Errata

Page 2, Eq. 2:

$$V_n^o = V^c(t_N + \tau) \ldots$$

should read

$$V_n^o = V^c(t_n + \tau) \ldots$$

page 3, Eq. 3:

$$\frac{\left[-\left(\frac{\tau}{T_o} + \frac{t_N}{T_1}\right)\right]}{T_1} \ldots$$

should read

$$\ldots \frac{\left[-\left(\frac{\tau}{T_o} + \frac{t_n}{T_1}\right)\right]}{T_1} \ldots$$

page 3, Eq. 4:

$$\ldots \left[\frac{-(n+1)\tau}{T_o}\right] \ldots$$

should read

$$\ldots \left[\frac{-(N+1)\tau}{T_o}\right] \ldots$$

THE UNIVERSITY OF MICHIGAN RESEARCH INSTITUTE
ANN ARBOR

ANALYSIS OF A COMMUTATED FILTER FOR RDF

Technical Memorandum No. 84

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Approved by: H. W. Farris

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1. The study reported was not exhaustive.
2. The results presented concern one phase of a continuing study.
3. The study reported was judged to have insufficient scope.

Project 3697

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ABSTRACT

The use of a capacitive-memory, integrating, commutated switch filter in spinning-goniometer radio direction finders is discussed. The effects of various input and output time constants on the bandwidth and phase shift are considered.

ANALYSIS OF A COMMUTATED FILTER FOR RDF

1. STATEMENT OF PROBLEM

The problem treated in this report concerns the utilization of a commutated memory circuit in conjunction with a spinning-goniometer radio direction finder.

2. DESCRIPTION

A schematic diagram of the unit is shown in Fig. 1. The commutator is rotated in synchronism with the goniometer. The RDF re-

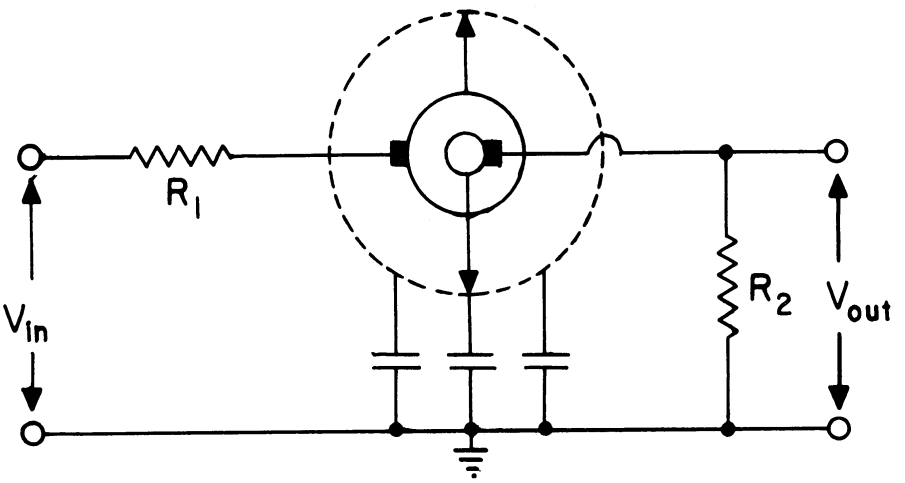

Fig. 1. Schematic diagram of commutated filter.

ceiver output is the signal to be processed. Ideally, this is the full-wave-rectified sine wave shown in Fig. 2, where T equals the time required for one rotation of the goniometer. The receiver output is the potential V_{in} and is connected to the capacitors through one of the rotating brushes. The capaci-

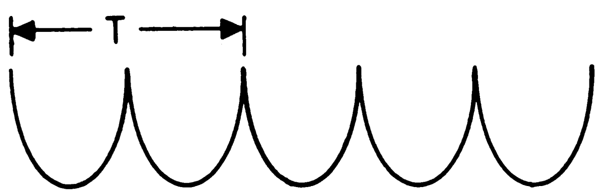

Fig. 2. Filter input.

tor voltage is sampled by means of a second brush on the same armature and forms the output which thus appears across the load resistor R_2.

