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Project M784

OFFICE OF NAVAL RESEARCH, U. S. NAVY DEPARTMENT
CONTRACT N8onr-55800, ONR PROJECT NR021-064

August 6, 1951

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ABSTRACT

The Fermi model for nucleon--nucleon collisions at high energies has been used to calculate the energy distribution, the energy dependence of the angular distribution, and the number of emitted pions as a function of primary energy and impact parameter. Both the case of purely pion emission and the case of nucleon and pion emission are considered. The results approach agreement with air-shower observations.

*Supported in part by the joint program of the ONR and AEC.

I. INTRODUCTION

Experimental observations of the density structure of air showers[1,2] seem to indicate that the density does not vary appreciably over distance of about one meter or less from the axis of the shower. This result might be explained either by a peculiar density distribution for a single shower core or by a multiplicity of cores with average separations somewhat less than one meter. The latter seems to be the more reasonable explanation.

The most promising model[3] is the customary one in which π° mesons are emitted in nuclear interactions of the primary cosmic rays and subsequently decay into photons. If one assumes that the emission is nearly isotropic in the center of mass system and merely exploits the relativistic contraction in explaining the observed separation of the air-shower cores, it is necessary to attribute events of shower energy 10^{13}-10^{14} ev to primaries of energy $\sim 10^{17}$ ev. The objection to this result is that only 0.01-0.1 per cent of the energy goes into π° mesons, in contradiction with estimates from observations at lower energies and with general arguments of equipartition of energy.

Fermi[4] has improved the situation by an order of magnitude in the energy. In this model, conservation of angular momentum already dictates a

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1. R. W. Williams, Phys. Rev. 74, 1689 (1948); J. M. Blatt, Phys. Rev. 75, 1584 (1948).

2. W. E. Hazen, in press.

3. H. W. Lewis, J. R. Oppenheimer, and S. A. Wouthuysen, Phys. Rev. 73, 127 (1948).

4. E. Fermi, Phys. Rev. 81, 683 (1951).

concentration of mesons near the collision axis in the center of mass system. The present discussion seeks to show that a more detailed analysis of the Fermi model of single nucleon--nucleon collisions results in an additional gain of nearly another order of magnitude. The angular distribution and number of π° mesons emitted "during" a collision will be calculated as a function of energy of the primary nucleons and of the mesons.

Another question that we shall seek to answer is whether or not a reasonable number of mesons of sufficient energy are emitted at the energy required by considerations of the angular separation.

The calculations of the angular distribution and spectral distribution of pi-mesons and nucleons produced in a high energy collision of two nucleons follows and extends the work done by E. Fermi[4] on the statistical theory of multiple meson production for extremely high energies. Only a bare outline of the method is contained here; for a fuller discussion of the ideas and limitations Fermi's paper should be consulted. The results contained in this paper are mainly obtained by an extension of the ideas presented in Fermi's paper.

We shall consider the question of the angular distribution and the number of particles produced as a function of energy of the primary nucleon and the secondary particles. In most of the work, we assume that statistical equilibrium is attained only by the pi-mesons and that the impact parameter is the median one. For comparison, however, calculations have been made for the case in which it is assumed that the incident nucleon energy is high enough to bring the mesons and the nucleon--antinucleon pairs into statistical equilibrium and also for one case with the impact parameter greater than median.

It is found that the angular distribution as a function of the energy of the secondaries is only very weakly dependent upon whether or not one assumes nucleon--antinucleon production in addition to meson production. Of course, the number of mesons produced is smaller if nucleons are produced. If the impact parameter is increased, the angular distribution is found to be more peaked, as Fermi has stated[4], and the proportion of higher-energy mesons is increased.

II. CALCULATIONS

We shall first review briefly the work of Fermi, using for the most part, his notation. Unprimed quantities refer to the center-of-mass system and primed ones to the laboratory system. The total energy in the center-of-mass system is to be W, deposited initially into a sphere of radius R, which we choose to be

$$R = \hbar/\mu c = 1.4 \times 10^{-13} \text{ cm} \quad ,$$

which has been Lorentz-contracted, due to the relative motion in the center-of mass system. Its volume, V, is taken as

$$V = (2Mc^2/W)\, \frac{4}{3}\, \pi R^3 \quad ,$$

where Mc^2 is the rest energy of a nucleon. In Fig. 1 is shown the flattened sphere and the initial directions of motion of the two nucleons (along a and b) imparting only an angular momentum, M_z, along the z-axis, which is perpendicular to the drawing and upward. For the extreme relativistic case, the volume is very flattened and the y dimension may be neglected in computing the z-component of angular momentum, Z, of a particle produced at the point x and z; so that

$$Z = xp \cos\theta = x \frac{w}{c} \eta \quad , \qquad (1)$$

$$\eta = \cos\theta \quad ,$$

where p is the momentum and w the energy of the emitted particle (also assumed to be relativistic) and θ is the angle between p and the y-axis.

