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TRANSIENT THERMAL STRESS DISTRIBUTION IN AN INTERNAL ENERGY GENERATING, CIRCULAR ROD RESULTING FROM A STEP CHANGE IN THE INLET TEMPERATURE OF THE COOLANT

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NOMENCLATURE

A = Fluid Cross Sectional Area

A^* = $2\pi R^2/A$

a = Thermal Diffusivity

C = Constant Pressure Specific Heat

c = $\frac{1 + \nu}{1 - \nu}\alpha$

E = Young's Modulus

G = Shear Modulus

H = 2 x Biot Number

I_v = Modified Bessel Function of the First Kind of Order v

i = $\sqrt{-1}$

J_v = Bessel Function of the First Kind of Order v

K_v = Modified Bessel Function of the Second Kind of Order v

k = Index for Roots of Equation (28)

K = Thermal Conductivity

M = $A^* \frac{\rho_R}{\rho_F} \frac{C_R}{C_F}$ = Dimensionless Number

P = Peclet Number

q' = Rate of Internal Heat Generation per Unit Volume of Rod

R = Radius of Rod

r^* = Radial Space Variable

r = Dimensionless Radial Space Variable

S	=	Laplace Transform Variable
T	=	Laplace Transformed Rod Temperature
T*	=	Rod Temperature
t*	=	Time
t	=	$\frac{at^*}{R^2}$ = Fourier Number
V	=	Velocity
Z*	=	Axial Space Variable
Z	=	Z*/RP = Graetz Number
$\bar{Z}$	=	Z*/R = Dimensionless Axial Space Variable
α	=	Thermal Coefficient of Expansion
θ^*	=	Fluid Temperature
θ	=	Laplace Transformed Fluid Temperature
λ_K	=	Roots of Equation (29)
ν	=	Poisson's Ratio
ρ	=	Density
σ	=	Stress
σ	=	$\bar{\sigma}$/2GC = Dimensionless Stress
Bi	=	Biot Number
Nu	=	Nusselt Number
Pr	=	Prandtl Number
Re	=	Reynolds Number
St	=	Stanton Number

CHAPTER I

INTRODUCTION

In the design of many machine parts it is often found that transients, which result from a change in the operating conditions, are more severe than the maximum steady state conditions. From the thermal stress viewpoint, this occurs in many cases because sudden changes in the temperature cause larger temperature gradients and hence more severe stresses than the steady state ones. If transients are suspected to be significant, they should be analyzed and accounted for in the design of the part.

In the design of nuclear reactor fuel rods a number of fluctuations such as those of neutron flux, thermal power, coolant flow, and coolant temperature are known to exist. They are caused by the on-off action of automatic controls used to maintain steady operating conditions or to change from one operating level to another.

The problem which will be considered in this investigation is to find the transient stress distribution in a nuclear fuel rod resulting from a sudden change in the inlet temperature of the coolant which flows axially over the rod. Although only a particular problem will be solved, it is hoped that the method of solution will also be applicable to similar types of problems.

A general formulation of the temperature problem can be found in Carslaw and Jaeger[(1)*] . A general formulation for both the temperature and stress problem is presented in Parkus[(2)] , Boley and Weiner[(3)], and Nowacki[(4)] . The thermal stress problem neglecting inertial and mechanical

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* Numbers in parentheses refer to references in the bibliography.

coupling is discussed in most texts dealing with the theory of elasticity. See for instance Timoshenko and Goodier[5] and Sokolnikoff[6] .

Although the general formulation for the transient stress problem exists, there does not appear to be any published solutions for the two domain-circular geometry problem being considered in this investigation. If the fluid temperature distribution were known or somehow completely specified, then there are methods available for determining the temperature distribution in the rod. One of these is the Duhamel's Theorem which is discussed by Carslaw and Jaeger[1] . Another method is to expand the known surrounding temperature distribution in an infinite series and use the method presented by Parkus[2].

Until recently, this type of problem, two domain-circular geometry, had not been solved. In 1960, Arpaci[7] obtained an approximate temperature solution for a similar problem using an asymptotic Laplace transform solution. The associated thermal stresses, however, were not derived. The purpose of this work is to obtain the transient temperature and stress distributions for a similar problem which is outlined at the beginning of this chapter.

The general procedure for solving this problem is to take the Laplace transform with respect to the time variable of the governing partial differential temperature equations. The resulting equations are then solved exactly. Unfortunately the inverse transforms cannot be obtained exactly, but can be obtained approximately be finding asymptotic expansions for the temperature equations for large and small values of time. These temperature distributions are then used to determine the particular solution of the thermo-elastic problem. This reduces the problem to an ordinary isothermal boundary value problem which is solved by classical methods.

CHAPTER II

TEMPERATURE SOLUTION

A. General Theory

The problem under consideration is to find the temperature in a circular rod which generates internal energy in proportion to its volume and which is cooled on the surface by an annulus of fluid. A transient effect is introduced by a step change in the inlet temperature of the coolant.

A sketch of the system along with the formulation of the problem is presented in APPENDIX A. The basic assumptions which were made are:

1. Inertial and mechanical coupling effects in the rod are negligible.
2. The temperature and velocity of the coolant do not vary in the radial direction.
3. The axial conduction in both the rod and coolant is negligible.
4. The convection heat tranfer coefficient and physical properties of the rod and coolant are constant.
5. The outer wall of the coolant passage is insulated.
6. The coolant is an incompressible fluid.
7. Internal energy is generated in proportion to the volume at a constant rate.

The first assumption is classical and is discussed in detail in References[2,3,4].

The third assumption is related to the Peclet number of the coolant. In general, if the Peclet number is greater than 100 the axial conduction may be neglected. One might note at this time that for fully turbulent flow (Re > 10,000) that most of the common liquid metals are covered by this solution. In the transition region (2,100 < Re < 10,000) the Peclet number for liquid metals is less than 100 and hence the validity of the solution becomes doubtful.

Assumption seven is most likely not a valid one in the case of an actual fuel rod. The primary reason for this assumption was to simplfy the experimental apparatus.

Using the above assumptions and applying the First Law of Thermodynamics to a control volume for the coolant and a system for the rod the basic differential equations of the problem are obtained. In APPENDIX A the problem is further simplified by separating it into a steady state part which deals with the internal energy generation and a transient part which deals with a rod, initially at zero temperature, subjected to a unit step change in the temperature of the inlet coolant.

The resulting steady state formulation is:

$$\nabla_r^2 T_{ss} + 1 = 0 \tag{1}$$

$$\frac{d\theta_{ss}}{dz} = -M \frac{\partial T_{ss}}{\partial r}(1,z) \tag{2}$$

where

$$\nabla_r^2 \phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \phi}{\partial r}\right) \tag{3}$$

subject to the following boundary conditions:

$$\theta_{ss}(0) = 0 \tag{4a}$$

$$\frac{\partial T_{ss}}{\partial r}(0, z) = 0 \tag{4b}$$

$$\frac{\partial T_{ss}}{\partial r}(1, z) = H\left[\theta_{ss}(z) - T_{ss}(1, z)\right] \tag{4c}$$

T_{ss} and θ_{ss} are the dimensionless temperatures of the rod and coolant respectively, H a modified Biot number, Z the Graetz number, M a dimensionless number, and r the dimensionless radius.

The resulting transient formulation is:

$$\nabla_r^2 T^* = \frac{\partial T^*}{\partial t} \tag{5}$$

$$\frac{\partial \theta^*}{\partial z} + \frac{\partial \theta^*}{\partial t} = -M \frac{\partial T^*}{\partial r}(1, z, t) \tag{6}$$

Subject to the following initial and boundary conditions:

$$\theta^*(0,t) = 1 \tag{7a}$$

$$\frac{\partial T^*}{\partial r}(0,z,t) = 0 \tag{7b}$$

$$\frac{\partial T^*}{\partial r}(1,z,t) = H\left[\theta^*(z,t) - T^*(1,z,t)\right] \tag{7c}$$

$$T^*(r,z,0) = 0 \tag{7d}$$

$$\theta^*(z,0^+) = 0 \tag{7e}$$

where T^* and θ^* are dimensionless rod and fluid temperatures respectively and t the Fourier number. The other terms are defined above.

B. <u>Steady State Solution</u>

The steady state temperature solution is obtained by a direct integration of the two equations. The resulting steady state rod temperature is:

$$T_{ss} = \frac{1}{2}\left[\frac{1}{H} + \frac{1}{2} + Mz - \frac{r^2}{2}\right] \tag{8}$$

and for the fluid

$$\theta_{ss} = \frac{Mz}{2} \tag{9}$$

C. Transient Solution

1. Exact Solution

The general solution to the problem can be found by taking the Laplace transform with respect to time of Equations (5), (6), and (7). Thus multiplying these equations by e^{-St}, where S is the complex Laplace parameter, and integrating with respect to time from 0 to ∞, the following equations are obtained:

$$\nabla_r^2 T = \frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = ST \tag{10}$$

$$\frac{d\theta}{dz} + S\theta = -M\frac{dT}{dr}(1, z, S) \tag{11}$$

subject to:

$$\theta(0, S) = 1/S \tag{12a}$$

$$\frac{\partial T}{\partial r}(0, z, S) = 0 \tag{12b}$$

$$\frac{\partial T}{\partial r}(1, z, S) = H\left[\theta(z, S) - T(1, z, t)\right] \tag{12c}$$

$$\lim_{s\to\infty}\left[s\,T(r,z,s)\right] = 0 \tag{12d}$$

$$\lim_{s\to\infty}\left[s\,\theta(z,s)\right] = 0 \tag{12e}$$

where

$$T(r,z,s) = \int_0^\infty T^*(r,z,t)\,e^{-st}\,dt \tag{13}$$

and

$$\theta(z,s) = \int_0^\infty \theta^*(z,t)\,e^{-st}\,dt \tag{14}$$

The solution to Equation (10) is well known and can be written as:

$$T(r,z,s) = f(s,z)\,I_0(\sqrt{s}\,r) + g(s,z)\,K_0(\sqrt{s}\,r) \tag{15}$$

where $I_0(x)$ and $K_0(x)$ are modified Bessel functions of the first and second kind of order zero. Using Equation (12b) and noting that $K_0(o)$ and $\frac{dK_0}{dr}(0)$ are not finite one finds that

$$g(s,z) \equiv 0 \tag{16}$$

Substituting the remaining part of Equation (15) into Equation (12c) yields

$$T = \frac{H\,\theta\,I_0(\sqrt{s}\,r)}{\sqrt{s}\,I_1(\sqrt{s}) + H I_0(\sqrt{s})} \tag{17}$$

If Equation (17) is differentiated with respect to r and substituted into Equation (11), a simple integration of this later equation yields

$$\theta = A e^{-sz} e^{-\frac{MH\sqrt{s}\,I_1(\sqrt{s})\,z}{\sqrt{s}\,I_1(\sqrt{s}) + H I_0(\sqrt{s})}} + B \tag{18}$$

Making use of Equations (12a) and (12e) one finds that

$$A = 1/s \qquad\qquad B = 0 \tag{19}$$

Thus

$$\theta = \frac{e^{-sz} e^{-\frac{MH\sqrt{s}\,I_1(\sqrt{s})\,z}{\sqrt{s}\,I_1(\sqrt{s}) + H I_0(\sqrt{s})}}}{s} \tag{20}$$

Since there are no branch points the exact temperature solutions can be written as

$$T(r,z,t) = \oint_c \frac{H I_0(\sqrt{s}\,r)\, e^{-sz} e^{-\frac{MH\sqrt{s}\,I_1(\sqrt{s})\,z}{\sqrt{s}\,I_1(\sqrt{s}) + H I_0(\sqrt{s})}} e^{+st}}{s\left[\sqrt{s}\,I_1(\sqrt{s}) + H I_0(\sqrt{s})\right]}\, ds \tag{21}$$

$$\theta(z,t) = \oint_c \frac{e^{-sz} e^{-\frac{MH\sqrt{s}\,I_1(\sqrt{s})\,z}{\sqrt{s}\,I_1(\sqrt{s}) + H I_0(\sqrt{s})}} e^{+st}}{s}\, ds \tag{22}$$

where C is a path which lies to the right of all singularities. Because of the complexity of Equations (21) and (22) an exact solution of the problem was not obtained.

Various asymptotic solutions of Equations (21) and (22), however, can be obtained by noting that large values of S correspond to small values of time and that small values of S correspong to large values of time. Therefore, if asymptotic expansions of the various functions for large and small values of the Laplace variable, S, are used the inverse transforms of these expansions will be asymptotic to the exact time solution.

2. Short Time Temperature Solution

There are a number of ways in which asymptotic solutions can be obtained from the exact equations. For example one could expand only a single term of the total function and obtain a solution or alternatively expand a different or even all terms of the function and get a different solution. Depending on the approach used, various degrees of approximation are obtained.

Initially an attempt was made to solve this problem by treating all terms of equal importance and expanding the total function into several relatively simple terms. It was found that this solution was valid only for extremely small (or large) values of time. Since the peak stresses do not occur at either of these extremes, this type of solution was abandoned.

The approach used here is to retain as much as possible of the basic form and expand only those terms which would prevent a solution. Thus the exponential term involving the Bessel functions is expanded into a suitable form. In particular this is written as

$$e^{+X} = 1 - X + \frac{X^2}{2} - \cdots \tag{23}$$

Equation (20) can be rewritten as

$$\theta = \frac{e^{-MHz}\, e^{-sz}\, e^{+\frac{MH^2 I_0(\sqrt{s})\,z}{\sqrt{s}\,I_1(\sqrt{s}) + H I_0(\sqrt{s})}}}{s} \tag{24}$$

As shown in APPENDIX B the term $\dfrac{I_0(\sqrt{s})}{\sqrt{s}\,I_1(\sqrt{s}) + H I_0(\sqrt{s})}$

for large values of S (small time) approaches zero. If the solution is restricted to finite values of S and small values of time, and only two terms of Equation (23) are used, the Equations (17), (23), and (24) can then be used to write the rod temperature as

$$T(r, z, s) \cong \frac{H I_0(\sqrt{s}\,r)\, e^{-MHz}\, e^{-sz}}{s\left(\sqrt{s}\,I_1(\sqrt{s}) + H I_0(\sqrt{s})\right)} \left[1 + \frac{MH^2 z\, I_0(\sqrt{s})}{\sqrt{s}\,I_1(\sqrt{s}) + H I_0(\sqrt{s})} \right] \tag{25}$$

The solution can be simplified by defining T as

$$T = T_1 + T_2 \tag{26}$$

where

$$T_1 = \frac{H I_0(\sqrt{s}\,r)\, e^{-sz}\, e^{-MHz}}{s\left(\sqrt{s}\,I_1(\sqrt{s}) + H I_0(\sqrt{s})\right)} \tag{27}$$

$$T_2 = \frac{MH^3 I_0(\sqrt{S}\,r) I_0(\sqrt{S}) Z\, e^{-SZ} e^{-MHZ}}{S\left[\sqrt{S}\, I_1(\sqrt{S}) + H I_0(\sqrt{S})\right]^2} \tag{28}$$

The inverse transform of Equation (27) is not new and can be found in Carslaw and Jaeger[1]. For sake of completeness a brief account of the procedure is presented here. Using the appropriate contour and noting that T_1 has poles at $S = o$, and $\sqrt{S} = i\,\lambda_K$ where $i = \sqrt{-1}$ and λ_K are the roots of the equation:

$$\sqrt{S}\, I_1(\sqrt{S}) + H I_0(\sqrt{S}) \tag{29}$$

$T_1{}^*$ can be written as

$$T_1^*(r,z,t) = \mathrm{Res}\left[T_1(r,z,s)e^{st}, 0\right] + \sum_{K=1}^{\infty} \mathrm{Res}\left[T_1 e^{st}, -\lambda_K^2\right] \tag{30}$$

$$= \lim_{s\to 0}\left[sT_1 e^{st}\right] + \lim_{s\to -\lambda_K^2}\left[\sum_{K=1}^{\infty}(s+\lambda_K^2)T_1 e^{st}\right]$$

which after some reorganization can be written as

$$T_1^*(r,z,t) = e^{-MHz}\left[1 - 2H\sum_{K=1}^{\infty}\frac{J_0(\lambda_K r)\, e^{-\lambda_K^2(t-z)}}{(H^2+\lambda_K^2)\, J_0(\lambda_K)}\right] \tag{31}$$

for $t - Z > o$ and where $J_o(x)$ is the Bessel function of the first kind of order zero. Although it will not be explicitly stated in later equations, it will always be assumed that the equations are identically zero for $t - Z < o$.

In proceeding to solve for T_2 it is noted that there are at least three possible methods of solution. One of these is to expand the second term in the bracket of Equation (25) asymptotically, take its inverse transform, and apply the convolution theory to combine it with the T_1 solution. The result of such an approach leads to a power series in terms of the Error function:

$$Erf\,(n) = \frac{2}{\sqrt{\pi}} \int_0^n e^{-x^2}\,dx \tag{32}$$

Since this will not lend itself readily to a solution of the stress problem this form is not pursued further. A second method would be to take the inverse transform of the second term in Equation (25) and again use the convolution principle to combine it with T_1. This leads to a double power series solution which is identical, mathematically, to the next solution which is presented.

The method which is used here is to solve Equation (28) directly by determining the necessary residues. Using Equation (28) it is found that T_2 has a simple pole at $S = o$, and a second order pole at $\sqrt{S} = i\,\lambda_K$ where λ_K is defined by Equation (29). The residue at the origin presents no problem and is simply

$$\operatorname{Res}\left[T_2 e^{st}, 0\right] = MH^3 z\, e^{-MHz} \tag{33}$$

The residue of the second order poles can be obtained by noting that

$$\operatorname{Res}\left[f(S), Z_0\right] = \frac{6A'B'' - 2AB'''}{3\,[B'']^2} \tag{34}$$

where $f(S) = \dfrac{A(S)}{B(S)}$, $A(S)$ and $B(S)$ are analytic in the neighborhood of Z_0 , $A(Z_0) \neq 0$ and $B(S)$ has a second order pole at Z_0.

In the present problem

$$A = \frac{I_0(\sqrt{S}\,r)\, I_0(\sqrt{S})\, e^{st}}{S}$$

(35)

$$B = \left[\sqrt{S}\, I_1(\sqrt{S}) + H I_0(\sqrt{S})\right]^2$$

Performing the required differentiations and evaluating the derivatives at $\sqrt{S} = i\,\lambda_K$ gives

$$A = -\,\frac{J_0(\lambda_K r)\, J_0(\lambda_K)\, e^{-\lambda_K^2 t}}{\lambda_K^2} \tag{36}$$

$$A' = -\frac{J_0(\lambda_K)e^{-\lambda_K^2 t}}{\lambda_K^4}\left[\left(1+\frac{H^3}{2}+\lambda_K^2 t\right)J_0(\lambda_K r)+\frac{\lambda_K r\, J_1(\lambda_K r)}{2}\right]$$

$$B'' = \frac{(H^2+\lambda_K^2)^2\, J_0(\lambda_K)^2}{2\,\lambda_K^4}$$

$$B''' = \frac{3H^2(H^2+\lambda_K^2)\, J_0(\lambda_K)^2}{2\,\lambda_K^6} \tag{36}$$

If these are then substituted into Equation (34) the second approximation of the temperature is found to be

$$T_2^*(r,z,t) = MHz\, e^{-MHz} - 4MH^3 z\, e^{-MHz}\sum_{K=1}^{\infty}\left[\alpha_1(t-z,K)\, J_0(\lambda_K r)\right. \tag{37a}$$

$$\left. + \alpha_2(K)\, r\, J_1(\lambda_K r)\right] e^{-\lambda_K^2 (t-z)}$$

where

$$\alpha_1(t-z,K) = \frac{\lambda_K^2 + (H^2+\lambda_K^2)(H/2 + \lambda_K^2(t-z))}{(H^2+\lambda_K^2)^3\, J_0(\lambda_K)} \tag{37b}$$

$$\alpha_2(K) = \frac{\lambda_K}{2(H^2+\lambda_K^2)^2\, J_0(\lambda_K)} \tag{37c}$$

It is interesting to note that T_1^* could be obtained in a much more direct manner by simply using the fact that $T^* << \theta^*$ for small values of time. Thus Equation (7c) could be written as

$$\frac{\partial T^*}{\partial r}(1,z,t) = H\left[\theta^*(z,t) - T^*(1,z,t)\right] \cong H\theta^*(z,t) \tag{38}$$

Proceeding in a manner similar to that for T_1^* and T_2^* the fluid temperature is found to be

$$\theta_1^*(z,t) = e^{-MHz} \tag{39}$$

$$\theta_2^*(z,t) = MHz\, e^{-MHz}\left[1 - 2H\sum_{K=1}^{\infty}\frac{e^{-\lambda_K^2(t-z)}}{(H^2+\lambda_K^2)}\right] \tag{40}$$

3. Large Time Temperature Solution

To obtain a solution for the temperature problem, which will be valid for large values of time, a procedure similar to the short time solution is used. In this case the Laplace variable, S , now assumes small values. As is shown in APPENDIX B, for large values of time Equation (20) must be used instead of Equation (24) in order to be able to truncate the expansion of the exponential term.