3. ANALYSIS

The analysis of the circuit is as follows. The situation during the time that a given capacitor is connected to the input is depicted in Fig. 3(a). The switch is closed at time t_n and opened again at time $t_n + \tau$, where τ is the length of time the brush remains in contact with one segment. The potential V^c at the instant the switch is opened is (see Appendix):

$$V^c_{(t_n+\tau)} = V^c_{(t_n)} \exp\left[\frac{-\tau}{T_1}\right] + \exp \frac{\left[-(t_n + \tau)\right]}{T_1} \int_{t_n}^{t_n+\tau} \exp\left[\frac{\tau'}{T_1}\right] V_{in}(t')dt' \quad (1)$$

where $T_1 = R_1C_1$ the charging time constant, and $V^c_{(t_n)}$ is, of course, the voltage across the capacitor just prior to closing the switch.

The situation one half period later, when the given capacitor is being discharged, is shown in Fig. 3(b). At the end of this discharge

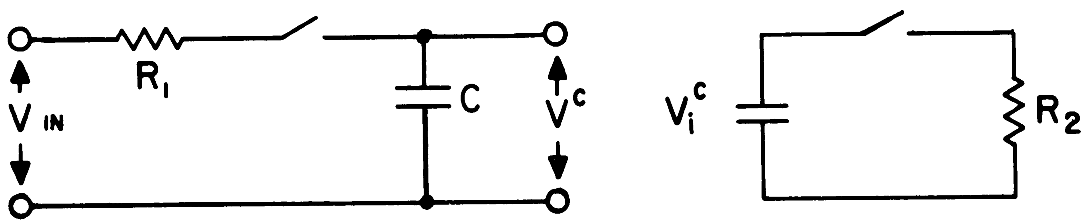

Fig. 3. Charge (a) and discharge (b) circuits.

period the potential (V^o) across the load resistor R_2 is:

$$V^o_n = V^c_{(t_N+\tau)} \exp\left[\frac{-\tau}{T_2}\right] = \exp \frac{\left\{-\left[\frac{\tau}{T_o} + \frac{t_n}{T_1}\right]\right\}}{T_1} \int_{t_n}^{t_n+\tau} \exp\left[\frac{t'}{T_1}\right] V_{in}(t')dt'$$

$$+ V^c_{(t_n)} \exp\left[\frac{-\tau}{T_o}\right] \quad (2)$$

where $1/T_o = 1/T_1 + 1/T_2$. Noting that $V^c_{(t_n)} = V^o_{n-1}$, that is, the initial voltage on the capacitor for the nth charge cycle is the potential remaining after the previous discharge period, the equation becomes:

$$V^o_n = \exp \frac{\left[-\left(\frac{\tau}{T_o} + \frac{t_N}{T_1}\right)\right]}{T_1} \int_{t_n}^{t_n+\tau} \exp\left(\frac{\tau'}{T_1}\right) V_{in}(t')dt' + \exp\left(\frac{-\tau}{T_o}\right) V^o_{n-1} \quad (3)$$

Now suppose that there are in all, ℓ segments or capacitors numbered consecutively and that at the beginning of the epoch ($t = 0$) capacitor number one was connected to the input. Suppose further that the capacitor in question is the mth one. Then for this capacitor $t_o = mT/\ell$, recalling that the rotation period is T. Further, it is seen that, in general, $t_N = t_o + NT$. Making use of these facts and solving the recurrence relation represented by Eq. 3 it is found that:

$$V^o_N = \exp\left[\frac{-(n+1)\tau}{T_o}\right] \sum_{k=0}^{N} \exp \frac{\left\{-\left[\frac{KT}{T_1} + \frac{mT}{\ell T_1} - \frac{K\tau}{T_o}\right]\right\}}{T_1} \int_{\frac{mT}{\ell+nT}}^{\frac{mT}{\ell+nT+\tau}} \exp\left(\frac{t'}{T_1}\right) V_{in}(t')dt' \quad (4)$$

Equation 4 thus represents the output from the mth capacitor after N revolutions (assuming zero initial conditions).

