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ELASTICITY LAW FOR COMPOSITE RUBBER-TEXTILE BODIES

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NOTE: The German practice of using δ to
signify ∂ has been followed in this trans-
lation.

THE ELASTICITY LAW FOR COMPOSITE RUBBER-TEXTILE BODIES

W. Hofferberth

The composite rubber-textile body finds application in place of rubber wherever the tensile strength of the rubber is too small or its deformation is too large. To give rubber structural elements the capacity to withstand large tensile stresses with small deformations, suitable textile reinforcements are inserted. The resultant composite rubber-textile bodies then share the loads in a manner depending on the relative strains of the components.

Examples of composite bodies of this kind are conveyer belts, pneumatic tires and the textile-reinforced "rubber bellows" used in air springs. The elastic properties of these composite bodies arise from the elastic properties of rubber and textile and their interaction as determined by their particular structural geometry. Further development of these interesting physical properties of the composite rubber-textile body depends on finding accurate analytical solutions in order to calculate the static stresses and to allow the largest number of fatigue stress cycles.

The conditions for determining the magnitudes of stress and strain are extremely complex because of the complexity of the various influences. Still, the relative sizes of the components for a composite body of known geometry may be determined approximately if certain idealizations are assumed in the solution. Under such an idealization, a simplified expression for the determination of stress and strain under static load can be obtained by assuming a linearly elastic composite body.

I. INTRODUCTION

The stresses and deformations of a typical elastic element, and therefore also of the whole composite body of rubber and textile, may be determined for a fixed load and an assumed set of edge conditions. The following assumptions are made: the conditions of equilibrium for the elastic body must be fulfilled, and the relation between the stress and strain must be known. Then, with the aid of classical elasticity theory, the geometric and stress states may be mathematically determined as a function of the external forces. That is, the stresses and strains of the elastic body, at a known load and known edge conditions, may be determined.

In carrying out this analysis, the elastic constants which describe the stress state of the elastic materials are assumed to be known. Similarly the

geometry is assumed given. Then the relations between the elastic and geometric dimensions can be determined. The basic equations are to be solved by considering the boundary conditions. These equations are derived from mathematical statements. The solution is simplified, or even made possible, if, in establishing the basic equations, certain idealizations are anticipated which concern the kind and size of stresses, the kind and size of deformations, and the relation between stresses and deformations. The elasticity law for many materials is taken as linear, while for other materials, such as rubber and textile, a nonlinear relation exists between the stress and strain. This, however, may be linearized under certain assumptions.

In the classical elasticity theory, we have, in this respect, a much more simplified case since we limit ourselves to homogeneous and isotropic materials and to the consideration of very small deformations. We omit the nonlinear products of the elastic displacements and their derivatives to get a linear expression in these values. We fulfill the equilibrium conditions on the undeformed body instead of on the deformed one and assume the deformations proportional to the resultant forces. Under these assumptions the superposition law of mechanics is valid.

The superposition law is no longer valid for a more common case in the theory of elasticity. For example, whenever a linearization of the elasticity law or the assumption of small deformations can no longer be assumed, or if the homogeneity and the isotropy can no longer be approximately assumed as valid, then superposition no longer holds. How much of a generalization of the elasticity theory is required depends on the actual elasticity problem.

We do not wish to occupy ourselves here with general formulations of the elasticity theory for the composite rubber-textile body; that will be reported elsewhere. The classical theory of elasticity permits the creation of a set of equations sufficient for an approximate determination of the static rigidity of the composite rubber-textile body. Since investigations of the possibility of determining equations of equilibrium for certain problems such as pneumatic tires have already been made, then the assumptions necessary to the determination of stress and strain will be taken up here, as well as the essential relationship between stress and deformation, under the assumption of linear elasticity.


