;\vec{r}',0) = -\,\delta_{\ell i}\delta(\vec{r} - \vec{r}') \tag{14}$$

Thus, in close analogy to (I.4), we define

$$\Sigma^0_{3\ell i}(\vec{r},\vec{r}') \equiv \frac{1}{2}[\Sigma_{3\ell i}(\vec{r},0+;\vec{r}',0) + \Sigma_{3\ell i}(\vec{r},0-;\vec{r}',0)] \tag{15}$$

Now we let $x_3 \to 0$ in (12). Using (14) and (15), the result may be written as

$$\frac{1}{2}g_{ij}(\vec{r},0;\underset{\sim}{r}_o) = G_{ij}(\vec{r},0;\underset{\sim}{r}_o) + S^0_{ij}(\vec{r},\underset{\sim}{r}_o) \tag{16}$$

where

$$S^0_{ij}(\vec{r},\underset{\sim}{r}_o) \equiv \int d\vec{r}'\; g_{\ell j}(\vec{r}',0;\underset{\sim}{r}_o)\; \Sigma^0_{3\ell i}(\vec{r}',\vec{r}) \tag{17}$$

(12) and (16) are the basic equations by means of which our problem is to be solved.

2. SOLUTION OF THE INTEGRAL EQUATION (16)

We define

$$F_{\ell i}(\vec{k},x_3) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} dk_3\; e^{-ik_3x_3}\; e^{-i\underset{\sim}{k}\cdot\underset{\sim}{r}'}\; \tilde{\Sigma}_{3\ell i}(\underset{\sim}{k},\underset{\sim}{r}') \tag{18}$$

Using the known[13] fact that

$$\tilde{G}_{ij}(\underset{\sim}{k},\underset{\sim}{r}') = \frac{e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}'}}{\omega^2\rho}\left[\frac{k_ik_j}{k^2 - k_\ell^2} + \frac{k_t^2\delta_{ij} - k_ik_j}{k^2 - k_t^2}\right] \tag{19}$$

where

$$k_\ell^2 = \frac{\omega^2\rho}{\lambda + 2\mu} \quad ; \quad k_t^2 = \frac{\omega^2\rho}{\mu} \tag{20}$$

it requires only a straightforward computation to obtain

$$\begin{aligned} 2\pi i\omega^2\rho F_{kj} = {} & \lambda k_\ell^2\delta_{3k}[(k_j - \delta_{3j}k_3)\, I_{o\ell} + \delta_{3j}I_{1\ell}] + \mu\Big\{2(k_\ell^2 - k_t^2)[(k_k - \delta_{3k}k_3) \\ & \times (k_j - \delta_{3j}k_3)I_2 + (k_j - \delta_{3j}k_3)\delta_{3k}I_3 + (k_k - \delta_{3k}k_3)\delta_{3j}I_3 + \delta_{3k}\delta_{3j}I_4] \\ & + k_t^2[\delta_{3j}(k_k - \delta_{3k}k_3)I_{ot} + \delta_{3k}\delta_{3j}I_{1t} + \delta_{kj}I_{1t}]\Big\} \end{aligned} \tag{21}$$

where

$$I_{0^{\ell}_{t}}(\vec{k},x_3) = \frac{\pi}{\kappa_{\ell_t}} e^{-\kappa_{\ell_t}|x_3|}$$

$$I_{1^{\ell}_{t}}(\vec{k},x_3) = \mp i\pi e^{-\kappa_{\ell_t}|x_3|} \qquad x_3 \gtrless 0$$

$$I_2(\vec{k},x_3) = \frac{i\pi}{\kappa_\ell^2 - \kappa_t^2}(e^{-\kappa_\ell|x_3|} - e^{-\kappa_t|x_3|})$$

$$I_3(\vec{k},x_3) = \frac{\pi}{\kappa_\ell^2 - \kappa_t^2}(\kappa_\ell e^{-\kappa_\ell|x_3|} - \kappa_t e^{-\kappa_t|x_3|})$$

$$I_4(\vec{k},x_3) = \frac{\mp i\pi}{\kappa_\ell^2 - \kappa_t^2}(\kappa_\ell^2 e^{-\kappa_\ell|x_3|} - \kappa_t^2 e^{-\kappa_t|x_3|}) \quad ; \ x_3 \gtrless 0 \qquad (22)$$

In (22), we have introduced the abbreviations

$$\kappa^2_{\ell_t} = \vec{k}^2 - k^2_{\ell_t} \qquad (23)$$

Note that $F_{\ell i}(\vec{k},x_3)$ is independent of x_3':

$$F_{\ell i}(\vec{k},x_3) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk_3 e^{-ik_3x_3} e^{-i\vec{k}\cdot\vec{r}'} \tilde{\Sigma}_{3\ell i}(\underset{\sim}{k};\vec{r}',0) \qquad (24)$$

Hence if we let

$$F^o_{\ell i}(\vec{k}) \equiv \frac{1}{2}[F_{\ell i}(\vec{k},0+) + F_{\ell i}(\vec{k},0-)] \qquad (25)$$

then the two-dimensional Fourier transform of (16) clearly takes the form [cf., equation (8)]

$$[\frac{1}{2}\delta_{\ell i} + F^o_{\ell i}(\vec{k})]\, \hat{g}_{\ell j}(\vec{k},0;\underset{\sim}{r}_o) = \hat{G}_{ij}(\vec{k},0;\underset{\sim}{r}_o) \qquad (26)$$

and we need merely find the inverse of the matrix $\|A\|$, where

$$\|A\|_{i\ell} \equiv \frac{1}{2}\delta_{\ell i} + F^{o}_{\ell i}(\vec{k}) \tag{27}$$

Now it follows from (21) and (22) that

$$F^{o}_{\ell i}(\vec{k}) = \frac{a(\vec{k}^2)}{2}\,\delta_{3\ell}(k_i - \delta_{3i}k_3) + \frac{b(\vec{k}^2)}{2}\,\delta_{3i}(k_\ell - \delta_{3\ell}k_3) \tag{28}$$

where

$$a(\vec{k}^2) = -i\,\frac{\lambda\kappa_t - (\lambda + 2\mu)\kappa_\ell}{\kappa_\ell(\kappa_\ell + \kappa_t)(\lambda + 2\mu)} \tag{29}$$

$$b(\vec{k}) = i\,\frac{\lambda\kappa_t - (\lambda + 2\mu)\kappa_\ell}{\kappa_t(\kappa_\ell + \kappa_t)(\lambda + 2\mu)} \tag{30}$$

whence

$$\|A\| = \frac{1}{2}\begin{bmatrix} 1 & 0 & ak_1 \\ 0 & 1 & ak_2 \\ bk_1 & bk_2 & 1 \end{bmatrix} \tag{31}$$

and we easily find

$$\|A\|^{-1} = \frac{2}{1 - ab\vec{k}^2}\begin{bmatrix} 1 - abk_2^2 & abk_1k_2 & -ak_1 \\ abk_1k_2 & 1 - abk_1^2 & -ak_2 \\ -bk_1 & -bk_2 & 1 \end{bmatrix} \tag{32}$$

Now

$$\hat{g}_{kj}(\vec{k},0;\underset{\sim}{r}_o) = \|A^{-1}\|_{ki}\;\hat{G}_{ij}(\vec{k},0;\underset{\sim}{r}_o) \tag{33}$$

and, since the two-dimensional Fourier transform of equation (12) is

$$\hat{g}_{ij}(\vec{k},x_3;\underset{\sim}{r}_o) = \hat{G}_{ij}(\vec{k},x_3;\underset{\sim}{r}_o) - F_{\ell i}(\vec{k},x_3)\,\hat{g}_{\ell j}(\vec{k},0;\underset{\sim}{r}_o) \qquad (34)$$

our problem is solved [cf., equations (21) and (33)]. The function $\hat{G}_{ij}$ occurring in (33) and (34) is, of course, obtained from (19):

$$\hat{G}_{ij}(\vec{k},x_3;\underset{\sim}{r}_o) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk_3 e^{-ik_3x_3}\,\frac{e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}_o}}{\omega^2\rho}\left[\frac{k_ik_j}{k^2 - k_\ell^2} + \frac{k_t^2\delta_{ij} - k_ik_j}{k^2 - k_t^2}\right]$$

$$= \frac{e^{i\vec{k}\cdot\vec{r}_o}}{\omega^2\rho}\left\{(\kappa_t^2 - \kappa_\ell^2)\, I_{ij}(\vec{k},x_3) - k_t^2\delta_{ij}\,\frac{e^{-\kappa_t|x_3-x_{o3}|}}{2\kappa_t}\right\} \qquad (35)$$

Here, $I_{ij} = \|I\|_{ij}$, with

$$\|I\| \equiv \begin{bmatrix} k_1k_1 & k_1k_2 & ik_1\frac{\partial}{\partial x_3} \\ k_1k_2 & k_2k_2 & ik_2\frac{\partial}{\partial x_3} \\ ik_1\frac{\partial}{\partial x_3} & ik_2\frac{\partial}{\partial x_3} & -\frac{\partial^2}{\partial x_3^2}\end{bmatrix} J(\vec{k},x_3) \qquad (36)$$

where

$$J(\vec{k},x_3) \equiv \frac{1}{2}\,\frac{\kappa_t e^{-\kappa_\ell|x_3-x_{o3}|} - \kappa_\ell e^{-\kappa_t|x_3-x_{o3}|}}{\kappa_\ell\kappa_t(\kappa_t^2 - \kappa_\ell^2)} \qquad (37)$$

The inverse transform with respect to $\vec{k}$ of (34) is obviously tedious and will not be discussed here ($\hat{g}_{ij}$ may be directly useful for certain applications; cf., for example, Part 6 of Section IV). We merely draw attention to the fact that there is no suggestion, in equation (34), of an "image" source at $-\underset{\sim}{r}_o$. In fact, it is not hard to show from the differential equations (1) and (4), that no solution of (1) by images is possible.

IV. LINEAR TRANSPORT THEORY

1. INTRODUCTION

In this section we solve a number of simple boundary value problems associated with the time-independent, one-speed transport equation with isotropic scattering [equation (2.1)]. One-dimensional solutions to this equation are already well-known[14]; the extension here lies in the fact that although we do not consider boundary surfaces any more complicated than parallel planes, our boundary conditions will be such that the full three-dimensional form of the equation must be used. Essentially, we deal with point sources rather than plane sources.

As in previous sections, the method used here relies on the Fourier transformation. We first obtain two basic equations which are valid for any configuration of boundaries and sources. The first of these, equation (2.6), gives the Fourier transform of the angular density in terms of a certain quantity in transform space, $\tilde{\rho}_V(k)$, where the subscript V refers to the region enclosed by the boundary surfaces; the second equation, (2.10), can, at least for the simple geometries considered here, be solved for $\tilde{\rho}_V(k)$. Thus (2.6) and (2.10) provide a general formulation by means of which the Fourier transforms of the solutions of all the problems considered can be found.

The form of equation (2.10) obviously depends upon the particular region V, and on the boundary and/or external source contributions. Thus for the infinite medium the equation is trivial; for half-space problems we must use the Wiener-Hopf and related techniques; and for slab problems we must combine the

Wiener-Hopf technique with a Fredholm-like iterative procedure.

It is not possible to express the inverse Fourier transform integrals in closed form; in particular, the dependence of the transformed solutions on the "transverse" transform variables, $\vec{k}$, is very complicated. (Of course, by setting $\vec{k} = 0$, we obtain the known one-dimensional solutions, although, since our method is not the standard one, in somewhat unconventional form.) It is nonetheless found possible to express various significant physical quantities in fairly tractable form, generally by means of approximation techniques; these points are briefly discussed at the end of the section.

We note here some notations which, in addition to those defined in Section I, will be used in this section:

$$\hat{f}(x,\vec{k}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\,\tilde{f}(\underset{\sim}{k})dk = \int e^{i\vec{k}\cdot\vec{r}}\, f(\underset{\sim}{r})d\vec{r} \qquad (1.1)$$

is the two-dimensional Fourier transform of $f(\underset{\sim}{r})$, the latter being defined for all $\underset{\sim}{r}$ in three-dimensional space. Here

$$\vec{k} \equiv (k_y, k_z) \quad ; \vec{r} \equiv (y,z) \qquad (1.2)$$

and $\tilde{f}(\underset{\sim}{k})$ is the three-dimensional transform, defined by trivial extension of (I.1).

We will also use the function ω, defined by

$$\omega(\underset{\sim}{\Omega}) \equiv -\frac{i}{\mu}(1 - i\vec{k}\cdot\vec{\Omega}) \qquad (1.3)$$

where $\vec{\Omega}$ refers to the y and z components of the three-dimensional, normalized, velocity vector, and $\mu = \Omega_x$:

$$\underset{\sim}{\Omega} = \frac{\underset{\sim}{v}}{|\underset{\sim}{v}|} = (\mu, \sqrt{1 - \mu^2} \cos \varphi, \sqrt{1 - \mu^2} \sin \varphi) \tag{1.4}$$

(φ is measured counter-clockwise from the positive y-axis). Note that

$$1 - i\underset{\sim}{k}\cdot\underset{\sim}{\Omega} = -i\mu(k_x - \omega) \tag{1.5}$$

2. GENERAL FORMULATION

Let V be some (bounded or unbounded) region of three-dimensional space, with boundary S. Our problem is to find the function $\varphi(\underset{\sim}{r},\underset{\sim}{\Omega})$ for $\underset{\sim}{r} \in V$, $|\underset{\sim}{\Omega}| = 1$, which satisfies

$$(\underset{\sim}{\Omega}\cdot\underset{\sim}{\nabla} + 1)\ \varphi(\underset{\sim}{r},\underset{\sim}{\Omega}) = \frac{c}{4\pi}\ \rho(\underset{\sim}{r}) + q(\underset{\sim}{r},\underset{\sim}{\Omega}) \qquad \underset{\sim}{r} \in V$$

$$\varphi(\underset{\sim}{r}_s,\underset{\sim}{\Omega}) = \varphi_s(\underset{\sim}{r}_s,\underset{\sim}{\Omega}) \qquad \underset{\sim}{r}_s \in S,\ \underset{\sim}{\Omega} \text{ inward} \tag{2.1}$$

Here

$$\rho(\underset{\sim}{r}) \equiv \iint d\underset{\sim}{\Omega}\ \varphi(\underset{\sim}{r},\underset{\sim}{\Omega}) \tag{2.2}$$

while $q(\underset{\sim}{r},\underset{\sim}{\Omega})$ for $\underset{\sim}{r} \in V$ and $\varphi_s(\underset{\sim}{r}_s,\underset{\sim}{\Omega})$ for $\underset{\sim}{r}_s \in S$, $\underset{\sim}{\Omega}$ inward to V, are assumed to be given.