Using Equations (17), (20), and (23) the rod temperature can be written as

$$T(r,z,S) \cong \frac{H I_0(\sqrt{S}\,r)\,e^{-Sz}}{S\left[\sqrt{S}\,I_1(\sqrt{S}) + H I_0(\sqrt{S})\right]}\left[1 - \frac{MHz\sqrt{S}\,I_1(\sqrt{S})}{\sqrt{S}\,I_1(\sqrt{S}) + H I_0(\sqrt{S})}\right] \tag{41}$$

The solution is simplified by defining T as

$$T = T_3 + T_4 \tag{42}$$

where

$$T_3 = \frac{H I_0(\sqrt{s}\,r)\, e^{-sz}}{s\left[\sqrt{s}\, I_1(\sqrt{s}) + H I_0(\sqrt{s})\right]} \tag{43}$$

$$T_4 = -\frac{M H^2 z \sqrt{s}\, I_1(\sqrt{s})\, I_0(\sqrt{s}\,r)\, e^{-sz}}{s\left[\sqrt{s}\, I_1(\sqrt{s}) + H I_0(\sqrt{s})\right]^2} \tag{44}$$

In a similar manner the fluid temperature can be written as

$$\theta \cong \frac{e^{-sz}}{s}\left[1 - \frac{M H z \sqrt{s}\, I_1(\sqrt{s})}{\sqrt{s}\, I_1(\sqrt{s}) + H I_0(\sqrt{s})}\right] \tag{45}$$

$$\theta = \theta_3 + \theta_4 \tag{46}$$

$$\theta_3 = \frac{e^{-sz}}{s} \tag{47}$$

$$\theta_4 = -\frac{M H z \sqrt{s}\, I_1(\sqrt{s})\, e^{-sz}}{s\left[\sqrt{s}\, I_1(\sqrt{s}) + H I_0(\sqrt{s})\right]} \tag{48}$$

By comparing Equations (27) and (43) one notes that they differ only be a term e^{-MHZ}. Thus T_3 can be obtained directly by multiplying Equation (31) by e^{+MHZ}.

$$T_3^*(r,z,t) = 1 - 2H \sum_{K=1}^{\infty} \frac{J_0(\lambda_K r)e^{-\lambda_K^2(t-z)}}{(H^2+\lambda_K^2)\,J_0(\lambda_K)} \tag{49}$$

The inverse transform of Equation (47) is simply

$$\theta_3^*(z,t) = 1 \tag{50}$$

The inverse transform of Equation (48) is also readily obtained by noting that this equation is simply

$$-Mz \frac{\partial T_3}{\partial r}(1,z,t) \tag{51}$$

where T_3 is defined by Equation (43). Therefore θ_4^* can be found by differentiating Equation (49), evaluating this at r = 1, and multiplying this result by $-MZ$. Thus one obtains for the inverse transformation of Equation (48)

$$\theta_4^*(z,t) = -2MHz \sum_{K=1}^{\infty} \frac{\lambda_K J_1(\lambda_K)e^{-\lambda_K^2(t-z)}}{(H^2+\lambda_K^2)\,J_0(\lambda_K)} \tag{52}$$

This can be further simplfied by noting from Equation (29) that

$$\lambda_K J_1(\lambda_K) = H J_0(\lambda_K) . \tag{53}$$

Thus Equation (52) reduces to

$$\theta_4^*(z,t) = -2MH^2 z \sum_{K=1}^{\infty} \frac{e^{-\lambda_K^2(t-z)}}{(H^2+\lambda_K^2)} \tag{54}$$

The second term of the rod temperature may be obtained by using the method of residues as was done for the short-time solution. T_4 has second order poles at $\sqrt{S} = i\,\lambda_K$ where λ_K is defined by Equation (29). Unlike the previous cases however there is no pole at the origin. Thus T_4^* can be written as

$$T_4^*(r,z,t) = \sum_{K=1}^{\infty} Res\left[T_4 e^{+st}, -\lambda_K^2\right] \tag{55}$$

Using the same nomenclature as is found in Equation (34) one finds for this term that

$$A = \frac{\sqrt{s}\, I_1(\sqrt{s})\, I_0(\sqrt{s}\, r)\, e^{st}}{S} \tag{56}$$

$$B = \left[\sqrt{s}\, I_1(\sqrt{s}) + H I_0(\sqrt{s})\right]^2$$

Performing the required differentiations and evaluating these derivatives at $\sqrt{S} = i\,\lambda_K$ yields

$$A = \frac{H\,J_0(\lambda_K)\,J_0(\lambda_K r)\,e^{-\lambda_K^2 t}}{\lambda_K^2}$$

$$A' = \frac{J_0(\lambda_K)\,e^{-\lambda_K^2 t}}{\lambda_K^2}\left[\left\{(1+\lambda_K^2 t)\frac{H}{\lambda_K^2} - \frac{1}{2}\right.\right. \tag{57}$$

$$B'' = \frac{(H^2+\lambda_K^2)^2\,J_0(\lambda_K)^2}{2\,\lambda_K^4}$$

$$B''' = \frac{3H^2\,(H^2+\lambda_K^2)\,J_0(\lambda_K)^2}{2\,\lambda_K^6}$$

Using Equations (34) and (57) yields

$$T_4^*(r,z,t) = MHZ\sum_{K=1}^{\infty}\left[\alpha_3(t-z,K)\,J_0(\lambda_K r) + \alpha_4(K)\,r\,J_1(\lambda_K r)\right]e^{-\lambda_K^2(t-z)} \tag{58}$$

where

$$\alpha_3(t-z,K) = -\frac{4H^2\lambda_K^2\left[1+\left\{(t-z)-\frac{1}{2H}\right\}\left\{H^2+\lambda_K^2\right\}\right]}{(H^2+\lambda_K^2)^3\,J_0(\lambda_K)} \tag{59}$$

$$\alpha_4(K) = -\frac{2H^2\lambda_K}{(H^2+\lambda_K^2)^2\,J_0(\lambda_K)} \tag{60}$$

D. Discussion of the Temperature Solution

The transient temperature solution shows that the basic shape of the temperature distribution is controlled by the following four parameters: r , $t - Z$, H , and MHZ . Figures 1 and 2 show the characteristic shape of the rod temperature for various values of the r and $t - Z$ parameters. As is expected the rod temperature at any r increases from zero and approaches asymptotically the new steady state fluid temperature. One also sees that the maximum temperature gradient occurs at the surface and at small values of time. For large values of time the gradient approaches zero as the rod reaches its new steady condition.

The third parameter, H , is two times the classical Biot number. This term represents the ratio of internal to external thermal resistance. As is well known, if the external resistance is very large ($Bi < .1$), one may assume a thermally lumped mass in the radial direction. If on the other hand the internal resistance is large ($Bi > .1$) one must take into account the radial heat conduction. Since this report deals with the latter case and the emphasis is on significant thermal stresses, relatively large values of H , say $H > 2$ ($Bi > 1$) are of primary interest. One might also note at this point that the radial temperature gradients are much more significant than the axial ones. Hence H is the primary factor which determines the magnitude of the thermal stresses. Figure 3 shows the effect of H on the surface temperature at various times.

The fourth parameter, MHZ , can be analyzed more easily by defining new terms. Let A^* = two times the ratio of the rod to the coolant

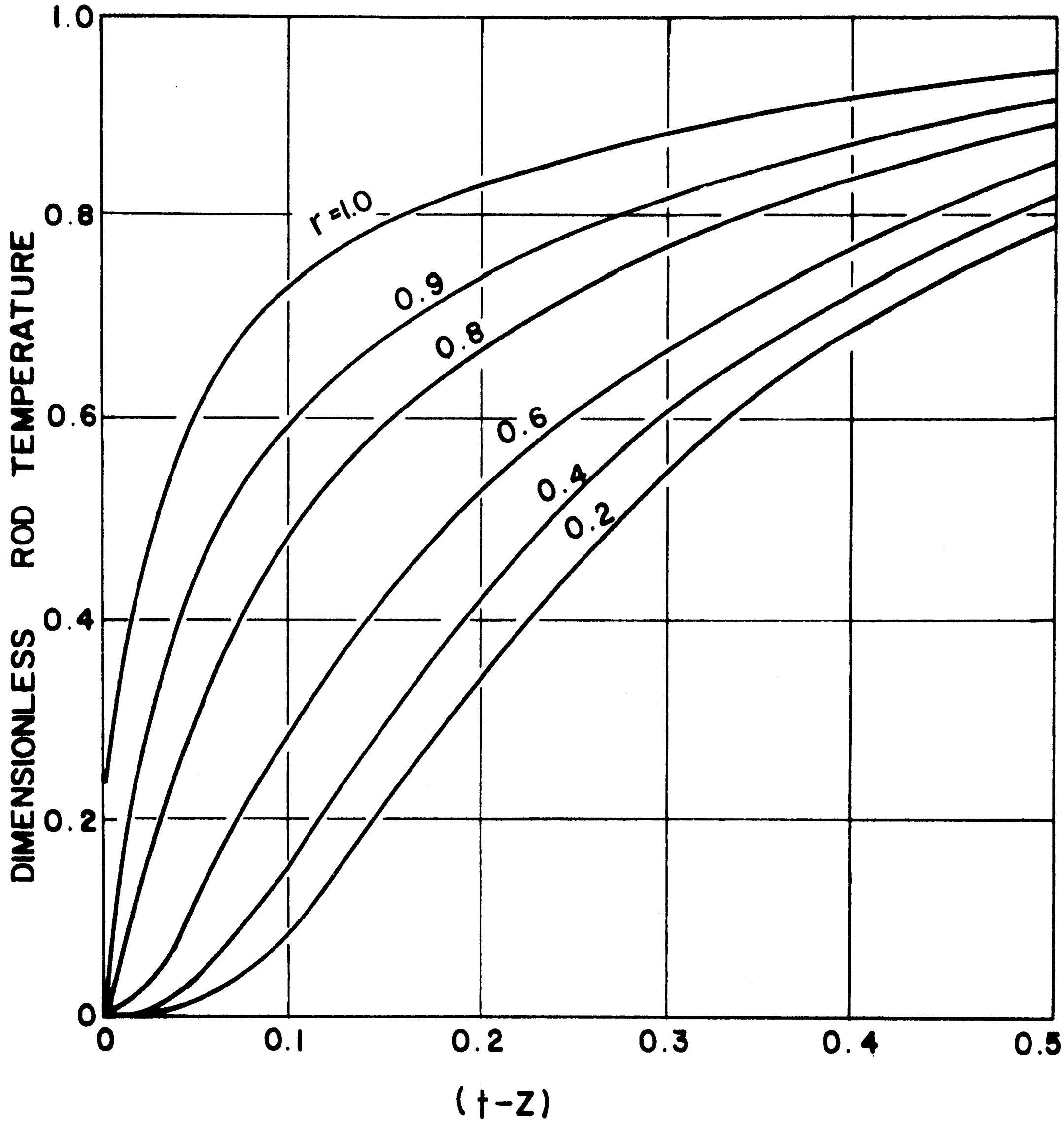

Figure 1. Transient Rod Temperature Distribution I H = 4.7, Z = 0.

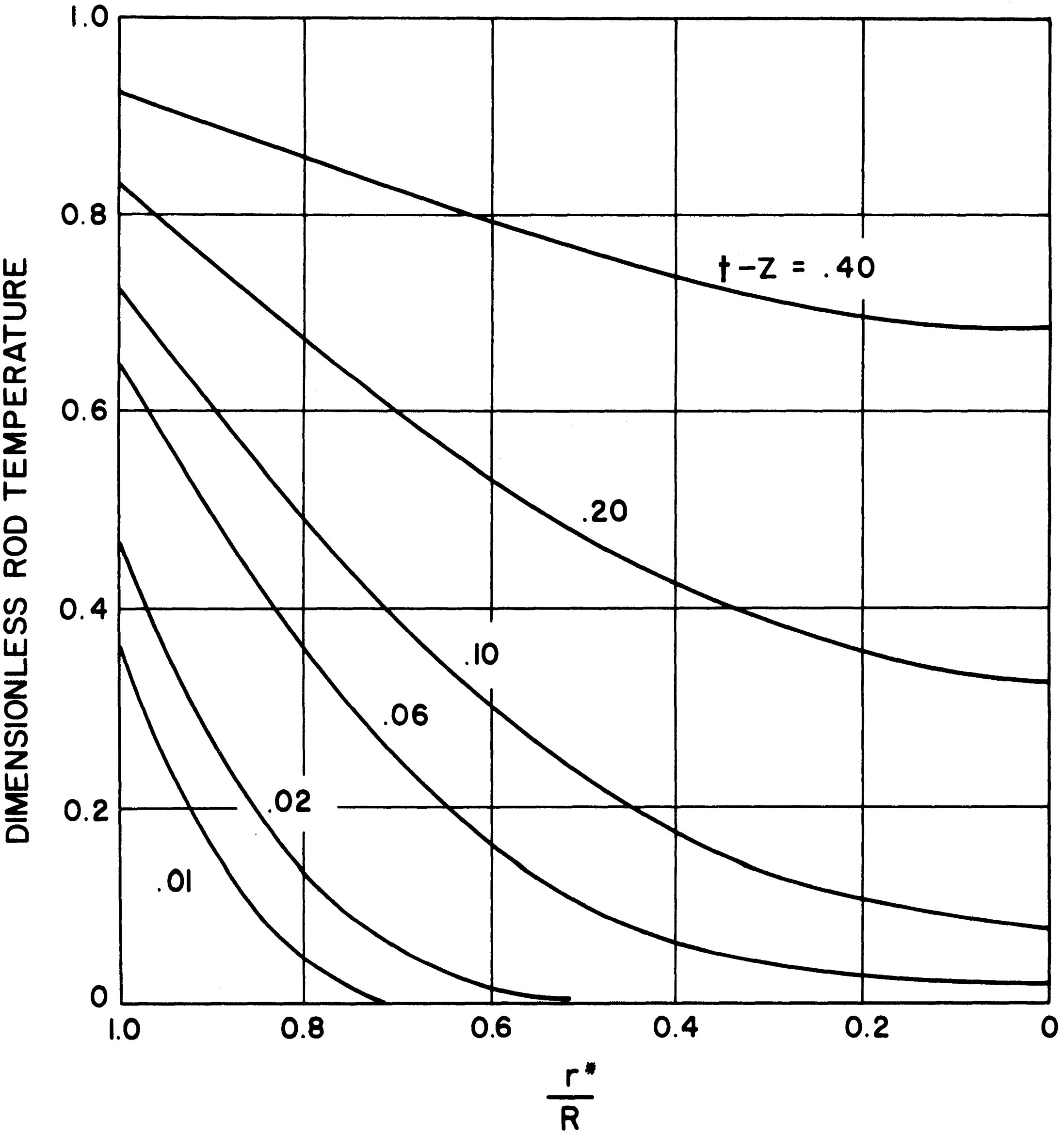

Figure 2. Transient Rod Temperature Distribution II H = 4.7, Z = 0.

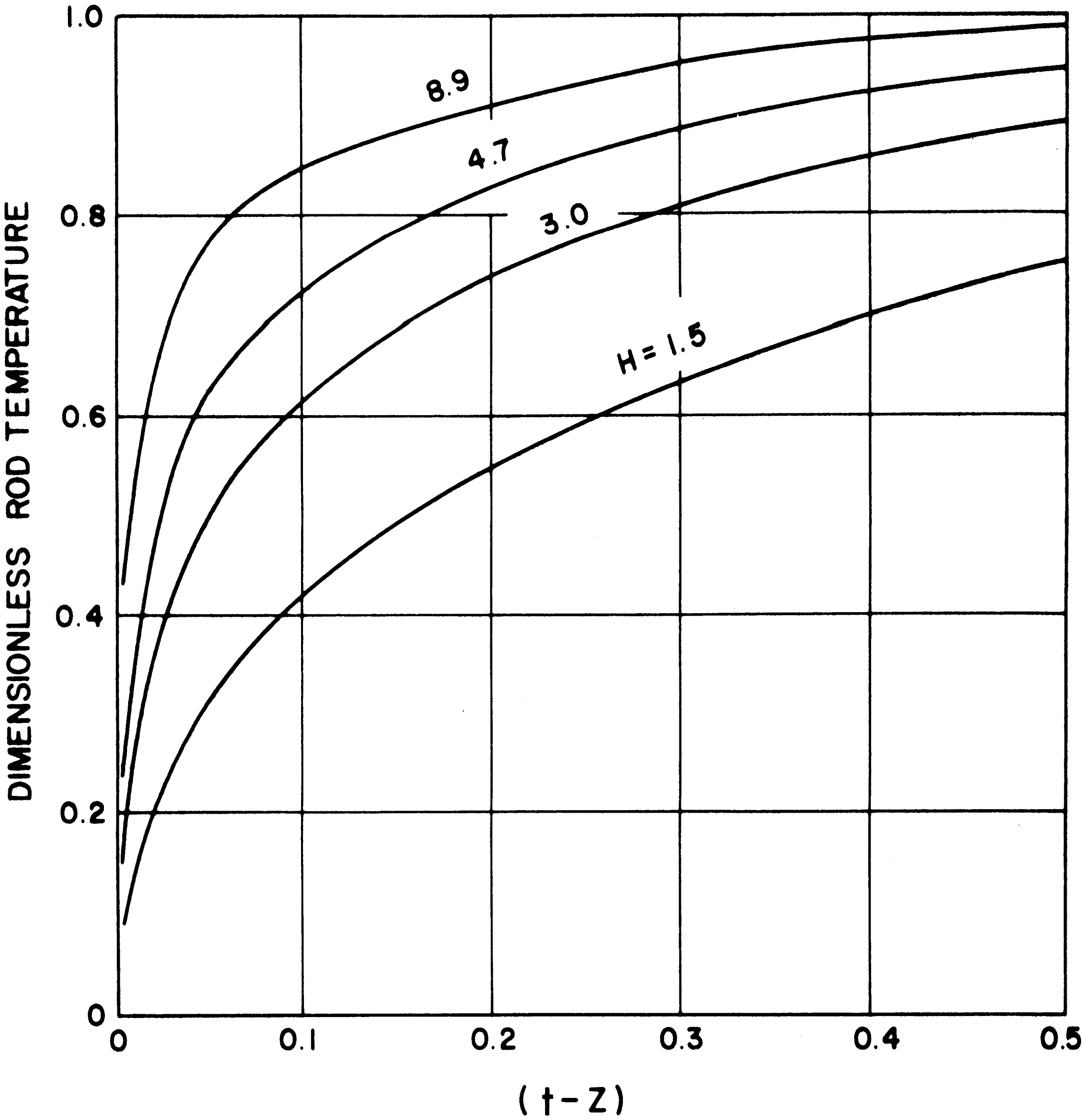

Figure 3. Effect of the Biot Number on the Transient Rod Temperature. R = 1.0, Z = 0.

cross-sectional areas, $\mathcal{Z} = \left(\frac{Z^*}{R}\right)\frac{1}{P} = \frac{\bar{Z}}{P}$, P = Peclet number based on the rod, and St = Stanton number of the fluid. Then with reference to APPENDIX A one finds that

$$MHZ = A^*(St)\bar{\mathcal{Z}} \tag{61}$$

The maximum value of $\bar{Z}$ is determined primarily by physical size limitations and will be treated as if it could be infinite. A^* , besides being limited by physical size, also is bounded to some extent by pumping costs and flow conditions. In general the clearance space is .1-.2x the rod diameter. Thus A^* has a range of roughly 2-4. The Stanton number, defined as

$$St = \frac{\text{Nusselt No.}}{\text{Reynolds No. x Prandtl No.}} = \frac{N_u}{\text{Re Pr}} = \frac{\text{Nu}}{P_{fluid}} \tag{62}$$

has a very large range of values. It can assume values from .0001 (Oils with high Prandtl numbers) to .04 (Liquid metals with low Prandtl numbers). Therefore the term A^* (St) is restricted to roughly

$$0 < A^*(St) < .2 \tag{63}$$

Figure 4 shows the effect of the fourth parameter on the surface temperature. Note for MHZ $\leq$ 1.0 that almost an exact solution to the problem is obtained. Note also that in the case of water (Pr $\cong$ 3), that

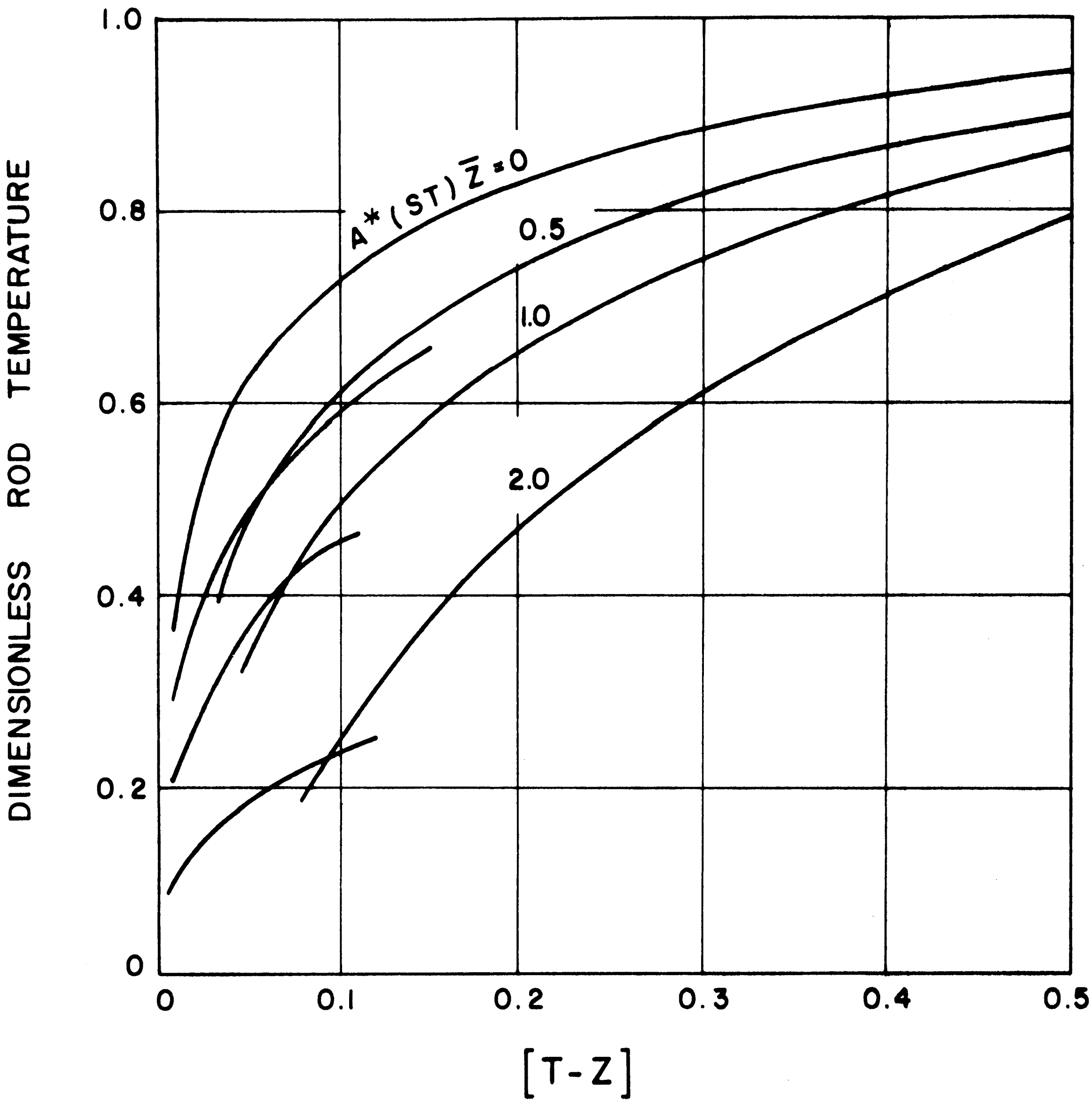

Figure 4. **Effect** of the Stanton Number on the Transient **Rod Temperature**
R = 1.0, H = 4.7.

$St \cong .002$ and $A^* (St) \cong .005-.01$. Thus $\bar{Z}$ could be as large as 100-200 and one would still have essentially an exact solution. On the other hand with the limiting case, $A^* (St) = .2$, $\bar{Z}$ would only be 5.

Figure 4 also illustrates that for small values of MHZ , one could just as well treat the problem as a step change in the ambient temperature rather than introduce axial effects at all.

Figure 5 shows the relative order of magnitudes of the first and second approximations. As one would expect, the second approximations have their primary importance in the mid-time range.

Figure 6 shows the fluid temperature curve for various values of the Stanton number. Since neither the large or short time fluid temperature first order approximations are functions of $t - Z$, this curve represents primarily the second order terms.

Before proceeding to the stress solution the convergence of the series is considered. It was found that only 3 terms are required in the large time solution when values as low as $t - Z = .1$ are used. The third term under these limiting conditions generally contributes only about 5-10% of the total solution. At $t-Z = .5$, only one term is required. The short time solution can be used with three terms down to about $t-Z = .02-.03$. For smaller values of time a larger number of terms is required. In general the peak stresses occur at $t-Z > .03$. Thus there is little need to consider the problem for the smaller values of time.

The steady state temperature solution is treated throughly in most elementary heat transfer texts. Thus no analysis of these results are presented here.