II. BASIS OF SIMPLIFIED CALCULATION

Composite bodies made of rubber and textile are mostly thin-walled, regarded from the point of view of their construction. Their sizes are indicated by the geometry of their middle surface, that is, the surface which passes through the mid-point of the wall thickness at every point. The middle surface can be simple or doubly curved, or it can be a combination of plane and curved surfaces.
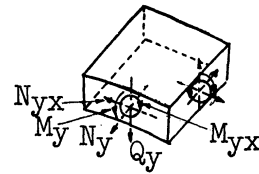
If we cut an element out of such a composite body in such a way that the cut faces are vertical to the mid-plane and parallel to the co-ordinate axes x and y (see Fig. 1), then components of stress have an effect on the cut surface which we can consider per unit of cutting length, in regard to forces and moments on the cut surface.

For the cut surface, x = const., the forces are: N_x (longitudinal force), N_{xy} (shearing force), and Q_x (transverse force); the moments are: M_x (bending moment), and M_{xy} (torsion moment).

Correspondingly for the cut surface, y = const., the forces on the cut face are: N_y, N_{yx}, and Q_y; and the moments are: M_y and M_{yx}.
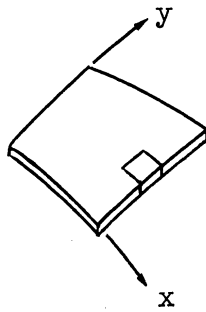
(a) Element of the composite body.   (b) Composite element.

Fig. 1.   The forces and moments acting on the cut surface.

To determine exactly the stress components, or alternately, the cutting forces and moments acting on the cut surface, the application of the general theory of elasticity to this problem would be necessary. If we avoid a rigorous method of solution and carry out a simplified calculation for the compound rubber-textile body by means of a classical elasticity theory, then the idealized assumptions previously discussed must be used.

The assumption of a linear strain law and the assumptions for the homogeneity and the isotropy of the materials are unrealistic for the compound rubber-textile body, for these imply the use of an ideal material, which we do not have. However, the stress-strain law may be linearized for each of the materials to be used in our compound body. This may be done within certain strain limitations which will be determined and stated. However, the conditions of homogeneity and especially isotropy may be fulfilled only approximately because of the construction of the compound body. The applicability of the assumption of isotropy for the compound body may be judged from the results of experiments reported below. It can then be decided whether the analysis using the classical theory of elasticity adequately approximates real conditions, and whether the results of this analysis permit sufficiently accurate determination of the stresses in the composite rubber-textile body.

The fundamentals of elasticity theory will be briefly reviewed, especially those aspects necessary for the approximate calculation of stress and strain.

1. THE STATE OF STRESS

If we imagine an element in the form of a square cut out of an elastic composite body (see Fig. 2), whose edges run parallel to the axes of a rectangular co-ordinate system, x,y,z, then we may establish the following stress components according to the theory of elasticity:

> Normal stress components: σ_x, σ_y, σ_z, and
> Shearing stress components: τ_{xy}, τ_{yx}, τ_{yz}, τ_{zy}, τ_{zx}, τ_{xz} .

Nine components of stress are present on each square element. As a result of equilibrium in rotation, that is to say $\tau_{xy} = \tau_{yx}$, $\tau_{yz} = \tau_{zy}$, $\tau_{zx} = \tau_{xz}$, these nine are reduced to six stress components: σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, τ_{zx}; these we can call parameters of the stress at every point.

If we set up equilibrium for the square element, then equilibrium must exist for each coordinate direction, that is, the sum of the forces due to the normal and shearing stresses in each direction must be set equal to the exterior forces acting in the same direction. This is the effect of the normal stresses σ_x and $\sigma_x + d\sigma_x$ and the shearing stresses τ_{yx} and $\tau_{yx} + d\tau_{yx}$, τ_{zx} and $\tau_{zx} + d\tau_{zx}$ in x-direction.

Since in general the state of stress is a function of the coordinates x,y,z, similar forces acting on opposite cut faces change only by differentials, for whose calculation Taylor series can be applied.