To express (2.1) as an integral equation, let $G(\underset{\sim}{r} - \underset{\sim}{r}',\underset{\sim}{\Omega})$, defined for all $\underset{\sim}{r}$, $\underset{\sim}{r}'$, satisfy

$$(-\underset{\sim}{\Omega}\cdot\underset{\sim}{\nabla} + 1)\ G(\underset{\sim}{r} - \underset{\sim}{r}',\underset{\sim}{\Omega}) = \delta(\underset{\sim}{r} - \underset{\sim}{r}') \tag{2.3}$$

Then, by the conventional argument* (2.1), and (2.3) imply

*cf., Section III.

$$\varphi(\underset{\sim}{r},\underset{\sim}{\Omega}) = \int_V dV'\, G(\underset{\sim}{r}' - \underset{\sim}{r},\underset{\sim}{\Omega})[\frac{c}{4\pi}\,\rho(\underset{\sim}{r}') + q(\underset{\sim}{r}',\underset{\sim}{\Omega})] + \underset{\sim}{\Omega}\cdot\int_S \hat{n}_i dS'\, G(\underset{\sim}{r}'_s - \underset{\sim}{r},\underset{\sim}{\Omega})\, \varphi_s(\underset{\sim}{r}'_s,\underset{\sim}{\Omega}) \tag{2.4}$$

where $\hat{n}_i$ is the inward normal to V. Note we have put the known function $\varphi_s(\underset{\sim}{r}_s,\underset{\sim}{\Omega})$ in the integrand, instead of $\varphi(\underset{\sim}{r}_s,\underset{\sim}{\Omega})$; this is justified by the easily verified fact that

$$G(\underset{\sim}{r}_s - \underset{\sim}{r},\underset{\sim}{\Omega}) = 0 \quad \text{for } \underset{\sim}{r} \in V,\ \underset{\sim}{\Omega} \text{ outward} \tag{2.5}$$

Equation (2.4) holds of course only for $\underset{\sim}{r} \in V$, the domain of definition of $\varphi(\underset{\sim}{r},\underset{\sim}{\Omega})$. We now extend this domain by assuming equation (2.4) to hold for all $\underset{\sim}{r}$, so that we may take its three-dimensional Fourier transform. [Note that it is not obvious how to extend the original differential equation (2.1) without contradicting the boundary conditions.] This takes the form

$$\tilde{\varphi}(\underset{\sim}{k},\underset{\sim}{\Omega}) = \frac{c}{4\pi}\,\frac{1}{1 - i\underset{\sim}{k}\cdot\underset{\sim}{\Omega}}\,\tilde{\rho}_V(\underset{\sim}{k}) + \frac{\tilde{q}(\underset{\sim}{k},\underset{\sim}{\Omega})}{1 - i\underset{\sim}{k}\cdot\underset{\sim}{\Omega}} + \frac{\underset{\sim}{\Omega}}{1 - i\underset{\sim}{k}\cdot\underset{\sim}{\Omega}}\cdot\int_S \hat{n}_i dS' e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}'_s}\,\varphi_s(\underset{\sim}{r}'_s,\underset{\sim}{\Omega}) \tag{2.6}$$

where

$$\tilde{\rho}_V(\underset{\sim}{k}) \equiv \int_V dV' e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}'}\,\rho(\underset{\sim}{r}') \tag{2.7}$$

In deriving (2.6), we used the representation

$$G(\underset{\sim}{r} - \underset{\sim}{r}',\underset{\sim}{\Omega}) = \frac{1}{(2\pi)^3}\int d^3k\, \frac{e^{-i\underset{\sim}{k}\cdot(\underset{\sim}{r}-\underset{\sim}{r}')}}{1 + i\underset{\sim}{k}\cdot\underset{\sim}{\Omega}} \tag{2.8}$$

which follows easily from (2.3), and made the convenient definition

$$q(\underset{\sim}{r},\underset{\sim}{\Omega}) = 0 \quad \underset{\sim}{r} \notin V \tag{2.9}$$

Upon integrating (2.6) over all directions $\underset{\sim}{\Omega}$, we obtain our basic equation

$$\tilde{\rho}(\underset{\sim}{k}) = [1 - \tilde{\Lambda}(\underset{\sim}{k})]\, \tilde{\rho}_V(\underset{\sim}{k}) + \tilde{B}(\underset{\sim}{k}) + \tilde{Q}(\underset{\sim}{k}) \tag{2.10}$$

where

$$\tilde{\Lambda}(\underset{\sim}{k}) \equiv 1 - \frac{c}{4\pi} \iint \frac{d\underset{\sim}{\Omega}}{1 - i\underset{\sim}{k}\cdot\underset{\sim}{\Omega}} \tag{2.11}$$

is the three-dimensional dispersion function and

$$\tilde{B}(\underset{\sim}{k}) \equiv \int_S dS' e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}'_s} \iint d\underset{\sim}{\Omega} \frac{\hat{n}_i \cdot \underset{\sim}{\Omega}}{1 - i\underset{\sim}{k}\cdot\underset{\sim}{\Omega}}\, \varphi_s(\underset{\sim}{r}'_s, \underset{\sim}{\Omega}) \tag{2.12}$$

$$\tilde{Q}(\underset{\sim}{k}) \equiv \iint d\underset{\sim}{\Omega} \frac{\tilde{q}(\underset{\sim}{k}, \underset{\sim}{\Omega})}{1 - i\underset{\sim}{k}\cdot\underset{\sim}{\Omega}} \tag{2.13}$$

result of course from the (given) boundary and source contributions, respectively.

We see from (2.6) that our problem in $\underset{\sim}{k}$-space is solved once we find the function $\tilde{\rho}_V(\underset{\sim}{k})$; this function will be determined from (2.10).

Our method of solving (2.10) consists essentially in the observation that

$$\tilde{\rho}_V(\underset{\sim}{k}) = \int d\underset{\sim}{k}'\, \tilde{\rho}(\underset{\sim}{k}')\Delta_V(\underset{\sim}{k} - \underset{\sim}{k}') \tag{2.14}$$

where

$$\Delta_V(\underset{\sim}{k} - \underset{\sim}{k}') \equiv \frac{1}{(2\pi)^3} \int_V dV' e^{i(\underset{\sim}{k}-\underset{\sim}{k}')\cdot\underset{\sim}{r}'} \tag{2.15}$$

so that (2.10) can be written, in general, as an integral equation for $\tilde{\rho}(\underset{\sim}{k})$. The geometry of a particular problem enters solely through the kernel, $\Delta_V(\underset{\sim}{k})$; this will be, for the simple problems we consider, highly singular, i.e., a generalized function [when V = all space, for example, $\Delta_V(\underset{\sim}{k}) = \delta^{(3)}(\underset{\sim}{k})$], with the result that our solutions of (2.10) will depend more on analyticity arguments than on the Fredholm theory. Therefore, before proceeding further, we briefly examine the properties of $\tilde{\Lambda}(\underset{\sim}{k})$ as an analytic function of, say, k_x.

The Dispersion Function

The definition (2.11) may be written

$$\tilde{\Lambda}(\underset{\sim}{k}) = 1 - \frac{c}{4\pi}\int_{-1}^{1} d\mu \int_{0}^{2\pi} d\varphi \, \frac{1}{1 - ik_x\mu - iB\cos(\varphi - \Delta)} \tag{2.16}$$

where

$$B \equiv |\vec{k}| = \sqrt{k_y^2 + k_z^2} \tag{2.17}$$

and

$$\Delta \equiv \tan^{-1}\frac{k_z}{k_y} \tag{2.18}$$

The integrals are known* and we find, for real k_x and B,

$$\tilde{\Lambda}(\underset{\sim}{k}) = \Lambda_3(k_x, B) \qquad \mathrm{Im}(k_x) = 0 \tag{2.19}$$

where

$$\Lambda_3(k_x, B) \equiv 1 + \frac{ic}{2\sqrt{k_x^2 + B^2}} \ln \frac{1 + i\sqrt{k_x^2 + B^2}}{1 - i\sqrt{k_x^2 + B^2}} \tag{2.20}$$

Note that for $\mathrm{Im}(k_x)$ sufficiently large,

$$\tilde{\Lambda}(\underset{\sim}{k}) \neq \Lambda_3(k_x, B)$$

$\tilde{\Lambda}(\underset{\sim}{k})$ is in fact quite pathological as a function of k_x outside a certain neighborhood of the real k_x axis; since our "basic" equations are true for real k_x,

*See, for example, Dwight No. 446 and No. 380.001.

we always can, and will, use $\Lambda_3(k_x,B)$, which can be taken to be defined by (2.20) for complex k_x also.

We will use the notation

$$\Lambda_3(k_x,B) = \Lambda_3(k)$$

where

$$k \equiv k_x$$

$\Lambda_3(k)$ has fairly simple analytic properties: its only singularities are branch points at

$$k = \pm i\beta$$

where

$$\beta \equiv \sqrt{B^2 + 1} \qquad (2.21)$$

and we will take the branch cuts, $\ell_{\pm}$, as extending to $\pm$ i∞ along the imaginery axis (cf., Figure 3). Thus $\Lambda_3(k)$ is analytic in the plane cut along ℓ_+ and ℓ_-.

Using well-known[(14)] properties of the one-dimensional dispersion function,

$$\Lambda_1(k) \equiv \Lambda_3(k,B=0) \qquad (2.22)$$

it is easy to show that $\Lambda_3(k)$ has only two simple zeroes. If we define ν_o in the conventional way*

$$\Lambda_1(\pm\, i/\nu_o) = 0 \qquad (2.23)$$

*Note $\Lambda_1(i/\nu) = \Lambda(\nu)$, where $\Lambda(\nu)$ is the function defined by Case.[(14)]

then the roots of $\Lambda_3(k)$ are clearly given by

$$\Lambda_3(\pm i\kappa_0) = 0$$

where

$$\kappa_0 \equiv \sqrt{B^2 + 1/\nu_0^2} \qquad (2.24)$$

It also follows from the one-dimensional theory that

(i) $c < 1 \Rightarrow |\kappa_0| > B$ and $\mathrm{Im}(\kappa_0) = 0$

(ii) $c > 1$ and $|B| > |1/\nu_0| \Rightarrow |\kappa_0| < B$ and $\mathrm{Im}(\kappa_0) = 0$

(iii) $c > 1$ and $B < |1/\nu_0| \Rightarrow |\kappa_0| < B$ and $\mathrm{Re}(\kappa_0) = 0$

From the known fact that $|\nu_0| > 1$ for any c, we may conclude that the roots of $\Lambda_3(k)$ are never located on its cuts, i.e.,

$$\mathrm{Im}(\kappa_0) = 0 \Rightarrow |\kappa_0| < \beta \qquad (2.25)$$

The only other properties of $\Lambda_3(k)$ which we will need are both evident from (2.20):

$$\Lambda_3(k) = \Lambda_3(-k) \qquad (2.26)$$

and

$$\Lambda_3(k) \xrightarrow[|k|\to\infty]{} 1 \qquad (2.27)$$

Factorization of $\Lambda_3(k)$

In Part 1 of Section II, we performed on the function H(k) a factorization appropriate to the Wiener-Hopf technique, essentially by inspection. Here we obtain an analagous representation for $\Lambda_3(k)$ by a more complicated but conventional[12] method.

It is clear that (any branch of) the function

$$L(k) \equiv \ln\left[\frac{\Lambda_3(k)(k^2 + \beta^2)}{(k^2 + \kappa_o^2)}\right] \tag{2.28}$$

is analytic in the region R, as shown in Figure 3, and that

$$L(k) \xrightarrow[\substack{k\to\infty \\ k\in R}]{} 0 \tag{2.29}$$

Hence, by Cauchy's theorem,

$$L(k) = L_+(k) + L_-(k) \qquad k \in R \tag{2.30}$$

where

$$L_{\pm}(k) = \frac{1}{2\pi i} \int_{\gamma_{\mp}} \frac{L(k')}{k' - k} dk' \tag{2.31}$$

Here the contours $\gamma_{\mp}$ are close to, but not coincident with, the cuts $\ell_{\mp}$ [see Figure 3; with this definition, the $L_{\pm}(k)$ are well defined even for $k \in \ell_{\mp} \not\in R$].

Equations (2.28) and (2.30) imply that

$$\Lambda_3(k) = \frac{k^2 + \kappa_o^2}{k^2 + \beta^2} e^{L_+(k)+L_-(k)} \qquad k \in R \tag{2.32}$$

so that we may write

$$\Lambda_3(k) = \frac{\Lambda_+(k)}{\Lambda_-(k)} \qquad k \in R \tag{2.33}$$

where the $\Lambda_{\pm}(k)$ are given by

$$\Lambda_+(k) = \frac{k + i\kappa_o}{k + i\beta} e^{L_+(k)} \tag{2.34}$$

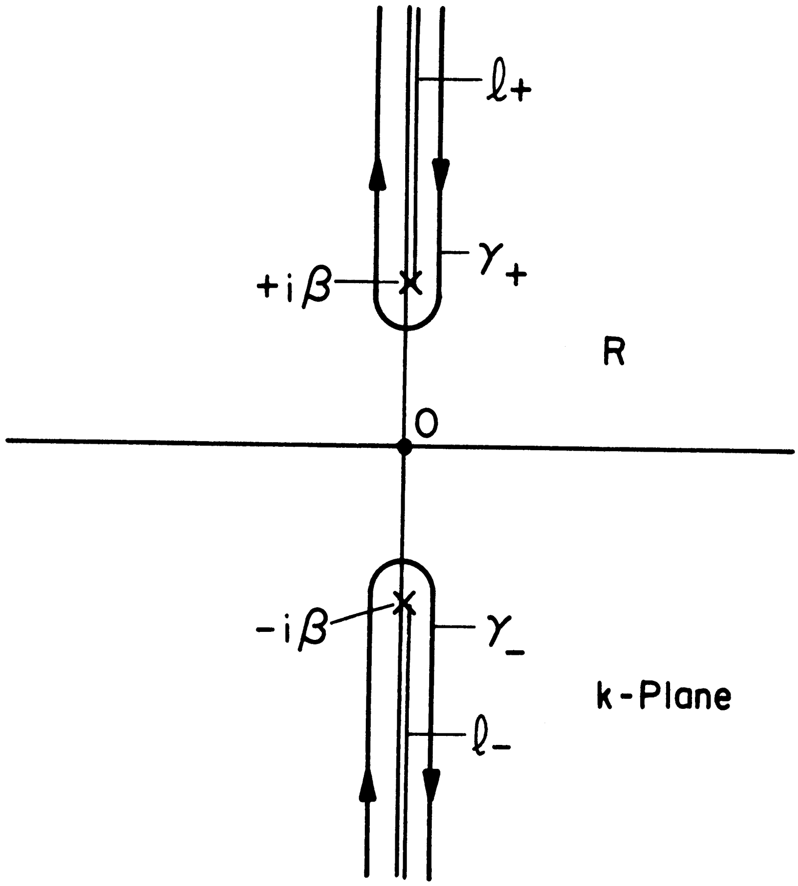

Figure 3. The region R of Section IV.2. The region R includes the entire plane except for the small neighborhoods of $\ell_\pm$ enclosed by $\gamma_\pm$.

$$\Lambda_-(k) = \frac{k - i\beta}{k - i\kappa_o} e^{-L_-(k)} \tag{2.35}$$

and clearly have the analyticity properties implied by the subscripts; equation (2.33) comprises a Wiener-Hopf factorization of $\Lambda_3(k)$.

Note that

$$\Lambda_\pm(k) \xrightarrow[k\to\infty]{} 1 \tag{2.36}$$

and, as follows easily from (2.26),

$$\Lambda_+(-k) = \frac{1}{\Lambda_-(k)} \tag{2.37}$$

Except for the path $\gamma_{\mp}$ [on which $L_{\pm}(k)$ is not defined] and the obvious poles, $\Lambda_{\pm}(k)$ is defined by equations (2.34) and (2.35) for all complex k. We now derive representations which are valid only in specified regions, but which we will eventually find useful. The argument consists wholly of applications of Cauchy's theorem.

For $k \in R$, it is evident that

$$\frac{1}{2\pi i} \int_{\gamma_{\mp}} \frac{\ln\left[\dfrac{k'^2 + \beta^2}{k'^2 + \kappa_o^2}\right] dk'}{k' - k} = 0 \qquad (2.38)$$

whence

$$L_{\pm}(k) = \Gamma_{\pm}(k) \qquad k \in R \qquad (2.39)$$

where

$$\Gamma_{\pm}(k) = \frac{1}{2\pi i} \int_{\gamma_{\mp}} \frac{\ln \Lambda_3(k')}{k' - k} dk' \qquad (2.40)$$

On the other hand, for $k \notin R$ (i.e., $k \in \ell_+$ or $k \in \ell_-$),

$$L_+(k) + L_-(k) = 0 \qquad k \notin R \qquad (2.41)$$

$$\Rightarrow \frac{\Lambda_+(k)}{\Lambda_-(k)} = \frac{k^2 + \kappa_o^2}{k^2 + \beta^2} \qquad k \notin R \qquad (2.42)$$

In particular,

$$L_+(k) = -L_-(k) = \Gamma_+(k) \quad \text{for } k \in \ell_+ \qquad (2.43)$$

while

$$L_+(k) = -L_-(k) = \Gamma_+(k) - \ln\left[\frac{k^2 + \beta^2}{k^2 + \kappa_o^2}\right] \text{ for } k \in \ell_- \qquad (2.44)$$

From (2.40) through (2.44) we deduce the relations

$$\Lambda_-(k) = \frac{k + i\kappa_o}{k + i\beta} e^{\Gamma_+(k)} \qquad k \in \ell_- \qquad (2.45)$$

$$\Lambda_+(k) = \frac{k + i\kappa_o}{k + i\beta} e^{\Gamma_+(k)} \qquad k \in \ell_+ \qquad (2.46)$$

Incidentally and finally, we note the identities

$$X(z) = \nu_o \frac{\Lambda_+(-i/z)e^{-\Gamma_+(0)}}{z - \nu_o} \qquad (B = 0) \qquad (2.47)$$

$$X(\nu_o) = \frac{e^{\Gamma_+(-i/\nu_o)-\Gamma_+(0)}}{1 - \nu_o} \qquad (B = 0) \qquad (2.48)$$

where X(z) is the function defined by Case.[14]

We are now prepared to apply our general method to the solution of specific transport problems, the simplest of which is clearly