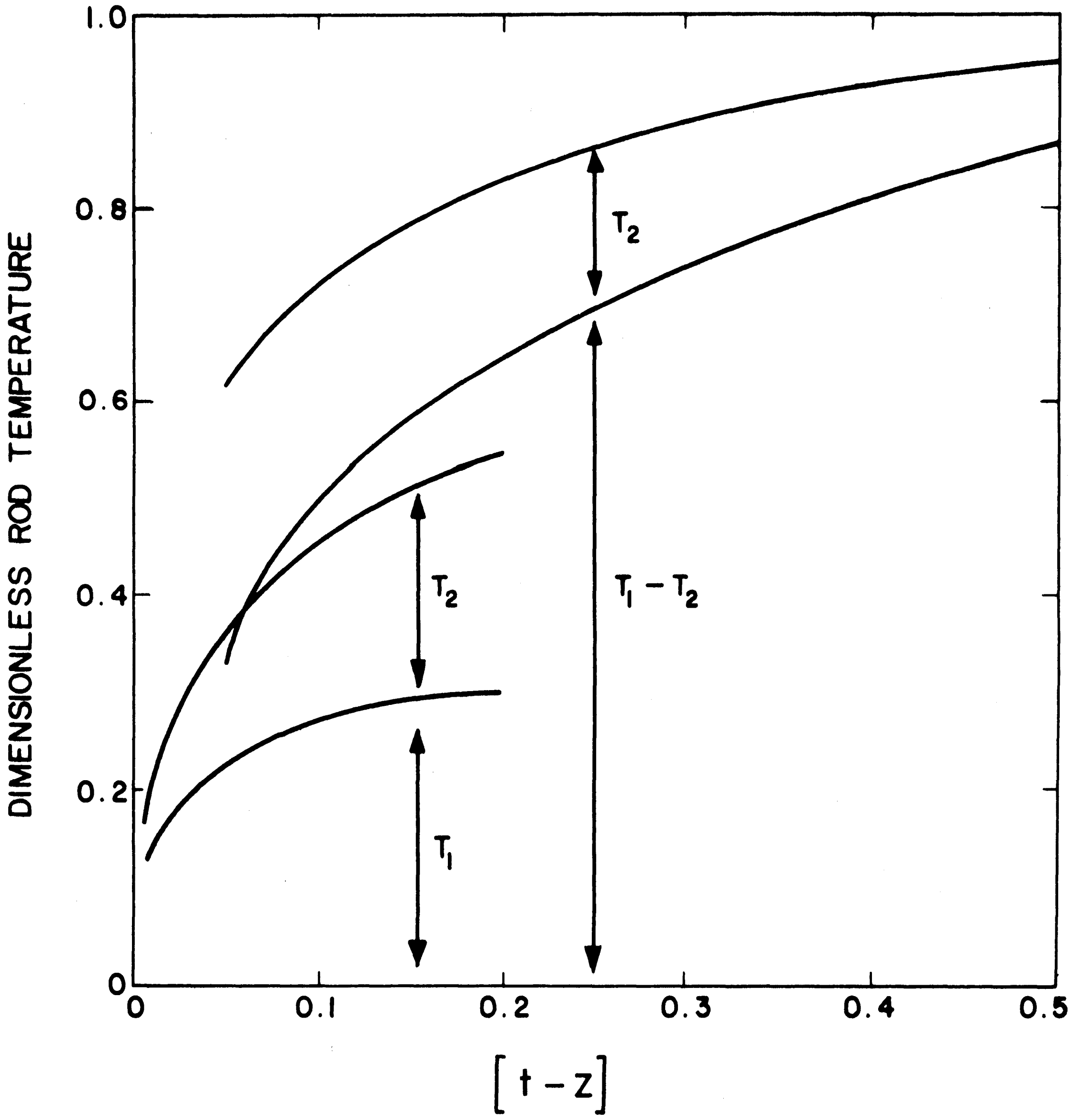

Figure 5. Order of Magnitude of the First and Second Temperature Approximations. H = 4.7, A*(St)Z = 1.0, $r = 1.0$

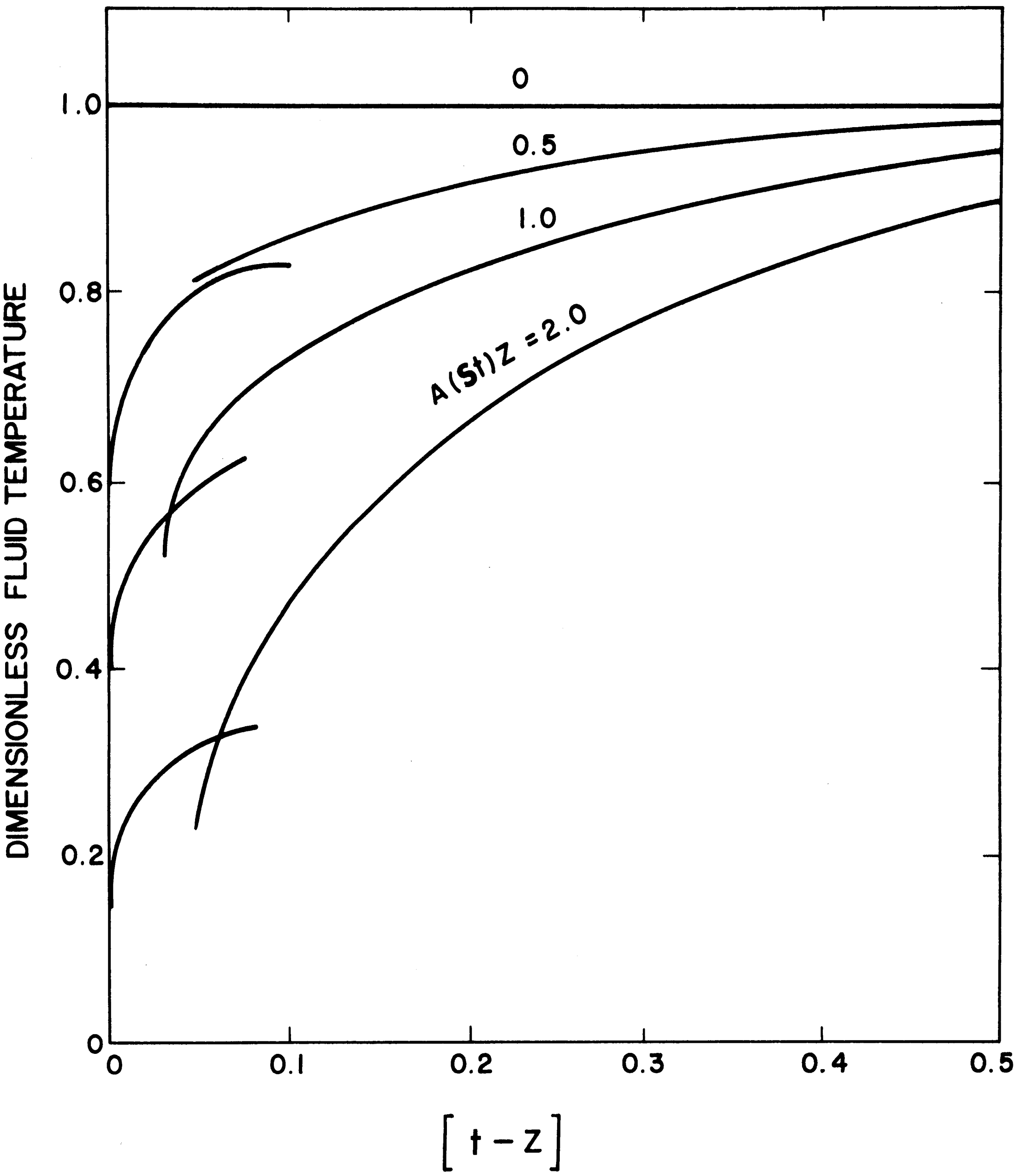

Figure 6. Effect of the Stanton Number on The Fluid Temperature. H = 4.7.

CHAPTER III

STRESS

A. General Theory

A brief summary of the various methods available for solving thermoelastic problems is presented in Reference[3]. The basic approach used in this report is Goodier's method which consists of reducing the thermoelastic problem to an isothermal problem with no body forces by first determining a temperature dependent particular solution. The isothermal problem is then solved by classical methods.

The particular solution is found by using Goodier's thermoelastic displacement potential, ψ , which is defined as $u = \frac{\partial \psi}{\partial r}$ and $w = \frac{\partial \psi}{\partial \bar{z}}$ where u and w are the radial and axial displacements respectively. The function ψ satisfied the equations of equilibrium if the following condition is satisfied.

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial \bar{z}^2} = \frac{\alpha (1+\nu) T}{(1-\nu)} = CT \tag{64}$$

where $\bar{Z} = Z/R$, α is the thermal coefficient of expansion, ν is Poisson's ratio, and C is a constant.

The isothermal problem is solved by deducing appropriate Love functions, L , (Reference[12]) which satisfy the differential equation

$$\nabla^2 \nabla^2 L = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial \bar{z}^2}\right]\left[\frac{\partial^2 L}{\partial r^2} + \frac{1}{r}\frac{\partial L}{\partial r} + \frac{\partial^2 L}{\partial \bar{z}^2}\right] = 0 \tag{65}$$

APPENDIX C contains a tabulation of the various Love functions and the corresponding stresses which are used in this report.

The stresses as determined by the particular stress solution ψ are

$$\sigma_{rr}^{\psi} = \frac{\bar{\sigma}_{rr}^{\psi}}{2GC} = \frac{\partial^2}{\partial r^2}\left(\psi/c\right) - T \tag{65a}$$

$$\sigma_{\theta\theta}^{\psi} = \frac{\bar{\sigma}_{\theta\theta}^{\psi}}{2GC} = \frac{1}{r}\frac{\partial}{\partial r}\left(\psi/c\right) - T \tag{65b}$$

$$\sigma_{zz}^{\psi} = \frac{\bar{\sigma}_{zz}^{\psi}}{2GC} = \frac{\partial^2}{\partial z^2}\left(\psi/c\right) - T \tag{65c}$$

$$\sigma_{rz}^{\psi} = \frac{\bar{\sigma}_{rz}^{\psi}}{2GC} = \frac{\partial^2}{\partial r\,\partial z}\left(\psi/c\right) \tag{65d}$$

where G = shear modulus, $\bar{\sigma}$ = stress, and σ = dimensionless stress. The stresses as determined by the Love functions are

$$\sigma_{rr}^{L} = \frac{\partial}{\partial z}\left[\nu\nabla^2 L - \frac{\partial^2 L}{\partial r^2}\right] \tag{66a}$$

$$\sigma_{\theta\theta}^{L} = \frac{\partial}{\partial z}\left[\nu\nabla^2 L - \frac{1}{r}\frac{\partial L}{\partial r}\right] \tag{66b}$$

$$\sigma_{zz}^{L} = \frac{\partial}{\partial z}\left[(1-\nu)\nabla^2 L - \frac{\partial^2 L}{\partial z^2}\right] \tag{66c}$$

$$\sigma_{rz}^{L} = \frac{\partial}{\partial r}\left[(2-\nu)\nabla^2 L - \frac{\partial^2 L}{\partial \bar{z}^2}\right] \tag{66d}$$

The boundary conditions for the given problem are

$$\sigma_{rr}(1,\bar{z},t) = 0 \tag{67a}$$

$$\sigma_{rz}(1,\bar{z},t) = 0 \tag{67b}$$

$$\int_0^1 2\pi r\,\sigma_{zz}(r,\bar{z},t)\,dr = 0 \tag{67c}$$

$$\sigma_{ij}(0,\bar{z},t) \neq \infty \tag{67d}$$

Throughout most of the stress solution the axial parameter $\bar{Z} = Z^*/R$ will be used in place of the Graetz number, $\bar{Z} = Z^*/R(P)$, which was used in the temperature solution. The initial formulation of the stress problem indicates that $\bar{Z}$ is the natural parameter. However as will be seen the Graetz number reappears as one would expect in the final stress solution.

B. Steady State Stress Solution

The temperature distribution as defined by Equation (8) is

$$T_{ss} = \frac{1}{2}\left[\frac{1}{H} + \frac{1}{2} + MZ - \frac{r^2}{2}\right] \tag{8}$$

The constant term $\left[\frac{1}{H} + \frac{1}{2}\right]$ and also the linear axial variation, MZ , cause no stress in the rod. Therefore the following temperature distribution may be used for calculating the steady stress distribution

$$T'_{ss} = -K_1 r^2 \tag{68}$$

where

$K_1 = q'R^2/4K$, q' = rate of energy generation per volume of rod, R = rod radius, and K = thermal conductivity of the rod.

Since there is no Z variation the following plain strain solution which is presented in Timoshenko(5) is used

$$\sigma_{rr} = \int_0^1 Tr\,dr - \frac{1}{r^2}\int_0^r Tr\,dr \tag{69a}$$

$$\sigma_{\theta\theta} = \int_0^1 Tr\,dr + \frac{1}{r^2}\int_0^r Tr\,dr - T \tag{69b}$$

For the case of free axial expansion

$$\sigma_{zz} = \sigma_{rr} + \sigma_{\theta\theta} \tag{69c}$$

$$\sigma_{rz} = 0 \tag{69d}$$

Substituting Equation (68) into (69) yields

$$\sigma_{rr} = -\frac{K_1}{4}\left[1 - r^2\right] \tag{70a}$$

$$\sigma_{\theta\theta} = -\frac{K_1}{4}\left[1 - 3r^2\right] \tag{70b}$$

$$\sigma_{zz} = -\frac{K_1}{2}\left[1 - 2r^2\right] \tag{70c}$$

$$\sigma_{rz} = 0 \tag{70d}$$

C. Transient Stress Solution

1. Exact Stress Solution

The first step in obtaining a solution to the stress problem is to determine a function ψ such that Equation (64) is satisfied. Using Equations (17) and (20) one sees that a solution to the following equation is required

$$\nabla^2\left(\frac{\psi}{c}\right) = \frac{HI_0(\sqrt{s}\,r)\,e^{-\left[s + \frac{MH\sqrt{s}\,I_1(\sqrt{s})}{\sqrt{s}\,I_1(\sqrt{s}) + HI_0(\sqrt{s})}\right]\frac{z}{P}}}{s\left[\sqrt{s}\,I_1(\sqrt{s}) + HI_0(\sqrt{s})\right]} \tag{71}$$

A direct substitution will verify that

$$\psi = \frac{\nabla^2\psi}{s + \left[s + \frac{MH\sqrt{s}\,I_1(\sqrt{s})}{\sqrt{s}\,I_1(\sqrt{s}) + HI_0(\sqrt{s})}\right]^2\frac{1}{P^2}} \tag{72}$$

satisfies Equation (71). Using the following definitions

$$\lambda(s) = \left[s + \frac{MH\sqrt{s}\,I_1(\sqrt{s})}{\sqrt{s}\,I_1(\sqrt{s}) + HI_0(\sqrt{s})} \right]^{1/P} \tag{73}$$

$$\gamma(s) = \frac{H}{s\left[\sqrt{s}\,I_1(\sqrt{s}) + HI_0(\sqrt{s})\right]} \tag{74}$$

it is found from Equations (65a) and (65d) that

$$\sigma_{rr}^{\psi}(1, z, s) = -\frac{\gamma(s)\, e^{-\lambda(s)\bar{z}}}{(s + \lambda(s)^2)} \left[\lambda(s)^2 I_0(\sqrt{s}) - \sqrt{s}\, I_1(\sqrt{s}) \right] \tag{75a}$$

$$\sigma_{rz}^{\psi}(1, z, s) = -\frac{\gamma(s)\, e^{-\lambda(s)\bar{z}}}{(s + \lambda(s)^2)} \left[\sqrt{s}\, I_1(\sqrt{s})\, \lambda(s) \right] \tag{75b}$$

Since these stresses do not satisfy the boundary conditions of Equations (67a) and (67b), an isothermal solution to the problem must be superimposed such that

$$\begin{aligned} \sigma_{rr}^{L}(1, z, s) + \sigma_{rr}^{\psi}(1, z, s) &= 0 \\ \sigma_{rz}^{L}(1, z, s) + \sigma_{rz}^{\psi}(1, z, s) &= 0 \end{aligned} \tag{76}$$

Using APPENDIX C one finds that the following function satisfies Equation (76);

$$L = \left[A(s)\, r\, J_1(\lambda(s) r) + B(s)\, J_0(\lambda(s) r) \right] e^{-\lambda(s)\bar{z}} \tag{77}$$

From a formal point of view this completes the development of the exact stress solution to the problem. One can see at this point that the inverse transform of the exact solution would be even more complex than the temperature solution because $\lambda(S)$, which was the difficult parameter in the temperature solution, is now also encountered as the argument of the Bessel functions in the Love functions. As was the case for the temperature problem an exact solution for the stress distribution does not appear to be feasible.

2. Discussion of Some Approximate Stress Solutions

Similar to the temperature problem, there are also many possible asymptotic solutions for the stress distribution. For example in some cases it may be more advantageous to use the exact form of $\psi(S)$ and in other cases some approximate form of $\psi(S)$ or $\psi(t)$ may be more desirable. One must also decide between first approximating the displacement function and then substituting into the stress equations or substituting the displacement function first and then approximating the individual stress components. Several of the more promising approaches were investigated and the results of some of these are presented here.

One approach is to treat all terms of the temperature solution equally and expand these into an asymptotic solution which is valid for small values of time (large S). One form of such an approach is

$$T \cong \frac{H e^{-MHz} e^{-Sz}}{\sqrt{r}} \left[\frac{e^{-\sqrt{S}(1-r)}}{S^{3/2}} \right] \tag{78}$$

or

$$T^{*} = \frac{H e^{-MHz}}{\sqrt{r}} \left[\frac{2\sqrt{t-z}}{\sqrt{\pi}} e^{-\frac{(1-r)^2}{4(t-z)}} - (1-r)\,\mathrm{Erfc}\left(\frac{1-r}{2\sqrt{t-z}}\right) \right] \tag{79}$$

where $\mathrm{Erfc}(x) = 1 -$ Error function. The primary concern in attempting to use this for a stress solution is the $\sqrt{r}$ term. In the temperature problem the solution is known at the centerline for small values of time and hence, the fact that the solution is not valid there, is not critical. The stress problem, however, depends on what happens at the center for this controls the surface stresses and vice-versa. Therefore this is not a good form from which to begin the stress solution. The $1/\sqrt{r}$ term results from the expansion of the $I_\nu(\lambda r)$ type of term.

If instead of approximating $T(S)$, one starts with $\psi(S)$ and proceeds just as in the previous example, one form of the solution is

$$\sigma_{rr}^{\psi} \cong -\frac{H e^{-MHz}}{\sqrt{r}} \left[\frac{2\sqrt{t-z}}{\sqrt{\pi}} e^{-\frac{(1-r)^2}{4(t-z)}} - (1-r)\,\mathrm{Erfc}\left(\frac{1-r}{2\sqrt{t-z}}\right) \right] \tag{80}$$

Evaluating this at $r = 1$ yields

$$\sigma_{rr}^{\psi}(1,z,t) \cong -\frac{2H\sqrt{t-z}}{\sqrt{\pi}}\, e^{-MHz} \tag{81}$$

This is better than the previous approach but offers some difficulty as a result of the $\sqrt{t-Z}$ term. The $\sqrt{t}$ type of term appears because this is characteristic of the diffusion process. This, coupled with the time shift, $t \Longrightarrow t - Z$, causes difficulty in many of the various possible solutions.

Another basic difficulty is encountered in attempting to work with the classical Love functions at small values of time in terms of S. In general, the lowest form of the argument of the Bessel functions will include an αS term. Since S is large for small values of time, the Bessel functions may be written as

$$\begin{aligned} J_0(x) &\cong \left[\cos\left(x - \pi/4\right)\right]\left[\frac{2}{x\pi}\right]^{1/2} \\ J_1(x) &\cong \left[\sin\left(x - \pi/4\right)\right]\left[\frac{2}{x\pi}\right]^{1/2} \end{aligned} \tag{82}$$

After determining the unknown coefficients of the Love function, it is found that the inverse transform would require knowing the roots of a second order trigonometric equation. This causes some definite problems and leads to a power series solution.

Another solution results from the similarity between $T(S)$ and $\psi(S)$. As is seen in Equation (72), $\psi(S)$ differs from $T(S)$ only by a

$1/(S + \lambda(S)^2)$ term. Since the numerator, $T(S)$, was handled easily by the method of residues, one is tempted to try approximating the denominator and use the inversion theorem directly to obtain $\psi(t)$. The simplest possible form for the denominator for small values of time is

$$S + (MH + S)^2 \frac{1}{P^2} \tag{83}$$

As will be shown later mH is a small quantity. Making use of this fact Equation (83) reduces to

$$\psi/C \cong \frac{T(S)}{S[1 + S/P^2]} \tag{84}$$

Thus it is seen that this would yield a direct solution to the displacement function. It would, however, require going through a procedure similar to the temperature problem. This method was not used because it was easier to deduce the required ψ rather than solve for it by direct means.

Solutions, which are valid for large values of time, are in general easier to obtain than those for small values of time. The reason for this is primarily that no $\sqrt{r}$ or $\sqrt{t}$ type of terms appear. Whether or not a good solution is obtained with a few terms depends on the order of the pole at the origin and hence on the basic nature of the problem itself. The higher the order of the pole at the origin the better the solution with less terms. In the present problem only a first order pole for the temperature problem exists and hence one would not expect to get an exceptionally good solution with only a few terms. This is the basic reason why the modified Bessel functions were left in the denominator of the temperature solution. They represent more of the poles in the vicinity of the origin and hence improve the solution for intermediate values of time.

3. Approximate Stress Solution

a. First Approximation for Large Values of Time

The first approximation of the stress problem for large values of time is found by using the temperature distribution defined by Equation (49). The first term of this equation is constant and causes no stresses. Therefore only a particular solution to the following equation is required.

$$\nabla^2\left(\psi/c\right) = -\sum_{K=1}^{\infty} \beta_K J_0(\lambda_K r)\, e^{\frac{\lambda_K^2 \bar{z}}{P}} \tag{85}$$

where

$$\beta_K = \frac{2He^{-\lambda_K^2 t}}{(H^2+\lambda_K^2)\, J_0(\lambda_K)} \tag{86}$$

A direct substitution into Equation (85) will verify that

$$\frac{\psi(r,z,t)}{c} = -\sum_{K=1}^{\infty} \frac{\beta_K J_0(\lambda_K r)\, e^{\frac{\lambda_K^2 \bar{z}}{P}}}{\left[\left(\frac{\lambda_K^2}{P}\right)^2 - \lambda_K^2\right]} \tag{87}$$

is the particular solution of the problem.

Equation (87) can be simplified by showing that the $\left(\frac{\lambda_K^2}{P}\right)^2$ term in the denominator can be eliminated since it is negligible in comparison to λ_K^2 . The first six roots, λ_K , are tabulated in Carslaw and Jaeger[1] for various values of H . Assuming H to be greater than one and limiting the solution to three terms, (recall the convergence of the temperature solution), it is found that

$$1.5 < \lambda_K^2 < 75 \tag{88}$$

The order of magnitude of P can be determined by rewriting it as follows:

$$P = \frac{VR}{\alpha_R} \frac{P_f}{\frac{V D_H}{\alpha_f}} = \frac{1}{2}\left(\frac{\alpha_f}{\alpha_R}\right)\left(\frac{P_f}{D_H/D}\right) \tag{89}$$

where α_R and α_f are the thermal diffusivities of the rod and fluid respectively, $^{D_H}/_D$ the ratio of the hydraulic diameter to the rod diameter, and P_f the Peclet number of the fluid. As was stated before $^{D_H}/_D$ has a maximum value of about .4 and P_f is required to be greater than 100. Assuming the limiting case is fully turbulent liquid metal, the ratio $^{\alpha_f}/_{\alpha_R}$ has an order of magnitude of unity. If these values are substituted into Equation (89), one finds that P could also be about 100 in a very limiting case.

Under the given conditions, then $(\lambda_K^2/P)^2 \cong .6$ which is negligible in comparison to $\lambda_K^2 = 75$ for this case.