For example, this may be written:

$$\sigma_x + d\sigma_x = \sigma_x + \frac{\delta\sigma_x}{\delta x} dx + \cdots \text{ (terms of a higher order)} \quad .^*$$

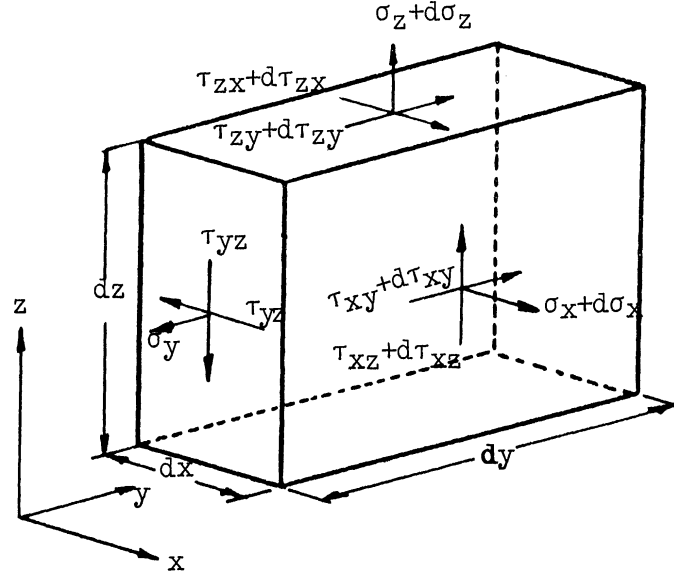*Note: δ signifies ∂ throughout.

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Fig. 2. Stresses on the square-shaped element.

If the terms of higher order are omitted, the result is

$$\sigma_x + d\sigma_x = \sigma_x + \frac{\delta\sigma_x}{\delta x} dx \quad .$$

We find similar developments for the remaining stress components in x-, y-, and z-direction. If the stress components operating on the element as a result of external forces are p_x, p_y, and p_z, then the conditions for the equilibrium in x-, y-, and z-directions are as follows:

$$\frac{\delta\sigma_x}{\delta x} + \frac{\delta\tau_{yx}}{\delta y} + \frac{\delta\tau_{zx}}{\delta z} + p_x = 0$$

$$\frac{\delta\tau_{xy}}{\delta x} + \frac{\delta\sigma_y}{\delta y} + \frac{\delta\tau_{zy}}{\delta z} + p_y = 0 \qquad\qquad (1)$$

$$\frac{\delta\tau_{xz}}{\delta x} + \frac{\delta\tau_{yz}}{\delta y} + \frac{\delta\sigma_z}{\delta z} + p_z = 0 \quad .$$

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2. THE STATE OF DISPLACEMENT AND STRAIN

We indicate the displacement components with u,v,w and postulate a rectangular cartesian coordinate system. Since the displacement components are functions of the coordinates x,y,z, displacements only change by differentials at adjoining points. If, for the point with coordinates x,y,z the displacements are u,v,w, and for the neighboring point with the coordinates x + dx, y + dy, z + dz the displacements are u + du, v + dv, w + dw, then these displacements may be expressed using the Taylor series in terms of variables at x,y,z. We then obtain, omitting the terms of higher order,

$$u + du = u + \frac{\delta u}{\delta x} dx + \frac{\delta u}{\delta y} dy + \frac{\delta u}{\delta z} dz \quad ,$$

and therefore

$$du = \frac{\delta u}{\delta x} dx + \frac{\delta u}{\delta y} dy + \frac{\delta u}{\delta z} dz \quad .$$

Correspondingly,

$$dv = \frac{\delta v}{\delta x} dx + \frac{\delta v}{\delta y} dy + \frac{\delta v}{\delta z} dz \quad ;$$

$$dw = \frac{\delta w}{\delta x} dx + \frac{\delta w}{\delta y} dy + \frac{\delta w}{\delta z} dz \quad .$$

(2)

Changes of the length of its diagonal and of the angle enclosed by the elements are related to the deformation of the elastic body. For two neighboring points with coordinates x, y, z and x + dx, y + dy, z + dz, the length of diagonal before deformation is

$$ds = \sqrt{dx^2 + dy^2 + dz^2}$$

and after deformation is

$$ds' = \sqrt{(dx+du)^2 + (dy+dv)^2 + (dz+dw)^2} \quad .$$

If the relationships (2) are inserted for the differentials of the displacements, the result after some manipulations is:

$$ds' = \sqrt{(1+2\epsilon_x)dx^2+(1+2\epsilon_y)dy^2+(1+2\epsilon_z)dz^2+2\gamma_{xy}dxdy+2\gamma_{yz}dydz+2\gamma_{zx}dzdx}$$

where ϵ_x, ϵ_y, ϵ_z = strains in the direction of the coordinate axes, and γ_{xy}, γ_{yz}, γ_{zx} = changes of the right angles formed by the sides (dx, dy, dz) of the element.