3. THE INFINITE SPACE GREEN'S FUNCTION

Here, the region V includes all space, there is a point source at $\underset{\sim}{r}_o$ with direction $\underset{\sim}{\Omega}_o$,

$$q(\underset{\sim}{r},\underset{\sim}{\Omega}) = \delta(\underset{\sim}{r} - \underset{\sim}{r}_o)\, \delta(\underset{\sim}{\Omega} - \underset{\sim}{\Omega}_o)$$

and no boundary contribution. Our basic equation (2.10) thus takes the form

$$\tilde{\rho}_G(\underset{\sim}{k}) = [1 - \tilde{\Lambda}(\underset{\sim}{k})]\, \tilde{\rho}_G(\underset{\sim}{k}) + \frac{e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}_o}}{1 - i\underset{\sim}{k}\cdot\underset{\sim}{\Omega}_o}$$

or

$$\tilde{\rho}_G(\underset{\sim}{k}) = \frac{1}{\tilde{\Lambda}(\underset{\sim}{k})} \frac{e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}_o}}{1 - i\underset{\sim}{k}\cdot\underset{\sim}{\Omega}_o} \qquad (3.1)$$

so that, from (2.6),

$$\tilde{\varphi}_G(\underset{\sim}{k},\underset{\sim}{\Omega}) = \frac{c}{4\pi} \frac{e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}_o}}{(1 - i\underset{\sim}{k}\cdot\underset{\sim}{\Omega})(1 - i\underset{\sim}{k}\cdot\underset{\sim}{\Omega}_o)\,\tilde{\Lambda}(\underset{\sim}{k})} + \frac{e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}_o}\,\delta(\underset{\sim}{\Omega} - \underset{\sim}{\Omega}_o)}{(1 - i\underset{\sim}{k}\cdot\underset{\sim}{\Omega}_o)} \qquad (3.2)$$

and we have merely (!) to perform the inverse Fourier transformation. For the present, we examine only the inverse transformation with respect to $k_x = k$. Using the notations discussed in III.1, and equation (2.19), we have

$$\hat{\varphi}_G(x,\vec{k};\underset{\sim}{\Omega}) = -\frac{e^{i\vec{k}\cdot\vec{r}_o}}{2\pi} \int_{-\infty}^{\infty} dk\, \frac{e^{-ik(x-x_o)}}{\mu\mu_o} \left\{\frac{c}{4\pi} \frac{1}{(k - \omega)(k - \omega_o)\,\Lambda_3(k)} + \frac{\delta(\underset{\sim}{\Omega} - \underset{\sim}{\Omega}_o)}{k - \omega_o}\right\}$$

(3.3)

where

$$\omega_o \equiv \omega(\vec{\Omega}_o) \qquad (3.4)$$

This integral is easily reduced to a useful form. Assuming for definiteness that $x > x_o$, we close the contour in the lower half plane by means of a path which excludes the cut ℓ_- (cf., Figure 4), and apply Cauchy's theorem. By the relation (2.25), our path will always enclose the pole of $[\Lambda_3(k)]^{-1}$ at $-i\kappa_o$; we may have additional pole contributions from $\omega(\omega_o)$, depending upon the sign of $\mu(\mu_o)$, since

$$\mathrm{Im}(\omega) \gtrless 0 \Leftrightarrow \mu \lessgtr 0 \qquad (3.5)$$

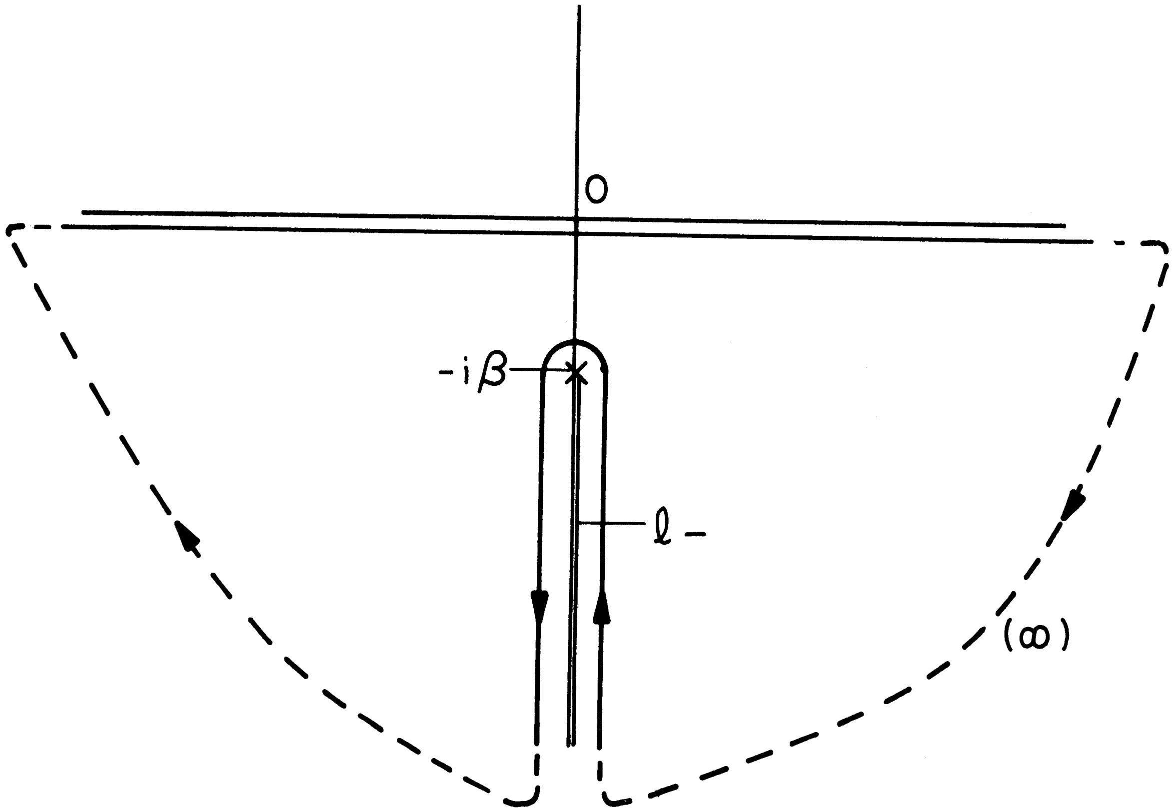

Figure 4. An inverse Fourier transform integration contour for Section IV.

as is clear from (1.3), and provided that $\omega(\omega_o) \notin \ell_-$. The result is that

$$\begin{aligned}
\hat{\phi}_G(x,\vec{k};\underset{\sim}{\Omega}) = \frac{e^{i\vec{k}\cdot\vec{r}_o}}{\mu\mu_o}\Bigg\{ & \frac{ic}{4\pi}\,\frac{e^{-\kappa_o(x-x_o)}}{(i\kappa_o+\omega)(i\kappa_o+\omega_o)\left.\frac{\partial\Lambda_3}{\partial k}\right|_{-i\kappa_o}} \\
& - \frac{1}{2\pi}\,\frac{c}{4\pi}\int_{\gamma_-} dk\,\frac{e^{-ik(x-x_o)}}{(k-\omega)(k-\omega_o)\Lambda_3(k)} \\
& + \left[\textcircled{H}(\mu)\,\frac{ic}{4\pi}\,\frac{e^{-i\omega(x-x_o)}}{(\omega-\omega_o)\,\Lambda_3(\omega)} + \textcircled{H}(\mu_o)\,\frac{ic}{4\pi}\,\frac{e^{-i\omega_o(x-x_o)}}{(\omega_o-\omega)\,\Lambda_3(\omega_o)} \right] \\
& + \textcircled{H}(\mu_o)\,\mu_o\,\delta(\underset{\sim}{\Omega}-\underset{\sim}{\Omega}_o)\,e^{-i\omega_o(x-x_o)} \Bigg\}
\end{aligned} \qquad (3.6)$$

for $x > x_o$, ω, $\omega_o \in R$.

We remark that

(i) if $\omega(\omega_0) \in \ell_-$, which occurs, in particular, when $\vec{k} = 0$, the only change in the above expression is that the first (second) term in brackets does not appear.

(ii) the last term represents merely the uncollided source contribution and is of little interest while

(iii) the first and second terms give the so-called[14] discrete and continuum "modes," respectively. Note that, by (2.25), the former always dominate for large x. Hence, for an asymptotic approximation, we could ignore the unwieldy integral term.

(iv) we can most easily compute an analagous representation for $\hat{\rho}_G(x,\vec{k})$, not from the definition (2.2), but directly from equation (3.1). In fact the integrals over $\underset{\sim}{\Omega}$ of our solutions $\hat{\varphi}$ will in general be rather complicated, but it is an obvious feature of the present method that such integrals need never be explicitly evaluated.

4. HALF-SPACE PROBLEMS

If the region V is the half-space lying to the right of the x = 0 plane, then*

$$\tilde{\rho}_V(\underset{\sim}{k}) = \frac{1}{(2\pi)^3} \int_0^\infty dx' \, e^{ik_x x'} \int_{-\infty}^\infty d\vec{r}' \, e^{i\vec{k}\cdot\vec{r}'} \tilde{\rho}(x',\vec{r}') = \tilde{\rho}_+(k) \qquad (4.1)$$

*Note that equation (4.1) could also have been obtained by noting that, for the half-space,

$$\Delta_V(\underset{\sim}{k} - \underset{\sim}{k}') = \frac{\delta(k_y - k_y')\,\delta(k_z - k_z')}{2\pi i(k_x' - k_x - i0)}$$

and using equation (I.5).

where $k \equiv k_x$ and we omit reference to the transverse variables $\vec{k}$. (Throughout this section subscripts will refer to regions of analyticity with respect to the $k_x = k$ variable; k_y and k_z are always assumed to be real.) Our basic equation (2.10) becomes in this case

$$\tilde{\rho}(k) = [1 - \Lambda_3(k)]\,\tilde{\rho}_+(k) + \tilde{B}(k) + \tilde{Q}(k) \qquad (k \text{ real}) \qquad (4.2)$$

Decomposing $\tilde{\rho}(k)$ according to (I.2), and using the factorization of $\Lambda_3(k)$ provided by equations (2.34) and (2.35), we can write (4.2) in the form

$$\tilde{\rho}_+(k)\,\Lambda_+(k) + \tilde{\rho}_-(k)\,\Lambda_-(k) = [\tilde{B}(k) + \tilde{Q}(k)]\,\Lambda_-(k) \qquad (k \text{ real}) \qquad (4.3)$$

Assuming that $\tilde{B}$ and $\tilde{Q}$ satisfy a Hölder condition, equations of the form of (4.3) can always be solved by a well-known[3] generalization of the Wiener-Hopf technique. We let

$$f(k) \equiv \begin{cases} \tilde{\rho}_+(k)\,\Lambda_+(k) & \text{for } \mathrm{Im}(k) > 0 \\ -\tilde{\rho}_-(k)\,\Lambda_-(k) & \text{for } \mathrm{Im}(k) < 0 \end{cases} \qquad (4.4)$$

so that (4.3) is the statement

$$f^+(k) - f^-(k) = [\tilde{B}(k) + \tilde{Q}(k)]\,\Lambda_-(k) \qquad (k \text{ real}) \qquad (4.5)$$

where we use the notation of (I.6). We see that f(k) is analytic in the plane cut along the real axis, and has a discontinuity across the cut given by (4.5). It follows that

$$f(k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} dk'\,\frac{[\tilde{B}(k') + \tilde{Q}(k')]\,\Lambda_-(k')}{k' - k} + A \qquad [\mathrm{Im}(k) \neq 0] \qquad (4.6)$$

where the constant A is arbitrary. From (4.4) we have

$$\tilde{\rho}_+(k) = \frac{1}{2\pi i\, \Lambda_+(k)} \int_{-\infty}^{\infty} dk' \, \frac{[\tilde{B}(k') + \tilde{Q}(k')]\, \Lambda_-(k')}{k' - k - i0} + \frac{A}{\Lambda_+(k)} \tag{4.7}$$

and the general half-space problem is solved.

Note, from equation (2.36), that $\tilde{\rho}_+(\infty) = A$. Hence, for nonzero A, the density $\hat{\rho}(x,\vec{k})$ will have a δ-function singularity at x = 0, and the choice

$$A = 0 \tag{4.8}$$

clearly corresponds to taking the least singular solution as discussed in Section II.1. Aside from remarking that for $\tilde{B} = \tilde{Q} = 0$ there is no (nontrivial) least singular solution, we will generally confine out attention below to the case of equation (4.8).

The Half-Space Albedo Problem

The above remarks have their simplest application to the albedo problem, in which

$$q(\underset{\sim}{r},\underset{\sim}{\Omega}) = 0 \Rightarrow \tilde{Q}(k) = 0$$

and

$$\varphi_s(0,y,z,\underset{\sim}{\Omega}) = \delta(y)\,\delta(z)\,\delta(\underset{\sim}{\Omega} - \underset{\sim}{\Omega}_o) \qquad \mu > 0,\ \mu_o > 0$$

$$\Rightarrow \tilde{B}(k) = \frac{-i}{\omega_o - k} \tag{4.9}$$

where ω_o is defined by (3.4). Note that

$$\mu_o > 0 \Rightarrow \mathrm{Im}(\omega_o) < 0 \tag{4.10}$$

We now have merely to substitute (4.9) into the formula (4.7), and perform an elementary integral. However, in this case it is perhaps more instructive to work directly from equation (4.3), which can here be written as

$$(\omega_o - k)\, \tilde{\rho}_{a+}(k)\, \Lambda_+(k) = -\Lambda_-(k)[(\omega_o - k)\, \tilde{\rho}_{a-}(k) + i] \tag{4.11}$$

This equation, each side of which is analytic in an appropriate half-plane, is of the same form as (II.1.19). By precisely the same argument as was used in Section II, we conclude that both sides must equal a constant; and by setting $k = \omega_o$, and noting relation (4.10), we see that the constant is $-i\Lambda_-(\omega_o)$. Thus,

$$\tilde{\rho}_{a+}(k) = \frac{-i\Lambda_-(\omega_o)}{(\omega_o - k)\, \Lambda_+(k)} \tag{4.12}$$

as we obviously would also have obtained from (4.7).

With (4.12) and (2.34), we can now compute, for example

$$\hat{\rho}_a(x,\vec{k}) = \frac{i\Lambda_-(\omega_o)}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \frac{(k + i\beta)e^{L_+(k)}}{(k + i\kappa_o)(k - \omega_o)}\, dk \tag{4.13}$$

By means of the same argument as was applied to $\hat{\varphi}_G$, above, we find

$$\hat{\rho}_a(x,\vec{k}) = i\Lambda_-(\omega_o)(\beta - \kappa_o)e^{-\Gamma_+(-i\kappa_o)}\, e^{-\kappa_o x}$$
$$- \frac{i\Lambda_-(\omega_o)}{2\pi} \int_{\gamma_-'} e^{-ik'x} \frac{(k' + i\beta)}{(k' + i\kappa_o)}\, e^{-L_+(k')}\, dk' + \left[\frac{e^{-i\omega_o x}}{\Lambda_3(\omega_o)}\right] \qquad x > 0 \tag{4.14}$$

where, again, the term in brackets does not occur when $\omega_o \in \ell_-$. γ_-' refers to a path just outside the path γ_-, across which $L_+(k)$ is discontinuous.

Similarly, by (2.6),

$$\hat{\varphi}_a(x,\vec{k},\underset{\sim}{\Omega}) = e^{-i\omega_o x}\,\delta(\underset{\sim}{\Omega} - \underset{\sim}{\Omega}_o)$$

$$- \frac{1}{2\pi}\,\Lambda_-(\omega_o)\,\frac{c}{4\pi}\int_{-\infty}^{\infty} \frac{(k + i\beta)e^{-L_+(k)}e^{-ikx}}{(k - \omega_o)(k - \omega)(k + i\kappa_o)}\,dk \qquad (4.15)$$

and the integral can clearly be reduced to discrete and continuum modes, etc., in the usual manner. We will discuss the content of (4.15) in Part 6.

A Half-Space Green's Function

With the boundary condition of zero incidence at $x = 0$,

$$\varphi_s(0,\vec{r},\underset{\sim}{\Omega}) = 0 \Rightarrow \tilde{B}(k) = 0 \qquad (4.16)$$

and an isotropic point source at $(x_o,0,0)$,

$$q(\underset{\sim}{r},\underset{\sim}{\Omega}) = \frac{q}{4\pi}\,\delta(x - x_o)\,\delta(y)\,\delta(z)$$

$$\Rightarrow \tilde{Q}(\underset{\sim}{k}) = qe^{ikx_o}\,\frac{1 - \Lambda_3(k)}{c} \qquad (4.17)$$

our general half-space formula (4.7) yields

$$\tilde{\rho}_{g+}(k) = \frac{q}{2\pi ic}\,\frac{1}{\Lambda_+(k)}\int_{-\infty}^{\infty} dk'\,\frac{e^{ik'x_o}[\Lambda_-(k') - \Lambda_+(k')]}{k' - k - i0} = \frac{-q}{c}\,e^{ikx_o} + \frac{q}{c}\,e^{ikx_o}\,\frac{1}{\Lambda_3(k)}$$

$$+ \frac{q}{c}\,\frac{e^{-\kappa_o x_o}}{\left.\frac{\partial\Lambda_3}{\partial k}\right|_{i\kappa_o}}\,\frac{\Lambda_+(i\kappa_o)}{\Lambda_+(k)(i\kappa_o - k)} - \frac{q}{c}\,\frac{1}{\Lambda_+(k)}\,\frac{1}{2\pi i}\int_{\gamma_+'} dk'\,\frac{e^{ik'x_o}}{k' - k - i0}\,\Lambda_-(k') \qquad (4.18)$$

where we have closed the contour in the upper half-plane [γ_+' refers, of course, to a path bordering on γ_+, analagously to γ_-'; by affixing appropriate superscripts on $L_{\pm}(k)$, we could use the path $\gamma_{\mp}$ itself], and used the identity

$$\operatorname*{Res}_{i\kappa_0}[\Lambda_-(k)] = \operatorname*{Res}_{i\kappa_0}\left[\frac{\Lambda_+(k)}{\Lambda_3(k)}\right] = \frac{\Lambda_+(i\kappa_0)}{\left.\frac{\partial\Lambda_3}{\partial k}\right|_{i\kappa_0}} \tag{4.19}$$

The inverse transform of (4.18) contains little that is new: the first two terms are essentially the isotropic form of (3.1) and the third term is very similar to (4.12). The remaining integral term is clearly $O(e^{-\beta x_0})$ and, in the important case of large x_0, comparatively insignificant. We now consider this case in detail.