In view of this discussion Equation (87) can be written as

$$\psi/c = \sum_{K=1}^{3} \frac{\beta_K J_0(\lambda_K r)\, e^{\frac{\lambda_K^2 \bar{z}}{P}}}{\lambda_K^2} \tag{90}$$

Using Equations (65) and (90) one finds the following particular stress solution.

$$\sigma_{rr}^{\psi} = +\sum_{K=1}^{3} \beta_K \left[\frac{J_1(\lambda_K r)}{\lambda_K r} \right] e^{\frac{\lambda_K^2 \bar{z}}{P}} \tag{91a}$$

$$\sigma_{\theta\theta}^{\psi} = -\sum_{K=1}^{3} \beta_K \left[\frac{J_1(\lambda_K r)}{\lambda_K r} - J_0(\lambda_K r) \right] e^{\frac{\lambda_K^2 \bar{z}}{P}} \tag{91b}$$

$$\sigma_{zz}^{\psi} = +\sum_{K=1}^{3} \beta_K J_0(\lambda_K r)\, e^{\frac{\lambda_K^2 \bar{z}}{P}} \tag{91c}$$

$$\sigma_{rz}^{\psi} = -\sum_{K=1}^{3} \beta_K \left[\frac{\lambda_K J_1(\lambda_K r)}{P} \right] e^{\frac{\lambda_K^2 \bar{z}}{P}} \tag{91d}$$

$$\int_0^1 r\,\sigma_{zz}^{\psi}\, dr = -\sum_{K=1}^{3} \frac{\beta_K H J_0(\lambda_K)}{\lambda_K^2}\, e^{\frac{\lambda_K^2 \bar{z}}{P}} \tag{91e}$$

Using the identity

$$\lambda_K J_1(\lambda_K) = H J_0(\lambda_K) \tag{92}$$

one finds

$$\sigma_{rr}^{\psi}(1, \bar{z}, t) = \sum_{K=1}^{3} \alpha_1(\lambda_K) e^{-\frac{\lambda_K^2 \bar{z}}{P}}$$

where (93)

$$\alpha_1(\lambda_K) = \frac{\beta_K H J_0(\lambda_K)}{\lambda_K^2}$$

and

$$\sigma_{rz}^{\psi}(1, \bar{z}, t) = \sum_{K=1}^{3} \alpha_4(\lambda_K) e^{\frac{\lambda_K^2 \bar{z}}{P}}$$

where (94)

$$\alpha_4(\lambda_K) = -\beta_K H J_0(\lambda_K)$$

Using APPENDIX C a suitable homogeneous solution for the isothermal problem is found to be

$$L = \left[A_K r J_1\left(\frac{\lambda_K^2 r}{P}\right) + B_K J_0\left(\frac{\lambda_K^2 r}{P}\right)\right] e^{-\frac{\lambda_K^2 \bar{z}}{P}} \tag{95}$$

Equations (67a) and (67b) show that the boundary conditions are satisfied if

$$\alpha_1(\lambda_K) + A_K a_1(1) + B_K b_1(1) = 0$$
$$\alpha_4(\lambda_K) + A_K a_4(1) + B_K b_4(1) = 0 \qquad (96)$$

where a_1 , b_1 , a_4 , and b_4 are defined in APPENDIX C. Solving for A_K and B_K would complete the solution. The remaining two boundary conditions are automatically satisfied because L and ψ satisfy the equilibrium equations.

It is easy to solve for the coefficients A_K and B_K in this case. However, in the second order approximation, six Love functions are involved which requires solving a 6x6 determinate for the unknown coefficients. If the solution is to be practical the problem must be reduced to a simpler form. This can be done by using an order of magnitude analysis on the stresses determined by L . An examination of Equation (95) shows that the maximum value of the argument of the Bessel functions is λ_K^2/P . As was shown before this is small. For $H = \infty$ (maximum values of λ_K) one finds that λ_K^2/P = .06, .3, .75 for K = 1, 2,3. If in this limiting case one were to assume that

$$J_0(X) = 1 - \frac{X^2}{4} + \cdots \cong 1$$

and (97)

$$J_1(X) = \frac{X}{2} - \frac{X^3}{16} + \cdots \cong \frac{X}{2}$$

errors of about 1%, 3%, and 16% respectively for $K = 1$, 2,3 would be found in the $J_0(x)$ term. Even less error is involved in $J_1(x)$. However, from the temperature solution it is known that the third term rarely contributes more than 5-10% of the solution in its desired time range. Thus no large error is introduced if Equation (97) is used in the stresses of the Love functions. For small values of time this is also true because the argument for these cases is $\left(\frac{mH}{P} - \frac{\lambda_K^2}{P}\right)$ and mH/P has already been shown to be less than .2. The approximate forms of all the stresses derived from the Love functions are also presented in APPENDIX C.

From APPENDIX C three very basic observations can be made for each Love function:

(1) $$\sigma_{rr}^L , \sigma_{\theta\theta}^L , \sigma_{zz}^L \neq f(r) \tag{98a}$$

(2) $$\sigma_{rz}^L (r, z, t) = r\, \sigma_{rz}^L (1, z, t) \tag{98b}$$

and

(3) $$\sigma_{rr}^L = \sigma_{\theta\theta}^L \tag{98c}$$

Referring to Equation (67a) one finds that

$$\sigma_{rr}^{\psi} (1, z, t) = - \sigma_{rr}^L (1, z, t) = - \sigma_{rr}^L (r, z, t) \tag{99a}$$

Thus

$$\sigma_{rr} = \sigma_{rr}^{\psi} + \sigma_{rr}^{L} = \sigma_{rr}^{\psi}(r,z,t) - \sigma_{rr}^{\psi}(1,z,t) \tag{99b}$$

From Equation (98c) one finds

$$\sigma_{\theta\theta} = \sigma_{\theta\theta}^{\psi} + \sigma_{\theta\theta}^{L} = \sigma_{\theta\theta}^{\psi} + \sigma_{rr}^{L} \tag{100a}$$

Thus

$$\sigma_{\theta\theta} = \sigma_{\theta\theta}^{\psi}(r,z,t) - \sigma_{rr}^{\psi}(1,z,t) \tag{100b}$$

From Equations (98b) and (67b)

$$\sigma_{rz}^{\psi}(1,z,t) = -\sigma_{rz}^{L}(1,z,t) = -\frac{\sigma_{rz}^{L}(r,z,t)}{r} \tag{101a}$$

and

$$\sigma_{rz} = \sigma_{rz}^{\psi} + \sigma_{rz}^{L} = \sigma_{rz}^{\psi}(r,z,t) + r\,\sigma_{rz}^{L}(1,z,t) \tag{101b}$$

Thus

$$\sigma_{rz} = \sigma_{rz}^{\psi}(r,z,t) - r\,\sigma_{rz}^{\psi}(1,z,t) \tag{101c}$$

From Equations (67c) and (98a) one finds

$$\int_0^1 \sigma_{zz}^{L}(r,z,t)\,r\,dr = \frac{\sigma_{zz}^{L}(r,z,t)}{2} = \frac{\sigma_{zz}^{L}(1,z,t)}{2} \tag{102a}$$

and

$$\int_0^1 \sigma_{zz}^{\psi}\, r\, dr = -\int_0^1 \sigma_{zz}^{L}(r,z,t)\, r\, dr \tag{102b}$$

Thus

$$\sigma_{zz} = \sigma_{zz}^{\psi} + \sigma_{zz}^{L} = \sigma_{zz}^{\psi}(r,z,t) - 2\int_0^1 r\,\sigma_{zz}^{\psi}(r,z,t)\, dr \tag{102c}$$

As a result of this analysis it is shown that if the argument of $J_o(x)$ and $J_1(x)$ is small in the Love functions the whole problem is solved once the particular solution is known. For sake of easy reference Equations (99b), (100b), (101c) and (102c) are rewritten here

$$\sigma_{rr} = \sigma_{rr}^{\psi}(r,z,t) - \sigma_{rr}^{\psi}(1,z,t) \tag{99b}$$

$$\sigma_{\theta\theta} = \sigma_{\theta\theta}^{\psi}(r,z,t) - \sigma_{rr}^{\psi}(1,z,t) \tag{100b}$$

$$\sigma_{zz} = \sigma_{zz}^{\psi}(r,z,t) - 2\int_0^1 r\,\sigma_{zz}^{\psi}(r,z,t)\, dr \tag{102c}$$

$$\sigma_{rz} = \sigma_{rz}^{\psi}(r,z,t) - r\,\sigma_{rz}^{\psi}(1,z,t) \tag{101c}$$

Equations (91) substituted into Equations (99b), (100b), (101c), and (102c) completely define the first approximation to the large time stress solution

b. First Approximation for Small Values of Time

For this stress distribution a solution to the following equation is required

$$\nabla^2(\psi/c) = e^{-\frac{MH\bar{z}}{P}} - \sum_{K=1}^{\infty} \beta_K \, J_0(\lambda_K r)\, e^{-\left[\frac{MH}{P} - \frac{\lambda_K^2}{P}\right]\bar{z}} \tag{103}$$

A direct substitution will verify that

$$\psi/c = \frac{e^{-\frac{MH\bar{z}}{P}}}{\left(\frac{MH}{P}\right)^2} - \sum_{K=1}^{\infty} \frac{\beta_K \, J_0(\lambda_K r)\, e^{-\left[\frac{MH}{P} - \frac{\lambda_K^2}{P}\right]\bar{z}}}{\left[\frac{MH}{P} - \frac{\lambda_K^2}{P}\right]^2 - \lambda_K^2} \tag{104}$$

satisfies this condition. Equations (65a) and (65b) show that if $\psi \neq f(r)$, then $\sigma_{rr}^{\psi} = \sigma_{\theta\theta}^{\psi} = -T$. Also Equations (99b), (100b), (101c) and (102c) show that if $\psi \neq f(r)$ then all of the stresses are identically zero. Therefore the first term of Equation (104) produces no stresses and may be ignored.

The denominator of the second term in Equation (104) can be simplified by restricting the solution to three terms of the series and assuming $H > 1$. Assuming a limiting case where $MH/P = .2$, $P = 100$, $H = 1$, and $\lambda_K^2 = 1.7$, one finds that $.04 << 1.7$ or $(MH/P - \lambda_K^2/P) << \lambda_K^2$.

Using the results of the discussion a new function, ψ', for the particular stress solution can be written as

$$\frac{\psi'}{c} = + e^{-\frac{MH\bar{z}}{P}} \sum_{K=1}^{3} \frac{\beta_K \, J_0(\lambda_K r)\, e^{\frac{\lambda_K^2 \bar{z}}{P}}}{\lambda_K^2} \tag{105}$$

The resulting particular stress solution is

$$\sigma_{rr}^{\psi} = e^{-MHz} \sum_{K=1}^{3} \frac{\beta_K J_1(\lambda_K r)\, e^{\lambda_K^2 z}}{\lambda_K r} \tag{106a}$$

$$\sigma_{\theta\theta}^{\psi} = -e^{-MHz} \sum_{K=1}^{3} \beta_K \left[\frac{J_1(\lambda_K r)}{\lambda_K r} - J_0(\lambda_K r) \right] e^{\lambda_K^2 z} \tag{106b}$$

$$\sigma_{zz}^{\psi} = e^{-MHz} \sum_{K=1}^{3} \beta_K\, J_0(\lambda_K)\, e^{\lambda_K^2 z} \tag{106c}$$

$$\sigma_{rz}^{\psi} = e^{-MHz} \sum_{K=1}^{3} \beta_K \left[\frac{MH}{P} - \frac{\lambda_K^2}{P} \right] \frac{J_1(\lambda_K r)}{\lambda_K}\, e^{\lambda_K^2 z} \tag{106d}$$

$$\int_0^1 r\, \sigma_{zz}^{\psi}\, dr = -e^{-MHz} \sum_{K=1}^{3} \frac{\beta_K H\, J_0(\lambda_K)\, e^{\frac{\lambda_K^2 \bar{z}}{P}}}{\lambda_K^2} \tag{106e}$$

This set of equations completely defines the required stress distribution.

c. Second Approximations

Using the similarity between Equations (36a), (36b), (36c), (58), (59), and (60) both second order temperature distributions can be represented as

$$T_{2-4}^{*} = \sum_{K=1}^{\infty}\left[(\Delta_1 \bar{z} + \Delta_2 \bar{z}^2) J_0(\lambda_K r) + \bar{z}\Delta_3 r J_1(\lambda_K r)\right] e^{+n\bar{z}} \tag{107}$$

where for small values of time

$$\Delta_1 = -\frac{4MH^3}{P} \frac{\left[\lambda_K^2 + (H^2+\lambda_K^2)(H/2 + \lambda_K^2 t)\right]}{(H^2+\lambda_K^2)^3 \, J_0(\lambda_K)} e^{-\lambda_K^2 t} \tag{108a}$$

$$\Delta_2 = \frac{4MH^3 \lambda_K^2 \, e^{-\lambda_K^2 t}}{P^2 (H^2+\lambda_K^2)^2 \, J_0(\lambda_K)} \tag{108b}$$

$$\Delta_3 = -\frac{2MH^3 \lambda_K \, e^{-\lambda_K^2 t}}{P (H^2+\lambda_K^2)^2 \, J_0(\lambda_K)} \tag{108c}$$

$$n = \frac{\lambda_K^2}{P} - \frac{MH}{P} \tag{108d}$$

and where for large values of time

$$\Delta_1 = -\frac{4MH^3}{P} \frac{\left[1 + (t - \frac{1}{2H})(H^2+\lambda_K^2)\right] \lambda_K^2 \, e^{-\lambda_K^2 t}}{(H^2+\lambda_K^2)^3 \, J_0(\lambda_K)} \tag{109a}$$

$$\Delta_2 = - \frac{4MH^3 \lambda_K^2 e^{-\lambda_K^2 t}}{P^2 (H^2 + \lambda_K^2)^2 J_0(\lambda_K)} \tag{109b}$$

$$\Delta_3 = - \frac{2MH^3 \lambda_K e^{-\lambda_K^2 t}}{P (H^2 + \lambda_K^2)^2 J_0(\lambda_K)} \tag{109c}$$

$$n = \frac{\lambda_K^2}{P} \tag{109d}$$

The problem can be simplified by defining ψ as $\psi = \psi_1 + \psi_2 + \psi_3$ where ψ_1 corresponds to the term involving Δ_1, ψ_2 to Δ_2, and ψ_3 to Δ_3. The first sub-problem to be solved is

$$\nabla^2\left(\frac{\psi_1}{c}\right) = \bar{Z} \sum_{K=1}^{\infty} \Delta_1 J_0(\lambda_K r) e^{+n\bar{Z}} \tag{110}$$

A direct substitution will verify that

$$\frac{\psi_1}{c} = \sum_{K=1}^{\infty} \Delta_1 \left[\bar{Z} - \frac{2n}{n^2 - \lambda_K^2}\right] \frac{J_0(\lambda_K r) e^{+n\bar{Z}}}{(n^2 - \lambda_K^2)} \tag{110}$$

satisfies this condition.

This can be simplified further by limiting the expansion to three terms and requiring that $H > 1$. As has been previously shown for these conditions, $(mH/P - \lambda_K^2/P)$ and λ_K^2/P are negligible in comparison to λ_K^2. Therefore $\eta^2 << \lambda_K^2$ and Equation (111) can be rewritten as

$$\frac{\psi_1}{C} = -\sum_{K=1}^{3} \Delta_1 \left[\bar{z} + \frac{2n}{\lambda_K^2}\right] \frac{J_0(\lambda_K r)}{\lambda_K^2} e^{+n\bar{z}} \tag{112}$$

Performing the required differentiations and substituting these into the stress equations, the following particular stress solution is obtained

$$\sigma_{rr}^{\psi} = \sum_{K=1}^{3} \Delta_1 \left[\frac{2n}{\lambda_K^2} J_0(\lambda_K r) - \left(\bar{z} + \frac{2n}{\lambda_K^2}\right) \frac{J_1(\lambda_K r)}{\lambda_K r} \right] e^{+n\bar{z}} \tag{113a}$$

$$\sigma_{\theta\theta}^{\psi} = \sum_{K=1}^{3} \Delta_1 \left[\left(\bar{z} + \frac{2n}{\lambda_K^2}\right) \frac{J_1(\lambda_K r)}{\lambda_K r} - \bar{z} J_0(\lambda_K r) \right] e^{+n\bar{z}} \tag{113b}$$

$$\sigma_{zz}^{\psi} = -\sum_{K=1}^{3} \Delta_1 \left(\bar{z} + \frac{2n}{\lambda_K^2}\right) J_0(\lambda_K) e^{+n\bar{z}} \tag{113c}$$

$$\sigma_{rz}^{\psi} = \sum_{K=1}^{3} \Delta_1 \left(1 + n\bar{z} + \frac{2n^2}{\lambda_K^2}\right) \frac{J_1(\lambda_K r)}{\lambda_K} e^{+n\bar{z}} \tag{113d}$$

$$\int_0^1 r\,\sigma_{zz}^{\psi}\,dr = -\sum_{K=1}^{3} \Delta_1 \left(\bar{z} + \frac{2n}{\lambda_K^2}\right) \frac{H\,J_0(\lambda_K)}{\lambda_K^2} e^{+n\bar{z}} \tag{113e}$$

The particular solution for the Δ_2 term of the second approxiamtion can be shown to be

$$\frac{\psi_2}{c} = \sum_{K=1}^{\infty} \Delta_2 \left[\bar{z}^2 - \frac{4n\bar{z}}{n^2-\lambda_K^2} - \frac{2}{n^2-\lambda_K^2} + \frac{8n^2}{(n^2-\lambda_K^2)^2}\right] \frac{J_0(\lambda_K r)e^{+n\bar{z}}}{(n^2-\lambda_K^2)} \tag{114}$$

After eliminating negligible terms $(\eta^2 << \lambda_K^2)$, Equation (114) can be re-written as

$$\frac{\psi_2}{c} = -\sum_{K=1}^{3} \Delta_2 \left[\bar{z}^2 + \frac{4n\bar{z}}{\lambda_K^2} + \frac{2}{\lambda_K^2} + \frac{8n^2}{\lambda_K^4}\right] \frac{J_0(\lambda_K r)e^{+n\bar{z}}}{\lambda_K^2} \tag{115}$$

The resulting particular stress solution is

$$\sigma_{rr}^{\psi} = \sum_{K=1}^{3} \Delta_2 \Big[\Big(\frac{4n\bar{z}}{\lambda_K^2} + \frac{2}{\lambda_K^2} + \frac{8n^2}{\lambda_K^4} \Big) J_0(\lambda_K r)$$

$$- \Big(\bar{z}^2 + \frac{4n\bar{z}}{\lambda_K^2} + \frac{2}{\lambda_K^2} + \frac{8n^2}{\lambda_K^4} \Big) \frac{J_1(\lambda_K r)}{\lambda_K r} \Big] e^{+n\bar{z}} \tag{116a}$$

$$\sigma_{\theta\theta}^{\psi} = \sum_{K=1}^{3} \Delta_2 \Big[\Big(\bar{z}^2 + \frac{4n\bar{z}}{\lambda_K^2} + \frac{2}{\lambda_K^2} + \frac{8n^2}{\lambda_K^4} \Big) \frac{J_1(\lambda_K r)}{\lambda_K r}$$

$$- \bar{z}^2 J_0(\lambda_K r) \Big] e^{+n\bar{z}} \tag{116b}$$

$$\sigma_{zz}^{\psi} = -\sum_{K=1}^{3} \Delta_2 \Big[\bar{z}^2 + \frac{4n\bar{z}}{\lambda_K^2} + \frac{8n^2}{\lambda_K^4} + \frac{2}{\lambda_K^2} \Big] J_0(\lambda_K r) e^{+n\bar{z}} \tag{116c}$$

$$\sigma_{rz}^{\psi} = \sum_{K=1}^{3} \Delta_2 \Big[n\bar{z}^2 + 2\Big(1 + \frac{2n^2}{\lambda_K^2}\Big)\bar{z}$$

$$+ \frac{2n}{\lambda_K^2}\Big(3 + \frac{4n^2}{\lambda_K^2}\Big) \Big] \frac{J_1(\lambda_K r)}{\lambda_K} e^{+n\bar{z}} \tag{116d}$$

$$\int_0^1 r\,\sigma_{zz}^{\psi}\,dr = -\sum_{K=1}^{3} \Delta_2 \left[\bar{z}^2 + \frac{4n\bar{z}}{\lambda_K^2} + \frac{2}{\lambda_K^2} + \frac{8n^2}{\lambda_K^4} \right] \frac{H J_0(\lambda_K)}{\lambda_K^2}\, e^{+n\bar{z}} \tag{116e}$$

The particular solution for the Δ_3 term of the second approximation is

$$\frac{\psi_3}{c} = \sum_{K=1}^{\infty} \Delta_3 \Big[\left(\bar{z} - \frac{2n}{n^2-\lambda_K^2} \right) r J_1(\lambda_K r) - 2\lambda_K \left(\bar{z} - \frac{4n}{n^2-\lambda_K^2} \right) \frac{J_0(\lambda_K r)}{(n^2-\lambda_K^2)} \Big] \frac{e^{+n\bar{z}}}{(n^2-\lambda_K^2)} \tag{117}$$

Assuming $\eta^2 << \lambda_K^2$ this can be rewritten as

$$\frac{\psi_3}{c} = -\sum_{K=1}^{3} \Delta_3 \Big[\left(\bar{z} + \frac{2n}{\lambda_K^2} \right) r J_1(\lambda_K r) + 2\left(\bar{z} + \frac{4n}{\lambda_K^2} \right) \frac{J_0(\lambda_K r)}{\lambda_K} \Big] \frac{e^{+n\bar{z}}}{\lambda_K^2} \tag{118}$$

The corresponding particular stress solution is

$$\sigma_{rr}^{\psi} = \sum_{K=1}^{3} \Delta_3 \Big[\left(\bar{z} + \frac{6n}{\lambda_K^2} \right) \lambda_K J_0(\lambda_K r) + \left\{ 2nr - \frac{2}{r}\left(\bar{z} + \frac{4n}{\lambda_K^2} \right) \right\} J_1(\lambda_K r) \Big] \frac{e^{+n\bar{z}}}{\lambda_K^2} \tag{119a}$$

$$\sigma_{\theta\theta}^{\psi} = -\sum_{K=1}^{3} \Delta_3 \Big[\left(\bar{z} + \frac{2n}{\lambda_K^2}\right) J_0(\lambda_K r)\lambda_K$$

$$+ \left\{ \bar{z}\, r \lambda_K^2 - 2\left(\bar{z} + \frac{4n}{\lambda_K^2}\right)\frac{1}{r} \right\} J_1(\lambda_K r) \Big] \frac{e^{+n\bar{z}}}{\lambda_K^2} \tag{119b}$$

$$\sigma_{zz}^{\psi} = -\sum_{K=1}^{3} \Delta_3 \Big[\left(\bar{z} + \frac{2n}{\lambda_K^2}\right) \lambda_K^2 r\, J_1(\lambda_K r)$$

$$+ 2\left(n^2 \bar{z} + 2n + \frac{4n^3}{\lambda_K^2}\right) \frac{J_0(\lambda_K r)}{\lambda_K} \Big] \frac{e^{+n\bar{z}}}{\lambda_K^2} \tag{119c}$$

$$\sigma_{rz}^{\psi} = -\sum_{K=1}^{3} \Delta_3 \Big[\left(1 + n\bar{z} + \frac{2n^2}{\lambda_K^2}\right) \lambda_K r\, J_0(\lambda_K r)$$

$$- 2\left(1 + n\bar{z} + \frac{4n^2}{\lambda_K^2}\right) J_1(\lambda_K r) \Big] \frac{e^{+n\bar{z}}}{\lambda_K^2} \tag{119d}$$

$$\int_0^1 r\, \sigma_{zz}^{\psi}\, dr = -\sum_{K=1}^{3} \Delta_3 \Big[\left\{ 2H(\lambda_K^2 + 2n^2) - \lambda_K^4 \right\} \bar{z}$$

$$+ \left(8Hn - 2n\lambda_K^2\right) \Big] \frac{J_0(\lambda_K)}{\lambda_K^5} e^{+n\bar{z}} \tag{119e}$$

As was shown for the first approximations, the particular stress solution, Equations (113), (116), and (119), when substituted into Equations (99b), (100b), (101c), and (102c) completely defines the stress solution for the given problem.