For infinitesimal strains, the squares and products of the derivatives of displacements can be omitted and we obtain the following linear relationships

for the strain components:

$$\epsilon_x = \frac{\delta u}{\delta x}, \qquad \epsilon_y = \frac{\delta v}{\delta y}, \qquad \epsilon_z = \frac{\delta w}{\delta z} \qquad (3)$$

and

$$\gamma_{xy} = \frac{\delta u}{\delta y} + \frac{\delta v}{\delta x}, \qquad \gamma_{yz} = \frac{\delta v}{\delta z} + \frac{\delta w}{\delta y}, \qquad \gamma_{zx} = \frac{\delta w}{\delta x} + \frac{\delta u}{\delta z}. \qquad (4)$$

The geometric conditions (3) and (4) give the relationship between the strain components and the derivatives of displacements.

3. RELATIONSHIP BETWEEN STATE OF STRESS AND STATE OF STRAIN

The relationship between stress and strain is established from a law determined experimentally. The simplest elasticity law is Hooke's Law, which states that the strain is proportional to the force causing it.

According to Hooke's Law the longitudinal strain for the uniaxial state of stress—for example, the elongation of a rod in the direction of its axis—is $\epsilon = \sigma/E$, where E = modulus of elasticity. In the presence of transverse elongation, this becomes $\epsilon_q = -\nu(\sigma/E)$, where ν = coefficient of transverse expansion of Poisson's Ratio. E and ν are constant for all directions in isotropic materials.

For the 3-dimensional state of stress, as a result of the linearity of Hooke's Law for the x-, y-, and z-direction, the following relationships between the normal stresses and the strains exist:

$$\begin{aligned}
\epsilon_x &= 1/E \, [\sigma_x - \nu \, (\sigma_y + \sigma_z)] \\
\epsilon_y &= 1/E \, [\sigma_y - \nu \, (\sigma_x + \sigma_z)] \\
\epsilon_z &= 1/E \, [\sigma_z - \nu \, (\sigma_x + \sigma_y)]
\end{aligned} \qquad (5)$$

as well as those between the shearing stress and the shearing strain:

$$\begin{aligned}
\gamma_{xy} &= 2 \, (1+\nu) \, \frac{\tau_{xy}}{E} = \frac{\tau_{xy}}{G} \\
\gamma_{yz} &= 2 \, (1+\nu) \, \frac{\tau_{yz}}{E} = \frac{\tau_{yz}}{G} \\
\gamma_{zx} &= 2 \, (1+\nu) \, \frac{\tau_{zx}}{E} = \frac{\tau_{zx}}{G}
\end{aligned} \qquad (6)$$

whereby the modulus of shear $G = E[2(1+\nu)]$.

With the help of the conditions of equilibrium (1), of the geometric conditions (3) and (4), and of the relationships (5) and (6) of the elasticity law, the magnitudes of stress and strain of a loaded composite substance can be calculated. Residual stresses and strains are present in the unloaded state and these are superimposed on those stresses due to external loads. Some other, more suitable coordinate system can be chosen in place of the rectangular coordinates, for certain specific problems where this is convenient. Equations (1) through (6) must then be rewritten.

We once again point out that these relationships have only a limited scope of validity.

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III. LINEARIZED ELASTICITY LAW FOR THE COMPOSITE RUBBER-TEXTILE BODY

The rubber-textile system is understood to be rubber with textile inserts especially suited to absorb the tensile stresses present in the loaded structure. Indeed, the rubber has the properties of being essentially incompressible and also of producing shearing stresses with shear deformations. Still, its tensile strength is either very small or else associated with large deformations. To avoid these disadvantages of rubber, some kind of rigid reinforcements made of textile are arranged in the tensile area of the rubber body so that the composite body can carry the stresses in a definite way.