The Milne Problem

Here, we have the same boundary condition as above, but the source has been displaced to infinity, in such a way that the quantity

$$q_0 \equiv \lim_{\substack{x_0 \to\infty \\ q\to\infty}} q e^{-\kappa_0 x_0} \tag{4.20}$$

is finite and nonzero.

Using a certain amount of care with regard to the first two terms, we could read off the solution directly from (4.18); but it is somewhat more instructive to approach the problem from a different point of view, as follows.

We replace the source at ∞ by the requirement that our solution, $\rho_m(\underset{\sim}{r})$, be $O(e^{\kappa_0 x})$ for large x. This suggests the decomposition

$$\rho_m(\underset{\sim}{r}) = \rho(\underset{\sim}{r}) + \rho_\infty(\underset{\sim}{r}) \tag{4.21}$$

Here

$$\rho_\infty(\underset{\sim}{r}) = f(\vec{r})e^{\kappa_0 x} \qquad (4.22)$$

where $f(\vec{r})$, which could be determined, for example, from the infinite space Green's function, is independent of x, and

$$\rho(\underset{\sim}{r}) \xrightarrow[x\to\infty]{} 0 \qquad (4.23)$$

Our basic equation, (2.10), now may be written

$$\tilde{\rho}(\underset{\sim}{k}) + \tilde{\rho}_\infty(\underset{\sim}{k}) = [1 - \Lambda_3(\underset{\sim}{k})][\tilde{\rho}_+(k) + \tilde{\rho}_{\infty+}(k)] \qquad (4.24)$$

In view of the definition (4.22), the quantities $\tilde{\rho}_\infty(\underset{\sim}{k})$ and $\tilde{\rho}_{\infty+}(\underset{\sim}{k})$ are not obviously meaningful; indeed, in the generalized function space S' which we have hither-to implicitly used, they do not exist. But $\rho_\infty(x,\vec{r})$ is locally summable in x, and hence a well-defined generalized function in the space K' of Gel'fand.[1] It follows [assuming $f(\vec{r})$ is suitably behaved] that the Fourier transform $\tilde{\rho}_\infty(\underset{\sim}{k})$ exists as a generalized function in the space Z', and in fact

$$\tilde{\rho}_\infty(\underset{\sim}{k}) = 2\pi\, \tilde{f}(\vec{k})\, \delta(k - i\kappa_0) \qquad (k = k_x) \qquad (4.25)$$

Note that not only is the usual formula (I.1) meaningless here, but that the inverse transform

$$\hat{\rho}_\infty(x,\vec{k}) \stackrel{?}{=} \int_{-\infty}^{\infty} e^{-ikx}\, \tilde{f}(\vec{k})\, \delta(k - i\kappa_0)\, dk \qquad (4.26)$$

might seem to give zero, since the "integral" is over real k, and $i\kappa_0$ is not (in general) real*. The correct formulae are, of course, obtained from the definition

$$(\rho,\psi) = \frac{1}{2\pi}(\tilde{\rho},\tilde{\psi}) \qquad (4.27)$$

where $\psi(\tilde{\psi})$ is a test function in K(Z).

It happens in our case that $\tilde{\rho}_{\infty-}(\underset{\sim}{k})$ exists in the classical sense:

$$\tilde{\rho}_{\infty-}(\underset{\sim}{k}) = \tilde{f}(\vec{k}) \int_{-\infty}^{0} e^{i(k-i\kappa_0)x} dx = \frac{\tilde{f}(\vec{k})}{i(k - i\kappa_0)} \qquad (4.28)$$

so we may conclude that

$$\tilde{\rho}_{\infty+}(\underset{\sim}{k}) = \tilde{f}(\vec{k})[2\pi\,\delta(k - i\kappa_0) + \frac{i}{k - i\kappa_0}] \qquad (4.29)$$

Now (4.24) may be written in the form

$$[\tilde{\rho}_{\infty+}(k) + \tilde{\rho}_{+}(k)]\,\Lambda_{+}(k) = -\,\Lambda_{-}(k)[\tilde{\rho}_{\infty-}(k) + \tilde{\rho}_{-}(k)] \qquad (4.30)$$

or, using (4.29),

$$[(k - i\kappa_0)\,\tilde{\rho}_{+}(k) + i\tilde{f}(\vec{k})]\,\Lambda_{+}(k) = -\,\Lambda_{-}(k)[(k - i\kappa_0)\,\tilde{\rho}_{-}(k) - i\tilde{f}(\vec{k})] \qquad (4.31)$$

$$= i\tilde{f}(\vec{k})\,\Lambda_{+}(i\kappa_0) \qquad (4.32)$$

*The point here is that $\tilde{\rho}_{\infty}(k)$ is an analytic generalized function, the "support" of which is ambiguous. For example, it can be shown(1) that

$$\delta(k - i\kappa_0) = \sum_n \frac{\delta^{(n)}(k)(-i\kappa_0)^n}{n!} \qquad \text{(in Z')}$$

where we have obtained (4.32) by precisely the same Wiener-Hopf argument as was applied to (4.11). The result is

$$\tilde{\rho}_+(k) = i\tilde{f}(\vec{k}) \frac{\Lambda_+(i\kappa_o) - \Lambda_+(k)}{(k - i\kappa_o)\,\Lambda_+(k)} \tag{4.33}$$

whence

$$\tilde{\rho}_{m+}(k) = i\tilde{f}(\vec{k}) \left[\frac{\Lambda_+(i\kappa_o)}{(k - i\kappa_o)\,\Lambda_+(k)} - 2\pi i\, \delta(k - i\kappa_o)\right] \tag{4.34}$$

To find $\tilde{f}(\vec{k})$, we note from (4.18) that, for $x < x_o$,

$$\hat{\rho}_g(x,\vec{k}) = i\frac{q}{c} \frac{e^{-\kappa_o x_o}}{\left.\frac{\partial\Lambda_3}{\partial k}\right|_{i\kappa_o}} \left[e^{\kappa_o x}\right] + O(e^{-\kappa_o x}) + O(e^{-\beta x_o}) \tag{4.35}$$

Thus (4.34) will be consistent with the application of (4.20) to (4.18) only if

$$\tilde{f}(\vec{k}) = \frac{iq_o}{c} \frac{1}{\left.\frac{\partial\Lambda_3}{\partial k}\right|_{i\kappa_o}} \tag{4.36}$$

$$\Rightarrow \tilde{\rho}_m(k) = -\frac{q_o}{c} \frac{1}{\left.\frac{\partial\Lambda_3}{\partial k}\right|_{i\kappa_o}} \left[\frac{\Lambda_+(i\kappa_o)}{(k - i\kappa_o)\,\Lambda_+(k)} - 2\pi i\, \delta(k - i\kappa_o)\right] \tag{4.37}$$

Note that we could just as easily have obtained the above expression for $\tilde{f}(\vec{k})$ from the infinite space, isotropic source, Green's function; the method used above for finding $\tilde{\rho}_m$ is actually independent of that used for $\tilde{\rho}_g$.

From (4.37), we find, in the usual way,

$$\hat{\rho}_m(x,\vec{k}) = \frac{-iq_o}{c} \frac{1}{\left.\frac{\partial\Lambda_3}{\partial k}\right|_{i\kappa_o}} \left[\frac{\Lambda_+(i\kappa_o)(\beta - \kappa_o)e^{-\kappa_o x}}{2\kappa_o e^{L_+(-i\kappa_o)}} - e^{\kappa_o x} - \frac{1}{2\pi}\int_{\gamma_-} \frac{\Lambda_+(i\kappa_o)e^{-ikx}}{(k - i\kappa_o)\,\Lambda_+(k)}\, dk\right] \tag{4.38}$$

It is clear that there exists a point x_m such that

$$\hat{\rho}_m(x_m, \vec{k}) = O(e^{-\beta x}) \qquad (4.39)$$

i.e., the asymptotically dominant component of ρ_m vanishes at $x = x_m$. The definition (4.39) actually yields $x_m < 0$; it is conventionally[14] denoted by

$$x_m = -z_o \quad , \; z_o > 0 \qquad (4.40)$$

(z_o is called the extrapolated end-point.) Note that $\tilde{\rho}_{m-}(k)$, with which one would ordinarily compute $\hat{\rho}_m$ for $x < 0$, is not relevant here; since our original differential equation (2.1) holds only in V, only $\tilde{\rho}_V = \tilde{\rho}_+$ has physical significance. We have, from (4.38) through (4.40),

$$e^{-2\kappa_o z_o} = \frac{\Lambda_+(i\kappa_o)(\beta - \kappa_o)e^{L_+(-i\kappa_o)}}{2\kappa_o} \qquad (4.41)$$

which, with $\vec{k} = 0$, is equivalent to the known[14] one-dimensional formula.

5. SLAB PROBLEMS

When the region V is the slab defined by

$$V = (-\ell \leq x \leq \ell, \; -\infty \leq y \leq \infty, \; -\infty \leq z \leq \infty)$$

we easily find

$$\Delta_V(\underset{\sim}{k} - \underset{\sim}{k}') = \delta(\vec{k} - \vec{k}') \frac{1}{2\pi i}\left[\frac{e^{i(k-k')\ell} - e^{-i(k-k')\ell}}{k - k'}\right] \qquad (5.1)$$

$$= \delta(\vec{k} - \vec{k}') \frac{1}{2\pi i}\left[\frac{e^{i(k'-k)\ell}}{k' - k - io} - \frac{e^{-i(k'-k)\ell}}{k' - k - io}\right] \qquad (5.2)$$

where the equivalence of (5.1) and (5.2) is clear from (I.4)

We restrict our attention to the case in which the inhomogeneous term,

$$\tilde{S}(k) \equiv \tilde{B}(k) + \tilde{Q}(k)$$

satisfies

$$\tilde{S}(k) = \tilde{S}(-k) \tag{5.3}$$

so that [cf., equation (2.26)] it is permissible to assume

$$\tilde{\rho}(k) = \tilde{\rho}(-k) \tag{5.4}$$

Since, of course, any $\tilde{S}(k)$ can be decomposed into odd and even parts, and the problem solved for each part in essentially the same way, (5.3) is not a serious restriction.

Substituting this $\tilde{\rho}(k)$ into (2.14), we find, from (5.2) and a few manipulations using (I.4), that

$$\tilde{\rho}_V(k) = \tilde{\rho}(k) - e^{ik\ell} J_+(k) - e^{-ik\ell} J_+(-k) \tag{5.5}$$

where

$$J_+(k) \equiv \frac{1}{2\pi i} \int_{-\infty}^{\infty} dk' \frac{\tilde{\rho}_+(k')e^{-ik'\ell}}{k' - k - io} \tag{5.6}$$

The form of equation (5.5) suggests the following decomposition of $\tilde{\rho}(k)$:

$$\tilde{\rho}(k) = e^{-ik\ell} f(k) + e^{ik\ell} f(-k) \tag{5.7}$$

and similarly

$$\tilde{S}(k) = e^{-ik\ell}\,\sigma(k) + e^{ik\ell}\,\sigma(-k) \tag{5.8}$$

[Note that (5.8) is automatic whenever any sources present are concentrated at the boundaries.] We easily find

$$J_+(k) = f_-(-k) + \frac{1}{2\pi i}\int_{-\infty}^{\infty} dk' \, \frac{f_+(k')e^{-2ik'\ell}}{k' - k - io} \tag{5.9}$$

so that our basic equation (2.10) can be written in the form

$$\begin{aligned} f(k)e^{-ik\ell} + f(-k)e^{ik\ell} = [1 - \Lambda_3(k)]\Bigg\{ & e^{-ik\ell}\left[f_+(k) - \frac{1}{2\pi i}\int_{-\infty}^{\infty} dk' \, \frac{f_+(k')e^{-2ik'\ell}}{k' + k - io}\right] \\ & + e^{ik\ell}\left[f_+(-k) - \frac{1}{2\pi i}\int_{-\infty}^{\infty} dk' \, \frac{f_+(k')e^{-2ik'\ell}}{k' - k - io}\right]\Bigg\} \\ & + e^{-ik\ell}\,\sigma(k) + e^{ik\ell}\,\sigma(-k) \end{aligned} \tag{5.10}$$

By considering the coefficients of $e^{ik\ell}$ and $e^{-ik\ell}$ in (5.10), we conclude that if $f(k)$ satisfies

$$f(k) = [1 - \Lambda_3(k)]\left[f_+(k) - \frac{1}{2\pi i}\int_{-\infty}^{\infty} dk' \, \frac{f_+(k')e^{-2ik'\ell}}{k' + k - io}\right] + \sigma(k) \tag{5.11}$$

then a solution to the symmetric slab problem is given by (5.7).

Equation (5.11) is similar in form to (4.2); the ansatz (5.7) has, not surprisingly, reduced the slab problem to a half-space problem. We could solve (5.11) by our general half-space formula (4.7),

$$\begin{aligned} f_+(k) &= \frac{1}{2\pi i\,\Lambda_+(k)}\int_{-\infty}^{\infty} dk' \left\{\frac{\sigma(k') - [1 - \Lambda_3(k')]I_-(k')}{k' - k - io}\right\}\Lambda_-(k') \\ &= \frac{1}{2\pi i\,\Lambda_+(k)}\int_{-\infty}^{\infty} dk' \, \frac{\Lambda_-(k')\,\sigma(k') + \Lambda_+(k')\,I_-(k')}{k' - k - io} \end{aligned} \tag{5.12}$$

where

$$I_-(k) \equiv \frac{1}{2\pi i} \int_{-\infty}^{\infty} dk' \, \frac{f_+(k')e^{-2ik'\ell}}{k' + k - io} \tag{5.13}$$

except, of course, for the fact that $I_-(k')$ is not a known function. On the other hand, whenever $f_+(k)$ is analytic in a region $\{Im(k) > -\alpha; \alpha > 0\}$ it is evident that $I_-(k)$ is $O(e^{-2\alpha\ell})$, i.e., small for large ℓ. [We remark that the only important problem in which $f_+(k)$ does not have this property is the so-called critical problem; that the method to be outlined is applicable even to this "worst case" is demonstrated explicitly below.] What is clearly called for is a perturbation expansion of $f_+(k)$, the zeroth approximation being given by (5.12) with $I_- = 0$, and the nth correction term being $O(e^{-2n\alpha\ell})$. Thus we set

$$f_+(k) = \sum_{n=0}^{\infty} f_+^{(n)}(k) \tag{5.14}$$

where

$$f_+^{(o)}(k) = \frac{1}{2\pi i \, \Lambda_+(k)} \int_{-\infty}^{\infty} dk' \, \frac{\sigma(k') \, \Lambda_-(k')}{k' - k - io} \tag{5.15}$$

and

$$f_+^{(n)}(k) = \frac{1}{2\pi i \, \Lambda_+(k)} \int_{-\infty}^{\infty} dk' \, \frac{I_-^{(n-1)}(k') \, \Lambda_+(k')}{k' - k - io} \qquad n \geq 1 \tag{5.16}$$

where

$$I_-^{(n-1)}(k') \equiv \frac{1}{2\pi i} \int_{-\infty}^{\infty} dk' \, \frac{f_+^{(n-1)}(k')e^{-2ik'\ell}}{k' + k - io} \qquad n \geq 1 \tag{5.17}$$

and, assuming the series (5.14) converges, the symmetric slab problem is solved.

The iterative procedure of (5.14) through (5.17) is not as formidable as it may appear. In fact, the representation (5.14) can easily be reduced, in good approximation, to a simple power series, provided we stipulate

(i) reasonably well behaved $\sigma(k)$; for example, the "symmetric slab albedo" problem, for which $\sigma(k)$ is given by (4.9), is easily tractable.

(ii) a wide slab; explicitly, the power series obtained is correct to $O(e^{-\beta \ell})$, the means of obtaining greater accuracy being clear in theory but cumbersome in practice.

We illustrate these remarks by solving a problem the formulation of which is only slightly different.

The Critical Problem

Here we assume there is neither source nor boundary contribution,

$$\tilde{S}(k) = 0$$

and seek that value of ℓ such that there is a nonzero solution to the slab problem.