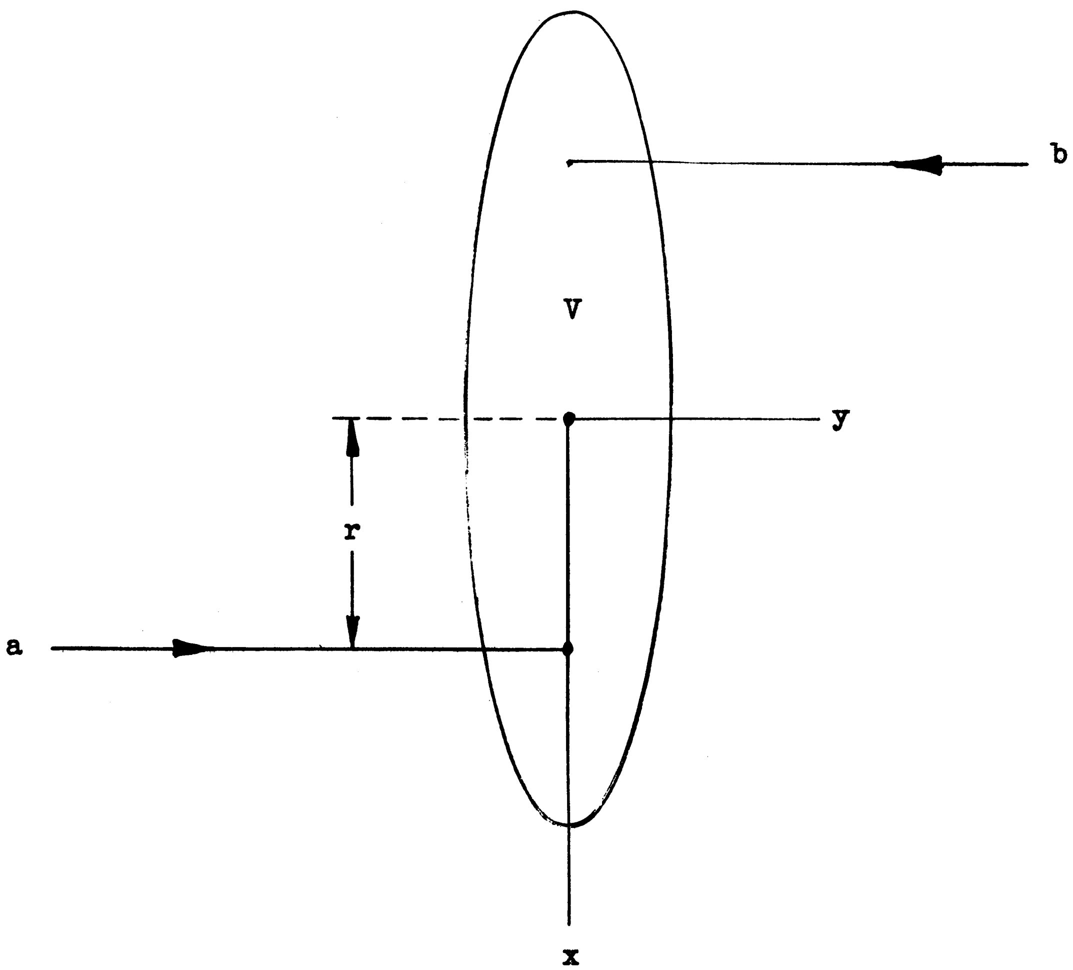

Fig. 1. Diagrammatic view of the interaction volume V of two colliding nucleons, a and b.

Using the thermodynamic approximation, allowable for high energies, and requiring conservation of energy, momentum, and angular momentum as described by Fermi, the average number of particles in a quantum state of energy, w, and angular momentum, Z, is

$$\frac{1}{e^{\beta w - \lambda Z} + 1} \tag{2}$$

if the particles obey Fermi-Dirac statistics, or

$$\frac{1}{e^{\beta w - \lambda Z} - 1} \tag{3}$$

if they obey Bose-Einstein statistics, where $\beta(= 1/kT)$ and λ are to be adjusted so that the total energy, W, and total angular momentum, M_z, have the correct values. We let

$$\begin{aligned} &\gamma = c\beta \;, \; \rho = \frac{\lambda R}{c\beta} \;, \; \frac{x}{R} = \xi \\ &\zeta = \gamma p \,(1 - \rho\eta\xi). \end{aligned} \tag{4}$$

Then using (1) and (4), the distribution laws (2) and (3) may be written as

$$F_{+} \;(\zeta) = \frac{1}{e^{\zeta} + 1} \;, \tag{5}$$

and

$$F_{-} \;(\zeta) = \frac{1}{e^{\zeta} - 1} \;,$$

respectively. The total number of particles in a volume element in phase space may then be written as

$$dn = \frac{Mc^2}{2\pi\hbar^3 W} \left[g_{+}\, F_{+}\,(\zeta) + g_{-}\, F_{-}\,(\zeta)\right] \,(R^2 - x^2)\; dx\; p^2 dp d\eta \;,$$

or

$$dn = A\,\frac{Mc^2}{W} \left[g_{+}\, F_{+}\,(\zeta) + g_{-}\, F_{-}\,(\zeta)\right] \;(1 - \xi^{\,2})\; d\xi\; p^2 dp i\eta \;, \tag{6}$$

where

$$A = \frac{R^3}{2\pi\hbar^3} \quad ,$$

and where g_+ and g_- are the statistical weights to be assigned to the particles obeying the Fermi-Dirac and Bose-Einstein statistics, respectively.