Fundamentally, the following essential parts may be established for a composite rubber-textile body:

(a) The carcass of the structure consists of one or several rubberized fabric layers which are arranged in a suitable way to attain the desired properties. The important task of the fabric foundation is to withstand the tensile stresses of the composite body.

(b) To protect the carcass of the structure against outside influences of an abrasive or chemical nature, it is provided with a rubber covering layer which is formed in a manner depending on the application.

(c) For the purpose of a proper support, or, as the case may be, orientation of the composite body, its edges must be built so that a good structural connection is made with the support.

In the case of the edges or supporting parts, it is important to make these more or less rigid according to the particular application of the composite body.

Individually, these main parts of the composite body can be made very differently according to what stresses are to be carried and which optimum properties are desired.

1. THE STATE OF STRESS OF THE COMPOSITE BODY

As already detailed above, the composite bodies of interest here are shell structures whose middle surfaces are plain or doubly curved. Under static load, these shell structures are essentially loaded so that the load is absorbed mainly by the extensional forces; thus, for an approximation, we may speak of a state of stress free from bending. In the case of small wall thicknesses in comparison to the other dimensions of the composite body, we may take as a basis for further derivations the assumptions of the membrane theory. That means, among other things, that the stresses are uniformly distributed over the wall thicknesses. No moments or transverse forces appear, and the longitudinal forces affect the middle surface of the structure in such a way that the middle surface

undergoes tension only. The bending stresses which eventually occur due to the changes in the curvature of the membrane are only secondary stresses and therefore may be omitted.

By omitting the moments and transverse forces we obtain a simplified state of stress, so that according to Fig. 1 for the cut face, x = constant, the forces acting on the surface of the cut are N_x and N_{xy}, and for the cutting surface, y = constant, the forces are N_y and N_{yx}. Forces on the cut faces are to be summed up for a fixed orientation of the composite body from the equilibrium conditions.

The state of stress can often be simplified still further if, for example, for a composite body, the middle surface and the load are axially symmetrical. It then becomes $N_{xy} = N_{yx} = 0$, so that out of the equilibrium conditions only the forces N_x and N_y are to be determined.

2. SIMPLIFIED ELASTICITY FOR THE COMPOSITE BODY

The elastic behavior of the composite rubber-textile body cannot be given without further explanation; it is first of all dependent on the known elasticity law of the rubber and textile. It also depends on the geometry of construction. A simplified relationship between the magnitude of stress and strain of the composite body for small deformations will now be given in the following derivations.

If we regard a composite element (see Fig. 3) that is stressed and strained in the x and y direction, simple linear relationship is valid between the stresses and strains under the assumption of Hooke's Law corresponding to Eq. (5). We base the further examination of the elastic properties of the composite body on the stress-strain laws of individual materials, for which Hooke's Law is valid in certain stress areas whose limits are to be established. Then the influence of geometrial factors on the stress-strain law may be established for the compound body if it is deformed only in its plane and if the materials used deform in the same way.

As is well known, the basic form of construction of the composite rubber-textile body remains essentially unchanged, so that here it is sufficient to treat the simple case when the wall of the composite body consists of two crossed, rubberized textile layers and a protective layer of rubber. The textile reinforcement is symmetrical to the symmetry lines x, y of the composite element with the intersecting angle $\omega = 2 \cdot \beta$. The results presented below are also valid for a general construction of the composite body.
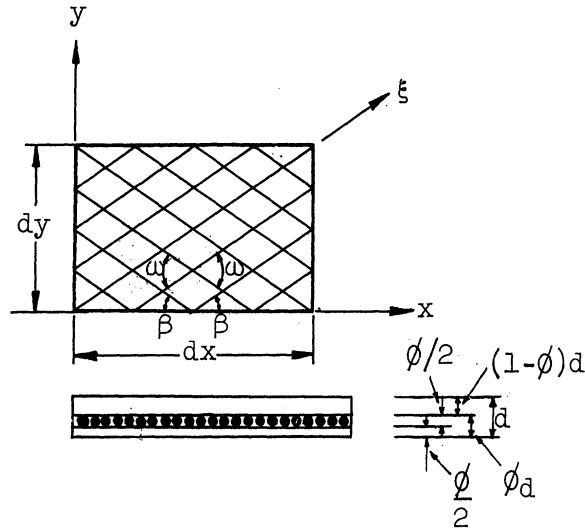
Fig. 3. Composite element.