It is perhaps easier to see the implications of the above result by transforming to the frequency domain. For this purpose let $V_{in}(t) = \exp(j\omega t)$. Equation 4 becomes:

$$V^o_N = \exp\left(\frac{mTj\omega}{\ell}\right)\left\{\exp\left[\tau\left(\frac{1}{T_1} + j\omega\right)\right] - 1\right\} \frac{\exp\left[-(N+1)\frac{\tau}{T_o}\right]}{1 + j\omega T_1} \sum_{k=0}^{N} \exp\left[K\left(\frac{\tau}{T_o} + j\omega T\right)\right] \quad (5)$$

The series on the right is a geometric series which can be summed to give:

$$V^o_N = \left\{\frac{\exp\left(\frac{mTj\omega}{\ell}\right)\left[\exp\left(\frac{\tau}{T_1} + \frac{j\omega}{T_1}\right) - 1\right]}{1 + j\omega T_1}\right\}\left\{\frac{\left[\exp\left(\frac{-N\tau}{T_o} - \frac{\tau}{T_o}\right) - \exp(Nj\omega T + j\omega T)\right]}{\left[1 - \exp\left(\frac{\tau}{T_o} + j\omega T\right)\right]}\right\} (6)$$

It is seen to be the product of two factors which will be considered separately in the next section.

4. INTERPRETATION

Referring to Fig. 4 it can be seen that the denominator of the second factor becomes small at the points $\omega T = 2n\pi$, or $f = n/T$. Thus its contribution represents a comb filter with pass bands at all harmonics of the rotation rate. In the steady state ($N \rightarrow \infty$) [$\exp(-N\tau/T_o) = 0$] the 3 db bandwidth can be computed as follows:

$$\left|\exp\left(\frac{-\tau}{T_o}\right) - \exp(j\omega T)\right| = \sqrt{2}\left|\exp\left(\frac{-\tau}{T_o}\right) - 1\right| \qquad (7)$$

From this relation it can be shown that the half-power bandwidth $\Delta\omega$ is the same for all passbands and is given by the smallest solution of:

$$\left|\sinh\left(\frac{\tau}{2T_o}\right)\right| = \left|\sin\left(\frac{\Delta\omega T}{2}\right)\right| \qquad (8)$$

$\Delta\omega T$ is plotted in Fig. 5 as a function of τ/T_o. Also the ratio of the bandpass peaks to the band-stop minima is plotted. It is to be noted here that the bandwidth is not a function of the RC time constant T_o alone, but rather of the ratio T_o/τ of the RC time constant to the length of time the brush spends on each commutator segment. Since, in general, τ may be less than one hundredth of T_o, this is an extremely significant effect. It is also worth noting that the bandwidth is not determined by either the charge or the discharge time constants alone; however, the smaller of

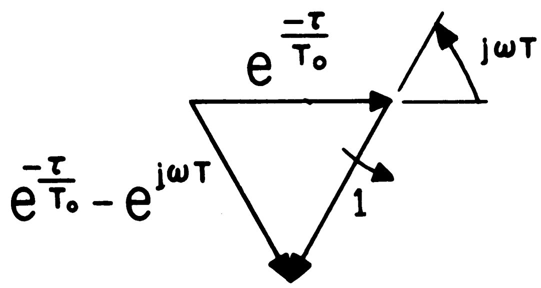

Fig. 4. Vector diagram of $e^{-\tau/T_o} - e^{j\omega T}$

the two will tend to dominate. Thus, when the input resistance R_1 becomes significantly greater than the output resistance R_2 its effect in further reduction of the bandwidth becomes negligible.

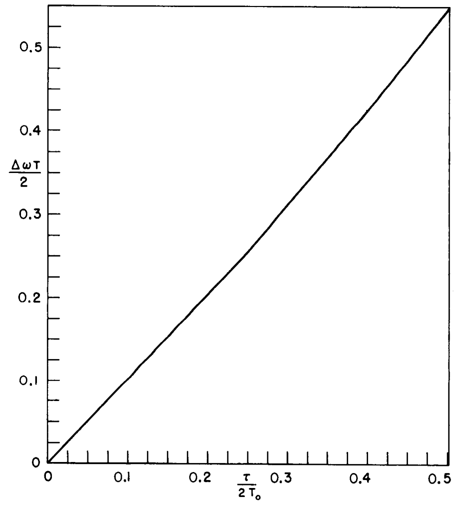

Fig. 5. Normalized bandwidth vs. time constant.