The prescription of (5.14) through (5.17) is evidently not directly applicable; in order to obtain a nontrivial solution for $\sigma = 0$, we must choose the constant A of equation (4.7) unequal to zero. With this minor modification, we have

$$f_+(k) = \sum_{n=0}^{\infty} f_+^{(n)}(k)$$

where,

$$f_{+}^{(o)}(k) = \frac{A}{\Lambda_{+}(k)} \tag{5.18}$$

and the $f_{+}^{(n)}(k)$ are given by (5.16) and (5.17).

Note that $f_{+}^{(o)}(k)$ is analytic for $\mathrm{Im}(k) > -\beta$ except for a pole at $-i\kappa_o$. [We anticipate that for criticality, $c \geq 1$, so that κ_o may be pure imaginery. Note that in this case the pole of (5.18) is to be interpreted as lying just below the real axis, as discussed in Section I.]. By induction, we assume that $f_{+}^{(n-1)}(k)$ also has this property, whence

$$I_{-}^{(n-1)}(k) = -\frac{R^{(n-1)}(\kappa_o)}{k - i\kappa_o - io} e^{-2\kappa_o \ell} + O(e^{-2\beta\ell}) \tag{5.19}$$

where

$$R^{(n-1)}(\kappa_o) \equiv \underset{-i\kappa_o}{\mathrm{Res}}\; f_{+}^{(n-1)}(k) \tag{5.20}$$

Now, from (5.16),

$$f_{+}^{(n)}(k) = -\frac{R^{(n-1)}(\kappa_o)e^{-2\kappa_o\ell}}{(k - i\kappa_o)\,\Lambda_{+}(k)}[\Lambda_{+}(k) - \Lambda_{+}(i\kappa_o)] + O(e^{-2\beta\ell}) \tag{5.21}$$

and we observe that: (a), the induction hypothesis is fulfilled to $O(e^{-2\beta\ell})$; and (b), the error term for each $I_{-}^{(n)}$ will be of the same form as that for $I_{-}^{(o)}$:

$$I_{-}^{(o)} = \frac{A}{2\pi i}\int_{\gamma-} dk' \frac{e^{-2ik'\ell}}{\Lambda_{+}(k')(k' - k - io)} \tag{5.22}$$

Hence a systematic means of obtaining more accurate solutions to our problem clearly lies in an appropriate asymptotic expansion of the integral (5.22).

With the abbreviation

$$R(\kappa_o) \equiv \underset{-i\kappa_o}{\mathrm{Res}}\left[\frac{1}{\Lambda_+(k)}\right] = i(\beta - \kappa_o)e^{-L_+(-i\kappa_o)} \tag{5.23}$$

it follows from (5.21) that

$$R^{(n)}(\kappa_o) = - e^{-2\kappa_o \ell}\,\frac{\Lambda_+(i\kappa_o)}{2i\kappa_o}\,R(\kappa_o)\;R^{(n-1)}(\kappa_o) \tag{5.24}$$

$$= r\,R^{(n-1)}(\kappa_o) \tag{5.25}$$

$$= r^n\,R^{(o)}(\kappa_o) = Ar^n\,R(\kappa_o) \tag{5.26}$$

where

$$r \equiv \frac{R^{(n)}}{R^{(n-1)}} = - \frac{e^{-2\kappa_o \ell}\;\Lambda_+(i\kappa_o)}{2i\kappa_o}\,R(\kappa_o) \tag{5.27}$$

so that, to $O(e^{-2\beta\ell})$, (5.14) is the power series given by

$$f_+(k) = f_+^{(o)}(k) - R^{(o)}(\kappa_o)\;e^{-2\kappa_o \ell}\left[\frac{\Lambda_+(k) - \Lambda_+ i\kappa_o)}{\Lambda_+(k)(k - i\kappa_o)}\right]\sum_{m=0}^{\infty} r^m \tag{5.28}$$

$$= \frac{A}{\Lambda_+(k)} - \frac{AR(\kappa_o)e^{-2\kappa_o \ell}}{1 - r}\left[\frac{\Lambda_+(k) - \Lambda_+(i\kappa_o)}{\Lambda_+(k)(k - i\kappa_o)}\right] \tag{5.29}$$

Of course (5.29) follows from (5.28) in general only for $|r| < 1$. However, it is not hard to verify by direct substitution that the $f_+(k)$ of (5.29) is in fact an appropriately approximate solution to (5.11) for any $r \neq 1$ [of course r must be given by (5.27)].

It is convenient to choose $A = (1 - r)$, so that

$$f_+(k) = \frac{(1 - r)}{\Lambda_+(k)} - R(\kappa_o)\;e^{-2\kappa_o \ell}\left[\frac{\Lambda_+(k) - \Lambda_+(i\kappa_o)}{\Lambda_+(k)(k - i\kappa_o)}\right] \tag{5.30}$$

It is now clear [cf., our discussion of equation (4.7) above] that the density obtained from (5.30) and (5.7) must have the form

$$\hat{\rho}(x,k) = (1 - r)[\delta(x + \ell) + \delta(x - \ell)] + \text{regular terms} \qquad (5.31)$$

and we conclude that the only nontrivial solutions to the homogeneous slab problem are singular, unless $r = 1$.

We now consider this last case, which is the critical case, in detail. When $r = 1$ we have the (nonsingular) critical solution given by (5.7) and*

$$f_{c+}(k) = (\text{const.})\,\frac{\Lambda_+(k) - \Lambda_+(i\kappa_o)}{\Lambda_+(k)(k - i\kappa_o)} \qquad (5.32)$$

Keeping in mind the fact that both multiplicative constants and, in our approximation, terms of $O(e^{-\beta\ell})$ are irrelevant, it is a simple matter to obtain the inverse ($k_x = k$) Fourier transform of the density as given by (5.7) and (5.32):

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}[e^{-ik\ell} f_{c+}(k)]dk \propto e^{-\kappa_o(x+\ell)} + O(e^{-\beta\ell}) \qquad (x > -\ell) \quad (5.33)$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}[e^{ik\ell} f_{c+}(-k)]dk \propto e^{\kappa_o(x-\ell)} + O(e^{-\beta\ell}) \qquad (x < \ell) \qquad (5.34)$$

whence

$$\rho_c(x,k) = (\text{const.}) \cos|\kappa_o|x \qquad (5.35)$$

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*Cf., equation (4.33).

The somewhat more interesting criticality condition on ℓ is obtained of course from (5.23) and (5.27):

$$r = 1 \Rightarrow -\frac{e^{-2\kappa_o \ell}\Lambda_+(i\kappa_o)(\beta - \kappa_o)}{2\kappa_o e^{L_+(-i\kappa_o)}} = 1 \tag{5.36}$$

Using (4.41), this can be written as

$$e^{-2\kappa_o(\ell+z_o)} = -1 \tag{5.37}$$

so that κ_o must be pure imaginery, as we have assumed, and the critical value of ℓ is

$$\ell_c = \frac{\pi}{2|\kappa_o|} - z_o \tag{5.38}$$

Both the results (5.35) and (5.38) are well-known.[14]

6. THE $\vec{k}$-DEPENDENCE

We have always been able, above, to bring the inverse Fourier transform with respect to k_x of our solutions into fairly workable form [cf., for example, equations (3.6) and (4.38)]. On the other hand, the dependence upon $\vec{k} = (k_y, k_z)$ of such functions as Λ_+ is evidently too complicated for the inverse transforms with respect to $\vec{k}$ to be performed by any simple analytical procedure. The fact that useful information can be nonetheless obtained, without recourse to numerical techniques, from the solutions given above in $(x,\vec{k})$ space, is here briefly illustrated.

Our remarks are based on the identity

$$\hat{f}(x,\vec{k}) = \int f(\underset{\sim}{r})e^{i\vec{k}\cdot\vec{r}}d\vec{r} = \sum_{n,m=0}^{\infty} \frac{(ik_y)^n(ik_z)^m}{n!\ m!} \int d\vec{r}\ y^n z^m\ f(\underset{\sim}{r}) \tag{6.1}$$

from which it follows that an expansion of the (known) function $\hat{f}$ in powers of k_y and k_z immediately yields the moments of f with respect to y and z. These moments, which uniquely determine $f(\underset{\sim}{r})$, are generally of physical interest. In particular, for n = m = 0 we have

$$\hat{f}(x,\vec{0}) = \bar{f}(x) \tag{6.2}$$

where

$$\bar{f}(x) \equiv \int d\vec{r}\ f(x,\vec{r}) \tag{6.3}$$

is the average of f over $\vec{r}$.

Of course the relation "inverse" to (6.1),

$$f(\underset{\sim}{r}) = \frac{1}{(2\pi)^2} \sum_{n,m=0}^{\infty} y^n z^m \int d\vec{k}\ \frac{(-ik_y)^n(-ik_z)^m}{n!\ m!}\ \hat{f}(x,\vec{k}) \tag{6.4}$$

would analagously furnish a power series for $f(\underset{\sim}{r})$, if we could compute the moments with respect to $\vec{k}$ of $\hat{f}$. Since, however, the latter computation is apparently no easier than computing $f(\underset{\sim}{r})$ directly, we do not pursue this point.

The formula (6.2) can be immediately applied, for example, to equation (4.15). After some routine calculations [as in the derivation of equation (3.6)], we find that the $\vec{r}$-average of the albedo angular density is asymptotically given by

$$\bar{\varphi}_a(x,\underset{\sim}{\Omega}) = \delta(\underset{\sim}{\Omega} - \underset{\sim}{\Omega}_o)e^{-x/\mu_o} + \frac{c}{4\pi}\frac{(\nu_o - 1)\mu_o e^{\gamma(\mu_o)-\gamma(\nu_o)}}{(1 - \mu_o)(\nu_o - \mu)}\ e^{-x/\nu_o} + O(e^{-x}) \tag{6.5}$$

where ν_o is defined by (2.23), and

$$\gamma(\mu) \equiv \Gamma_+(-i/\mu) - \Gamma_+(0) \qquad (B = 0) \tag{6.6}$$

The emergent angular density, $\overline{\varphi_a}(0,\underset{\sim}{\Omega})$, is even easier to compute, since for $x = 0$ we may close the contour of (4.15) in the upper half k-plane, with the (exact) result that

$$\overline{\varphi}(0,\underset{\sim}{\Omega}) = \delta(\underset{\sim}{\Omega} - \underset{\sim}{\Omega}_o) - \frac{c}{4\pi} \frac{\textcircled{H}(-\mu)\, \mu_o(\mu_o - \nu_o)(\mu - 1) e^{\gamma(\mu_o)-\gamma(\mu)}}{(\mu - \mu_o)(\mu_o - 1)(\mu - \nu_o)} \tag{6.7}$$

Here we have used the representations (2.45) and (2.46). (Note that $0 < \mu_o < 1 \Rightarrow -i/\mu_o \in \ell_-$.) The corresponding density is given by

$$\overline{\rho_a(0)} = \Lambda_-(\omega_o)\big|_{B=0} \tag{6.8}$$

$$= \frac{\mu_o - \nu_o}{\nu_o(\mu_o - 1)}\, e^{\Gamma_+(-i/\mu_o)} \qquad (B = 0) \tag{6.9}$$

Using (2.6) and (4.37) we similarly find, for the Milne problem

$$\overline{\varphi}(0,\underset{\sim}{\Omega}) = - \frac{iq_o\nu_o^2}{2\pi} \frac{\textcircled{H}(\mu)}{\left.\frac{\partial\Lambda_1}{\partial k}\right|_{i/\nu_o}} \frac{e^{\gamma(\nu_o)-\gamma(\mu)} \quad (\mu - 1)}{(\mu - \nu_o)(\mu + \nu_o)(\nu_o + 1)} \tag{6.10}$$

and

$$\overline{\rho}(0) = i\,\frac{q_o}{c}\, \frac{1}{\left.\frac{\partial\Lambda_1}{\partial k}\right|_{i/\nu_o}} \frac{e^{\Gamma_+(i/\nu_o)}}{(\nu_o + 1)} \tag{6.11}$$

The Emergent Current

We define

$$\underset{\sim}{J}(\underset{\sim}{r}) \equiv \int d\underset{\sim}{\Omega}\, \underset{\sim}{\Omega}\, \varphi(\underset{\sim}{r},\underset{\sim}{\Omega}) \tag{6.12}$$

The total emergent current for a half-space problem is clearly given by $\overline{J}_x(0)$ where

$$\overline{J}_x(x) \equiv \int J_x(\underset{\sim}{r})d\vec{r} \tag{6.13}$$

and is very simple to compute. From the easily verified continuity equation

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{J}(\underset{\sim}{r}) = (c - 1)\, \rho(\underset{\sim}{r}) \tag{6.14}$$

we find that

$$\frac{d}{dx}\, \overline{J}_x(x) = (c - 1)\, \overline{\rho}(x) \tag{6.15}$$

i.e.,

$$\overline{J}(x) = (c - 1) \int^x \overline{\rho}(x)dx \tag{6.16}$$

$$= (c - 1) \int^x dx' \left\{\frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{-ikx'}\, \tilde{\rho}(k,\vec{0})\right\} \tag{6.17}$$

$$= - \frac{c - 1}{2\pi i} \int_{-\infty}^{\infty} dk\, \frac{\tilde{\rho}(k,\vec{0})e^{-ikx}}{k - io} \tag{6.18}$$

whence

$$\overline{J}_x(0) = (1 - c)\, \tilde{\rho}_+(\underset{\sim}{0}) \tag{6.19}$$

Using (6.19), we find, for the half-space albedo,

$$\bar{J}_x(0) = (1 - c)\,\frac{\mu_o(\nu_o - \mu_o)}{(1 - \mu_o)}\, e^{\gamma(\mu_o)} \tag{6.20}$$

and for the Milne problem

$$\bar{J}_x(0) = - \frac{i(1 - c)\nu_o^2 q_o}{c \left.\dfrac{\partial\Lambda_1}{\partial k}\right|_{i/\nu_o}} \; \frac{e^{\gamma(-\nu_o)}}{\nu_o + 1} \tag{6.21}$$

All the above results are equivalent to those obtained from the one-dimensional theory. [Note that

$$\frac{\partial\Lambda_1}{\partial k} = - i\nu^2 \frac{\partial\Lambda}{\partial\nu} \quad ; \; \nu = -i/k \tag{6.22}$$

where $\Lambda(\nu)$ is the function defined by Case.[14]] Somewhat more interesting, therefore, are

The Higher Moments

For $B > 0$ but still small, we could proceed as follows. A general half-space equation can be written as

$$\tilde{\rho}_+(k)\, \Lambda_3(k) + \tilde{\rho}_-(k) = S(k) \tag{6.23}$$

We expand the unknown functions

$$\tilde{\rho}_{\pm}(k) = \rho_{0\pm}(k) + B\,\rho_{1\pm}(k) + B^2\,\rho_{2\pm}(k) + \dots \tag{6.24}$$

and the known functions

$$\Lambda_3(k) = \Lambda_1(k) + B^2 X_2(k) + B^4 X_4(k) + \dots$$

$$S(k) = S_0(k) + B\,S_1(k) + B^2\,S_2(k) + \dots \tag{6.25}$$

and find, by equating powers of B,

$$\rho_{0+}(k)\ \Lambda_1(k) + \rho_{0-}(k) = S_0(k)$$

$$\rho_{1+}(k)\ \Lambda_1(k) + \rho_{1-}(k) = S_1(k)$$

$$\rho_{2+}(k)\ \Lambda_1(k) + \rho_{2-}(k) = S_2(k) - \rho_{0+}(k)\ X_2(k) \tag{6.26}$$

etc. In general, we must solve

$$\rho_{n+}(k)\ \Lambda_1(k) + \rho_{n-}(k) = R_n(k) \tag{6.27}$$

where $R_n(k)$ is known in terms of $S_0,\ldots,S_n,\rho_{0+},\ldots,\rho_{(n-2)+}$ and $X_2,\ldots,X_n$. Of course (6.27) is of a familiar form and its solution presents no difficulties:

$$\rho_{n+}(k) = \frac{1}{2\pi i\ \Lambda_+(k)} \int_{-\infty}^{\infty} dk'\ \frac{R_n(k')\ \Lambda_-(k')}{k' - k - io} \qquad (B = 0) \tag{6.28}$$

To obtain the X_n, we begin with (2.16),

$$\Lambda_3(k) = 1 - \frac{c}{4\pi} \iint \frac{d\underset{\sim}{\Omega}}{1 - ik\mu - \alpha B}$$

where

$$\alpha \equiv i\sqrt{1 - \mu^2}\cos(\varphi - \Delta) \tag{6.29}$$

[cf., equation (2.18)]. For real k and $B < 1$, $|1 - ik\mu| > |\alpha B|$ so that we may expand

$$\Lambda_3(k) = 1 - \frac{c}{4\pi} \int_{-1}^{1} d\mu\ \frac{1}{1 - ik\mu} \int_{-\pi}^{\pi} d\varphi \sum_{n=0}^{\infty} \left(\frac{\alpha B}{1 - ik\mu}\right)^n \tag{6.30}$$

The φ-integral is trivial and we find that (6.25) holds, with

$$X_{2m}(k) = (-1)^{m+1} \frac{2m - 1}{2^m} \frac{c}{2} \int_{-1}^{1} d\mu \frac{(1 - \mu^2)^m}{(1 - ik\mu)^{2m+1}} \tag{6.31}$$