It is easily shown that the angular distribution for the emitted particles of all energies is independent of the type of statistics of the particles (except for a multiplicative constant), for by putting p^2dp in terms of ξ in (4), we see that

$$N(\eta) = \frac{AMc^2}{\gamma^3 W} (g_+ B_+ + g_- B_-) \int_{-1}^{+1} \frac{(1 - \xi^2)}{(1 - \rho\eta\xi)^3} d\xi \quad , \tag{7}$$

where

$$B_i = \int_0^\infty \xi^2 F_i (\xi) d\xi \quad , \quad i = +, - . \tag{8}$$

The angular distribution __does__ depend on the type of statistics when we consider the angular distribution of the emitted particles having an energy greater than a certain value.

The expression (6) above is identical with that of Fermi, except for the inclusion of the bracketed term to take into account the two possible types of statistics. The same is true of the Equations (7) through (16), since they are obtained from (6).

First, by integrating (7) we have,

$$N(\eta) = \frac{AMc^2}{\gamma^3 W} (g_+ B_+ + g_- B_-) f_4 (\rho\eta), \tag{9}$$

where

$$f_4(\alpha) = \frac{2}{\alpha^2(1 - \alpha^2)} - \frac{1}{\alpha^3} \ln \frac{1 + \alpha}{1 - \alpha} \tag{10}$$

Integrating (4) over all three variables, η between -1 and +1, the total number of emitted particles is found to be

$$N = \frac{A Mc^2}{\gamma^3 W} (g_+ B_+ + g_- B_-) f(\rho) \quad , \tag{11}$$

where

$$f(\rho) = \frac{1+\rho^2}{\rho^3} \ln \frac{1+\rho}{1-\rho} - \frac{2}{\rho^2} \quad . \tag{12}$$

Multiplying (4) by cp and integrating, the total energy is found to be

$$W = \frac{2}{3} \frac{A\ Mc^2}{\gamma^4 W} \quad (g_+ \beta_+ + g_- \beta_-)\ f_2\ (\rho) \quad , \tag{13}$$

where

$$f_2\ (\rho) = \frac{1}{\rho} \ln \frac{1+\rho}{1-\rho} + \frac{2}{1-\rho^2} \quad , \tag{14}$$

and

$$\beta_i = \int_0^\infty \zeta^3\ F_i\ (\zeta)\ d\zeta \quad , \quad i = +, -. \tag{15}$$

Multiplying (4) by $xp\eta = R\xi p\eta$ and integrating, the total angular momentum is found to be

$$M_z = \frac{ARMc^2}{\gamma^4 W} \quad (g_+ \beta_+ + g_- \beta_-)\ f_1\ (\rho)\ , \tag{16}$$

where

$$f_1\ (\rho) = \frac{2}{\rho^3} + \frac{4/3\rho}{1-\rho^2} - \frac{1+\rho^{2/3}}{\rho^4} \ln \frac{1+\rho}{1-\rho} \quad . \tag{17}$$

The quantity ρ, which is dependent upon the impact parameter, is then to be found by forming M_z/W (the ratio of (16) to (13)) and setting it equal to r/c (a relation for the total angular momentum found by referring to Fig. 1). Then it follows that

$$\frac{r}{R} = \frac{3}{2} \frac{f_1\ (\rho)}{f_2\ (\rho)} \quad . \tag{18}$$

Having found ρ for a particular collision, one finds the parameter γ, which is dependent upon ρ and W, from (13), (14), and (19).

The constants B_i and β_i, given by (8) and (15), are

$$B_+ = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} = 1.803,$$

$$B_- = 2 \sum_{n=1}^{\infty} \frac{1}{n^3} = 2.413, \tag{19}$$

$$\beta_+ = 6 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} = 5.682,$$
$$\beta_- = 6 \sum_{n=1}^{\infty} \frac{1}{n^4} = 6.494. \qquad (19)$$

Now, if in addition to the above, we wish to find the angular distribution of all particles having an energy greater than cp_o, we note that by expanding the functions F_i (ξ) (Equations (5) above), we may write (6) in the form,

$$dn = \frac{AMc^2}{W} \left(g_+ \sum_{n=1}^{\infty} (-1)^{n+1} e^{-n\xi} + g_- \sum_{n=1}^{\infty} e^{-n\xi} \right) (1 - \xi^2)\, d\xi\, p^2\, dp d\eta. \qquad (20)$$

Integrating over ξ from -1 to +1 and over p from p_o to ∞, we obtain

$$N(p_o,\eta) = \frac{2A}{\gamma^3} \frac{Mc^2}{W} \sum_{n=1}^{\infty} \left[g_+ (-1)^{n+1} + g_- \right] \left[\frac{\gamma p_o}{n^3 \rho^2 \eta^2} \left\{ \frac{e^{-na}}{a} + \frac{e^{-nb}}{b} \right\} - \frac{1}{n^3 \rho^3 \eta^3} \left\{ - Ei\,(-na) + Ei\,(-nb) \right\} \right], \qquad (21)$$

where

$$a = \gamma p_o\, (1 - \rho\eta) \text{ and } b = \gamma p_o\, (1 + \rho\eta).$$

The convergence of (21) is quite rapid for $\gamma p_o = cp_o/kT = 2$ or greater, and for $\gamma p_o = 50$ only the first term need be taken. Since the effect of different statistics is felt only in the even terms in the series in (21) which have the coefficient $(g_- - g_+)$ and since the first term in the series is the dominant one, the angular distribution as a function of γp_o is affected very little by the inclusion of particles obeying two types of statistics. Since the energy of an emitted particle corresponding to a given γp_o is a function of the temperature and therefore a function of the total energy, W, one must be careful about concluding that the angular distribution as a function of the energy of the emitted particles is insensitive to the use of two types of statistics. The question will be taken up in more detail in Part C.

