CAPACITY EXPANSION UNDER STOCHASTIC DEMANDS

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Abstract

We consider the problem of optimally meeting a stochastically growing demand for capacity over an infinite horizon. Under the assumption that demand for product follows either a nonlinear Brownian motion or a non-Markovian birth and death process, we show that this stochastic problem can be transformed into an equivalent deterministic problem. Consistent with earlier work by A. Manne, the equivalent problem is formed by replacing the stochastic demand by its deterministic trend and discounting all costs by a new interest rate that is smaller than the original, in approximate proportion to the uncertainty in the demand.

1.0 Introduction

We consider the problem of optimally planning for the periodic expansion of productive capacity to meet a stochastically growing demand for product over an infinite horizon. In practice, planners often ignore the stochastic nature of the demand process, solving deterministic versions of stochastic problems. Thus, within the modelling environment, a stochastic demand process is often replaced by a forecasted demand that grows deterministically over time. Other authors have shown that, under various conditions, it is possible to solve a deterministic equivalent in place of the original stochastic problem. For example, Manne [1967] gives results under which one must, in addition, replace the interest rate by a smaller, equivalent rate. Friedenfelds [1981] discusses how one can leave the interest rate unchanged and replace the forecasted means by larger quantities. Papers such as Giglio [1970], Skoog [1976], Luss [1982], Davis et al [1987], Buzacott and Chaouch [1988], and Erlenkotter et al. [1989] have established that simply replacing demands by their forecasted means is not satisfactory. Leiberman [1989] provides evidence that capacity decisions made by practitioners in the chemical industry deviate from those suggested by Manne's model, suggesting that the model may be inappropriate as a descriptive model for the chemical industry. In this paper, a more robust modeling framework is provided with a potentially greater area of application. In particular, we relax several of the assumptions imposed by Manne [1961], thereby allowing nonstationary demand processes that are either continuous or discrete in nature, as well as general cost structures. Consistent with Manne's work, the key to our transformation involves the replacement of the prevailing interest rate by an interest rate that is a decreasing function of the variance of the demand process. This paper generalizes the demand processes under which the Manne approach applies.

In modelling the capacity expansion problem with stochastic demand, we distinguish between demand for product and demand for capacity. That is, we begin with a right continuous stochastic process, $\{P(t), t \geq 0\}$, where $P(t)$ is the demand for product or service at time t. We assume that the available capacity must meet or exceed the demand for product or service. Thus from $\{P(t), t \geq 0\}$ we derive the stochastic process representing the demand for capacity, $\{D(t), t \geq 0\}$, where $D(t) = \sup\{P(s) | s \leq t\}$. We assume that demand for capacity may be satisfied by any of a set of durable facilities indexed by a set $I \subset \Re$, where a 'facility' is the physical entity providing capacity. For example, $D(t)$ may be the demand for manufacturing capacity through time t in units per period, representing the maximum production rate encountered through time t. The facility types would then include the technology set of machines available for the product's manufacture (Luss [1982]). Less conventionally, $D(t)$ may represent cumulative demand for circuits along a telephone transmission link. The facilities would then comprise various types and sizes of coaxial cables and microwave radio facilities (Yaged [1973]). Finally, we associate with facility $i \in I$ its available capacity $X_i > 0$ and its cumulative cost $C_i(x)$ of satisfying a demand for x units of capacity where $0 < x < X_i$ (independent of its installation time). For example, in a telephone industry application, X_i might represent the capacity in circuits of a coaxial cable of type i and $C_i(x)$ would be a fixed plus linear cost function representing the cost of the cable itself, the cost of the laying of the cable together with the rightaway, plus the cost of the repeater electronics required to enable x of the X_i circuits available (Yaged [1973]). All costs are continuously discounted using an interest rate $r > 0$.

We seek to expand capacity to meet demand over an infinite horizon at minimum expected discounted cost. The decisions include the timing, sizing, and type of capacity to be installed over time. We assume, for ease of exposition, that capacity is added only when existing spare capacity is exhausted. It should be noted, however, that in general it may be optimal to abandon a facility's unused capacity if the marginal cost of additional capacity utilization is sufficiently high (as, for example, in the presence of physical deterioration or technological obsolesence, Balcer and Lippman [1984]).