From the design stand point, $\sigma_{\theta\theta}$ and σ_{ZZ} at $R = 1$ are of primary interest because they are the maximum stresses. By further restriction of certain parameters, these surface stresses can be expressed by two relatively simple equations. This has been done and the results are presented in APPENDIX D.

D. Discussion of the Stress Solution

In discussing the temperature solution it was noted that axial effects are negligible for small values of η (Conventional fluids). In the stress solution for these cases the second order approximations are negligible and the general stress solution converges to the classical plain strain solution.

As η becomes larger (other parameters held constant) an axial effect primarily in the form of a shear stress appears. In these cases the second order terms become significant. However, the majority of the terms containing η in the second order approximation remain negligible. Thus the equations are still relatively simple.

When η approaches the upper limiting values of the problem all of the terms in the stress solution are significant in the general case. If some of the generality of the solution is sacrificed by requiring that

$\bar{Z} >> 4\eta/\lambda_K^2$ then once again most of the terms containing η can be eliminated. In this case these terms are insignificant because they are small in comparison to the $\bar{Z}$ and $\bar{Z}^2$ terms. In the previous case where η assumed intermediate values these same terms were eliminated because they were simply negligible.

If η were to be increased further (transition zone between laminar and turbulent flow of liquid metals), the formulation and the solution would have to be extended to include the effects of axial conduction.

In the remainder of this section the curves and discussion apply to the cases in which the range of η is intermediate to small. For these cases the stress solution is characterized by four parameters: r , t-Z , MHZ , and H .

The effects of the first two parameters are illustrated in Figures 7,8,9 and 10. Figure 8 is the data of Figure 7 replotted to better illustrate the magnitude of the stress gradient in the radial direction for small, intermediate, and large values of time. Curves of stress vs radius could also be plotted for the data in Figures 9 and 10 but were omitted since the general trend is illustrated by Figure 8.

Figure 11 illustrates the effect of the Biot number on the surface stresses. The larger the Biot number of the system the larger the temperature gradient and hence the larger the stresses.

Figure 12 shows the effect of the parameter mHZ (alternatively $A^*xStx\bar{Z}$ on the stresses. Note for MHZ $\leq$ 1.0 , that almost an exact solution to the problem is obtained. For the limiting case, A*xSt = .2 , the

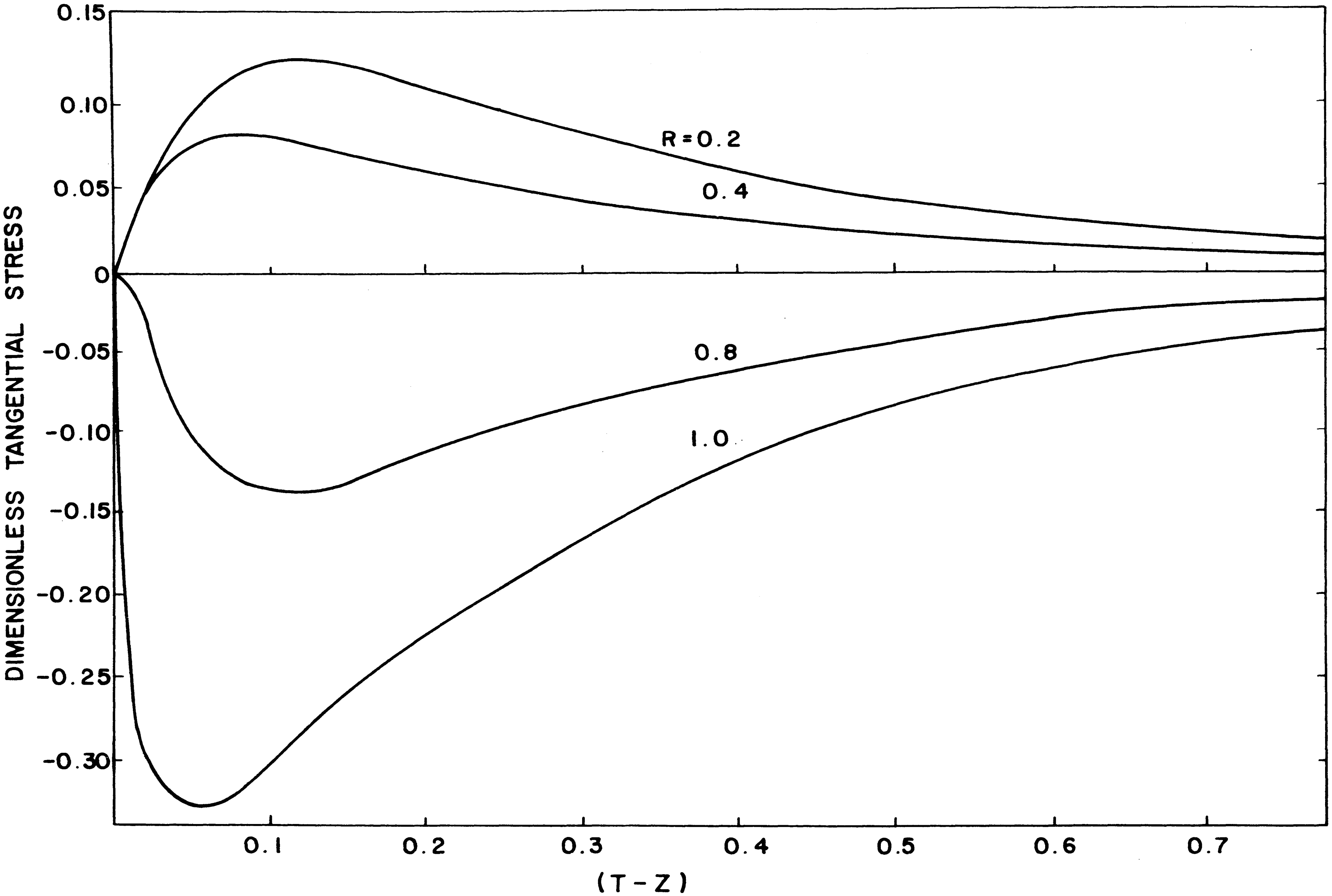

Figure 7. Transient Tangential Stress Distribution I $H = 3.6$, $Z = 0$.

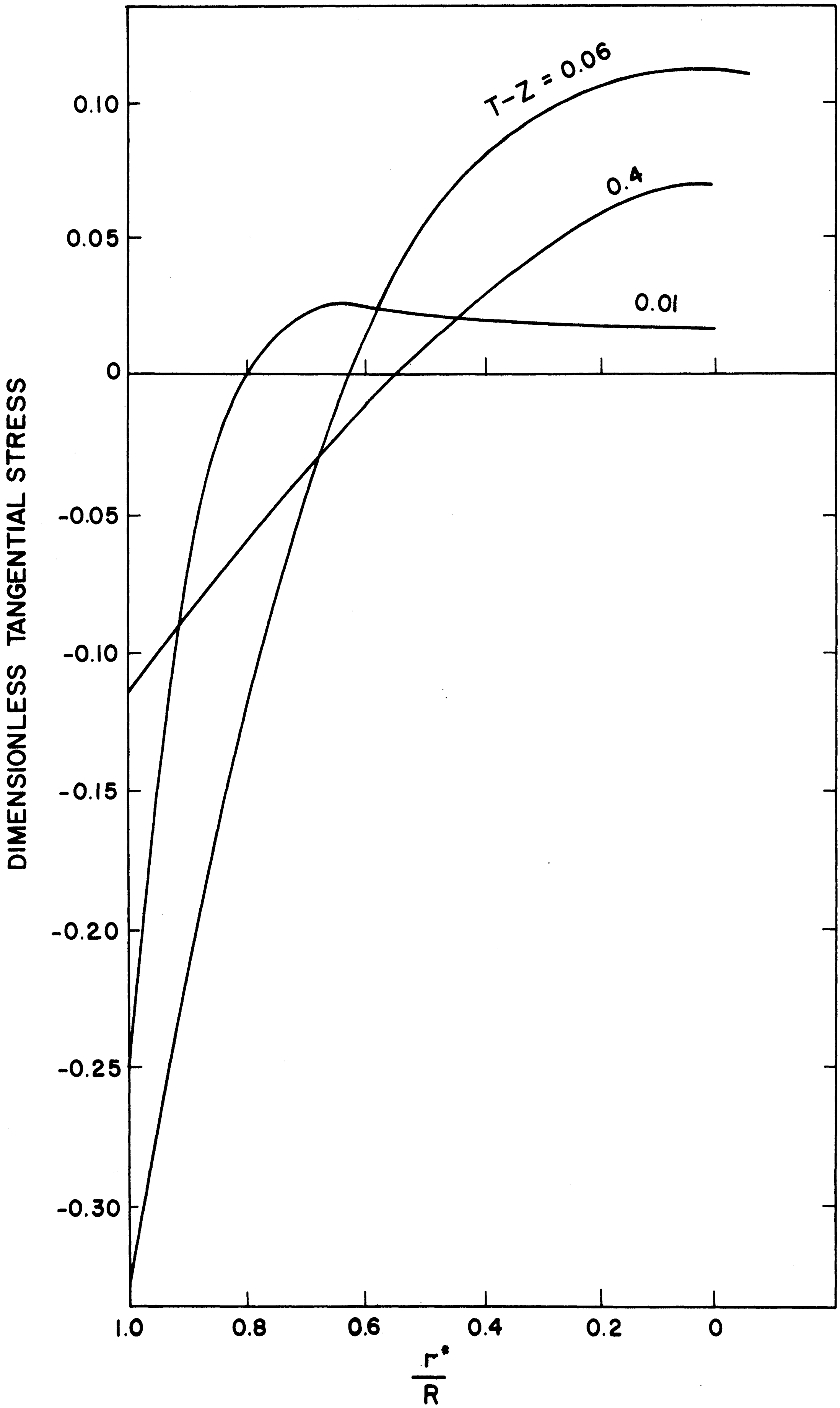

Figure 8. Transient Tangential Stress Distribution II H = 3.6, Z = 0.

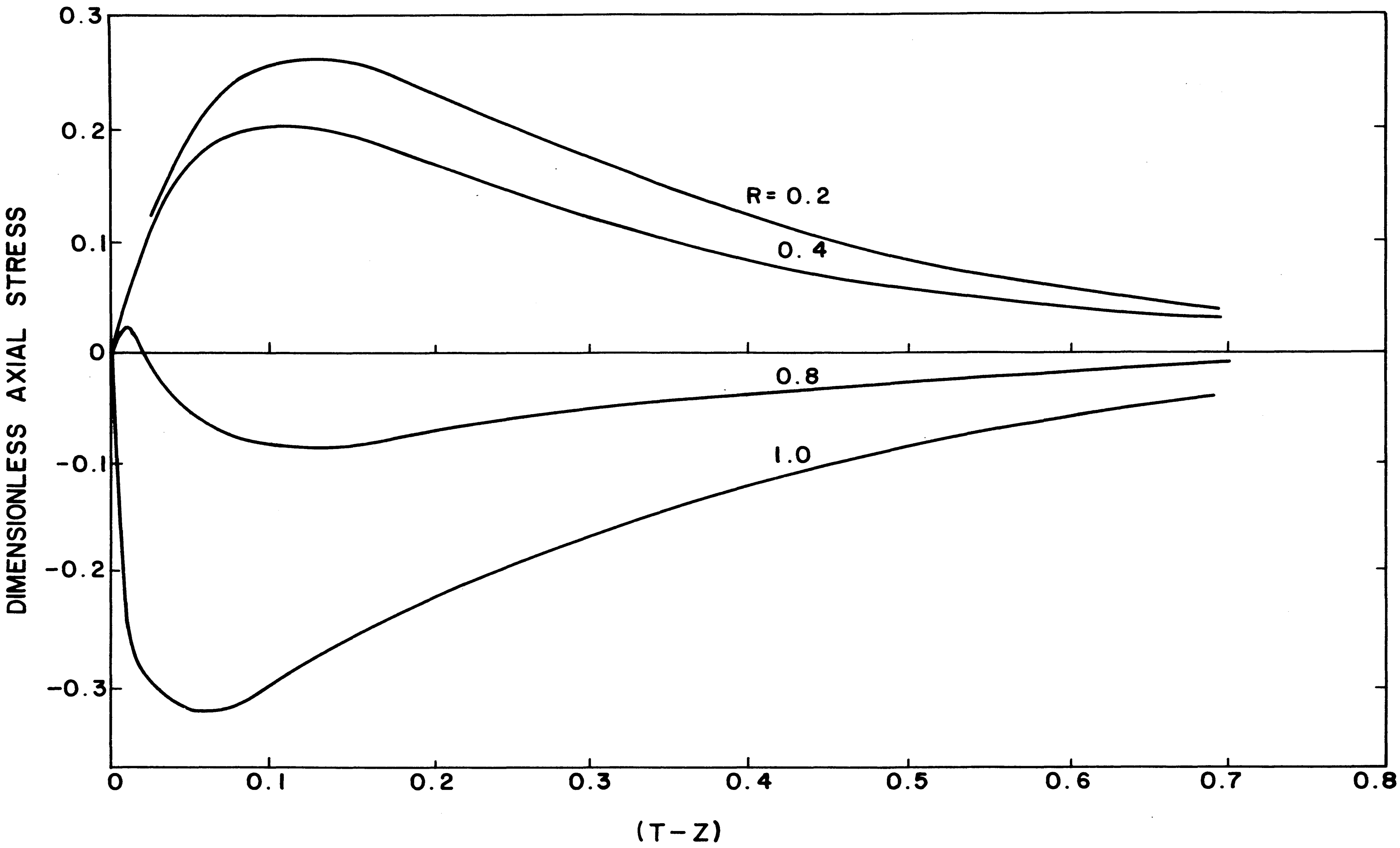

Figure 9. Transient Axial Stress Distribution $H = 3.6$, $Z = 0$.

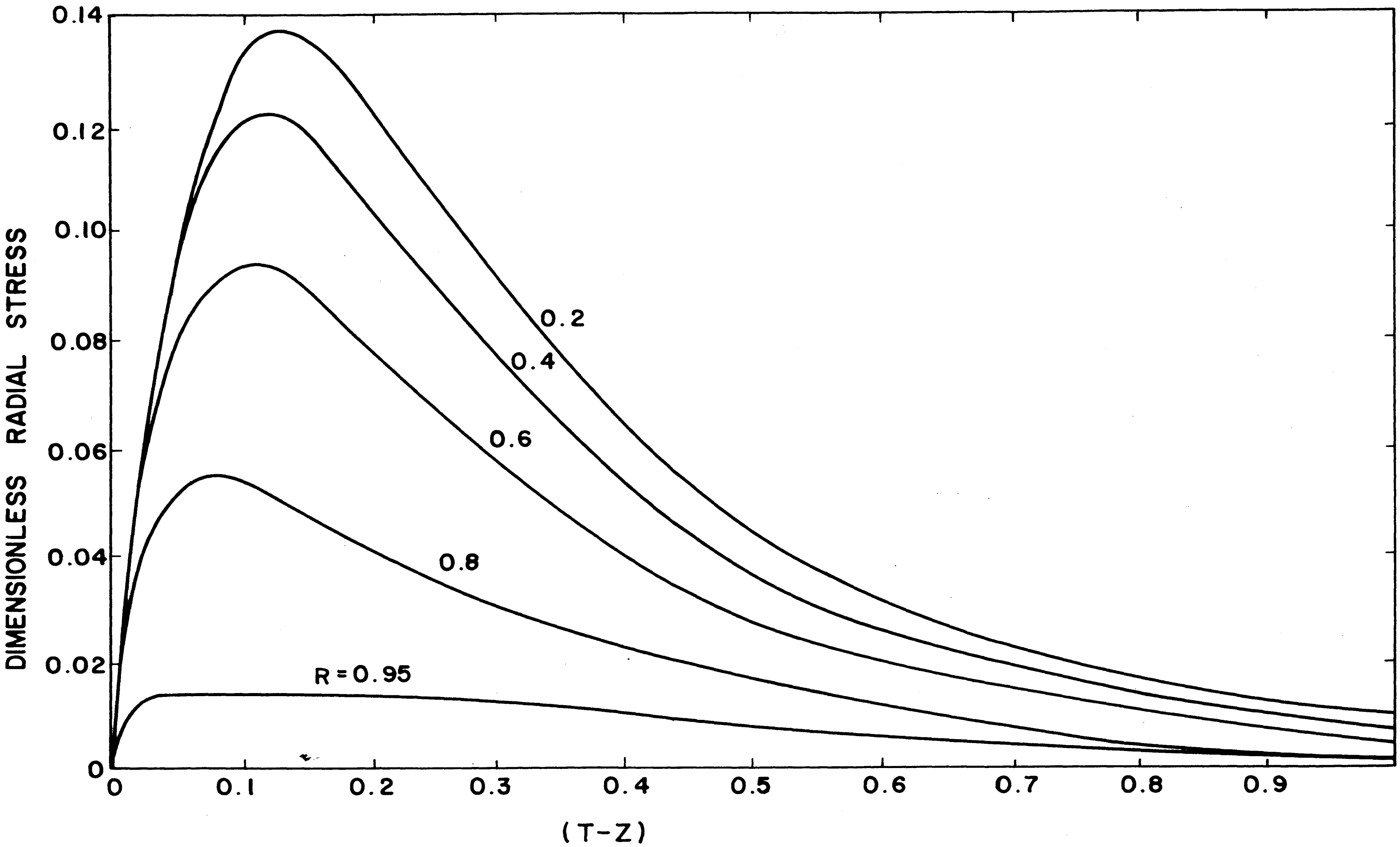

Figure 10. Transient Radial Stress Distribution $H = 3.6$, $Z = 0$.

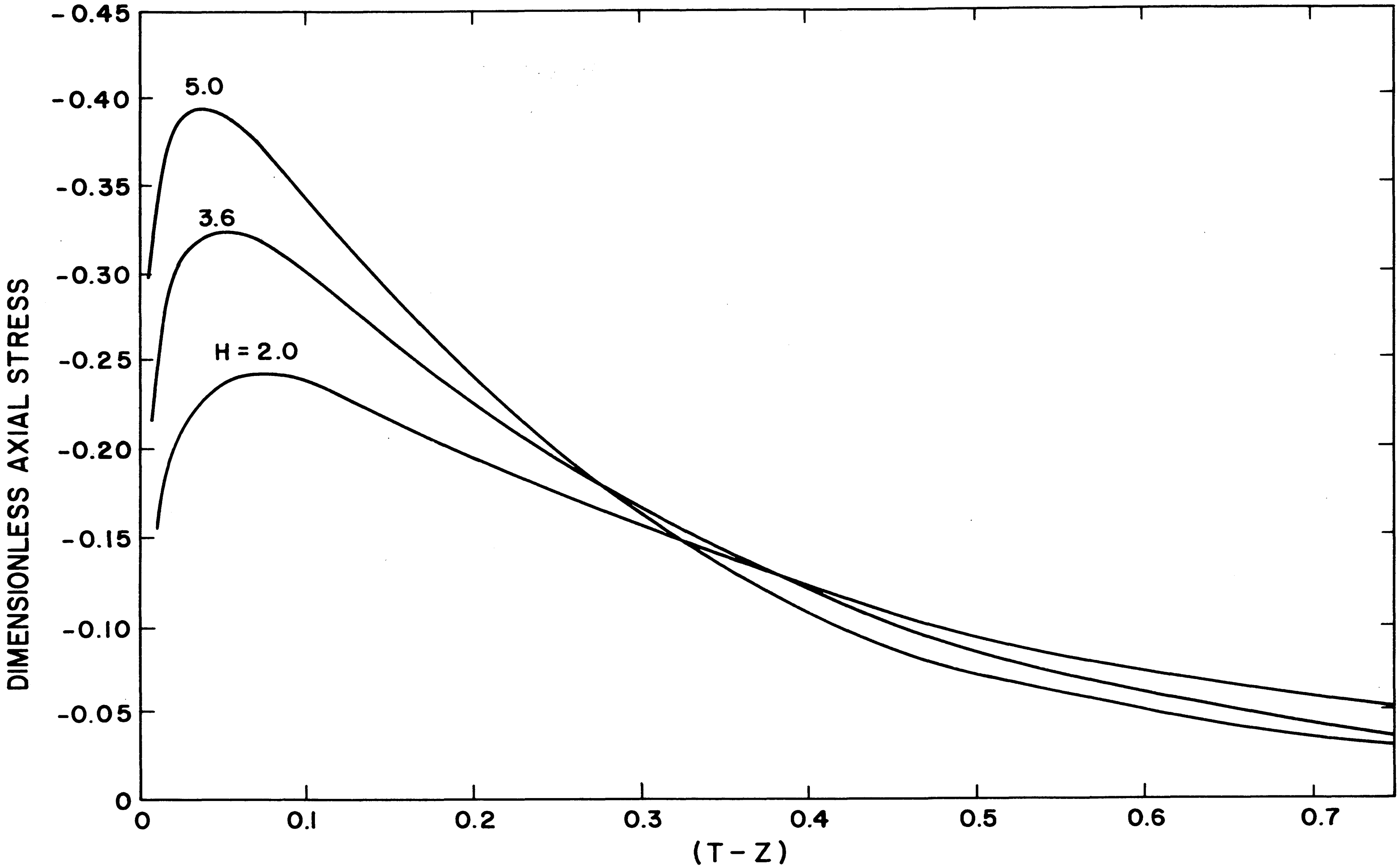

Figure 11. Effect of the Biot Number on the Axial Stress. R = 1.0, Z = 0.