Referring again to Fig. 1, it is easily shown that the probability, $P(r)$, of finding a collision in which the distance between nucleons is less than $2r$ is

$$P(r) = r^2/R^2.$$

For the **median** collision this probability is 1/2, so for this case

$$r = R/\sqrt{2}\ .$$

Thus, using (18) together with (17) and (14), one finds that $\rho = 0.959$.

Later, Part B, we wish to consider collisions for which we assume $\rho = 0.99$. This is not an exceptional case since one finds that $P(r) = 0.77$; so that there are still 23 per cent of the collisions whose impact parameters are larger.

For later reference we shall include here a few properties of the transformation from the center-of-mass system to the laboratory system for the extreme relativistic case.

If we let the energy of the incoming nucleon in the laboratory system be $W' = B\,Mc^2$, it is related to the total energy, W, in the center-of-mass system by the equation

$$W'/Mc^2 = B = 1/2(W/Mc^2)^2\ . \tag{22}$$

If v_o is the velocity with which the center-of-mass system is moving with respect to the laboratory system, let $B_o = 1/\sqrt{1-(v_o/c)^2}$. Then it can be shown that

$$B_o = \sqrt{\frac{B+1}{2}} \sim \sqrt{\frac{B}{2}}\ , \tag{23}$$

since $B >> 1$ for the extreme relativistic case. To transform the energy of the secondary particles to the laboratory system, one uses

$$cp_o' = B_o\,(1+\eta)cp_o\ . \tag{24}$$

The transformation of the angle of emission of a particle into the laboratory system may be shown to take place through

$$\theta' = \frac{1}{B_0}\sqrt{\frac{1-\eta}{1+\eta}} \tag{25}$$

for $\theta' \ll 1$ and $\eta > -0.9$.

A. Pi-meson excitation only; median impact parameter, $\rho = 0.959$.

If only meson production is assumed to take place, then the terms in all the preceding equations containing the statistical weight g_+ are dropped. The mesons obey Bose-Einstein statistics so that $g_- = 3$.

First one finds γ from (13) as described previously, and in this case

$$\gamma = 93.3 \ (Mc^2/W)^{1/2} \quad c/ev. \tag{26}$$

Using this value in (11), we find the total number of emitted particles, 1/3 of which are neutral mesons. Their number is given by

$$N_{\pi^0} = 0.115 \ (W/Mc^2)^{1/2} \quad . \tag{27}$$

The calculation of the angular distribution of particles whose energies are greater than a given cp_0 is made by using (9) and (10) if $cp_0 = 0$ or by using (21) if $cp_0 > 0$. The quantity $\gamma p_0 = cp_0/kT$ is used to specify the energy, cp_0, and is a function of the total energy through the temperature. The distributions for $\gamma p_0 = 0$, 2, 10, and 50 have been plotted in Fig. 2, where the ordinates are the number of mesons $N(p_0,\eta)$, in units of $0.054(W/Mc^2)^{1/2}$, and the absicissas are the corresponding values of η, the cosine of the angle of emission. The figure shows only the range in η from 0 to +1, the range from 0 to -1 being a mirror image of the former. The normalization is such that the areas under the curves are the numbers of any one of the three types of mesons, charged or neutral. The shape of the distributions is energy-independent, but the number of emitted particles