2.0 Deterministic Equivalent Problem

In order to transform the stochastic capacity expansion problem described in the previous section into an equivalent deterministic problem, we begin by making the following assumptions regarding the demand process $\{P(t), t \geq 0\}$.

Assumptions:

A) The demand for product or service, $\{P(t), t \geq 0\}$, is a semi-Markov process (and hence the demand for capacity $\{D(t), t \geq 0\}$ is a semi-Markov process) with state space S comprising either a closed countable subset of the set of all reals (a discrete demand process) or the set of all reals (a continuous demand process);

B) Sample paths of $\{P(t), t \geq 0\}$, and hence $\{D(t), t \geq 0\}$ must be connected in that a transition from state P^1 to state P^2 must pass through all intermediate feasible states $P \in S$ with $P^1 < P < P^2$.

These assumptions are quite general and allow for substantial flexibility in the modeling of the demand process. For example, in the case of discrete demand, arbitrarily distributed transition times between order arrivals are allowed. This differs substantially from the commonly assumed discrete-state Markov demand processes in which transition times are necessarily exponentially distributed (as in Freidenfelds [1980] and Davis et al [1987]).

To derive the deterministic model from the stochastic model, consider a situation in which a sequence of facilities $i_1, i_2, \ldots i_k$ whose capacity is first exhausted at time t_k have been installed. By definition, at each transition epoch, the future states of a semi-Markov process are conditionally independent of the past states when given the present state. Thus by Assumption A, the cumulative demand upon exhaust, $D(t_k)$, (or equivalently, $P(t_k)$) provides all of the information regarding the past evolution of the demand process $\{D(s), s \le t_k\}$ that is relevant to the optimal determination of the $(k+1)^{th}$ facility to be installed. It follows from Assumption B that if the installed capacity is first exhausted at time t_k, then

$$P(t_k) = D(t_k) = \text{Min}\{s \in S | s \ge \sum_{n=1}^{k} x_{i_n}\}.$$

Thus, the deterministic variable, installed capacity, captures all of the information regarding the historical evolution of the problem that is relevant to the optimal determination of the next facility. We may therefore state the problem under consideration as a deterministic dynamic program with the state variable representing installed capacity at expansion epochs.

We now establish the mathematical notation necessary to formally define the optimality equations for our problem. Let

$$T(x) = \inf\{t | D(t) \ge x\} = \inf\{t | P(t) \ge x\}$$

be the time at which a total capacity x is exhausted, and let $f(x)$ denote the minimum expected cost, discounted to time $T(x)$, of expanding capacity to meet future demand, given that the accumulated capacity x is currently exhausted. The optimal value function, $f(x)$, satisfies the following condition for all $x \ge 0$.

$$\begin{aligned} f(x) &= \underset{i \in I}{\text{Min}} \left\{ E\left[\int_0^{X_i} e^{-r\{T(x+s)-T(x)\}} dC_i(s)\right] + E[e^{-r\{T(x+X_i)-T(x)\}}] f(x+X_i) \right\} \\ &= \underset{i \in I}{\text{Min}} \left\{ \int_0^{X_i} E\left[e^{-r\{T(x+s)-T(x)\}}\right] dC_i(s) + E\left[e^{-r\{T(x+X_i)-T(x)\}}\right] f(x+X_i) \right\} \end{aligned}$$

The task at hand, which we refer to as problem (P), involves the determination of $f(0)$. Because it is formulated in terms of the deterministic variable x, representing a cumulative capacity level, we will refer to (P) as the "deterministic equivalent problem" (for a discussion of deterministic equivalent problems in a more general setting, the reader is referred to Higle, Bean, and Smith [1990]). Note that in specifying the deterministic formulation, $e^{-rT(x)}$ is replaced by its expectation $E[e^{-rT(x)}]$. Thus, to construct a determinstic problem that is equivalent to the stochastic problem, one need only provide a combination of a demand function and an interest rate that results in discount factors identical to $E[e^{-rT(x)}]$. As an example, Freidenfelds [1980] suggests (for the special case of a Markovian birth-death process) (P) be solved using a demand function which has an inverse given by $\frac{-1}{r} ln\{E[e^{-rT(x)}]\}$. Although potentially useful, this particular approach to the specification of the deterministic problem is somewhat unsatisfactory. For example, with the indicated demand function (which in general must be constructed in a stepwise manner), one cannot easily isolate the qualitative effects of changes in the interest rate from changes in the variability of the demand process.