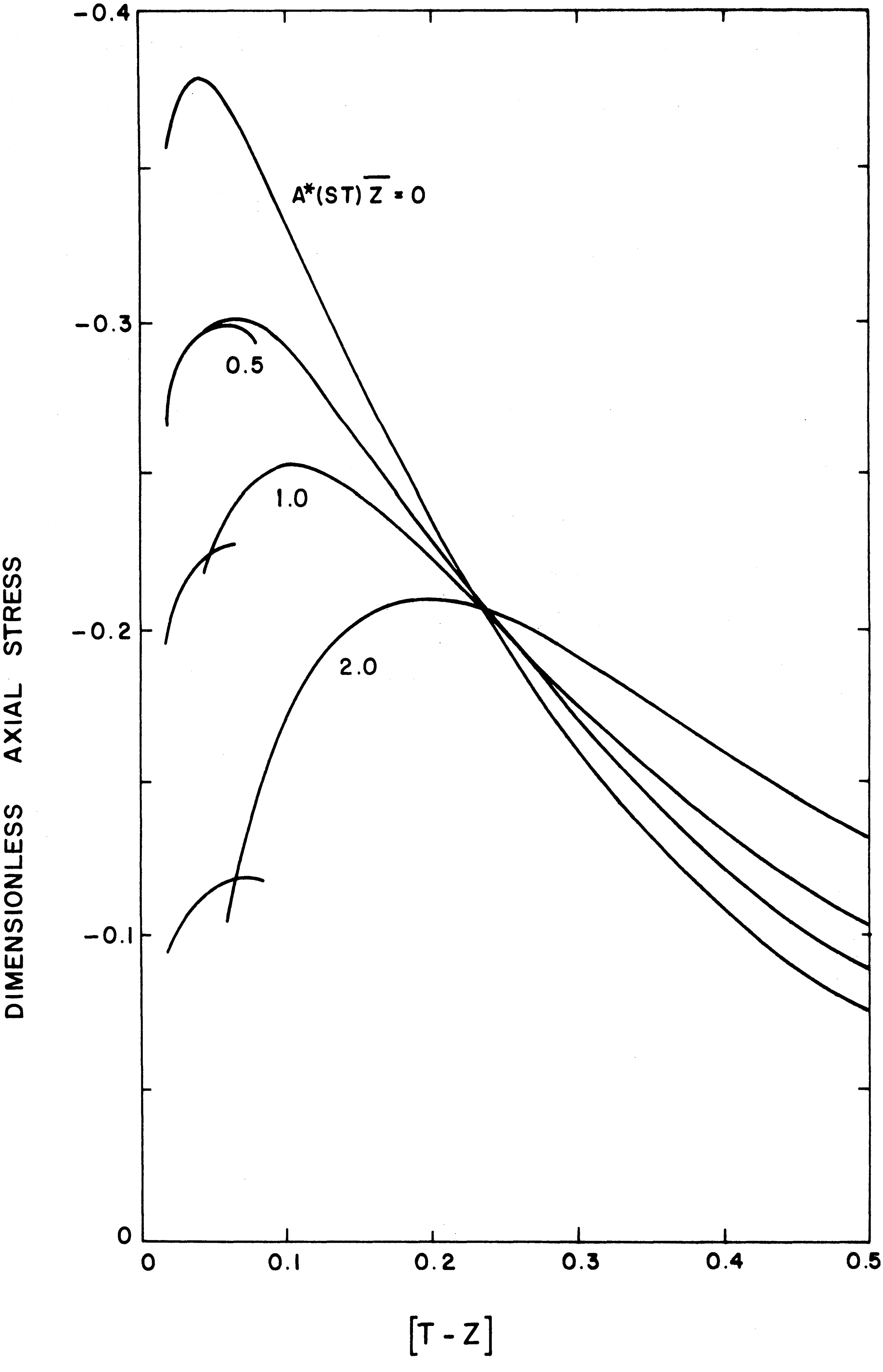

Figure 12. Effect of the Stanton Number on the Axial Stress. H = 4.7, R = 1.0.

"exact" solution would be restricted to $\bar{Z} < 5$. This would correspond approximately (large η effect not included in curves) to a fully turbulent liquid metal case.

Figure 13 shows the effect of the first and second terms of the approximate solution. As expected, the second order terms are most significant for intermediate values of time.

The steady state stresses, Equations (70a-d), are not shown graphically. All of the normal stresses vary parabolically with the radius and are directly proportional to the rate of internal energy generation per unit volume of rod. The stresses at the surface are tensile and at the center compressive. From the transient solution one finds that a step change from a hot to a cold fluid would also produce tensile stresses of the surface. Hence this is the most severe stress situation encountered by the rod for the given problem.

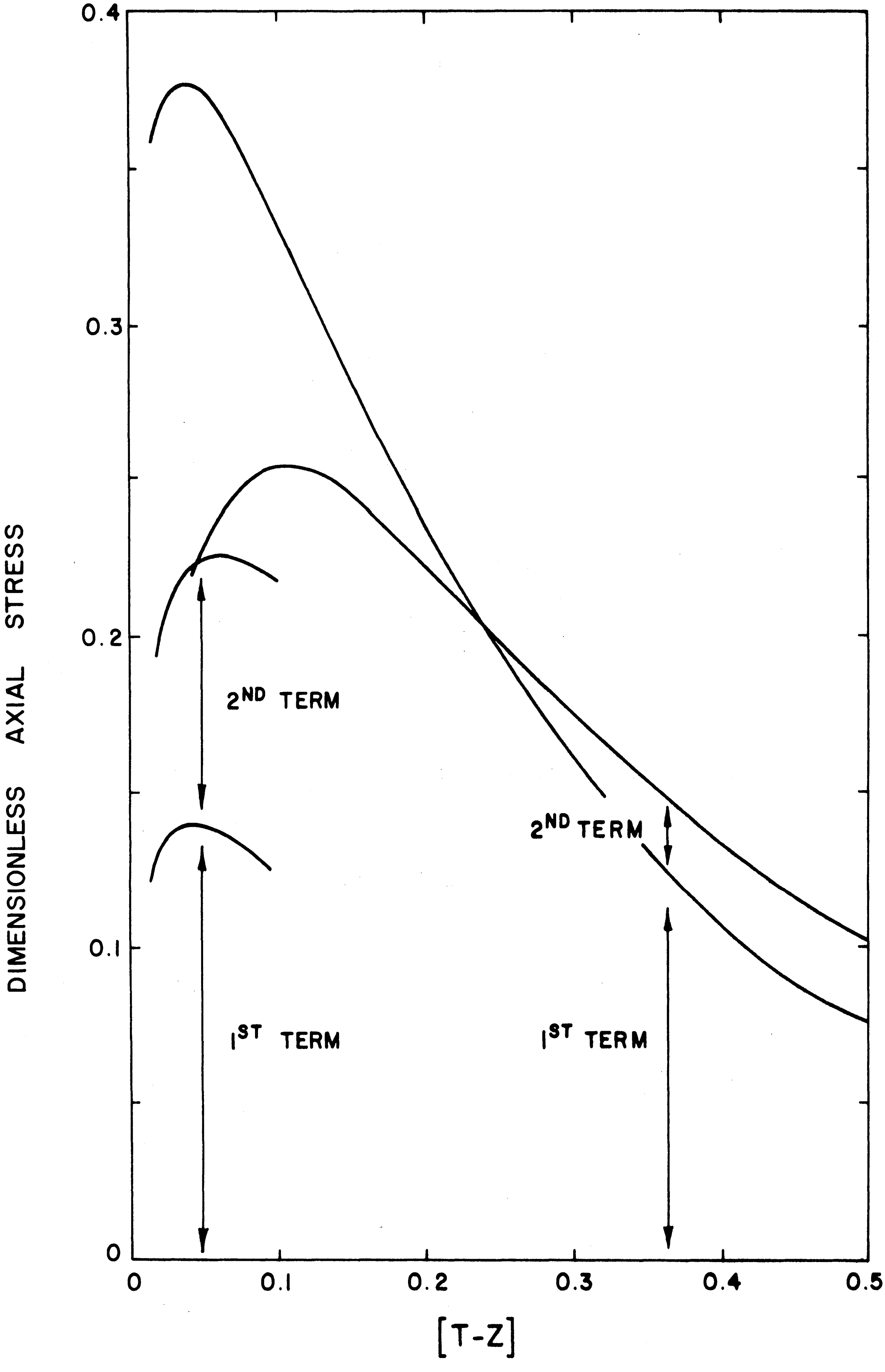

Figure 13. Order of Magnitude of the First and Second Axial Stress Approximations. H = 4.7, R = 1.0, A*(St)Z = 1.0.

CHAPTER IV

EXPERIMENT

A. Experimental Apparatus

The experimental apparatus was designed and the data for this chapter taken before even contemplating the extension of the solution to include the fully turbulent liquid metal cases. As such the experimental results presented here do not directly verify the second term approximations.

The case which was studied experimentally consisted of a solid stainless steel bar which was cooled on the surface with ordinary tap water. The transient effect was introduced into the system by suddenly changing from a hot (125°F) to a cold (65°F) water supply (or vice versa). A relatively high water velocity was used to assure a fully turbulent condition. As a result the axial effects were negligible.

Figure 14 is a schematic of the test apparatus. Figure 15 is a photograph of the actual test setup. The test equipment consisted of four sections: the step change section, inlet section, test section, and the exit section. The step change section centers around a four-way cock valve. The two inlet ports of the cock valve were connected via valves to the hot and cold water supplies of the building. One of the two exit ports of the cock valve led to the inlet section. The other exit port let to a turbine-type flow meter. The exit from the flow meter was connected to a drain via 1/2" plastic water pipe.

The test inlet section consisted of a reducer, a section of 1 1/4" plastic water pipe, and an insulated electrode. The reducer was used to

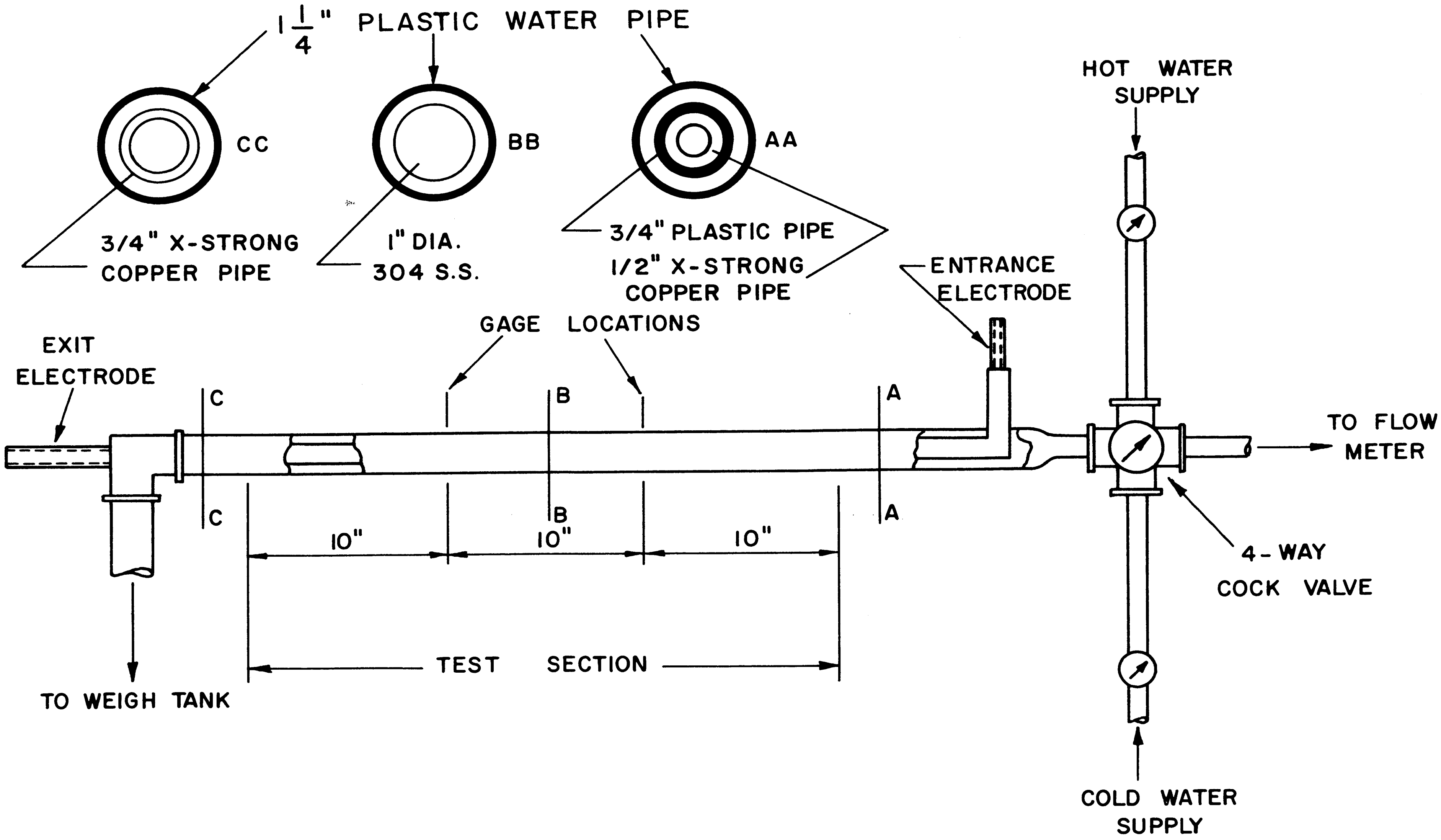

Figure 14. Schematic of the Test Apparatus.

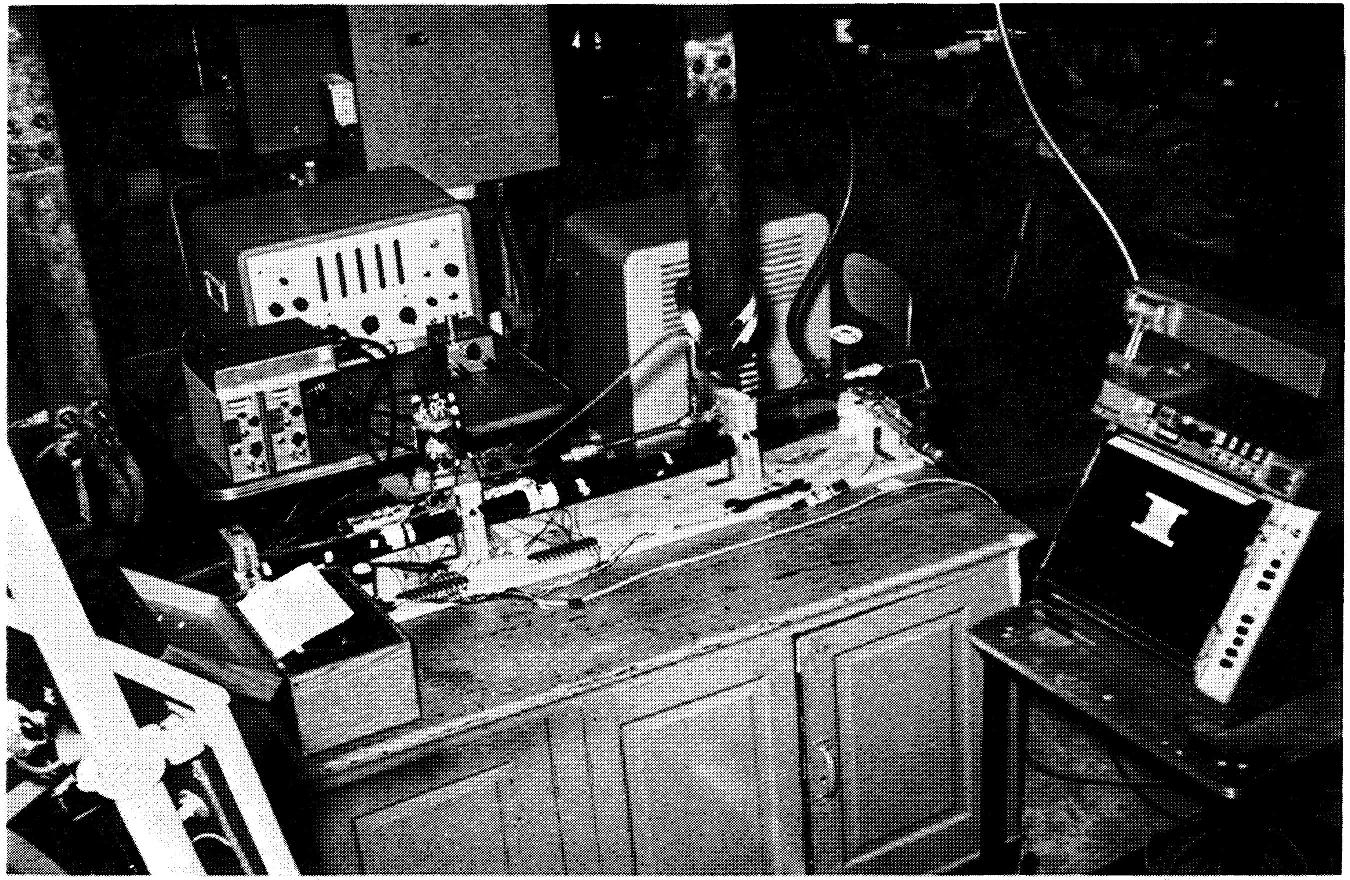

Figure 15. Photograph of the Test Apparatus.

match the 1/2" pipe fitting on the cock valve to the 1 1/4" pipe. This plastic pipe served as the casing and as an insulater for the outer surface of the annulus of water. The electrode, which was located concentrically inside of the 1 1/4" pipe, was made of 1/2" extra strong copper pipe covered with 3/4" plastic water pipe. The plastic pipe was used to insulate the copper from the water in order to avoid a heat sink prior to coming into contact with the actual test section. Copper was used as an electrode to keep I^2R power dissipation to a minimum. Pipe rather than a solid rod was used to enable water to be circulated in the electrode to remove the I^2R heat.

The test section consisted of a 30" section of solid 1" diameter 304 stainless steel. A male thread on one end connected the test section to the exit electrode. A female thread on the other end connected this end to the inlet electrode. Plastic water pipe (1 1/4") formed the outer casing and insulation for the annulus of coolant water. Temperature and stress measurements were taken at 10" and 20" from the inlet of the test section.

The exit section consisted of an electrode, a section of 1 1/4" plastic pipe, an elbow, a reducer, and a section of 1/2" plastic pipe. The electrode was 3/4" extra strong copper pipe (1.005"dia) which had a female thread on one end to connect it to the test section. The 1 1/4" pipe formed the outer casing for the annulus of water. A 1 1/4" 90° elbow was used to turn the water from the direction of the electrode. The electrode protruded through the corner of the elbow and was connected via buss bars to a 3000 amp motor generator set. The water exiting the elbow went through a reducer.

and into 1/2" plastic water pipe. This in turn led to a weigh tank.

B. Instrumentation

1. Temperature Measurement

The surface temperature data was obtained by using nickel foil temperature sensors (STG-50, Micro-Measurements Inc.). The gages resemble a foil-type strain gage in appearance, measure approximately 1/4 x 1/4 x .002 inches, and were cemented to the surface of the bar with a high temperature strain gage cement. The sensing element is laminated with a high-temperature epoxy resin and glass fiber matrix.

Since the nickel-foil was insulated electrically from both the bar and water no further protective coatings were used on the gage itself. This is not a recommended procedure for long-time applications since hot water will affect the laminating material. The primary reason for not using protective coating was to avoid thermally insulating the surface of the bar at the point of the temperature measurement. All internal lead wires were electrically insulated from the water with one or more coats of a waterproof enamel, Glyptal, and a coat of Silastic RTV 731 (Dow Corning Corp.).

Half inch lengths of No. 34 Soldereze wire were used as jumpers between the gage solder tabs and the terminal strips. No. 27 Soldereze wire was used from the terminal strip to the outside of the plastic water pipe. From here No. 20 solid hook-up wire was used to complete the rest of the wiring in the circuit. The temperature sensor was connected to a series-parallel circuit of precision resistors (LST-100, produced by Vishay) which converts the system to an equivalent two gage strain gage circuit with a nominal gage resistance of 120 ohms. The other part of the bridge circuit was fabricated from two 100 ohms wire wound resistors, a 10^k potentiometer and a 1 1/2 volt dry cell. The output of the bridge circuit was recorded via a Honeywell Visicorder.

Water temperatures, when required, were measured with either 30-gage copper-constantan thermocouples or a mercury thermometer depending on the required accuracy.

2. Strain Measurement

Surface strains in the axial and tangential directions were measured with, temperature compensated, foil type resistance strain gages (MA-09-125AD-120, Micro-Measurements, Inc.). The jumper wires, lead-out wires, terminal strips, and electrical insulation were the same type as was used for the temperature sensors. Since the outputs of the strain gages were relatively small, Ellis Bridge Amplifiers were used. The strain was recorded via a Honeywell Visicorder.

Unlike the temperature sensors, the strain gages were not encapsulated and hence had to be insulated electrically (but ideally not thermally) from the water. Silastic RTV 731 was found to be too much of a thermal insulator for use in this project. When using this type of coating, readings were very low. This indicated a lack of the large expected thermal gradient as a result of this thermal insulation.

The strain data presented in this report was taken using only a Glyptal coating on the gages. It was found that the ground-to-gage resistance changed with temperature and caused a false strain reading about equal in magnitude to the actual peak strains. Thus rather than getting a reading which started at zero and ended at zero, a constant apparent strain was present at large values of time. By subtracting this strain reading from the readings at other values of time, the actual strain could be determined (except for very small t-Z where the transient resistance changes of the Glyptal were present).

3. Velocity Measurement

The velocity of the water was measured with a turbine type flow meter. As shown in Figure 14, the meter is not in the test portion of the system. It was, however, calibrated to read the equivalent test section velocity. Prior to taking test data, the hot and cold water flows were individually set equal at some specific velocity by using the flow meter.

C. Heat Transfer Coefficient

The Biot number (H/2) is probably the one single most important parameter in the total solution. Since H is directly proportion to the convective heat transfer coefficient, $\bar{h}$, and since different empirical equations give different values of $\bar{h}$, it is desirable to measure this quantity experimentally for the actual test apparatus. One way of doing this is to assume various values of H and by trial and error determine an analytical curve which best fits the experimental transient temperature data. Besides being a tedious job, this approach is not desirable because it uses only the stress measurements for a check on the whole solution.

The procedure used here was to determine the coefficient by using the steady state solution and assume $\bar{h}$ is the same for the transient case.

Once $\bar{h}$ has been determined then analytical curves for both temperature and stress could be calculated and compared with the experimental results. Several possibilities exist for solving the problem this way. One is to measure only $T(1,z) - \theta(0)$. Another is to measure only $T(1,z) - \theta(z)$. Also one could measure $\frac{\partial T}{\partial r}(1,z)$ and $T(1,z) - \theta(z)$. The latter of these is the most ideal since $\bar{h}$ could be determined directly from its definition. The determination of $\frac{\partial T}{\partial r}(1,z)$, however, is difficult to achieve experimentally, especially where one encounters a relatively high heat flux at the surface.

The second method could be performed easily but requires two measurements. The first method was used for this report since it was ideally suited for the resistance type temperature sensors already cemented to the surface of the bar. Prior to initiating the internal energy generation, the output of the temperature bridge was set equal to zero. This represents the $\theta(0)$ reference temperature. Since the sensor measures only a change in temperature, then with energy generation, the surface temperature relative to its initial temperature, $T(1,z) - \theta(0)$, is measured directly.

Using the steady state solution, Equation (8) and rearranging, the following relationship is obtained

$$H = \frac{\bar{h}R}{K} = \frac{1}{\dfrac{2K\Delta T}{q'R^2} - Mz} \tag{120}$$

By evaluating the other parameters at an average temperature, $\bar{h}$ is determined from the temperature sensor reading.

D. Experimental Results

Table I gives values of the convective heat transfer coefficient, $\bar{h}$, for various flow conditions. The theoretical values were calculated by using the average of the h's as determined by four different empirical equations found in Kreith[9] and McAdams[10]. The physical properties were taken from Kreith[9] for the water and from Metals Handbook[11] for the stainless steel.

The experimental values in Table I were determined as discussed earlier. Two different locations, $\bar{Z}$ = 20 and 40 , and six different rates of internal energy generation (current varied between 600-1600 amps) were used. The average of these 12 readings is used as an experimental $\bar{h}$.

TABLE I

$\bar{h}$	2 FT/SEC		3.5 FT/SEC	
	EXP.	THEORY	EXP.	THEORY
cold water	508	544	960	840
hot water	770	693	1360	1083
average	639	619	1160	962

The table shows that the experimental values of $\bar{h}$ tend to be higher than predicted by theory. The theoretical values are, however, within $\pm$ 20% of the experimental results. Scatter of this order of magnitude is not unusual for convective heat transfer coefficient data. Part of the reason why the experimental $\bar{h}$'s tends to be high is that the temperature difference measured by the temperature sensor is slightly low. This would be expected since the sensor protrudes into the boundary layer of the coolant and hence this side of the sensor is a little cooler than the side next to the bar.