The following values are known, or may be determined:

E_q' = Modulus of elasticity of the protective rubber

E_q'' = Modulus of elasticity of the rubber coating

ν_g = Coefficient of cross expansion for the rubber

E_t = Modulus of elasticity of the textile material

d = Wall thickness of the composite body

$\emptyset \cdot d$ = Thickness of all rubberized textile layers

$(1-\emptyset)d$ = Thickness of the rubber layer

$\emptyset$ = Relationship of the thickness of the textile foundation to the wall thickness

$1-\emptyset$ = Relationship of the thickness of the rubber layer to the wall thickness

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ψ = Relationship of the textile cross section to the cross section of the layer of fabric

$\psi\emptyset$ = Relationship of the textile cross section to the entire cross section.

The stresses are absorbed by the individual elements of the composite body and the partial stresses for the rubber layer in the x direction are:

$$\sigma_{x_g}{}' = \frac{E_g{}'}{1 - \nu_g{}^2}\left(\epsilon_x + \nu_g\,\epsilon_y\right);$$

In the y direction

$$\sigma_{y_g}{}' = \frac{E_g{}'}{1 - \nu_g{}^2}\left(\epsilon_y + \nu_g\,\epsilon_x\right).$$

For a layer of fabric the stress components are to be determined next in the case of a textile reinforcement in ξ direction. For the textile part the stress in ξ-direction is:

$$\sigma_{\xi_t} = E_t \cdot \epsilon_\xi$$

By transformation we find the stress components for strain in the x and y directions (see Fig. 4):

$$\sigma_{x_t} = E_t\left\{\frac{(1 + \cos\omega)^2}{4}\,\epsilon_x + \frac{\sin^2\omega}{4}\,\epsilon_y\right\}$$

$$\sigma_{y_t} = E_t\left\{\frac{\sin^2\omega}{4}\,\epsilon_x + \frac{(1 - \cos\omega)^2}{4}\,\epsilon_y\right\}$$

For protective rubber layer the stresses in the ξ direction are:

$$\sigma_{\xi_g}{}'' = \frac{E_g{}''}{1 - \nu_g{}^2}\left(\epsilon_\xi + \nu_g\,\epsilon_\eta\right)$$

In η direction

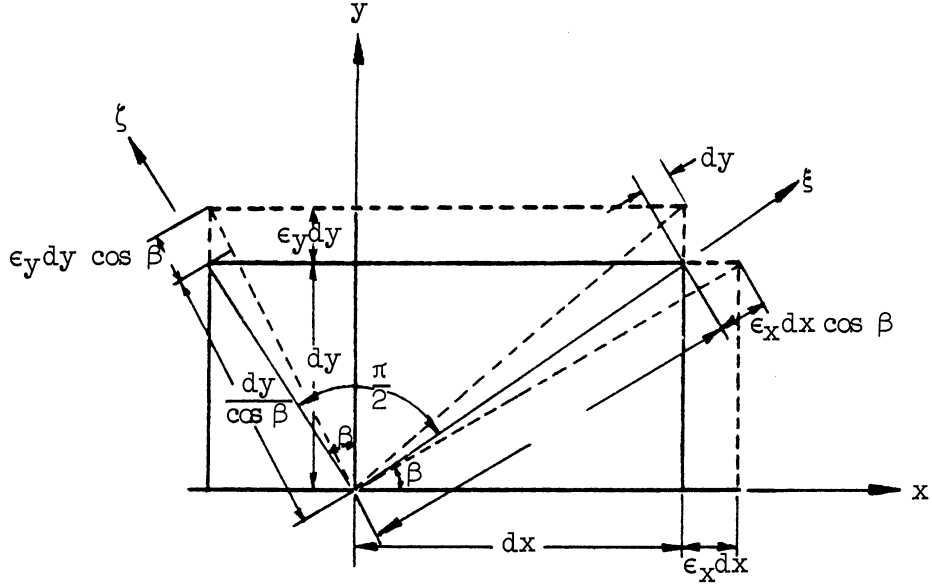$$\sigma_{\eta_g}{}'' = \frac{E_g{}''}{1 - \nu_g{}^2}\left(\epsilon_\eta + \nu_g\,\epsilon_\xi\right)$$