The equivalent interest rate suggested in Higle, Bean, and Smith [1990] simplifies the specification of the deterministic equivalent problem and lends insight into the effects of stochasticity on the optimal expansion policy. A nonnegative number r^* such that $E[e^{-rT(x)}] = e^{-r^* E[T(x)]}$ for all $x \geq 0$, is said to be an "equivalent interest rate" for (P). When an equivalent interest rate exits, it is unique and is given by

$$r^* = \frac{|\ln \Phi_{T(x)}(r)|}{E[T(x)]} \tag{1}$$

where $\Phi_{T(x)}(r)$ is the Laplace transform of $T(x)$. Furthermore, (P) may be rewritten in the following simplified form.

(P^*) Find $f(0)$, where

$$f(x) = \text{Min}\left\{ \int_0^{X_i} e^{-r^*\{E[T(x+s)-T(x)]\}} dC_i(s) + e^{-r^*\{E[T(x+X_i)-T(x)]\}} f(x+X_i) \right\}.$$

Note that (P^*) is merely a restatement of (P). The '*' is used here as an explicit reminder that the equivalent interest rate is being used.

We conclude that under Assumptions A and B, when an equivalent interest rate r^* exists, the original stochastic capacity expansion problem may be solved via a deterministic problem formulation in which

(a) the random expansion epochs $T(x)$ are replaced by their expected values; and
(b) the original interest rate is replaced by its equivalent r^*.

Note that since $E[e^{-rT(x)}] \geq e^{-rE[T(x)]}$ by Jensen's inequality, we have that $r^* \leq r$. Therefore, the qualitative effect of demand uncertainty is summarized by a drop in the effective rate of interest. For example, under economies of scale, as in the models of Manne [1961] and Srinivasan [1967], one would optimally install larger facilities than one would in the absence of demand uncertainty.

We now state and prove sufficient conditions on the product demand process, $\{P(t), t \geq 0\}$, that ensure the existence of r^*. We begin with the following definitions:

Definitions:

a) A continuous state stochastic process $\{P(t), t \geq 0\}$ is said to be "transformed Brownian motion" with underlying rate p and variance σ^2 if there exists a non-negative increasing deterministic transformation h such that $P(t) = h(p(t))$ where $\{p(t), t \geq 0\}$ is (linear) Brownian motion with drift $p > 0$ and variance $\sigma^2 > 0$. The function h is referred to as the transforming function;

b) A discrete state stochastic process $\{P(t), t \geq 0\}$ is said to be a "regenerative birth and death process" if there exists a non-negative increasing deterministic transformation h such that $P(t) = h(p(t))$, where $\{p(t), t \geq 0\}$ is a birth and death process with general distribution F for the first passage time in going from any state i to $i+1$. The Laplace transform $\Phi(r) = \int_0^\infty e^{-rt} dF(t)$ is referred to as the generator of the process, and the function h is referred to as the transforming function.

Lemma 1. *(a) In a continuous model if the demand for product $\{P(t), t \geq 0\}$ is transformed Brownian motion with underlying rate $p > 0$ and variance $\sigma^2 > 0$, the equivalent interest rate r^* corresponding to r exists and is given by*

$$r^* = (\frac{p}{\sigma})^2 \left(\sqrt{1 + 2r(\frac{\sigma}{p})^2} - 1 \right).$$

(b) In a discrete model if the demand for product $\{P(t), t \geq 0\}$ is a regenerative birth and death process with generator $\Phi(r)$, the equivalent interest rate corresponding to r exists and is given by

$$r^* = \ln \left(\Phi(r)^{\frac{1}{\Phi'(0)}} \right)$$

where $\Phi'(0) = \frac{d\Phi(r)}{dr}|_{r=0}$.