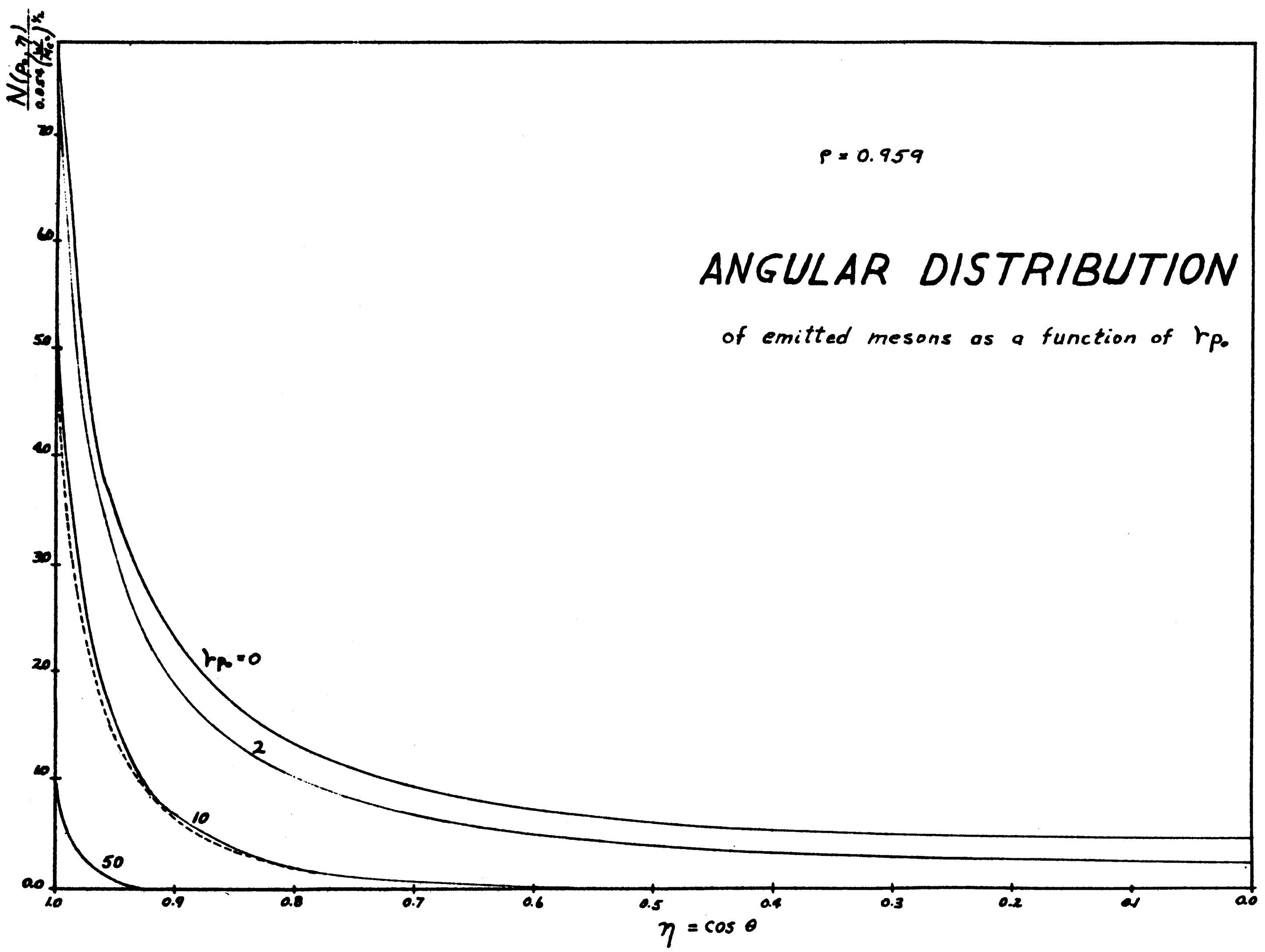

Fig. 2. The number of emitted mesons in units of 0.054 $(W/Mc^2)^{1/2}$ vs the cosine of the angle of emission (solid lines). See Section II-C for reference to the dotted-lined curve.

varies as the square root of the total energy in the center-of-mass system (as indicated by the scale of the ordinates of Fig. 2).

To interpret the curves from an experimental point of view we must translate γp_o into cp_o', the relativistic energy of an emitted particle in the laboratory system. After writing cp_o as $cp_o = (c/\gamma)\gamma p_o$, (24) may be used to give cp_o' as

$$cp_o' = \frac{B_o c}{\gamma} (1 + \eta)\gamma p_o \quad , \qquad (28)$$

which is a function of the angle of emission in the center-of-mass system. For the two groups of particles concentrated in the ranges of η from 0.82 to 1.0 and -0.82 to -1.0, Equation (28) shows that the latter group is more than an order of magnitude less energetic than the former group. This is the justification for considering only the former, more energetic group for the interpretations later in the paper.

The number of mesons having energy greater than cp_o, $N(\gamma p_o)$, was found by integrating the curves of Fig. 2 with a planimeter. The results are

$$N(50)/N(0) = 0.014, \; N(10)/N(0) = 0.23, \text{ and } N(2)/N(0) = 0.75. \qquad (29)$$

Then the cosines of the angles within which half the particles (considering only the range of η from 0 to +1) are emitted, $\eta_{1/2}$, was found by using the planimeter. They are given in Table I, together with the values of η for which the intensity drops to half that for $\eta = +1$.

Then, using (22), (23), and (26), Equation (28) becomes

$$cp_o' = 3.22 \times 10^8 (\gamma p_o) \; (W/Mc^2)^{3/2}, \qquad (30)$$

assuming $\eta = +1$, which is an approximation valid for the particles concentrated in the forward direction.

In Table II are listed the energies of the mesons in the laboratory system from (30) for the chosen values of γp_o and for several values of the total energy W. Also given are the total numbers of neutral pi-mesons from (27) and the energies of the initiating nucleon in the laboratory system, W', from (22).

B. Pi-meson excitation only; impact parameter larger than median, $\rho = 0.99$.

Qualitatively the angular distribution is more isotropic in the center of mass system for smaller than median impact parameters and more peaked at $\theta = 0°$ and 180° for larger impact parameters. As the impact parameter increases, more energy goes into rotation, lowering the temperature and thus decreasing the number of emitted particles.