The procedure of (6.24) through (6.31) may be applied, for example, to the half-space albedo problem. Here

$$S(k) = \frac{\mu_o}{1 - ik\mu_o - B\alpha_o} \quad ; \; \alpha_o \equiv i\sqrt{1 - \mu_o^2} \cos(\varphi_o - \Delta) \tag{6.32}$$

whence

$$S_n(k) = \frac{\alpha_o^n \mu_o}{(1 - ik\mu_o)^{n+1}} \tag{6.33}$$

Using the further result that

$$X_2(k) = \frac{1}{2k^2}\left[1 - \Lambda_1(k) - \frac{c}{1 + k^2}\right] \tag{6.34}$$

we find

$$R_1(k) = -i \frac{\sqrt{1 - \mu_o^2} \cos(\varphi_o - \Delta)}{\mu_o} \frac{1}{(k + i/\mu_o)^2} \tag{6.35}$$

$$R_2(k) = \frac{i(1 - \mu_o^2) \cos^2(\varphi_o - \Delta)}{\mu_o^2} \frac{1}{(k + i/\mu_o)^3} - \frac{\rho_{o+}(k)}{2k^2} \left[1 - \Lambda_1(k) - \frac{c}{1 + k^2}\right] \tag{6.36}$$

so that equation (6.28) gives

$$\rho_{o+}(k) = \frac{i \Lambda_-(-i/\mu_o)}{(k + i/\mu_o) \Lambda_+(k)} \tag{6.37}$$

$$\rho_{1+}(k) = \frac{-i\sqrt{1-\mu_o^2}\cos(\varphi_o - \Delta)}{\mu_o(k + i/\mu_o)\,\Lambda_+(k)} \left.\frac{\partial\Lambda_-}{\partial k}\right|_{k=-i/\mu_o} \tag{6.38}$$

$$\rho_{2+}(k) = \frac{i(1-\mu_o^2)\cos^2(\varphi_o - \Delta)}{\mu_o^2\,\Lambda_+(k)(k + i/\mu_o)} \frac{1}{2} \left.\frac{\partial^2\Lambda_-}{\partial k^2}\right|_{k=-i/\mu_o} - \frac{i\,\Lambda_-(-i/\mu_o)}{2\pi i\,\Lambda_+(k)} \int_{-\infty}^{\infty} \frac{dk'\,g(k')}{k' - k - io} \tag{6.39}$$

where

$$g(k) \equiv \frac{k^2 + 1 - c}{(k^2 + 1)\,\Lambda_1(k)} - 1$$

and, of course, the $\Lambda_{\pm}$ functions are to be evaluated at $B = 0$. Note that, because of the cosine factors in (6.38) and (6.39), the expansion (6.24) is of the same form as (6.1).

In order to extend the half-space procedure of (6.24) through (6.31), it is most convenient to return to the differential equation (2.1).[14] Let

$$\psi_n(x,\underset{\sim}{\Omega}) \equiv \int d\vec{r}\, z^n\, \varphi(\underset{\sim}{r},\underset{\sim}{\Omega})$$

$$f_n(x) \equiv \int d\underset{\sim}{\Omega}\, \psi_n(x,\underset{\sim}{\Omega}) \tag{6.40}$$

Similar functions could obviously be defined for the other types of moments; we consider only (6.40) for simplicity. By multiplying (2.1) by z^n and integrating over all $\vec{r}$,* we obtain (assuming, for convenience, $q = 0$),

$$\left(\mu \frac{d}{dx} + 1\right) \psi_n(x,\underset{\sim}{\Omega}) = \frac{c}{4\pi} f_n(x) + n\Omega_z\, \psi_{n-1}(x,\underset{\sim}{\Omega}) \qquad x \in V$$

$$\psi_n(x_s,\underset{\sim}{\Omega}) = \psi_{ns}(x_s,\underset{\sim}{\Omega}) \qquad \underset{\sim}{\Omega} \text{ inward} \tag{6.41}$$

*We must assume that the boundary of V is independent of $\vec{r}$.

where, of course,

$$\psi_{ns}(x_s,\underset{\sim}{\Omega}) \equiv \int d\vec{r}\, z^n\, \varphi_s(\underset{\sim}{r}_s,\underset{\sim}{\Omega}) \tag{6.42}$$

Hence, for each n we have only to solve a one-dimensional problem; this can be done by our general method of Part 2, with B = 0. Considering once again the example of the half-space albedo, we find

$$\tilde{\psi}_n(k,\underset{\sim}{\Omega}) = \frac{c}{4\pi} \frac{\tilde{f}_{n+}(k)}{1 - ik\mu} + \frac{\tilde{Q}_{n+}(k,\underset{\sim}{\Omega})}{1 - ik\mu} \tag{6.43}$$

where

$$\tilde{Q}_{o+}(k,\underset{\sim}{\Omega}) \equiv \frac{i}{k + i/\mu_o} \delta(\underset{\sim}{\Omega} - \underset{\sim}{\Omega}_o) \tag{6.44}$$

$$Q_{n+}(k,\underset{\sim}{\Omega}) \equiv n\Omega_z\, \tilde{\psi}_{(n-1)+}(k,\underset{\sim}{\Omega}) \qquad n \geq 1 \tag{6.45}$$

and

$$\tilde{f}_{n+}(k) = \frac{1}{2\pi i\, \Lambda_+(k)} \int_{-\infty}^{\infty} \frac{dk'\Lambda_-(k')}{k' - k - io} \int d\underset{\sim}{\Omega} \frac{\tilde{Q}_{n+}(k',\Omega)}{1 - ik'\mu} \tag{6.46}$$

which formulae are completely equivalent to (6.37) through (6.39).

V. A CONJECTURE OF KAC

Let

$$\rho(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-|k|^{\alpha}} e^{-ikx} dk \qquad (1)$$

where α is any positive number. For $T < \infty$, the integral equation

$$\lambda\varphi(x) = \int_{-T}^{T} \rho(x-x')\varphi(x')dx' \quad (0 \leq \lambda \leq 1) \qquad (2)$$

possesses discrete eigenvalues λ_n and eigenfunctions $\varphi_n(x)$. It has been conjectured(15) that, for T large,

$$\lambda_n = 1 - \frac{\mu_n}{T^{\alpha}} + o(T^{-\alpha}) \qquad (3)$$

where μ_n is independent of T. The purpose of this section is to prove the stronger result

$$\lambda_n = 1 - (\frac{n\pi}{2T})^{\alpha} + O(T^{-\alpha-1}) + O(T^{-2\alpha}) \qquad (4)$$

and also to obtain the eigenfunctions $\varphi_n(x)$ to $O(T^{-\alpha-1})$.

The similarity between (2) and the critical problem of section IV is evident, and will be exploited below. The differences between these two problems, therefore, might well be made explicit here:

(a) The function V(k) [equation (6)] which plays a role here corresponding to that of $\tilde{\Lambda}(k)$ in Section IV, has, unlike $\tilde{\Lambda}(k)$, both zeroes and branch points on the real axis, with the result that the Wiener-Hopf factorization of V(k) proceeds somewhat differently from Section IV. 1.

(b) We seek here, not just one critical width T, but an infinite sequence of eigenvalues λ_n. Since the validity of a Neumann series does not in general extend beyond the first eigenvalue, the argument of Section IV. 6 is no longer convincing and we must determine our eigenvalues by a different procedure.

With these two qualifications, the method of Section IV may be applied to the eigenvalue problem (2). For purposes of orientation, it is convenient first to briefly consider

1. THE T = ∞ CASE

When $T = \infty$, the Fourier transform of (2) may be written in the form

$$V(k)\tilde{\varphi}(k) = 0 \tag{5}$$

where

$$V(k) \equiv \lambda - \tilde{\rho}(k) = \lambda - e^{-|k|^{\alpha}} \tag{6}$$

V(k), which need not be defined for other than real k, has two real zeroes:

$$V(\pm k_o) = 0$$

where

$$k_o = [-\ln\lambda]^{1/\alpha} \tag{7}$$

It follows that the two linearly independent solutions to (2) may be chosen as

$$\left.\begin{aligned} \tilde{\varphi}_1(k;\infty) &= A\{\delta(k-k_o) + \delta(k+k_o)\} \\ \tilde{\varphi}_2(k;\infty) &= B\{\delta(k-k_o) - \delta(k+k_o)\} \end{aligned}\right\} \tag{8}$$

These satisfy the requirement

$$\tilde{\varphi}(-k) = \pm \tilde{\varphi}(k) \tag{9}$$

which may clearly be imposed for finite T also.

The inverse transform of (8) yields

$$\left.\begin{aligned} \varphi_1(x;\infty) &= (\text{constant}) \cos k_o x \\ \varphi_2(x;\infty) &= (\text{constant}) \sin k_o x \end{aligned}\right\} \tag{10}$$

Note that there exist solutions for each k_o of (7); the $T = \infty$ case is characterized by a continuous spectrum:

$$0 \leq \lambda \leq 1 \tag{11}$$

2. FACTORIZATION OF V(k)

We now return our attention to the case of finite T, for the solution of which a Wiener-Hopf factorization of V(k) [equation (6)] is crucial.

Observe that the function

$$\hat{B}(k) = \ln \left\{ \frac{V(k)(k^2+\beta^2)}{\lambda(k^2-k_o^2)} \right\} \tag{12}$$

where β is any real number, is continuous on the real k axis, and vanishes at ∞. Thus

$$B(k) \equiv \frac{1}{2\pi i} \int_{-\infty}^{\infty} dk' \, \frac{\hat{B}(k')}{k'-k} \tag{13}$$

is analytic in the plane cut along the real axis, and its boundary values satisfy

$$B^{+}(k) - B^{-}(k) = \hat{B}(k) \quad k \text{ real} \tag{14}$$

(14) is equivalent to

$$\frac{V(k)(k^2+\beta^2)}{\lambda(k^2-k_o^2)} = \frac{e^{B^{+}(k)}}{e^{B^{-}(k)}} \tag{15}$$

so that with the choice

$$V_{+}(k) = \frac{(k^2-k_o^2)}{(k+i\beta)} e^{B^{+}(k)} \tag{16}$$

$$V_{-}(k) = (k-i\beta)e^{B^{-}(k)} \tag{17}$$

we have

$$V(k) = \lambda \frac{V_{+}(k)}{V_{-}(k)} \tag{18}$$

where the $V_{\pm}(k)$ are analytic in appropriate half-planes, and

$$V_{\pm}(k) \sim k \quad k\to\infty \tag{19}$$

Note (cf., our discussion in Section I.) that the superscripts in the definitions (16), (17) are relevant only when k is real.

It is convenient to derive here three properties of $B^{+}(k)$ which will be required below.

(i) $B^{+}(k)$ can be analytically continued into a function cut, not along the real axis, but along the path Γ_{α}, below it (see Figure 5).

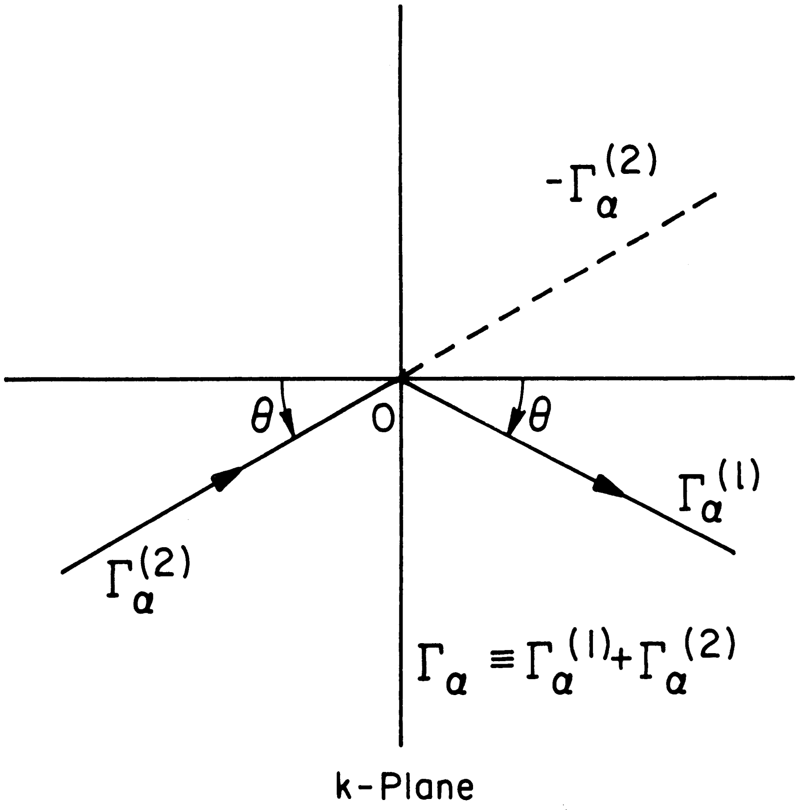

Figure 5. Integration contour for $B^+(k)$, Section V. $\Theta \equiv \max(\pi/2, \pi/\alpha)$.

We show that $B^+(k)$ can be considered analytic above Γ_α as follows. Let

$$\hat{B}_1(k) \equiv \ln\left[\frac{(\lambda - e^{-k^\alpha})(k^2+\beta^2)}{\lambda(k^2-k_o^2)}\right] \tag{20}$$

Then

$$B^+(k) = B_1^+(k) + B_2^+(k) \tag{21}$$

where

$$B_1^+(k) = \frac{1}{2\pi i}\int_o^\infty dk' \frac{\hat{B}_1(k')}{k'-k-io} \tag{22}$$

$$B_2^+(k) = \frac{-1}{2\pi i}\int_o^\infty dk' \frac{\hat{B}_1(k')}{k'+k-io} \tag{23}$$

It is clear that the integration path for $B_1^+(k)$ can be deformed down to the line $\Gamma_\alpha^{(1)}$ of Figure 5, by Cauchy's theorem. Similarly, that for $B_2^+(k)$ can be deformed up to $(-\Gamma_\alpha^{(2)})$. Thus the two functions are cut along $\Gamma_\alpha^{(1)}$ and $\Gamma_\alpha^{(2)}$, respectively $(\Gamma_\alpha \equiv \Gamma_\alpha^{(1)} + \Gamma_\alpha^{(2)})$.

(ii) a) If α is an integer, $B^+(k)$ and its first α derivatives are finite near $k = 0$.

b) If $\alpha = n + \beta$, $0 < \beta < 1$, n a positive integer, then $B^+(k)$ and its first n derivatives are finite at $k = 0$, while $\frac{d^{(n+1)}}{dk^{(n+1)}} B^+(k) \sim k^{\beta-1}$ near $k = 0$.

Since property (a) is fairly self-evident, we prove only (b). Here, the general statement becomes clear upon consideration of the case $n = 0$. From (22) and property (i),

$$\frac{dB_1^+}{dk} = \frac{-1}{2\pi i} \int_{\Gamma_\alpha^{(1)}} \hat{B}_1(k') \frac{d}{dk'} \left(\frac{1}{k'-k} \right) \tag{24}$$

We integrate by parts

$$\frac{dB_1^+}{dk} = \frac{-1}{2\pi i} \frac{\hat{B}_1(0)}{k} + \frac{1}{2\pi i} \int_{\Gamma_\alpha^{(1)}} dk' \frac{\frac{d\hat{B}_1(k')}{dk'}}{k'-k} \tag{25}$$

Similarly

$$\frac{dB_2^+}{dk} = \frac{\hat{B}_1(0)}{2\pi i k} + \frac{1}{2\pi i} \int_{-\Gamma_\alpha^{(2)}} dk' \frac{\frac{d\hat{B}_1(k')}{dk'}}{k'+k} \tag{26}$$

whence

$$\frac{dB^+}{dk} = \frac{1}{2\pi i}\int_{\Gamma_\alpha}{}_{(1)} dk' \frac{\frac{d\hat{B}_1}{dk'}}{k'-k} + \frac{1}{2\pi i}\int_{-\Gamma_\alpha}{}_{(2)} dk' \frac{\frac{d\hat{B}_1}{dk'}}{k'+k} \tag{27}$$

Now (for n = 0) $\frac{d\hat{B}_1}{dk}$ is singular at k = 0:

$$\frac{d\hat{B}_1}{dk} \sim k^{\alpha-1} \quad (k \sim 0) \tag{28}$$

from which it follows by a well-known[(3)] theorem that

$$\frac{dB^+}{dk} \sim k^{\alpha-1} \quad k \sim 0 \tag{29}$$

For $n \geq 1$, we can continue to integrate by parts so long as the boundary (k=0) contribution is finite and then proceed just as in equations (28), (29).

(iii) If $b(k_o)$ is defined by

$$2ib(k_o) = B^+(-k_o) - B^+(k_o) \tag{30}$$

then $b(k_o)$ is real.