Proof. In both cases, the existance of the equivalent interest rate follows from Theorem 5 of Higle, Bean, and Smith [1990]. For case (a) we have $P(t) = h(p(t))$ where $\{p(t), t \geq 0\}$ is linear Brownian motion with drift $p > 0$ and variance $\sigma^2 > 0$. Then

$$T(x) = \text{Min}\{t | P(t) \geq x\} = \text{Min}\{t | p(t) \geq h^{-1}(x)\}.$$

Since $\Phi(r)_{T(x)} = e^{\left\{(-\frac{p}{\sigma})\left\{\sqrt{1+2r(\frac{\sigma}{p})^2}-1\right\}\right\}h^{-1}(x)}$ and $E[T(x)] = h^{-1}(x)/p$ (Ross, p. 203), the indicated relationship between r^* and r follows from (1). Case (b) is shown in a similar manner. ■

To summarize the hypothesis of Lemma 1, the key property that any demand process must satisfy to have an associated equivalent interest rate is that it be an invertible transformation of a semi-Markov process that probabilistically renews at all state transition epochs. Assumption B, which requires that the demand process have connected sample paths, ensures that cumulative capacity and demand are uniquely determined at each expansion epoch. This assumption allows the use of the deterministic equivalent problem (P). For a discrete model, the requirement of connected sample paths restricts us to monotonically increasing transformations of birth and death transitions, while the renewal property forces the first passage times between adjacent ascending states to be independent and identically distributed random variables. Hence, we are reduced to a regenerative birth and death process. For a continuous model, we are reduced to a non-negative increasing transformation of a process with continuous sample paths and stationary and independent increments, which is transformed Brownian motion. Manne [1961] first introduced r^* for demand following ordinary (linear) Brownian motion. It is interesting to note that the same formulas for r^* apply to transformed Brownian motion; that is, r^* is independent of the transformation h.

To understand the implications of combining the equivalent interest rate, r^*, with the deterministic equivalent problem formulation, we must specify the nature of the expected times $E[T(x)]$. Toward that end, we offer the following theorem, in which we characterize the deterministic demand function $P^*(t)$, whose time to exhaust capacity x agrees with $E[T(x)]$. In doing so, we completely specify the nature of the deterministic equivalent problem, (P^*), thereby offering a simple procedure through which it may be easily obtained.

Theorem 2. *Consider the problem of selecting a sequence of capacity expansions from the set I to satisfy a stochastic demand for production $\{P(t), t \geq 0\}$ over an infinite horizon at minimum expected discounted cost using interest rate r, known as the 'stochastic problem'.*

a) If $\{P(t), t \geq 0\}$ is transformed Brownian motion with underlying rate $p > 0$ and variance $\sigma^2 > 0$, and transforming function h, then let $P^*(t) = h(pt)$ and

$$r^* = (\frac{p}{\sigma})^2 \left(\sqrt{1 + 2r(\frac{\sigma}{p})^2} - 1\right).$$

b) If $\{P(t), t \geq 0\}$ is a regenerative birth and death process with generator $\Phi(r)$ and transforming function h, then let $P^*(t) = h(\lceil pt \rceil)$, where

$$p = \frac{-1}{\Phi'(0)} \quad \text{and} \quad \lceil x \rceil = \text{Min}\{y | y \geq x, y \ integer\}$$

and let

$$r^* = \ln\left(\Phi(r)\right)^{\frac{1}{\Phi'(0)}}.$$

In either case, every optimal capacity expansion sequence for the deterministic problem with demand $P^*(\cdot)$ in which all costs are continuously discounted using the interest rate r^* is optimal for the stochastic problem.

Proof. In both cases, $\{P(t), t \geq 0\}$ is a semi-Markov process satisfying assumptions A and B of §2. Thus, we have that the cumulative installed capacity determines a dynamic programming state variable and it follows that the stochastic problem may be stated as the deterministic problem (P). Also, in both cases $\{P(t), t \geq 0\}$ satisfies the hypothesis of Lemma 1. Thus, we may transform (P) into the equivalent problem (P^*) with the indicated equivalent interest rate r^* and deterministic time to capacity exhaust given by $E[T(x)]$. It remains to show that $E[T(x)]$ agrees with the time to exhaust capacity x for the deterministic demand $P^*(t)$, or equivalently, that $E[T(x)] = \text{Min}\{t | P^*(t) \geq x\}$.
i) Under condition (a) of the hypothesis, $P^*(t) = h(pt)$. Consequently,

$$\text{Min}\{t | P^*(t) \geq x\} = \text{Min}\{t | h(pt) \geq x\} = \text{Min}\{t | pt \geq . h^{-1}(x)\} = \frac{h^{-1}(x)}{p}$$