The temperature is given by the new value of γ,

$$\gamma = c/kT = 129\ (Mc^2/W)^{1/2} \qquad c/ev, \tag{31}$$

which is to be compared with (26) for the previous case. The number of neutral mesons becomes

$$N_{\pi^\circ} = 0.0613\ (W/Mc^2)^{1/2}, \tag{32}$$

which is to be compared with (27).

The angular distributions for $\gamma p_o = 0$ and 50, found as in A, are shown as solid lines in Fig. 3, where for the sake of comparison the corresponding curves for the median impact parameter are shown as dotted lines. There is now a greater concentration of the particles in the forward direction, as shown by the increase of the cosine of the half angle $\gamma_{1/2}$ given in Table I, but the value of η for 1/2 intensity is changed only slightly.

The spectral distribution is changed in such a manner that there is a greater proportion of high-energy mesons. For example, by measuring the areas under the appropriate curves of Fig. 3, it is found that N(50)/N(0) = 0.12, compared to only 0.014 for the median collision. It is to be noted, however, that since γ has changed, the energy corresponding to $\gamma p_o = 50$ is not the same as found in Part A. We now have

$$cp_o' = 2.32 \times 10^8\ (\gamma p_o)\ (W/Mc^2)^{3/2} \quad ev, \tag{33}$$

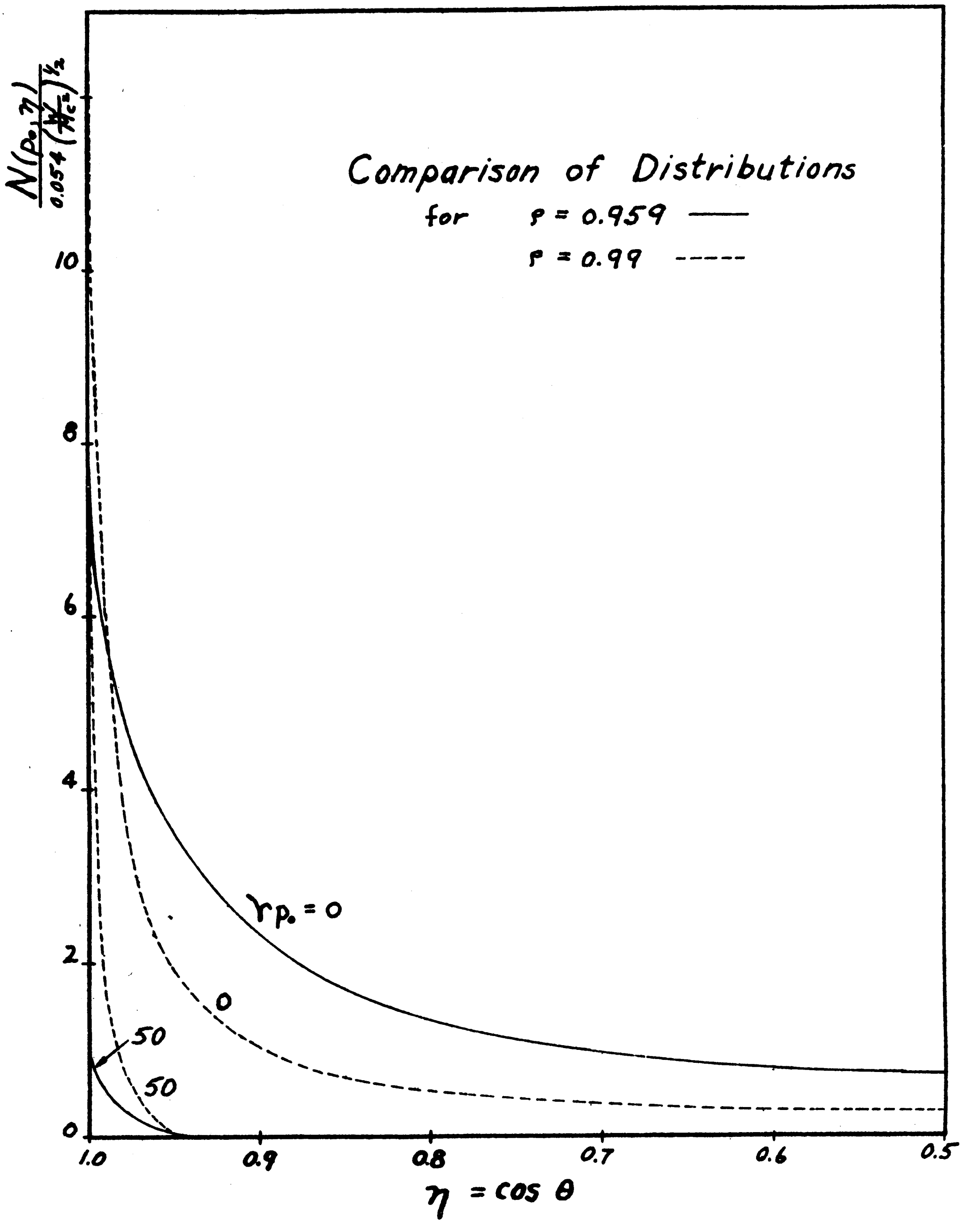

Fig. 3. Angular distributions of emitted particles assuming meson emission only (solid lines) and assuming meson and nucleon emission (dotted lines).

for which several numerical values are given in column 7 of Table II. Though these energies are lower than those found in column 4 of Table II, the above conclusion about the spectral distribution is still true.