Upon considering the numerator of the second factor, it is seen that steady state is approached when $(N + 1)\ \tau/T_o > 1$. In other words, as one might expect, a period of time on the order of the reciprocal of the bandwidth is required. Prior to this time the situation is similar to that in Fig. 4 except that the rotating vector rotates N + 1 times as fast. This introduces an additional N subsidiary peaks in the response spectrum between every two main peaks. These are evenly spaced between the harmonics of the fundamental rotation rate. The resulting situation is pictured qualitatively in Fig. 6 for N = 3.

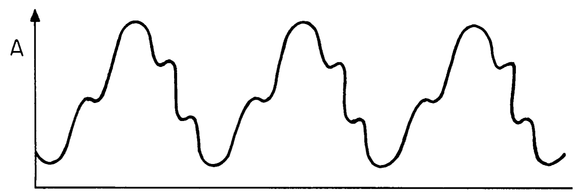

Fig. 6. Qualitative representation of filter response vs. ω.

Considering now the quantity inside the first set of brackets it is seen that the situation is different in that it is solely a function of T_1, the charging time constant. Considering first the denominator $(1 + j\omega T_1)$, if one is to keep the phase error within one degree, ωT_1 must be greater than the tangent of 89 degrees at the lowest frequency of interest, in this case 2/T. Thus:

$$\frac{4\pi T_1}{T} > 58 \quad (9)$$

or:

$$T_1 > 4.6T \quad (10)$$

This gives a minimum value. Increasing T_1 to extremely large values however also has a deleterious effect. Assuming the relation in Eq. 9, then $1 + j\omega T_1$ may, for all practical purposes, be replaced by $j\omega T_1$. Thus the amplitude is severely attenuated for large T_1. If the rotation rate were on the order of 30 revolutions per second, then a 30 second time constant would introduce close to 60 db attenuation at the lowest frequency of interest. If this were applied to a standard indicator unit without additional stages of gain the usual sharp propeller pattern would tend to expand and fill the whole screen. Also, from what has been said previously, if the discharge time constant T_2 were not increased to a similar value there would be no significant decrease in bandwidth. It would appear that since in any case sufficient gain has to be made available to compensate for the loss on the largest time constant, it would be simplest to maintain the input resistance R_1 fixed at its maximum value and control the bandwidth with the load resistor R_2. This would avoid having to change the gain of succeeding amplifier stages to prevent overloading the indicator unit on the shorter time constants.

The significant quantity in the numerator of the first factor of Eq. 6 is $\exp(\tau/T_1 + j\omega\tau) - 1$. Since in practice $\tau/T_1 \ll 1$ this is approximately equal to $-2j|\sin(\omega\tau/2)| \exp(j\omega\tau/2)$, which has a zero at the origin and at $\omega\tau = 2\pi$ and has constant delay. Since the zero at the origin is of no consequence and the upper zero is on the order of 50 or 60 times the fundamental frequency they do not seriously affect the result in practice with the exception of the introduction of a certain amount of blur. Since the entire first factor of Eq. 6 is of the form $|\sin(\omega\tau/2|/\omega T_1$ times a constant delay term the fall-off at the higher frequencies would tend to round off the "propeller tips." Since, in a crude sense, one would expect that for a tip width of the order of one degree significant harmonics up to the 360th should be present, it may be that a certain amount of nonlinear shaping might be desirable to "sharpen up" the pattern a bit.

Since for $a \ll 1$, $\sin a \cong a$, the rise in value of $|\sin(\omega\tau/2)|$ at the lower frequencies tends to compensate for the $1 + j\omega T_1$ term in the denominator and maintain a relatively flat response in this region.