Additionally,

$$T(x) = \text{Min}\{t | P(t) \geq x\} = \text{Min}\{t | p(t) \geq h^{-1}(x)\}$$
$$\Rightarrow E[T(x)] = \frac{h^{-1}(x)}{p},$$

since the expected first passage time to a for linear Brownian motion is a/p (Ross, p. 203). It follows that $E[T(x)] = \text{Min}\{t | P^*(t) \geq x\}$. Under condition (b) of the hypothesis, $P^*(t) = h(\lceil pt \rceil)$ and the result is established in a similar manner. ∎

Theorem 2 allows, without loss of optimality, the original stochastic problem to be replaced by a deterministic problem with a lower interest rate and a demand yielding the same expected times to exhaust capacity. Any optimal solution for this deterministic equivalent problem is known to be optimal for the stochastic problem. Note that the equivalent deterministic demand $P^*(t)$ is not equal to the average observed demand, i.e., $P^*(t) \neq E[P(t)]$. Instead, $P^*(t)$ is equal to the average observed value of $p(t)$ transformed by h, i.e. $P^*(t) = h(pt)$ for the continuous model and $P^*(t) = h(\lceil pt \rceil)$ for the deterministic model. Therefore, under the hypothesis of Theorem 2, an appropriate estimate of $P^*(t)$ is provided by a linear least squares fit to the transformed model $p(t) = h^{-1}(P(t))$.

Transformed Brownian motion, in particular, includes an interesting variety of non-stationary models with nonlinear demand trends The nonlinearity is represented by the function h, while the random effects are modelled as being induced through the rate perturbation process $\{p(t), t \geq 0\}$. An example of a transformed Brownian motion model for demand is $P(t) = P_0(e^{p(t)} - 1)$. This demand process satisfies the conditions of Lemma 1 since $h(x) = P_0(e^x - 1)$ is a non-negative increasing function of x for $P_0 > 0$. From Lemma 1, it is clear that this geometric Brownian motion model for demand has an equivalent interest rate $r^* = (p/\sigma)^2(\sqrt{1 - 2r(\sigma/p)^2} - 1)$ associated with it. By Theorem 2, the problem of optimally installing capacity to meet this demand is equivalent to the deterministic problem of meeting the demand $P^*(t) = P_0(e^{pt} - 1)$ with costs discounted at the rate r^*. To illustrate the advantages of solving the stochastic problem through its determinstic equivalent, let us suppose that capacity costs are fixed and only depend on the capacity according to a power cost law with economy of scale factor $0 < \alpha < 1$. That is, suppose the cost $K(x)$ of providing capacity x is given by $K(x) = kx^{1-\alpha}$, where $0 < x < \infty$ and $k > 0$. This problem of meeting a demand following geometric Brownian motion with a continuous technology obeying a power cost law was studied extensively in the telephone industry by Skoog [1976]. The demand for circuits on intercity transmission links has historically experienced growth proportional to itself, and transmission facility costs are known to follow the so-called Dixon-Clapp curve (a power cost law). The corresponding deterministic equivalent problem is a classic deterministic capacity problem analyzed by Srinivasan [1967] who showed that the optimal policy is to install facilities of increasing capacities so as to result in equal time intervals T^* between installation epochs. One obtains

$$T^* = \operatorname*{argmin}_{T>0} \frac{k(P_0(e^{pt} - 1))^{1-\alpha}}{1 - e^{-\nu T}}$$

where $\nu = r^* - (1 - \alpha)p$, and $\nu \leq 0$ is allowed (Smith [1980]). The optimal sequence of capacity additions to the original stochastic problem is given by

$$x_n^* = P_0 e^{p(n-1)T^*}(e^{pT^*} - 1)$$

for $n = 1, 2, 3, \ldots$. Note that the effect of demand uncertainty as measured by σ^2 is completely encapsulated within r^*, and thus within ν. Once again, the optimal hedge against uncertainty is to install larger facilities than expected. In fact, x_1^* diverges to infinity as σ^2 goes to infinity.

Although somewhat more restrictive than the transformed Brownian motion processes, regenerative birth and death processes include constant rate Markovian birth and death models as well as renewal processes as special cases. Of course, an example of such a demand model is the Poisson process. If $\{P(t), t \geq 0\}$ follows a Poisson process with mean rate $p > 0$, then its first passage time $T(x)$ follows an Erlang distribution with parameters $\lceil x \rceil$ and p. Its generator $\Phi(r)$ is therefore the Laplace transform of an exponential distribution with mean $1/p$. We have $\Phi(r) = p/(p+r)$ and $\Phi'(0) = -1/p$, so that $r^* = \ln(1+r/p)^p$. Note that r^* tends to r as $p \to \infty$ while r^* tends to 0 as $p \to 0$. This is consistent with the viewpoint that the magnitude of the departure of r^* from r is a measure of the uncertainty associated with the demand process. That is, the coefficient of variation γ of $P(t)/t$ is given by $\gamma = 1/\sqrt{p}$ which tends to 0 and ∞ as p tends to ∞ and 0, respectively.