Figure 16 shows experimental data of the transient surface temperature. The magnitude of the step change for the transient data was approximately 60°F. The solid line represents the analytical solution to the problem using the average $\bar{h}$ of the empirical equations and should lie half way between the two sets of data. The temperature sensor insulates the surface of the bar and in the presence of the high heat flux at small values of time causes a noticeable lag in the experimental temperature data. This same effect is seen in the transient stress data.

Figures 17 and 18 represent average data of the tangential and axial stresses. Each data point is an average of two readings. A continuous transient trace was recorded, however, only 20-25 data points at various time intervals were reduced for comparison with the analytical solution. Under the given test conditions the two surface stresses are theoretically equal and hence the two analytical curves are identical. As a result of the thermal insulation of the gage (plus coating), the measured peak stresses are somewhat low and lag the theoretical solution. The general trend is, however, readily apparent. Figures 17 and 18 show that in the initial part of the curves the hot water step change data agrees well with theory. This same trend is present in the surface temperature date in Figure 16.

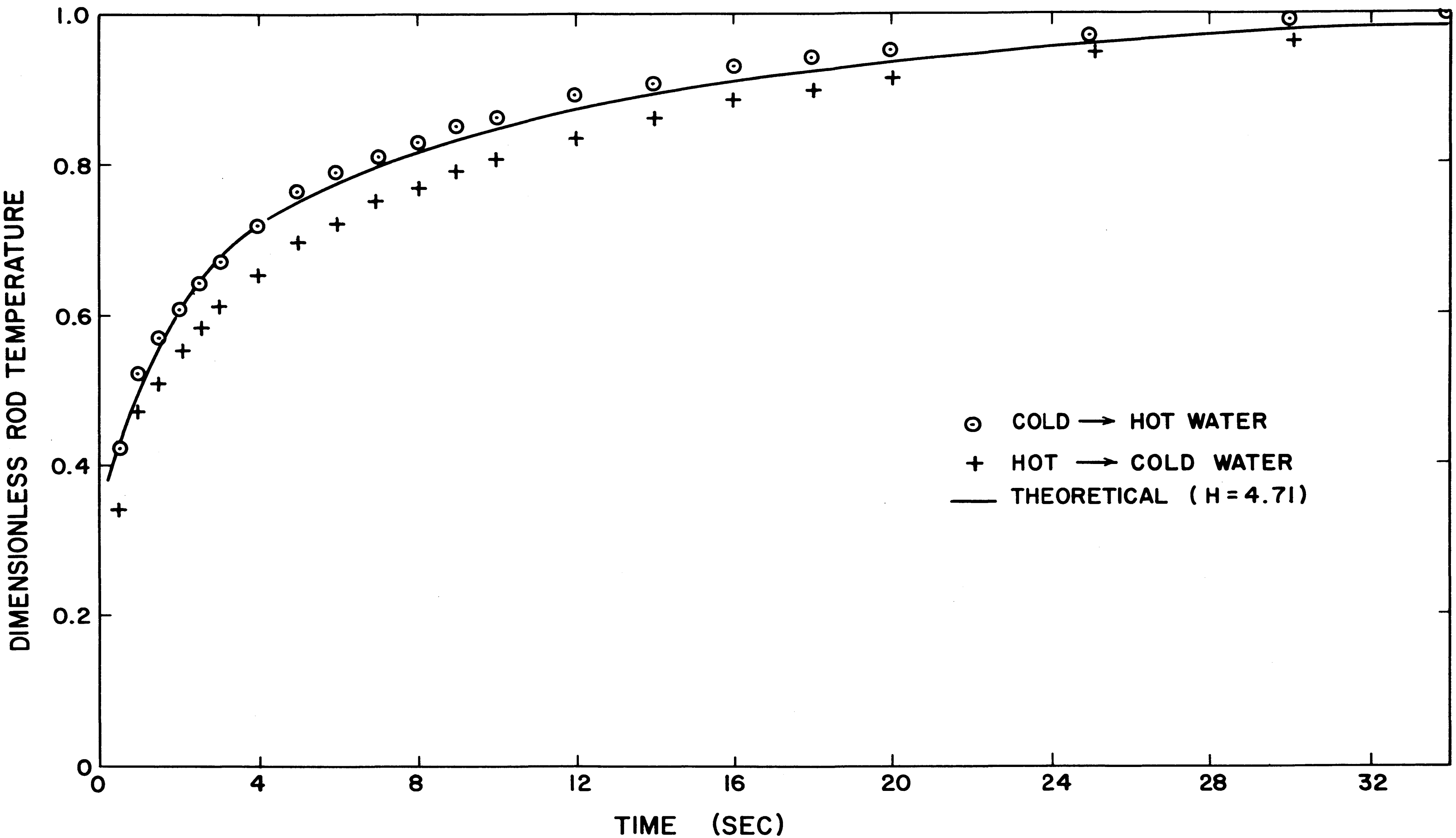

Figure 16. Experimental Transient Temperature Data. H = 4.7, R = 1.0.

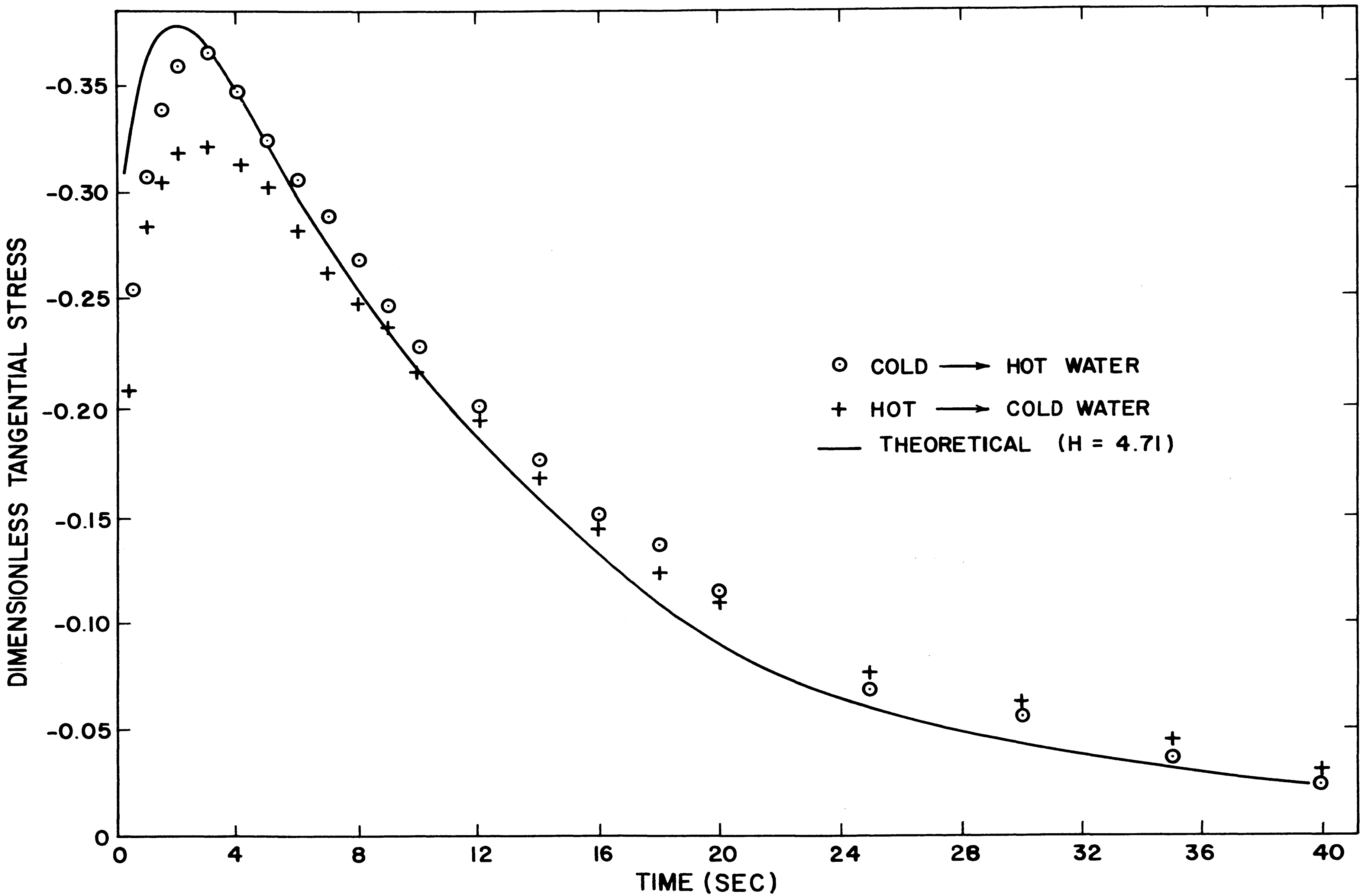

Figure 17. Experimental Transient Tangential Stress Data. H = 4.7, R = 1.0.

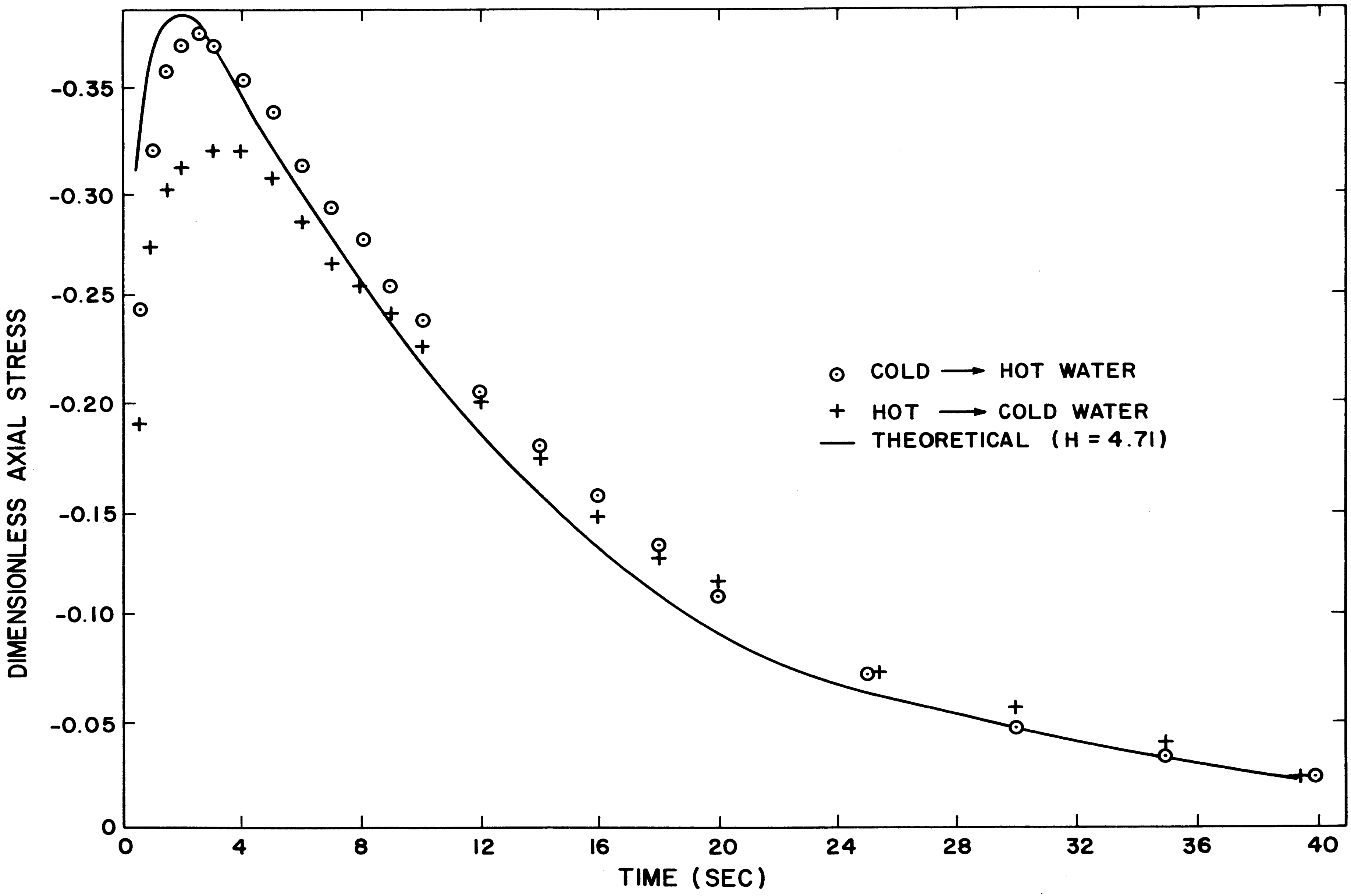

Figure 18. Experimental Transient Axial Stress Data. H = 4.7, R = 1.0.

Figure 19 represents the steady state stress data. Since the two surface stresses are theoretically equal, only one curve is plotted. No significant difference was noted between the measured axial and tangential strains. Although the general trend is illustrated, the data is on the high side. This is most likely a result of a change in the resistance of the Gylptal coating on the gages. This same effect was noted on the transient stress data and was compensated for in those cases. In this case, the temperature changes were relatively low and the data was taken early in the experimental program. Hence the effect slipped by unnoticed during the experimental phase.

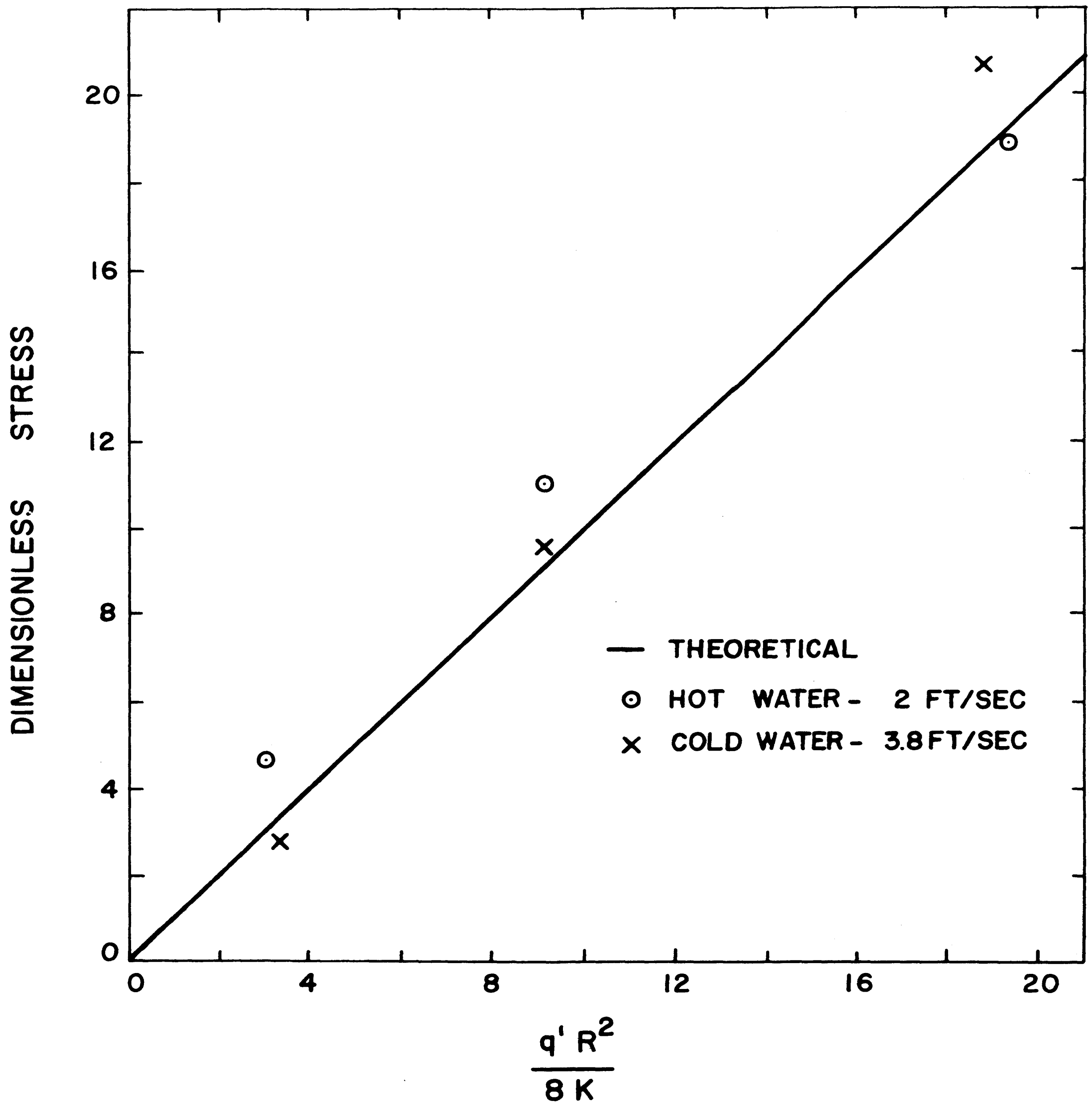

Figure 19. Experimental Steady State Stress Data.

CHAPTER V

SUMMARY

By deriving an exact solution to the thermoelastic problem in terms of the Laplace transform variable, S , it is found that an exact solution is not feasible. Making use of the fact that large values of S corresponding to small values of time and vice versa an approximate solution to the temperature problem is obtained. This is used in conjunction with Goodier's thermoelastic potential , ψ , to obtain an approximate, temperature dependent particular solution. The resulting isothermal stress problem is solved by deducing six appropriate Love functions. Using an order of magnitude analysis it is possible to express the stress components of the Love functions in terms of Goodier's potential , ψ . This eliminated the need for working directly with the Love functions.

The resultant approximate stress solution for the given problem is relatively complex. By sacrificing some of the generality of the solution such that $Z >> 4\eta/\lambda_K^2$ many terms in the solution can be eliminated. This can be observed by comparing the stresses in the general solution with the surface stresses presented in APPENDIX D. When conventional fluids are used as a coolant, a plain strain analysis is adequate.

The measurement of surface temperature and strain in the presence of a relatively high heat flux and an electrically conducting medium became a major problem area rather than a simple check on the theory. Essentially one was faced with the task of electrically, but not thermally, insulating a surface. The experimental surface temperature and heat transfer coefficient data were quite good. Strain measurements, however, were only moderately successful.

APPENDIX A

FORMULATION OF THE TEMPERATURE PROBLEM

Figure 1A is a schematic of the system under consideration. By using the assumptions given in Chapter II and applying the First Law of Thermodynamics to a control volume for the fluid and a system for the rod, one gets the following governing differential equations:

For the rod

$$\nabla_{r^*}^2 T + \frac{q'}{K} = \frac{\partial^2 T}{\partial r^{*2}} + \frac{1}{r^*}\frac{\partial T}{\partial r^*} + \frac{q'}{K} = \frac{1}{a}\frac{\partial T}{\partial t^*} \tag{A-1}$$

For the fluid

$$V\frac{\partial \theta}{\partial z^*} + \frac{\partial \theta}{\partial t^*} = -\frac{2\pi R K}{\rho_F A C_F}\frac{\partial T}{\partial r^*}(R, z^*, t^*) \tag{A-2}$$

subject to the following initial and boundary conditions

$$\theta(0, t^* < 0) = \theta_0 \tag{A-3}$$

$$\theta(0, t^* > 0) = \theta_0 + \Delta\theta \tag{A-4}$$

$$\frac{\partial T}{\partial r^*}(0, z^*, t^*) = 0 \tag{A-5}$$

$$K\frac{\partial T}{\partial r^*}(R, z^*, t^*) = \bar{h}\left[\theta(z^*, t^*) - T(R, z^*, t^*)\right] \tag{A-6}$$

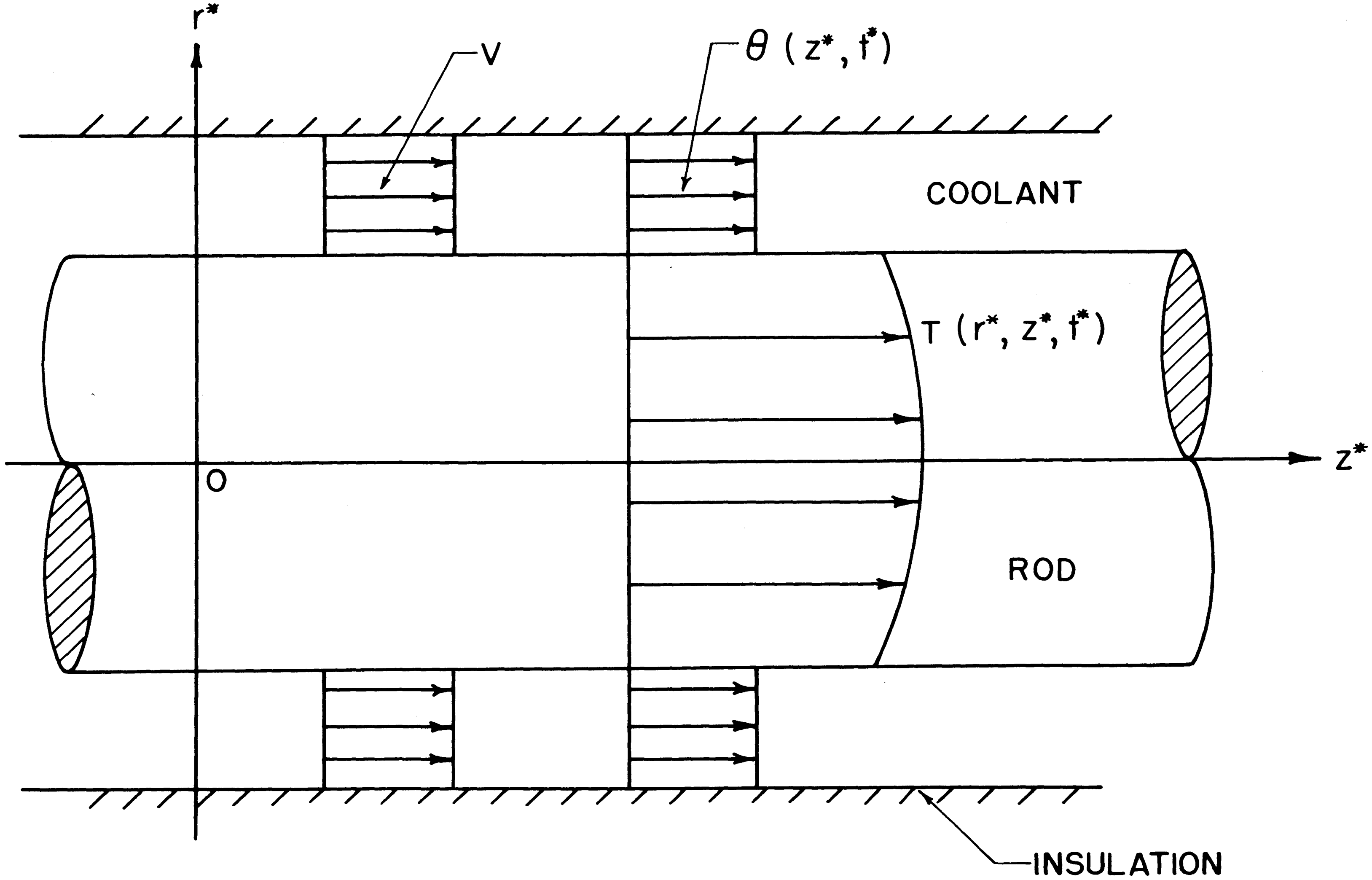

Figure 1A. Schematic of the System.