C. Effect of assuming nucleon--antinucleon production in addition to pi-meson production; $\rho = 0.959$ and $\rho = 0.99$.

In Equations (9) through (21), we must now keep the terms containing g_+ since there are particles obeying each type of statistics. The statistical weight for the nucleons is $g_+ = 8$, and the statistical weight for the mesons is again $g_- = 3$.

Then, in the same manner as in parts A and B, we find,

$\rho = 0.959$	$\rho = 0.99$	
$\gamma = 126\ (Mc^2/W)^{1/2}$	$\gamma = 174\ (Mc^2/W)^{1/2}$ c/ev	(34)
$N_{\pi^\circ} = 0.038\ (W/Mc^2)^{1/2}$	$N_{\pi^\circ} = 0.015\ (W/Mc^2)^{1/2}$	
$= 1.2$	$= 0.48,$	

Where $W/Mc^2 = 10^3$ has been assumed in calculating the actual numbers of neutral mesons emitted.

As indicated previously, the angular distribution for particles with $cp_o > 0$ is independent of the statistics. Even for the higher energy particles we find that in the case here discussed the distribution is not very different from that found for meson emission only. This is shown by Fig. 2, where the distribution (for $\rho = 0.959$) of particles having γp_o greater than 10 is plotted as a broken line. The scale of the ordinates has been multiplied by $(126/93.3)^3$ for easier comparison of the shapes of the curves. If we take $W/Mc^2 = 10^3$ and emission in the forward direction, (28) and (34) give us the energy corresponding to $\gamma p_o = 10$ as $cp_o' = 7.53 \times 10^{13}$ ev. The corresponding energy for meson emission only is 10.15×10^{13} ev, so that the dotted curve should be compared with a solid-line curve for $\gamma p_o \cong 7$ (not plotted). The shape of the distribution is changing so little from $\gamma p_o = 2$ to $\gamma p_o = 10$ that one may say that the angular distribution for mesons of $cp_o' \sim 7 \times 10^{13}$ ev is little changed by assuming nucleon emission in addition to meson emission.

III. DISCUSSION

Counter observations of the lateral structure of air showers show that there is no multiplicity of singularities of comparable strength separated by distances from a few meters to 200 meters. Ionization chamber[5] measurements show no multiplicity for distances from one meter to about ten meters but there is evidence that either the Moliere distribution is wrong or that there is a multiplicity of singularities within distances of about one meter[2]. Cloud-chamber observations show no distinctly resolved singularities for separations less than one meter, but they do confirm the ion chamber observation that there is a plateau region with very little variation in particle density near the shower axis[2]. Since the cloud-chamber observations should be able to resolve two Moliere singularities separated by more than about 20 cm, the cloud-chamber observations of particle densities imply either that there are usually more than two shower cores with separations less than one meter or that the Moliere singularities are too sharp.

There is some evidence of multiple cores in the cloud chamber pictures[2] as evidenced by cases where there are two separate concentration areas for rays of energy $>10^{10}$ ev. Since from the theory of lateral spread of cascade showers the probability is about one-half for rays of energy $>10^{10}$ to lie within 20 cm of their axes[6], concentrations of such rays can be used to identify cores with separations of the order of 50 cm or more.

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[5] G. Cocconi, V. Tongiorgi, and K. Greisen, Phys. Rev. 76, 1020 (1949).

[6] L. Eyges and Fernbach, Phys. Rev. 82, 23 (1951); Phys. Rev. 82, 287 (1951).

The shower energies involved can be estimated as follows: Both the ion-chamber and cloud-chamber observations are for cases in which the particle densities are ~ 500 m^{-2} in a region of about 0.2 m^2 surrounding the shower axis. If we use the Moliere distribution to obtain the total number of electrons at the observation level and the cascade theory for longitudinal development to obtain the initiating energy therefrom, we obtain 3×10^{13} ev for a single ray or 10^{13} ev for each of four initiating rays.

A satisfactory model should therefore give a multiplicity of 10^{13} ev rays with angular separations of roughly 10^{-4} radians or less. The decay of π° mesons in flight generates two gamma rays with an angular separation of about 1.5×10^{-5} radians for photon energies of 10^{13} ev. This mechanism alone would result in a saddle 20 cm long in the density distribution, but it fails to explain a plateau. In order to obtain a plateau, we require a multiplicity of π° mesons themselves with an angular spread less than 8×10^{-5} (one meter separation at the observation level of 3000 meters). The angle with the primary axis would be 4×10^{-5} radians.

The relativistic transformation from the rest system (θ) to the observation system (θ') is, for small values of the angles,

$$\theta' = \theta/(2W'/Mc^2)^{1/2}$$

from (25) for $\eta \sim 1$. Thus we need a model that will provide values for θ and W'. As mentioned in the introduction a model that assumes isotropic emission in the center of mass system is unsatisfactory.