As a final comment, we note that Bean and Smith [1985] discuss a general forward procedure for efficiently solving problems such as (P^*). Additionally, they provide a simple heuristic procedure through which a decision sequence with discounted costs arbitrarily close to optimality can be determined. Thus, under assumptions A and B and the conditions specified in Theorem 2, we see that elements of a general class of nonstationary stochastic capacity expansion problems are easily converted to equivalent deterministic problems that can be solved using readily available procedures.

3.0 Effects of Demand Uncertainty

The implications of Theorem 2 to the planner are clear. The protocol to account for demand uncertainty is to begin by collecting historical product demand data. For a continuous model, the next step is to transform the data to fit a linear regression model with time as the independent variable. The inverse of this transformation is the function h. The last step is to estimate the slope p and variance σ^2 associated with the linear regression fit. The planner may now solve for the optimal expansion policy for a problem with deterministic demand given by $P^*(t) = h(pt)$ and the reduced interest rate r^*. This procedure is remarkably close to that followed in practice with the exception of lowering the interest rate to r^*.

For a discrete model, the procedure is complicated by the need to estimate $\Phi(r)$, the Laplace transform of the first passage time to state 1 in the underlying birth-death process. In particular, this requires knowledge of the complete distribution of this first passage time. However, it is possible to obtain a second order approximation to r^* using only first and second moment information about the demand process. We begin with a definition.

Definition: Let the process $\{P(t), t \geq 0\}$ be given by $P(t) = h(p(t))$ where h is a non-negative increasing function and $\{p(t), t \geq 0\}$ is a semi-Markov process that probabilistically renews at all state transition epochs. The coefficient of variation, γ , of $\{P(t), t \geq 0\}$ is given by

$$\gamma = \frac{\sqrt{Var[p(1))]}}{E[p(1)]}. \tag{2}$$

Theorem 3. *If the demand for product $\{P(t), t \geq 0\}$ is given by transformed Brownian motion or a regenerative birth and death process, then the equivalent interest rate, r^*, is given by*

$$r^* = (1 - \frac{1}{2}\gamma^2 r)r + o(r^2)$$

where γ is the coefficient of variation, as defined in (2).

Proof. Let $\Phi(r) = E[e^{-rT(1)}]$ where $T(x) = \text{Min}\{t | p(t) \geq x\}$. Explicitly denoting the dependence of r^* on r, $r^*(r) = \ln \Phi(r)/\Phi'(0)$. From Theorem 8 of Higle, Bean, and Smith [1990], it follows that

$$r^*(r) = r - \left(\frac{Var[T(x)]}{2E[T(x)]}\right) r^2 + o(r^2).$$

Now $E[T(1)] = 1/p$ and $Var[T(1)] = \sigma^2/p^3$ where $p = E[p(1)]$ and $\sigma^2 = Var[p(1)]$ with $P(t) = h(p(t))$ (Ross, p. 62). Hence, since $\gamma = \sigma/p$ it follows that

$$r^* = (1 - \frac{1}{2}\gamma^2 r)r + o(r^2) \quad \blacksquare$$

From Theorem 3, the planner needs only to estimate the underlying mean and variance associated with the demand process to obtain an approximation of the equivalent interest rate.

This approximation formula also lends insight into the relative effects of variance on the interest rate. For example, a nominal interest rate of 10 percent is altered in magnitude by less than 5% for demand processes with coefficients of variation less than one. This lends some justification to the common practice of planners to ignore the effects of demand uncertainty. On the other hand, when demand uncertainty as measured by γ is high, the effects can be significant. For the class of Poisson demand processes, γ is given by $1/\sqrt{p}$. When the expected demand rate p is small, the effect on the equivalent interest can be sizeable. For example, for p equal to 1 unit per month, the equivalent interest rate associated with a nominal rate of 10% is given to second order by $r^* = 8.27\%$. The associated optimal increment of capacity will be correspondingly altered.