We have

$$B^+(-k_o) - B^+(k_o) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} dk' \, \hat{B}(k') \left\{ \frac{1}{k'+k_o-io} - \frac{1}{k'-k_o-io} \right\} \tag{31}$$

Using equation (I.4) and the fact that

$$\hat{B}(k_o) = \hat{B}(-k_o) \tag{32}$$

we find

$$B^+(-k_o) - B^+(k_o) = \frac{ik_o}{2\pi}\int_{-\infty}^{\infty} dk' \, \hat{B}(k') \, P\left[\frac{1}{k'^2-k_o^2}\right] \quad (k_o \neq 0) \tag{33}$$

so that the reality of $b(k_o)$ follows from that of $\hat{B}(k)$.

Note also that $b(k_o) = O(k_o)$ for $k_o \sim 0$, as is evident from the definition.

3. THE EIGENVALUE PROBLEM

We begin our solution of (2) by closely following the argument of Section IV. Assume

$$\tilde{\varphi}(k) = \tilde{\varphi}(-k) \tag{34}$$

in which case the Fourier transform of (2) takes the form

$$\lambda\tilde{\varphi}(k) = \tilde{\rho}(k)\{\tilde{\varphi}(k) - e^{ikT}J_+(k) - e^{-ikT}J_+(-k)\} \tag{35}$$

where

$$J_+(k) \equiv \frac{1}{2\pi i}\int_{-\infty}^{\infty} dk' \frac{\tilde{\varphi}_+(k')e^{-ik'T}}{k'-k-io} \tag{36}$$

and the ansatz

$$\tilde{\varphi}(k) = e^{-ikT}\psi(k) + e^{ikT}\psi(-k) \tag{37}$$

yields for $\psi(k)$ the integral equation

$$\lambda\psi(k) = \tilde{\rho}(k)\{\psi_+(k) - I_-(k)\} \tag{38}$$

where

$$I_-(k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} dk' \frac{\psi_+(k')e^{-2ik'T}}{k'+k-io}. \tag{39}$$

Equation (38) may be written in the form

$$V(k)\ \psi_+(k) = -\lambda\ \psi_-(k) - [\lambda - V(k)]I_-(k) \quad (k\ \text{real}) \tag{40}$$

or, using equation (18),

$$V_+(k)\ \psi_+(k) = -V_-(k)\ \psi_-(k) - [V_-(k) - V_+(k)]I_-(k) \tag{41}$$

We define the function f(k) by

$$f(k) = \begin{cases} V_+(k)\ \psi_+(k) & \text{Im}(k) > 0 \\ -V_-(k)[\psi_-(k) + I_-(k)] & \text{Im}(k) < 0 \end{cases} \tag{42}$$

so that (41) implies

$$\overset{+}{f}(k) - \overset{-}{f}(k) = V_+(k)\ I_-(k) \tag{43}$$

whence

$$f(k) = C + \frac{1}{2\pi i} \int_{-\infty}^{\infty} dk' \frac{V_+(k')I_-(k')}{k'-k-io} \qquad \text{Im } k \neq 0 \tag{44}$$

provided the integral exists. Now

$$\psi_+(k) = \frac{\overset{+}{f}(k)}{V_+(k)} = \frac{C}{V_+(k)} + \frac{1}{2\pi i V_+(k)} \int_{-\infty}^{\infty} dk' \frac{V_+(k')I_-(k')}{k'-k-io} \tag{45}$$

provides a relatively simple integral equation for $\psi_+(k)$. This is to be solved in the familiar way. Let

$$\psi_+(k) = \sum_{n=o}^{\infty} \psi_+^{(n)}(k) \tag{46}$$

where, choosing C = 1,

$$\psi_+^{(o)}(k) = \frac{1}{V_+(k)} \tag{47}$$

and

$$\psi_+^{(n)}(k) = \frac{1}{2\pi i V_+(k)} \int_{-\infty}^{\infty} dk' \frac{V_+(k') I_-^{(n-1)}(k')}{k'-k-io} \tag{48}$$

where

$$I_-^{(n-1)}(k) \equiv \frac{1}{2\pi i} \int_{-\infty}^{\infty} dk' \frac{\psi_+^{(n)}(k') e^{-2ik'T}}{k'+k-io} \tag{49}$$

We will carry out the program of (46)-(49) below; but first it is necessary to determine

4. THE EIGENVALUES(16)

It follows from (19) that the integral of (44) will exist only if

$$I_-(k) = o(k^{-1}) \tag{50}$$

which implies

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} dk\ \psi_+(k) e^{-2ikT} = 0 \tag{51}$$

Equation (51) is our eigenvalue equation. Since it must hold independently of T, we have, for each n,

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} dk\ \psi_+^{(n)}(k) e^{-2ikT} = 0 \tag{52}$$

so that (48) is meaningful. In particular,

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} dk \frac{e^{-2ikT}}{V_+(k)} = 0 \tag{53}$$

where the zeroes of $V_+(k)$ are to be considered as lying just below the real axis in the usual way. Because of property (i) noted above, (53) may be written in the form

$$-R(k_o)e^{-2ik_oT} - R(-k_o)e^{2ik_oT}$$

$$+ \frac{1}{2\pi i} \int_{\Gamma_\alpha} dk \, \frac{e^{-2ikT}}{V_+(k)} = 0 \qquad (54)$$

where

$$R(\pm k_o) \equiv \underset{\pm k_o}{\text{Res}} \left[\frac{1}{V_+(k)}\right] \qquad (55)$$

Now it follows easily from property (ii) that the integral term in the (exact) eigenvalue equation (54) is $O(T^{-\alpha-1})$. For example, in the case of property (ii b), we can integrate by parts n+1 times, obtaining

$$\frac{1}{2\pi i} \int_{\Gamma_\alpha} dk \, \frac{e^{-2ikT}}{V_+(k)} = \left(\frac{-1}{-2iT}\right)^{n+1} \frac{1}{2\pi i} \int_{\Gamma_\alpha} dk' e^{-2ik'T} \, \frac{d^{(n+1)}}{dk'^{(n+1)}} \left[\frac{1}{V_+(k')}\right] \qquad (56)$$

Now the integrand is singular at k=o, where it behaves like $k^{\beta-1}$ ($\beta = \alpha - n$). Considering the contributions from $\Gamma_\alpha^{(1)}$ and $\Gamma_\alpha^{(2)}$ separately, and using a well-known[9] "Tauberian" theorem, we see that the integral behaves for large T like $T^{-\beta}$, so that

$$\frac{1}{2\pi i} \int_{\Gamma_\alpha} dk \, \frac{e^{-2ikT}}{V_+(k)} \sim T^{-n-1-\beta} = T^{-\alpha-1} \qquad (57)$$

Hence for large T we have the eigenvalue equation

$$R(k_o)e^{-2ik_oT} + R(-k_o)e^{2ik_oT} = 0 \qquad (58)$$

which is correct to $O(T^{-\alpha-1})$. Using (16), it can be written as

$$e^{4ik_oT} = -\frac{k_o+i\beta}{k_o-i\beta}\, e^{-B^+(k_o)+B^+(-k_o)} \tag{59}$$

It is clear from our discussion of the T = ∞ case that, for large T, the first (i.e., largest) eigenvalues will be close to 1. Now

$$k_o = (+\ \ln \frac{1}{\lambda})^{1/\alpha}$$

$$= (1-\lambda)^{1/\alpha} + O((1-\lambda)^{1/\alpha+1}) \tag{60}$$

so that for sufficiently large T, $k_o \ll \beta$. Using also property (iii) we find, from (59)

$$4ik_oT = 2i\ b(k_o) + 2in\pi \tag{61}$$

But since $b(k_o)$ is $O(k_o)$ for small k_o, we have, for large T,

$$k_{on} = \frac{n\pi}{2T} + O(T^{-2}) \quad n = 1,2,\ldots \tag{62}$$

Equation (60) now gives

$$\lambda_n = 1 - (\frac{n\pi}{2T})^{\alpha} + O(T^{-\alpha-1}) + O(T^{-2\alpha}) \tag{63}$$

the desired formula.

5. THE EIGENFUNCTIONS

We find the eigenfunctions in the same approximation by the prescription of (46)—(49). First note that $\psi_+^{(o)}(k)$ is analytic above Γ_α except for poles at $\pm k_o$. By induction, we assume this is also true for $\psi_+^{(m-1)}(k)$ and let

$$R^{(m-1)}(\pm k_o) = \underset{\pm k_o}{\operatorname{Res}}\ \psi_+^{(m-1)}(k) \tag{64}$$

With the observation that

$$\frac{1}{k'+k-io} = \frac{1}{k} - \frac{k'}{k}\,\frac{1}{k'+k-io} \tag{65}$$

and equation (52) we find

$$I_-^{(m-1)}(k) = \frac{-1}{2\pi ik}\int_{-\infty}^{\infty} dk'\,\frac{k'\psi_+^{(m-1)}(k')e^{-2ik'T}}{k'+k-io} \cong \frac{1}{k}\left[\frac{k_oR^{(m-1)}(k_o)e^{-2ik_oT}}{k+k_o-io} - \frac{k_oR^{(m-1)}(-k_o)e^{2ik_oT}}{k-k_o-io}\right] \tag{66}$$

Here the integral along Γ_α, which is easily seen to be at most $O(T^{-\alpha-1})$, has been dropped. Another application of (52) yields

$$I_-^{(m-1)}(k) = \frac{2k_oR^{(m-1)}(k_o)e^{-2ik_oT}}{(k^2-k_o^2)_-} \tag{67}$$

where the subscript in the denominator reminds us that the poles are to be interpreted as lying just above the real axis. Now (48) gives

$$\psi_+^{(m)}(k) = \frac{2k_oR^{(m-1)}(k_o)e^{-2ik_oT}}{2\pi iV_+(k)}\int_{-\infty}^{\infty} dk'\,\frac{V_+(k')}{(k'-k-io)(k'^2-k_o^2)_-}$$

$$= \frac{2k_oR^{(m-1)}(k_o)e^{-2ik_oT}}{(k^2-k_o^2)} \tag{68}$$

as is clear from equation (16). We observe that (a) the induction hypothesis is fulfilled for $\psi_+^{(m)}$, and (b) $\psi_+^{(m)}$ is corsistent with equations (52) and (62). It is, however, clear from (68), and not surprising, that the series (46) will not converge for every eigenvalue:

$$R^{(m)}(k_o) = e^{-2ik_oT}\, R^{(m-1)}(k_o) \tag{69}$$

$$\Rightarrow R^{(m)}(k_o) = e^{-2imk_oT}\, R^{(o)}(k_o) = e^{-2imk_oT}\, R(k_o) \tag{70}$$

$$\Rightarrow \psi_+(k) = \frac{1}{V_+(k)} + \frac{2k_oR(k_o)}{(k^2-k_o^2)} \sum_{m=1}^{\infty} e^{-2imk_oT} \tag{71}$$

Restricting ourselves therefore to the first eigenvalue [n=1 in equation (62)], we have "convergence" in the sense of Cesàro:

$$\sum_{n=o}^{\infty}(-1)^n = \frac{1}{2} \tag{72}$$

and the eigenfunction is

$$\psi_+(k) = \frac{1}{V_+(k)} - \frac{k_oR(k_o)}{(k^2-k_o^2)} \quad . \tag{73}$$

It is now a simple matter to verify that this solution satisfies (51) [with the usual error of $O(T^{-\alpha-1})$] for <u>each</u> k_o given by equation (62), and not merely the first one. Furthermore, it is easy to check that, to $O(T^{-\alpha-1})$, the $\psi_+(k)$ of (73) is indeed a solution to equation (38), again for each k_o of (62), thus justifying the use of (72).

Having $\psi_+(k)$ [$\psi_-(k)$ is of course not needed for $|x| \leq T$], we may take the inverse Fourier transform of equation (37) to find the even eigenfunctions of (2). Dropping as usual contributions of $O(T^{-\alpha-1})$, we easily find

$$\varphi_n(x) \propto \sin k_{on} T \cos k_{on} x \tag{74}$$

i.e., the only nontrivial solutions obtained in this way are for odd n. The solutions with n even are found by replacing equation (34) by

$$\tilde{\varphi}(k) = -\tilde{\varphi}(-k) \tag{75}$$

and proceeding analagously to equations (34)—(38). (The only changes are sign changes in these equations). The result is not surprising:

$$\varphi_n(x) \propto \cos k_{on} T \sin k_{on} x \tag{76}$$

which vanishes for odd n.

We conclude that the normalized eigenfunctions of equation (2) are approximately given by

$$\varphi_{2n+1}(x) = \frac{1}{\sqrt{T}} \cos \frac{2n+1}{2T} \pi x \tag{77}$$

$$\varphi_{2n}(x) = \frac{1}{\sqrt{T}} \sin \frac{n\pi}{T} x \tag{78}$$

where the error is $O(T^{-\alpha-1})$. Note that these have the form of the $T = \infty$ solutions, except that k_o is chosen such that

$$\varphi_n(\pm T) = 0 \ . \tag{80}$$

We remark finally that another conjecture,[15] related to (3), is false. With φ_n given by (77) and (78), we define the functions

$$f_n(x) = \varphi_n(x;T = 1) \tag{81}$$

and the notation

$$(f_n,g) = \int_{-1}^{1} f_n(x)g(x)dx \tag{82}$$

Then the conjecture we wish to consider may be written in the form

$$\sum_{n=1}^{\infty} \frac{f_n(x)(f_n,1)}{\mu_n} \overset{?}{=} C(\alpha)(1-x^2)^{\alpha/2} \tag{83}$$

Here C depends only on α and we have found [cf. equations (3) and (4)]

$$\mu_n = \left(\frac{n\pi}{2}\right)^{\alpha} \tag{84}$$

Since

$$(f_n,f_m) = \delta_{mn} \tag{85}$$

(83) is equivalent to

$$\frac{(f_m,1)}{\mu_m} \overset{?}{=} C(\alpha)\left(f_m,(1-x^2)^{\alpha/2}\right) \quad . \tag{86}$$

Of course both sides vanish when m is even. For m = 2n+1, a trivial calulcation yields

$$\frac{(f_{2n+1},1)}{\mu_{2n+1}} = 2(-1)^n \left[(n+\frac{1}{2})\pi\right]^{-\alpha-1} \tag{87}$$

But it is known(12) that

$$\begin{aligned}\left(f_n,(1-x^2)^{\alpha/2}\right) &= \int_{-1}^{1} (1-x^2)^{\alpha/2} e^{i(n+\frac{1}{2})\pi x} dx \\ &= \sqrt{\pi}\ \Gamma(\frac{\alpha}{2}+1)\left[\frac{(n+\frac{1}{2})\pi}{2}\right]^{-\frac{\alpha+1}{2}} J_{\frac{\alpha+1}{2}}\left[(n+\frac{1}{2})\pi\right]\end{aligned} \tag{88}$$

whence the invalidity of (83) is evident. In fact (83) is true only in the special case $\alpha = 2$.

VI. SOME QUANTUM FIELD THEORY PROPAGATORS

We derive below generalized function representations for certain propagators of quantum field theory. Since most of our results have already been obtained by Gorgé and Jauch,[17] the significance of this section lies mainly in the method used. This consists of a systematic application of techniques due to Gel'fand.[1]

It is well-known that the generalized function definitions

$$(x \pm io)^{\lambda} \equiv x_+^{\lambda} + e^{\pm i\lambda\pi} x_-^{\lambda} \quad (\lambda \neq -n) \tag{1}$$

$$(x \pm io)^{-n} \equiv x^{-n} \mp \frac{i\pi(-1)^{n-1}}{(n-1)!} \delta^{(n-1)}(x) \quad (n=1,2,\ldots) \tag{2}$$

can be extended to the case in which x is replaced by a quadratic form in more than one variable. Consider the Lorentz invariant form

$$p^2 + m^2 \equiv \underset{\sim}{p} \cdot \underset{\sim}{p} - p_o^2 + m^2$$

The generalized functions

$$\tilde{u}_{\lambda}^{\pm}(p) \equiv (p^2 + m^2 \pm io)^{\lambda} \tag{3}$$

exist in S', and we have, in analogy to (1),

$$(p^2 + m^2 \pm io)^{\lambda} = (p^2 + m^2)_+^{\lambda} + e^{\pm i\pi\lambda}(p^2 + m^2)_-^{\lambda} \tag{4}$$

for $\lambda \neq -k$, and, corresponding to (2),

$$\tilde{u}_{-k}^{\pm}(p) \equiv (p^2 + m^2)^{-k} \mp \frac{i\pi(-1)^{k-1}}{(k-1)!} \delta^{(k-1)}(p^2 + m^2) \tag{5}$$

We will use in particular the functionals

$$\tilde{u}_{-k}(p) \equiv \frac{1}{2}\,[\tilde{u}^{+}_{-k}(p) + \tilde{u}^{-}_{-k}(p)] = (p^2 + m^2)^{-k} \tag{6}$$

$$\tilde{u}^{o}_{-k}(p) \equiv i(-1)^{k-1}[\tilde{u}^{+}_{-k}(p) - \tilde{u}^{-}_{-k}(p)] = \frac{2\pi}{(k-1)!}\,\delta^{(k-1)}(p^2 + m^2) \tag{7}$$

where the right-hand-sides are given by

$$(\tilde{u}_{-k}, \varphi) \equiv \lim_{\epsilon\to 0} \int_{|p^2+m^2| > \epsilon} d^4p\,(p^2+m^2)^{-k}\varphi(p) \tag{8}$$

and

$$(\tilde{u}^{o}_{-k}, \varphi) \equiv \frac{(-1)^{k-1}}{(k-1)!}\,\pi \int_0^\infty \rho^2 d\rho \left[\frac{\partial^{(k-1)}}{\partial Q^{(k-1)}}\left(\frac{\bar{\varphi}(\rho,\ \pm\sqrt{\rho^2+m^2-Q})}{\sqrt{\rho^2+m^2-Q}}\right)\right]_{Q\,=\,0} \tag{9}$$

Here $\rho \equiv |\underset{\sim}{p}|$ and

$$\bar{\varphi}(\rho,\ \pm\sqrt{\rho^2+m^2-Q}) \equiv \int d\underset{\sim}{\Omega}\ \varphi(\underset{\sim}{p},\ \pm\sqrt{\rho^2+m^2-Q}) \tag{10}$$

The definition (8) is an obvious four-dimensional extension of the ordinary principal value, while (9) results from the change of variable

$$Q = \rho^2 + m^2 - p_o^2\,,$$

whence

$$\int d^4p\ \delta^{(k)}(p^2+m^2)\varphi(\underset{\sim}{p},p_o)$$

$$= \int \frac{\rho^2 d\rho\, d\underset{\sim}{\Omega}\, dQ}{2\sqrt{\rho^2+m^2-Q}}\ \delta^{(k)}(Q)\varphi(\rho,\ \underset{\sim}{\Omega},\ \pm\sqrt{\rho^2+m^2-Q}) \tag{11}$$

This is easily seen to reduce to (9).