If we turn to Fermi's calculations, we find $\theta 1/2 = 0.6$ for the angle that includes one half of the forwardly-emitted mesons. We disregard the backwardly-emitted mesons because their energy in the laboratory system is an order of magnitude lower as previously shown. A primary energy of 10^{17} ev is now required in order to effect a contraction of the angle to 4×10^{-5} rad.

Since we really should be considering most probable events, the angle at 1/2 intensity is more appropriate. Fermi's distribution gives 0.28 for this angle and the corresponding primary proton energy is 2.4×10^{16} ev. Thus, we have a factor of ten reduction (from 3×10^{17} to 2.4×10^{16}) in the energy required to effect the required angular contraction. The fraction of initial energy going to π° mesons is therefore increased to 0.1--1 per cent.

Comparison with Detailed Analysis of the Fermi Model

The observed air-shower effects are generally attributable to the more energetic π° mesons ($E > 10^{13}$ ev), whereas the Fermi calculations were for mesons of all energies. Furthermore, the observed showers of a given minimum size are not necessarily caused predominantly by primary events whose average behavior corresponds to the minimum shower size; a more probable origin is one of the more abundant, lower-energy primaries that happens to make a collision with an impact parameter such that $\rho > 0.959$ with a consequent hardening of the average spectrum of emitted mesons, or a collision (with any impact parameter) in which the spectrum by chance is harder than average, with a consequent reduction in total number of emitted particles, or a collision in which π° mesons carry off more than their average share of the energy.

Thus we are justified in choosing $\rho \cong 0.99$ as the average impact parameter for collisions that contribute most strongly to a given-size shower when the emitted particles have an average energy distribution or in choosing a harder-than-average spectrum for collisions whose impact parameter distribution is normal.

Tables I and II show that for $\rho = 0.99$ we have an angle at half intensity of 0.14 ($\eta = 0.99$). The primary energy required to give $\Theta' = 4 \times 10^{-5}$ is 6×10^{15}. If pions only are excited, the number emitted at this primary energy in the forward direction in the CM system would be 1.8. Similarly, if nucleon emission is included, the average number would be one half. The average spectra are so hard

TABLE I

The Cosine of the Angle Within Which Half the Emitted Particles Fall ($\eta_{1/2}$) and the Cosine of the Angle at Half Intensity as a Function of γp_o and ρ.

γp_o	$\eta_{1/2}$	η at half intensity
	$\rho = 0.959$	
0	0.82	0.958
2	0.865	0.964
10	0.960	0.974
50	0.989	0.989
	$\rho = 0.99$	
0	0.89	0.991
50	0.991	0.994

TABLE II

Numerical Data Concerning the Emitted Particles for Collisions of Various Energies and Impact Parameters Predicted by the Fermi Theory.

			cp'_o (10^{13} ev) for				N_{π° (0)	
			$\rho = 0.959$ and γp_o =			$\rho = 0.99$ and		
B_o	W/Mc^2	W'(ev)	50	10	2	$\gamma p_o = 50$	$\rho = 0.959$	$\rho = 0.99$
50	10^2	$0.46 \cdot 10^{13}$	1.60	0.322	0.065	1.16	1.15	0.62
250	$5 \cdot 10^2$	$11.6 \cdot 10^{13}$	18.0	3.58	0.718	12.9	2.58	1.38
500	10^3	$46 \cdot 10^{13}$	50.8	10.15	2.03	36.5	3.64	1.95

that the numbers of particles of energy $>10^{13}$ or 10^{14} are not appreciably less than the above figures.

The second type of fluctuation that we mentioned above, namely, a fluctuation in the energy distribution, also leads to a smaller value of the angle of emission; for γp_o = 10 to 50, θ at half intensity is about 0.2 and the primary energy about 10^{16} ev. The number of particles would be about the same as before, but the pion energy for $\gamma p_o = 10$ has become rather high, i.e., 7×10^{14} ev.

IV. CONCLUSIONS

The detailed analysis of the Fermi model gives results that are almost compatible with the interpretation of existing observations. The main discrepancy is qualitatively similar to the case of the isotropic-emission model, i.e., the primary energy required to give the desired relativistic contraction in angle between π° mesons results in π° energies that are apparently too high. However, the estimates of the observed energies are probably too low because the Moliere density distribution is used to obtain the total number of electrons. The actual density distribution is probably somewhat flatter near the origin because the smaller showers are past their maxima.

The present discussion leads to an estimate of 10^{16} ev for the primary energy required to produce the smallest-size showers that were considered in the comparison with experiment (approximately 500 particles/m2 in a region of perhaps $0.2m^2$ surrounding the shower axis). The experimental intensity is about 2×10^{-8} $cm^{-2}sec^{-1}sterad^{-1}$. If we assume a power law for the integral primary spectrum between 1.5×1010 ev (where rocket measurements give 0.028 $cm^{-2}sec^{-1}sterad^{-1}$) and 10^{16} ev, there results for the primary spectrum $F(E) = 0.028(1.5 \times 1010/E)^{1.06}$ $cm^{-2}sec^{-1}sterad^{-1}$.

The exponent is large enough to effect an escape from an infinity in the total energy content even if the same exponent is assumed for greater energies. Actually, if one assumes that primary energies are linearly related with average shower energies, the exponent has increased to 1.5--1.9 for primary energies greater than $\sim 10^{16}$ ev.