4.0 Summary and Conclusions

In this paper, we have presented a capacity expansion model with general cost structures and demand that is either a transformed Brownian motion or a regenerative birth-death process. Working within the framework of this general model, we provide conditions for the existence of a deterministic equivalent problem and an equivalent interest rate. With the deterministic equivalent problem, the stochastic problem can be solved exactly with known deterministic techniques. The equivalent interest rate enables the effects of the variability of the demand process to be completely summarized in a single number r^*.

Although the use of a transformed Brownian process as a modeling tool seems non-traditional, we suggest that in practice it is often used implicitly, if not explicitly. For example, in forecasting demand, one often seeks a linear trend within some transformation of observed data. This procedure, combined with the assumptions of normality inherent to most regression models, yields a demand process that is implicitly assumed to be appropriately modeled by transformed Brownian motion. Recalling that the demand function used with the equivalent interest rate in the deterministic equivalent problem formulation can be obtained from such a regression model (see Theorem 2), it follows that the deterministic equivalent problem suggested in this paper is remarkably similar to the one used by many practitioners. That is, the transformation to the equivalent problem requires only a reduction of the interest rate from r to r^*.

References

Balcer, Y. and S.A. Lippman [1984], 'Technological Expectations and Adoption of Improved Technology', *Journal of Economic Theory*, vol. 34, pp. 292-318.

Bean, J.C. and R.L. Smith [1985], 'Optimal Capacity Expansion Over an Infinite Horizon', *Management Science*, vol. 31, no. 12, pp. 1523-1532.

Buzacott, J.A. and A.B. Chaouch [1988], 'Capacity Expansion with Interrupted Demand Growth', *European Journal of Operational Research*, vol. 34, pp. 19-26.

Davis, M.H.A., M.A.H. Dempster, S.P. Sethi, and D. Vermes [1987], 'Optimal Capacity Expansion Under Uncertainty', *Advances in Applied Probability*, vol. 19, pp. 156-176.

Erlenkotter, D., S.P. Sethi, and N. Okada [1989], 'Planning for Surprise: Water Resources Development under Demand and Supply Uncertainty. I. The General Model', *Management Science*, vol. 35, no. 2, pp. 149-163.

Freidenfelds, J. [1980], 'Capacity Expansion When Demand is a Birth-Death Random Process', *Operations Research*, vol. 28, no. 3, pp. 712-721.

Freidenfelds, J., [1981], *Capacity Expansion: Analysis of Simple Models with Applications*, North Holland Publishing, New York.

Giglio, R.J. [1970], 'Stochastic Capacity Expansion', *Management Science*, vol. 17, no. 3, pp. 174-184.

Higle, J.L., J.C. Bean, and R.L. Smith, [1990], 'Deterministic Equivalence in Stochastic Infinite Horizon Problems,' *Mathematics of Operations Research*, vol. 15, no. 3, pp.396-407.

Lieberman, M.B., [1989], 'Capacity Utilization: Theoretical Models and Empirical Results', *European Journal of Operational Research*, forthcoming.

Luss, J., [1982], 'Operations Research and Capacity Expansion Problems: A Survey', *Operations Research*, vol. 30, no. 5, pp. 907-947.

Manne, A.S., [1961], 'Capacity Expansion and Probabilistic Growth', *Econometrica*, vol. 29, no. 4, pp. 632-649.

Manne, A.S., [1967], editor, *Investments for Capacity Expansion: Size, Location, and Time-Phasing*, MIT Press.

Ross, S.M., [1983], *Stochastic Processes*, John Wiley and Sons, New York.

Skoog, R.A., [1976], 'Optimal Capacity Expansion with Stochastic Demands', Bell Laboratories Technical Memorandum TM 76-3411-2 (unpublished).

Smith, R.L., [1980], 'Optimal Expansion Policies for the Determinstic Capacity Problem', *The Engineering Economist*, vol. 25, no. 3, pp. 149-159.

Srinivasan, T.N, [1967], 'Geometric Rate of Growth of Demand', in *Investments for Capacity Expansion: Size, Location, and Time-Phasing*, A. Manne (ed.), MIT Press, pp. 151-159.

Yaged, B. [1973], 'Minimum Cost Routing for Dynamic Network Models', *Networks*, vol. 3, pp. 193-224.