Now consider the differential equations

$$(\Box - m^2)^k \; f(x) = (-1)^k \; \delta_4(x) \tag{12}$$

$$(\Box - m^2)^k \; f^o(x) = 0 \tag{13}$$

Here

$$\Box \equiv \underset{\sim}{\nabla} \cdot \underset{\sim}{\nabla} - \frac{\partial^2}{\partial x_o^2} ,$$

and we abbreviate

$$\delta_4(x) = \delta(x_o)\delta(x_1)\delta(x_2)\delta(x_3)$$

The generalized functions discussed above lead naturally to particular, Lorentz invariant solutions of (12) and (13); these "canonical" solutions are precisely the propagators of quantum field theory.

(12) and (13) have the Fourier Transforms [defined by four-dimensional extension of (I.1), with transform variables $\underset{\sim}{p}$, p_o]

$$(p^2+m^2)^k \; \tilde{f}(p) = 1 \tag{14}$$

$$(p^2+m^2)^k \; \tilde{f}^o(p) = 0 \tag{15}$$

It is clear that the generalized functions $\tilde{u}^{\pm}_{-k}$, $\tilde{u}_{-k}$ [equations (5), (6)] are solutions to (14), while the $\tilde{u}^o_{-k}$ of (7) is a solution to (15). The $\tilde{u}^{\pm}_{-k}$ can be expressed as linear combinations of $\tilde{u}_{-k}$ and $\tilde{u}^o_{-k}$, to which we therefore restrict our attention.

We compute the inverse Fourier transforms of

$$f(p) = \tilde{u}_{-k}(p)$$

and

$$f^{o}(p) = \tilde{u}^{o}_{-k}(p)$$

by means of the known fact that, for $\mathrm{Re}(\lambda) > 1$,

$$F^{-1}\,[\tilde{u}^{\pm}_{\lambda}] = \pm\, i \frac{m^{2+\lambda}}{(2\pi)^2 2^{-(\lambda+1)}\Gamma(-\lambda)} \frac{K_{2+\lambda}[m(x^2 \pm io)^{1/2}]}{(x^2 \pm io)^{1/2(2+\lambda)}} \tag{16}$$

where $x^2 = \underset{\sim}{x}\cdot\underset{\sim}{x} - x^{o2}$. (16) can be written in a more useful form by using the identity

$$K_{\nu}(-iz) = \frac{i\pi}{2} e^{-i\frac{\pi\nu}{2}} H^{(1)}_{-\nu}(z)$$

and equation (1). We find

$$K_{2+\lambda}[m(x^2 \pm io)] = \frac{K_{2+\lambda}[m(x^2)_+^{1/2}]}{(x^2)_+^{1/2(\lambda+2)}} \mp \frac{i\pi}{2} \frac{H^{(2)(1)}_{-(\lambda+2)}[m(x^2)_-^{1/2}]}{(x^2)_-^{1/2(\lambda+2)}} \tag{17}$$

whence

$$u_{\lambda}(x) = \frac{1}{2}\,[u^{+}_{\lambda} + u^{-}_{\lambda}] = i \frac{m^{2+\lambda}}{(2\pi)^2 2^{-(\lambda+1)}\Gamma(-\lambda)} \left[-\frac{\pi i}{2} \frac{J_{-(\lambda+2)}[m(x^2)_-^{1/2}]}{(x^2)_-^{1/2(\lambda+2)}} \right] \tag{18}$$

$$u^{o}_{\lambda}(x) = ie^{i\pi(\lambda-1)}[u^{+}_{\lambda} - u^{-}_{\lambda}] = \frac{m^{2+\lambda} e^{i\pi(\lambda-1)}}{(2\pi)^2 2^{-(\lambda+2)}\Gamma(-\lambda)} \left[\frac{K_{2+\lambda}[m(x^2)_+^{1/2}]}{(x^2)_+^{1/2(2+\lambda)}} - \frac{\pi}{2} \frac{N_{-(2+\lambda)}[m(x^2)_-^{1/2}]}{(x^2)_-^{1/2(2+\lambda)}} \right] \tag{19}$$

Equations (18) and (19) were derived by assuming $Re(\lambda) > -1$, in which case the transform can be computed by actually performing the integrals. However, by considering the usual series expansions of the Bessel functions, and the known analytic properties of $(x^2)_{\pm}^{\lambda}$, it is not hard to see that the expressions on the right-hand-sides are entire functions of λ. Thus, by analytic continuation, we may take (18) and (19) to be true for all λ. In particular, letting $\lambda \to -k$, we have our desired propagators:

$$u_{-k}(x) = \frac{\delta_{k1}\delta(x^2)}{4\pi} + \frac{m^{2-k}}{4\pi 2^k (k-1)!} (x^2)_-^{-\frac{1}{2}(2-k)} J_{k-2}[m(x^2)_-^{1/2}] \tag{20}$$

$$u^o_{-k}(x) = \frac{(-1)^{k-1} m^{2-k}}{\pi^2 2^k (k-1)!} \left[\frac{K_{k-2}[m(x^2)_+^{1/2}]}{(x^2)_+^{1/2(2-k)}} - \frac{\pi}{2} \frac{N_{k-2}[m(x^2)_-^{1/2}]}{(x^2)_-^{1/2(2-k)}} \right] \tag{21}$$

[The reason for the extra term in $u_{-1}(x)$ is as follows. Upon expanding the Bessel function in (20) in a power series, we find

$$u_\lambda(x) = N(\lambda) \frac{(x^2)_-^{-(\lambda+2)}}{\Gamma(-\lambda-1)} + \text{regular terms.} \tag{22}$$

Both $(x^2)_-^{-(\lambda+2)}$ and $\Gamma(-\lambda-1)$ have simple poles at $\lambda = -1$, with known residues. The first term of (20) results from an application of L'hopital's rule.]

The case in which $m = o$ is perhaps more interesting. We wish to find the canonical solutions to

$$\Box^k g(x) = (-1)^k \delta_4(x) \tag{23}$$

$$\Box^k g^o(x) = o \tag{24}$$

We denote these by $v_{-k}^{(o)}(p)$,

$$g(x) = v_{-k}(x)$$

$$g^{o}(x) = v_{-k}^{o}(x)$$

and require them to correspond in some natural way to the $u_{-k}^{(o)}(x)$ found above.

Observing that, from (21), $\lim_{m\to o} u_{-k}^{o}(x)$ does not exist in any ordinary sense for $k > 1$, we might conclude that the v_{-2}^{o}, v_{-3}^{o}, etc., cannot be defined. In p-space, however, the situation appears somewhat differently. The limit

$$\lim_{m\to o} \delta(p^2+m^2) = "\delta(p^2)"$$

does exist, in the sense that there are well-defined generalized functions $\tilde{g}^{o}(p)$ which satisfy

$$p^2\tilde{g}^{o}(p) = 0$$

In fact, there are two (linearly independent) such functions, which differ by $(\text{constant})\Box^{(k-2)}\delta(p)$. Thus the difficulty is actually one of uniqueness; we must insure that our solutions to (23) and (24) are the canonical ones.

This difficulty is easily surmounted. We require first that $\mathrm{Re}(\lambda) > -2$, and define

$$v_{\lambda}^{\pm}(x) \equiv \lim_{m\to o} u_{\lambda}^{\pm}(x) \quad \mathrm{Re}(\lambda) > -2 \tag{25}$$

The limit exists and we find

$$v_{\lambda}^{\pm}(x) = \mp i \frac{2^{2\lambda+2}}{(2\pi)^2\Gamma(-\lambda)} \Gamma(\lambda+2)(x^2\mp io)^{-(2+\lambda)} \tag{26}$$

It is clear that the prescription

$$v^{\pm}_{-k}(x) \stackrel{?}{\equiv} \lim_{\lambda\to -k} v^{\pm}_{\lambda}(x) \tag{27}$$

$$v_{-k}(x) \equiv \frac{1}{2}\,[v^{+}_{-k}(x) + v^{-}_{-k}(x)] \tag{28}$$

$$v^{(o)}_{-k}(x) \equiv i(-1)^{k}\,[v^{+}_{-k}(x) - v^{-}_{-k}(x)] \tag{29}$$

would yield the m = o analogues of u_{-k} and u^{o}_{-k}, and thus the desired canonical solutions to (23) and (24), provided the limit (27) exists. The fact that, for $k > 1$, this limit does not exist, requires only a slight and conventional[1] modification of (27).

Specifically, it is known that $(x^2 \mp io)^{-(\lambda+2)}$ and $\Gamma(\lambda+2)$ have their only singularities—poles—at $\lambda = 0,1,2,\ldots$, and $\lambda = -2, -3,\ldots$, respectively. Thus $v^{\pm}_{\lambda}$ is well-behaved at $\lambda = -1$, and we have, from (26),

$$v^{\pm}_{-1}(x) = \mp\, i\,\frac{1}{(2\pi)^2}\,(x^2 \mp io)^{-1} \tag{30}$$

For $\lambda = -2, -3,\ldots$, the formula (27) is useless and we use instead

$$v^{\pm}_{-k}(x) \equiv \lim_{\lambda\to -k} \frac{\partial}{\partial\lambda}\,[(\lambda+k)v^{\pm}_{\lambda}(x)] \quad k \geq 2 \tag{31}$$

Thus defined, $v^{\pm}_{-k}$ is the so-called "regular part" of $v^{\pm}_{\lambda}$ at $\lambda = -k$. It is clearly the constant term in the Laurent series for $v^{\pm}_{\lambda}$ near $\lambda = -k$.

Using (26), and the known Laurent series for $(x^2 \pm io)^{-(\lambda+2)}$, we easily find

$$v^{\pm}_{-k}(x) = \mp\, i(-1)^{k-2}\,\frac{2^{-2k}}{\pi^2 (k-1)!\,(k-2)!}\,(x^2)^{k-2}\,[\ln|x^2| \pm i\pi\Theta(-x^2)] \tag{32}$$

Equations (28)—(32) furnish the desired propagators:

$$v_{-1}(x) = \frac{\delta(x^2)}{4\pi} \tag{33}$$

$$v_{-k}(x) = \frac{(-1)^{k-2} 2^{-2k}}{\pi(k-1)!(k-2)!} (x^2)^{k-2} \Theta(-x^2) \quad k \geq 2 \tag{34}$$

$$\overset{o}{v}_{-1}(x) = \frac{1}{2\pi^2} (x^2)^{-1} \tag{35}$$

$$\overset{o}{v}_{-k}(k) = \frac{2^{-2k}}{\pi^2(k-1)!(k-2)!} (x^2)^{k-2} \ln|x^2| \quad k \geq 2 \tag{36}$$

We remark that

$$\left.\begin{matrix} v_{-k}(x) \\ \overset{o}{v}_{-1}(x) \end{matrix}\right\} = \lim_{m\to o} \left\{\begin{matrix} u_{-k}(x) \\ \overset{o}{u}_{-1}(x) \end{matrix}\right. \tag{37}$$

while for $\overset{o}{v}_{-k}(x)$, $k \geq 2$, we have only the correspondence of our definition:

$$\overset{o}{v}_{-k}(x) = \text{Regular part at } \lambda = -k \text{ of} \left\{ \lim_{\substack{m\to o \\ \lambda > -2}} \overset{o}{u}_{\lambda}(x) \right\} \quad k \geq 2 \tag{38}$$

APPENDIX

We show that, for $0 < \xi < \pi$,

$$\int_{-\infty}^{\infty} e^{-i\kappa r \sinh\tau + i\kappa r_o \sinh(\tau+i\xi)} d\tau = 2K_o(\kappa|\vec{r}-\vec{r}_o|_\xi) \tag{A.1}$$

as claimed in Section II.2 [equation (II.2.42)]. Beginning with the representation

$$K_o(\kappa|r-r_o|) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \frac{e^{i\vec{k}\cdot(\vec{r}-\vec{r}_o)}}{k_x^2 + k_y^2 + \kappa^2} \tag{A.2}$$

which follows directly from the fact that $K_o(\kappa|\vec{r}-\vec{r}_o|)$ is the infinite space Green's function for the two dimension Helmholtz equation,(12)

$$(\nabla^2-\kappa^2)K_o(\kappa|\vec{r}-\vec{r}_o|) = -2\pi\, \delta(\vec{r}-\vec{r}_o) \tag{A.3}$$

we "rotate" the integration variables: $(k_x, k_y) \rightarrow (k_1,k_2)$ where

$$k_x = k_1\cos\Theta - k_2\sin\Theta$$

$$k_y = k_1\sin\Theta + k_2\cos\Theta$$

so that, in polar coordinates, the exponent is

$$-i\vec{k}\cdot(\vec{r}-\vec{r}_o) = -ik_1 r + ik_1 r_o\cos(\Theta-\Theta_o) + ik_2 r_o \sin(\Theta_o-\Theta) \tag{A.4}$$

We now perform the integral over k_2,

$$K_o(\kappa|\vec{r}-\vec{r}_o|) = \frac{1}{2}\int_{-\infty}^{\infty} dk_1 \frac{e^{-ik_1 r + ik_1 r_o\cos(\Theta_o-\Theta) + i\sqrt{k_1^2+\kappa^2}\, r_o|\sin(\Theta_o-\Theta)|}}{\sqrt{k_1^2+\kappa^2}} \tag{A.5}$$

and let $k_1 = \kappa \sinh \tau$:

$$K_o(\kappa|\vec{r}-\vec{r}_o|) = \frac{1}{2} \int_{-\infty}^{\infty} d\tau \, e^{-i\kappa r \sinh \tau + i\kappa r_o \sinh \tau \cos(\Theta_o - \Theta)}$$

$$\cdot e^{i\kappa r_o \cosh \tau |\sin(\Theta_o - \Theta)|} \tag{A.6}$$

The desired equation (A.1) now follows from the identity

$$\sinh(\tau + i\xi) = \sinh \tau \cos \xi + i \cosh \tau \sin \xi$$

the definition of $|\vec{r}-\vec{r}_o|_\xi$ [equation (II.2.43)], and the fact that for $\xi \in (0,\pi)$, $\sin \xi = |\sin \xi|$.

We incidentally observe that according to equation (A.6), the τ-transform of $K_o(\kappa|\vec{r}-\vec{r}_o|)$ is, for general $\Theta-\Theta_o$,

$$\psi_o(\tau,\Theta) = \pi \, e^{i\kappa r_o \sinh[\tau + i\sigma(\Theta-\Theta_o)]} \tag{A.7}$$

where

$$\sigma \equiv \operatorname{sgn}[\sin(\Theta-\Theta_o)]$$

Taking the τ-transform of equation (A.3), we see that ψ_o should satisfy

$$\left(\frac{\partial^2}{\partial\tau^2} + \frac{\partial^2}{\partial\Theta^2}\right) \psi_o(\tau,\Theta) = -2\kappa r_o \cosh \tau \left\{ e^{i\kappa r_o \sinh \tau} \sum_n \delta(\Theta - 2n\pi) \right.$$

$$\left. + \, e^{-i\kappa r_o \sinh \tau} \sum_n \delta[\Theta - (2n+1)\pi] \right\} \tag{A.8}$$

provided equation (II.2.20) is correct; but the formula (A.8) is easily verified by performing the differentiations.

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