5. S/N RATIO AND VARIANCE REDUCTION

The uses of the commutation device in a direction finder can be split into two main categories. The first of these is as a standard bandwidth reduction filter for signal-to-noise ratio improvement. The second is as a means to average the bearing variations in the incident signal. These will be considered separately.

The first use would be of value in those oft-occurring cases where a signal is too weak to have any useful bearing information derived from it. For this purpose the system as it stands is not so useful as it might be. The useable information in the input to the unit is contained in only the even harmonics of the fundamental rotation rate. However, the filter has passbands at all harmonics. It would thus appear that roughly a two-fold increase in signal-to-noise ratio could be realized for a given time constant by removing the undesired odd-harmonic passbands. This could most easily be accomplished by connecting opposite commutator segments in parallel. This would be equivalent to rotating the armature twice as fast.

The second use, that of variance reduction, is more difficult to evaluate. To a first order the device may be said to "average" the bearings over a time period comparable in length to the reciprocal of the bandwidth. Whether or not the resultant number is in any sense a best estimate of the great circle bearing is, of course, open to question. The statistical properties of DF bearings are at present so ill defined that one does not really know what to do. In the face of such a situation he might as well use a mean as anything else; it has the property of giving a smaller standard deviation than the original data and thus gives the user a sense of confidence (misplaced or not). In any case, one can get an estimate of the variance reduction by using the results of a study by W. C. Bain.[1] In that paper the variance reduction is plotted as a function of averaging time. The reader is referred to the original paper for the actual reduction to be expected for a given time constant.

6. CONCLUSIONS

The commutated filter is essentially a comb filter with pass-bands at all harmonics of the rotation rate of the goniometer. The band-width as a function of the variable parameters is shown in Fig. 5. The signal-to-noise ratio improvement could be significantly increased by connecting opposite commutator segments to remove the odd-harmonic pass-bands. Since the integration time is a function of both the input and load resistances, while the attenuation through the device is a function only of the input resistance, it is recommended that the input time

[1] W. C. Bain, "On the Rapidity of Fluctuations in Continuous Wave Radio Bearings at High Frequencies," PIEE, (London) CII, 1955, pp. 541-543.

constant be held fixed at its maximum value and the integration time varied by charging the load resistance alone.

APPENDIX

Writing the loop equations

$$R\frac{dQ}{dt} + \frac{Q}{c} = E$$

$$\frac{dQ}{dt} + \frac{Q}{RC} = \frac{E}{R}$$

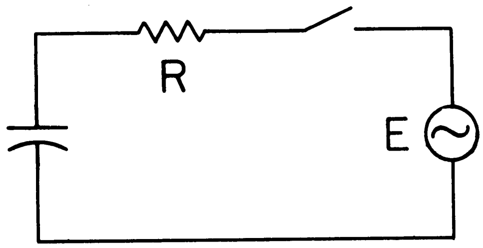

Fig. 7. Equivalent circuit during charge cycle.

Now

$$\int_{t_1}^{t_2} e^{\frac{t}{RC}} \frac{dQ}{dt}\,dt = Qe^{\frac{t}{RC}}\Big|_{t_1}^{t_2} - \frac{1}{RC}\int_{t_1}^{t_2} e^{\frac{t}{RC}} Q\,dt \ .$$

Thus

$$\int_{t_1}^{t_2} e^{\frac{t}{RC}} \left[\frac{dQ}{dt} + \frac{Q}{RC}\right] dt = Q_{(t_2)}\, e^{\frac{t_2}{RC}} - Q_{(t_1)}\, e^{\frac{t_1}{RC}} = \frac{1}{R}\int_{t_1}^{t_2} e^{\frac{t}{RC}} E\,dt$$

or

$$Q_{(t_2)} = Q_{(t_1)}\, e^{\frac{1}{RC}(t_1 - t_2)} + \frac{1}{R}\int_{t_1}^{t_2} e^{\frac{t}{RC}} E\,dt$$

Letting $V = \frac{Q}{c}$

$$V_{(t_2)} = V_{(t_1)}\, e^{\frac{1}{RC}(t_1 - t_2)} + \frac{1}{RC}\int_{t_1}^{t_2} e^{\frac{t}{RC}} E\,dt.$$