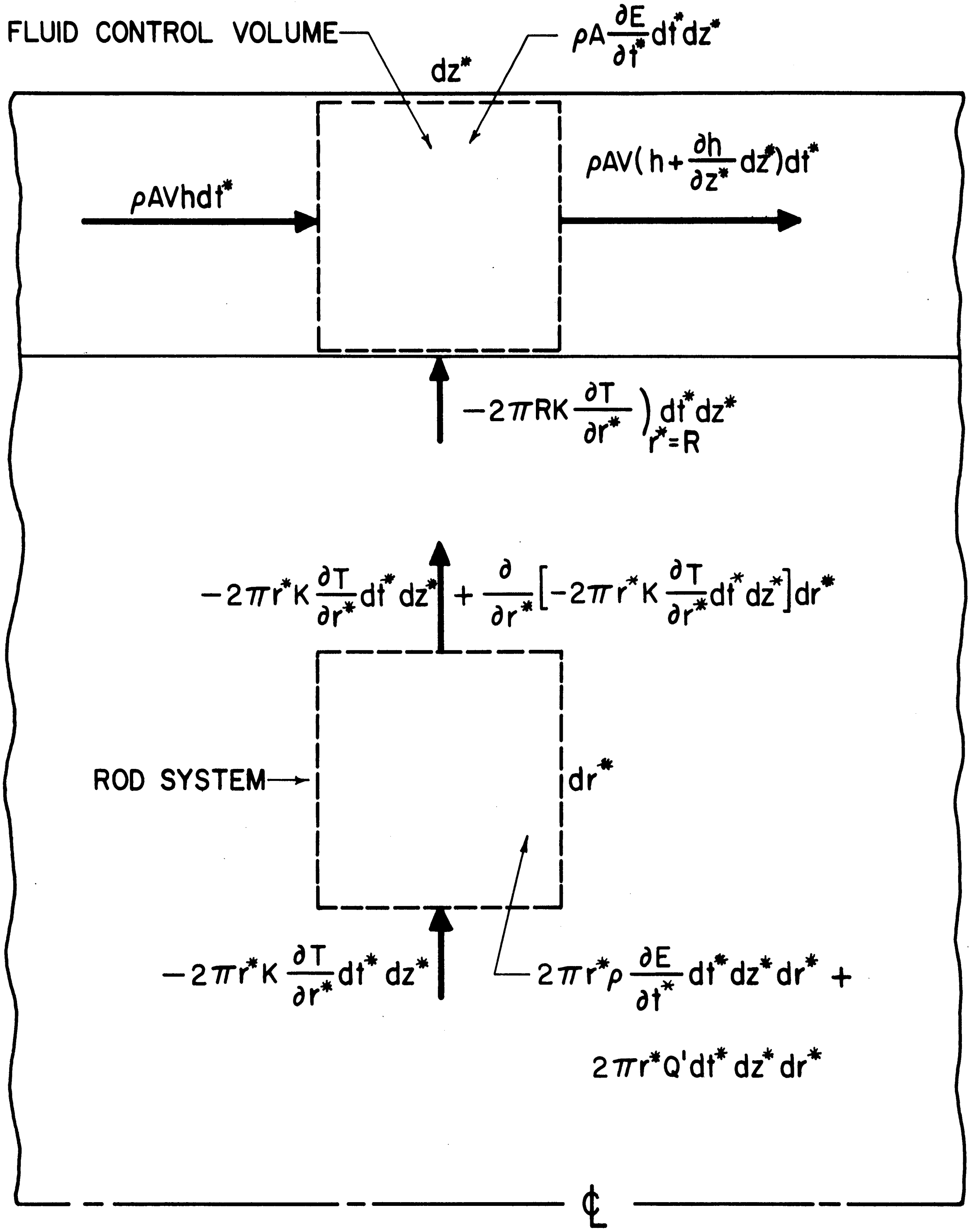

Figure 2A. First Law of Thermodynamics Applied to Rod and Fluid Elements.

where K is the thermal conductivity of the rod, a the thermal diffusivity of the rod, r the radius of the rod, q' the rate of internal energy generation per unit volume of rod, v the velocity of the coolant, ρ_F the density of the coolant, C_F the constant pressure specific heat of the coolant, A the cross sectional area of the coolant, and $\bar{h}$ the convective heat transfer coefficient.

By defining the rod temperature, T, as

$$T = T_{ss} \times \frac{q'R^2}{K} + T_{TR} \times \Delta\theta \tag{A-7}$$

and the fluid temperature, Θ, as

$$\theta = \theta_{ss} \times \frac{q'R^2}{K} + \theta_{TR} \times \Delta\theta \tag{A-8}$$

the same differential equations and initial and boundary conditions can be rewritten as follows:

$$\nabla_r^2 T_{ss} + 1 = 0 \tag{A-9}$$

$$\nabla_r^2 T_{TR} = \frac{\partial T_{TR}}{\partial t} \tag{A-10}$$

$$\frac{\partial \theta_{ss}}{\partial z} = -M \frac{\partial T_{ss}}{\partial r}(1, z) \tag{A-11}$$

$$\frac{\partial \theta_{TR}}{\partial z} + \frac{\partial \theta_{TR}}{\partial t} = -M \frac{\partial T_{TR}}{\partial r}(1, z, t) \tag{A-12}$$

$$\theta_{SS}(0) = 0 \tag{A-13}$$

$$\theta_{TR}(0, t) = 1 \tag{A-14}$$

$$\theta_{TR}(0, 0) = 0 \tag{A-15}$$

$$\frac{\partial T_{SS}}{\partial r}(0, z) = \frac{\partial T_{TR}}{\partial r}(0, z, t) = 0 \tag{A-16}$$

$$\frac{\partial T_{SS}}{\partial r}(1, z) = H\left[\theta_{SS}(z) - T_{SS}(1, z)\right] \tag{A-17}$$

$$\frac{\partial T_{TR}}{\partial r}(1, z, t) = H\left[\theta_{TR}(z, t) - T_{TR}(1, z, t)\right] \tag{A-18}$$

where $t = \frac{a t^*}{R^2}$ is the Fourier Number, $z = \frac{z^* a}{V R^2}$ the Graetz number, $H = \frac{\bar{h} R}{K}$ a modified Biot number, $r = \frac{r^*}{R}$ the dimensionless radius, $M = \left(\frac{2\pi R^2}{A}\right)\left(\frac{\rho_R}{\rho_F}\right)\left(\frac{c_R}{c_F}\right)$ a dimensionless number, $\theta_{SS} = \frac{\theta(0) - \theta_0}{\frac{q' R^2}{K}}$ the dimensionless steady fluid temperature, $\theta_{TR} = \frac{\theta_{TRANSIENT}}{\Delta\theta}$ the dimensionless transient fluid temperature, $T_{SS} = \frac{T(0) - \theta_0}{\frac{q' R^2}{K}}$ the dimensionless steady rod temperature, $T_{TR} = \frac{T_{TRANSIENT}}{\Delta\theta}$ the dimensionless transient rod temperature.

APPENDIX B

ASYMPTOTIC EXPANSIONS

In deriving the transient temperature solution, it was necessary to expand part of the exponential term in an infinite series in such a manner that the series could be truncated. To obtain this valid truncation, one form of the argument was required for small values of time and a different form was required for large values of time. The purpose of this section is to derive the limiting values of the two arguments in question (See Equations (20) and (24)) and show under which condition each is applicable. For sake of easy reference these arguements are rewritten here

$$\frac{\sqrt{s}\, I_1(\sqrt{s})}{\sqrt{s}\, I_1(\sqrt{s}) + H I_0(\sqrt{s})} \tag{B-1}$$

$$\frac{I_0(\sqrt{s})}{\sqrt{s}\, I_1(\sqrt{s}) + H I_0(\sqrt{s})} \tag{B-2}$$

One way of accomplishing the purpose of this section is to introduce two asymptotic expansions for the modified Bessel functions. For large values of x , one can write

$$I_\nu(x) = \frac{e^x}{\sqrt{2\pi x}} \left[1 - \frac{4\nu^2 - 1}{1!\,8x} + \cdots \right] \tag{B-3}$$

and for small values of x ,

$$I_{\nu}(x) \cong \left(\frac{x}{2}\right)^{\nu}\left[\frac{1}{\Gamma(1+\nu)} + \frac{x^2}{4\Gamma(2+\nu)} + \cdots\right] \tag{B-4}$$

Information concerning the derivation of these expansions can be found in Watson(8) .

If Equations (B-3) and (B-4) are substituted into Equations (B-1) and (B-2), the following results are obtained

For small values of S

$$\frac{\sqrt{S}\,I_1(\sqrt{S})}{\sqrt{S}\,I_1(\sqrt{S}) + H\,I_0(\sqrt{S})} = \frac{S}{2H}\left\{1 - \frac{S}{2H}\left(1+\frac{H}{4}\right)\right\} \Rightarrow 0 \tag{B-5}$$

$$\frac{I_0(\sqrt{S})}{\sqrt{S}\,I_1(\sqrt{S}) + H\,I_0(\sqrt{S})} = \left\{\frac{1}{H} - \frac{S}{2H^2}\right\} \Rightarrow \frac{1}{H} \tag{B-6}$$

For large values of S

$$\frac{\sqrt{S}\,I_1(\sqrt{S})}{\sqrt{S}\,I_1(\sqrt{S}) + H\,I_0(\sqrt{S})} = \left\{1 - \frac{1}{H\sqrt{S}}\right\} \Rightarrow 1 \tag{B-7}$$

$$\frac{I_0(\sqrt{S})}{\sqrt{S}\,I_1(\sqrt{S}) + H\,I_0(\sqrt{S})} = \frac{1}{\sqrt{S}}\left\{1 + \frac{1}{\sqrt{S}}\left(\frac{1}{2} - \frac{1}{H}\right)\right\} \Rightarrow 0 \tag{B-8}$$

In order to be able to truncate the series,

$$e^{Z} = 1 + \frac{Z}{1!} + \frac{Z^2}{2!} + \cdots , \qquad \text{(B-9)}$$

Z must be a small quantity. We see that Equations (B-5) and (B-8) satisfy this condition for small and large values of S respectively. It is concluded from this that for small values of time, use Equation (24) and for large values of time, use Equation (20).

APPENDIX C

LOVE FUNCTIONS

A number of displacement functions, called Love functions, have been used in the main text to derive the stress solution. In this section, the relationship between this function and the corresponding stress distribution is presented. Also included in this appendix is a tabulation of the various required Love functions and the corresponding stresses.

Besides the exact solution, an approximate form of each stress term is given which is valid for small values of the argument ($\frac{ARG^2}{4} << 1.0$) of the Bessel functions. Assuming the arguments are small, one can write

$$J_0(x) = 1 - \frac{x^2}{4} + \ldots \cong 1.0$$

$$J_1(x) = \frac{x}{2}\left[1 - \frac{x^2}{8} + \cdots\right] \cong \frac{x}{2}$$

These are the substitutions which were made to obtain the approximate solutions.

DEFINITION OF LOVE FUNCTIONS

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial \bar{z}^2}\right]\left[\frac{\partial^2 L}{\partial r^2} + \frac{1}{r}\frac{\partial L}{\partial r} + \frac{\partial^2 L}{\partial \bar{z}^2}\right] = \nabla^2\nabla^2 L = 0$$

where $\bar{z} = \frac{z^*}{R}$ and $r = \frac{r^*}{R}$

$$\sigma_{rr} = \frac{\partial}{\partial \bar{z}}\left[\nu\,\nabla^2 L - \frac{\partial^2 L}{\partial r^2}\right]$$

$$\sigma_{\theta\theta} = \frac{\partial}{\partial \bar{z}}\left[\nu \nabla^2 L - \frac{1}{r}\frac{\partial L}{\partial r}\right]$$

$$\sigma_{zz} = \frac{\partial}{\partial \bar{z}}\left[(2-\nu)\nabla^2 L - \frac{\partial^2 L}{\partial \bar{z}^2}\right]$$

$$\sigma_{rz} = \frac{\partial}{\partial r}\left[(1-\nu)\nabla^2 L - \frac{\partial^2 L}{\partial \bar{z}^2}\right]$$

TABULATION OF REQUIRED LOVE FUNCTIONS AND STRESSES

The following notation will be used in this section to simplify the writing of the exact stress terms.

$$J_0 = J_0(\lambda r)$$

$$J_1 = J_1(\lambda r)$$

$$\sigma_{rr} = A\left[a_1 + a_1'\bar{z} + a_1''\bar{z}^2\right]e^{\pm\lambda\bar{z}}$$

$$\sigma_{\theta\theta} = A\left[a_2 + a_2'\bar{z} + a_2''\bar{z}^2\right]e^{\pm\lambda\bar{z}}$$

$$\sigma_{zz} = A\left[a_3 + a_3'\bar{z} + a_3''\bar{z}^2\right]e^{\pm\lambda\bar{z}}$$

$$\sigma_{rz} = A\left[a_4 + a_4'\bar{z} + a_4''\bar{z}^2\right]e^{\pm\lambda\bar{z}}$$

where the capitol letter (A) is a constant, the lower case letters (a's) are functions of r as defined below and $n = \pm 1$ depending on the sign used in the exponential.

$$L_{A+B} = \left[A r J_1 + B J_0\right]\frac{e^{\pm\lambda\bar{z}}}{\lambda^2}$$

$$a_1 = (2\nu - 1) J_0 - \lambda r J_1 \quad \cong -(1-2\nu)$$

$$a_2 = -(1-2\nu) J_0 \quad \cong -(1-2\nu)$$

$$a_3 = 2(2-\nu) J_0 - \lambda r J_1 \quad \cong 2(2-\nu)$$

$$a_4 = \left[2(1-\nu) J_1 + \lambda r J_0\right] n \quad \cong (2-\nu)\lambda r n$$

$$b_1 = \lambda J_0 - \frac{J_1}{r} \quad \cong \quad \frac{\lambda}{2}$$

$$b_2 = \frac{J_1}{r} \quad \cong \quad \frac{\lambda}{2}$$

$$b_3 = -\lambda J_0 \quad \cong \quad -\lambda$$

$$b_4 = -\lambda J_1 n \quad \cong \quad -\frac{\lambda^2 r}{2}$$

all other a's and b's are zero.

$$L_{CD} = \left[\frac{C\bar{z}}{\lambda^2} J_0 + \frac{D}{\lambda} \left\{ \bar{z} r J_1 n + r^2 J_0 \right\} \right] e^{\pm \lambda \bar{z}}$$

$$C_1 = (1+2\nu) J_0 - \frac{J_1}{\lambda r} \quad \cong \quad \frac{1+4\nu}{2}$$

$$C_2 = 2\nu J_0 + \frac{J_1}{\lambda r} \quad \cong \quad \frac{1+4\nu}{2}$$

$$c_3 = (1-2\nu) J_0 \quad \cong \quad 1-2\nu$$

$$c_4 = 2\nu J_1 n \quad \cong \quad \nu \lambda r n$$

$$c_1' = \left[\lambda J_0 - \frac{J_1}{r} \right] n \quad \cong \quad \frac{\lambda}{2} n$$

$$c_2' = \frac{J_1}{r} n \quad \cong \quad \frac{\lambda}{2} n$$

$$c_3' = -\lambda J_0 n \quad \cong \quad -\lambda n$$

$$c_4' = \lambda J_1 \quad \cong \quad \frac{\lambda^2 r}{2}$$

$$d_1 = \left[\lambda^2 r^2 - 3(1-2\nu) \right] J_0 + 2(2-\nu) \lambda r J_1 \quad \cong \quad -3(1-2\nu)$$

$$d_2 = \lambda r J_1 (1-2\nu) - 3(1-2\nu) J_0 \quad \cong \quad -3(1-2\nu)$$

$$d_3 = (6(2-\nu) - \lambda^2 r^2) J_0 - \lambda r J_1 (7-2\nu) \qquad \cong 6(2-\nu)$$

$$d_4 = \{[\lambda^2 r^2 - 4(1-\nu)] J_1 - 2\lambda r J_0 (3-\nu)\} n \qquad \cong -4(2-\nu)\lambda r n$$

$$d_1' = \lambda n [\lambda r J_1 - (1-2\nu) J_0] \qquad \cong -\lambda n (1-2\nu)$$

$$d_2' = -\lambda n (1-2\nu) J_0 \qquad \cong -\lambda n (1-2\nu)$$

$$d_3' = \lambda n [2(2-\nu) J_0 - \lambda r J_1] \qquad \cong 2(2-\nu)\lambda n$$

$$d_4' = -\lambda [2(1-\nu) J_1 + \lambda r J_0] \qquad \cong -\lambda^2 r (2-\nu)$$

all other c's and d's are zero.

$$L_{EF} = \left\{E[r^2+z^2] J_0 + F\left[z^2 r J_1 - r^3 J_1 + \frac{r^2 J_0}{\lambda} + 2r^2 z J_0\right]\right\} e^{\pm\lambda z}$$

$$e_1 = [r^2\lambda^2 - 2(1-5\nu)]\lambda J_0 + \lambda^2 r J_1 (3-4\nu) \qquad \cong -2(1-5\nu)\lambda$$

$$e_2 = -2(1-5\nu)\lambda J_0 + (1-4\nu) r \lambda^2 J_1 \qquad \cong -2(1-5\nu)\lambda$$

$$e_3 = [2(7-5\nu) - \lambda^2 r^2]\lambda J_0 - 4(2-\nu)\lambda^2 r J_1 \qquad \cong 2(7-5\nu)\lambda$$

$$e_4 = \{[\lambda^2 r^2 - 2(2-3\nu)]\lambda J_1 - 2(3-2\nu)\lambda^2 r J_0\} n \cong -\lambda^2 n r (8-7\nu)$$

$$e_1' = \lambda^2\left[2(1+2\nu) J_0 - \frac{2 J_1}{\lambda r}\right] n \qquad = \lambda^2 n (1+4\nu)$$

$$e_2' = \lambda^2\left[\frac{2 J_1}{\lambda r} + 4\nu J_0\right] n \qquad = \lambda^2 n (1+4\nu)$$

$$e_3' = \lambda^2 [2(1-2\nu) J_0] n \qquad = \lambda^2 2(1-2\nu)$$

$$e_4' = 4\nu \lambda^2 J_1 \qquad = 2\nu\lambda^3 r$$

$$e_1'' = \lambda^3\left[J_0 - \frac{J_1}{\lambda r}\right] \cong \frac{\lambda^3}{2}$$

$$e_2'' = \lambda^3 \frac{J_1}{\lambda r} \cong \frac{\lambda^3}{2}$$

$$e_3'' = -\lambda^3 J_0 \cong -\lambda^3$$

$$e_4'' = \lambda^3 n J_1 \cong \frac{\lambda^4 r n}{2}$$

$$f_1 = -6(1-2\nu)J_0 + 2r^2\lambda^2(4-\nu)J_0 + (11-10\nu)\lambda r J_1 - \lambda^3 r^3 J_1 \cong -6(1-2\nu)$$

$$f_2 = -6(1-2\nu)J_0 + (1-2\nu)r^2\lambda^2 J_0 + 5(1-2\nu)r\lambda J_1 \cong -6(1-2\nu)$$

$$f_3 = 12(2-\nu)J_0 - (11-2\nu)r^2\lambda^2 J_0 - 2(13-5\nu)\lambda r J_1 + \lambda^3 r^3 J_1$$
$$\cong 12(2-\nu)$$

$$f_4 = \left[\left\{-2(11-5\nu) + \lambda^2 r^2\right\}\lambda r J_0 + \left\{-4(1-\nu) + (9-2\nu)\lambda^2 r^2\right\}J_1\right]n$$
$$\cong -12(2-\nu)\lambda n r$$

$$f_1' = \left[\left\{-6\lambda(1-2\nu) + 2r^2\lambda^3\right\}J_0 + (2-\nu)4\lambda^2 r J_1\right]n \cong -6(1-2\nu)\lambda n$$

$$f_2' = \left[-6\lambda(1-2\nu)J_0 + (1-2\nu)2\lambda^2 r J_1\right]n \cong -6(1-2\nu)\lambda n$$

$$f_3' = \left[2\lambda J_0(6(2-\nu) - \lambda^2 r^2) - (7-2\nu)2\lambda^2 r J_1\right]n \cong 12(2-\nu)\lambda n$$

$$f_4' = \left\{-4(1-\nu) + r^2\lambda^2\right\}2\lambda J_1 - 4(3-\nu)\lambda^2 r J_0 \cong -8(2-\nu)\lambda^2 r$$

$$f_1'' = -(1-2\nu)\lambda^2 J_0 + \lambda^3 r J_1 \cong -(1-2\nu)\lambda^2$$

$$f_2'' = -(1-2\nu)\lambda^2 J_0 \qquad \cong -(1-2\nu)\lambda^2$$

$$f_3'' = 2(2-\nu)\lambda^2 J_0 - \lambda^3 r J_1 \qquad \cong 2(2-\nu)\lambda^2$$

$$f_4'' = \left\{-(1-\nu)2\lambda^2 J_1 - \lambda^3 r J_0\right\} n \qquad \cong -(2-\nu)\lambda^3 r n$$

APPENDIX D

TANGENTIAL AND AXIAL SURFACE STRESS FOR NON-LIMITING CASES

In general, the primary interests of a stress solution are the location and magnitude of the maximum stress. In this case it is clear that the maximum stress will occur at the surface since this is the location of the largest thermal gradient. By letting $r = 1$ in the general stress solution and making use of the identity. $\lambda_K J_1(\lambda_K) = H J_0(\lambda_K)$, it is possible to consolidate terms and hence simplify the solution. If one also assumes that $\bar{z} >> \frac{4n}{\lambda_K^2}$, then the solution can be simplified even further. The resulting solution after making the above simplifications is shown below

$$\sigma_{\theta\theta}(1,z,t) = \sigma_{zz}(1,z,t) \cong \left[\sigma_1 + MHz\,\sigma_2\right] e^{-\lambda_K^2 (t-z)}$$

where for small values of time

$$\sigma_1 = \sum_{K=1}^{3} \frac{2H\left(1 - \frac{2H}{\lambda_K^2}\right) e^{-MHz}}{(H^2 + \lambda_K^2)}$$

$$\sigma_2 = \sum_{K=1}^{3} -4H^2 \left[\frac{\lambda_K^2\left(1 - \frac{2H}{\lambda_K^2}\right)}{(H^2+\lambda_K^2)^3} + \frac{\left(1 + H - \left\{\frac{H^2+2H}{\lambda_K^2}\right\}\right)}{(H^2+\lambda_K^2)^2} + \frac{\lambda_K^2 (t-z)\left(1 - \frac{2H}{\lambda_K^2}\right)}{(H^2+\lambda_K^2)^2} \right] e^{-MHz}$$

and where for large values of time

$$\sigma_1 = \sum_{K=1}^{3} \frac{2H\left(1 - \frac{2H}{\lambda_K^2}\right)}{(H^2 + \lambda_K^2)}$$

$$\sigma_2 = \sum_{K=1}^{3} 4H^2\left[\frac{\lambda_K^2\left(1 - \frac{2H}{\lambda_K^2}\right)}{(H^2 + \lambda_K^2)^3} + \frac{\left(2 + \frac{H}{2} - \frac{\lambda_K^2}{2H} - \frac{2H}{\lambda_K^2}\right)}{(H^2 + \lambda_K^2)^2} + \frac{\lambda_K^2 (t - Z)\left(1 - \frac{2H}{\lambda_K^2}\right)}{(H^2 + \lambda_K^2)^2} \right]$$

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