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Fig. 4. Strain as a result of distortions.

By transformation we find the stress components for strain in x and y directions (see Fig. 4):

$$\sigma_{x_g''} = \frac{E_g''}{1 - \nu_g^2} \left\{ \left[\frac{1 + \cos^2 \omega}{2} \epsilon_x + \frac{\sin^2 \omega}{2} \epsilon_y \right] \right.$$

$$+ \left. \nu_g \left[\frac{\sin^2 \omega}{2} \epsilon_x + \frac{1 + \cos^2 \omega}{2} \epsilon_y \right] \right\}$$

$$\sigma_{y_g''} = \frac{E_g''}{1 - \nu_g^2} \left\{ \left[\frac{\sin^2 \omega}{2} \epsilon_x + \frac{1 + \cos^2 \omega}{2} \epsilon_y \right] \right.$$

$$+ \left. \nu_g \left[\frac{1 + \cos^2 \omega}{2} \epsilon_x + \frac{\sin^2 \omega}{2} \epsilon_y \right] \right\}$$

If we account for the stresses of both fabric layers, the relationship yields

$$N_x = \sigma_x \cdot d = \sigma_{x_g'} \cdot (1 - \emptyset) \, d + \sigma_{x_t} \cdot \psi \cdot \emptyset \cdot d + \sigma_{x_g''} (1 - \psi) \, \emptyset d$$

or the equation:

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$$\sigma_x = \frac{E_g{}'(1 - \phi)}{1 - \nu_g{}^2} \left\{ \left[1 + \frac{n_1}{2}(1 + \cos \omega)^2 + n_2(1 + \cos^2 \omega + \nu_g \sin^2 \omega) \right] \epsilon_x \right.$$

$$\left. + \left[\nu_g + \frac{n_1}{2} \sin^2 \omega + n_2(\sin^2 \omega + \nu_g + \nu_g \cos^2 \omega) \right] \epsilon_y \right\} \tag{7}$$

and correspondingly

$$\sigma_y = \frac{E_g{}'(1 - \phi)}{1 - \nu_g{}^2} \left\{ \left[\nu_g + \frac{n_1}{2} \sin^2 \omega + n_2(\sin^2 \omega + \nu_g + \nu_g \cos^2 \omega) \right] \epsilon_x \right.$$

$$\left. + \left[1 + \frac{n_1}{2}(1 - \cos \omega)^2 + n_2(1 + \cos^2 \omega + \nu_g \sin^2 \omega) \right] \epsilon_y \right\} \tag{8}$$

where

$$\text{when} \quad n_1 = \frac{E_t \psi \phi}{E_g{}' \dfrac{1 - \phi}{1 - \nu_g{}^2}} \qquad \text{und} \quad n_2 = \frac{E_g{}''(1 - \psi)\,\phi}{E_g{}'(1 - \phi)} \tag{9}$$

Equations (7), (8), and (9) give the linear relations for small deformations between the stresses and strains for the composite rubber textile body as a function of the elastic constants of the individual materials and their geometry. By transforming the above equations, the anisotropy of the composite body may be determined as a function of the directions of the textile material. For this reason we rewrite Eqs. (7) and (8) as follows:

$$\epsilon_x = \frac{\sigma_x}{E_{xx}^{xx}} \quad \frac{\sigma_y}{E_{xx}^{yy}}$$

$$\epsilon_y = \frac{\sigma_x}{E_{yy}^{xx}} \quad \frac{\sigma_y}{E_{yy}^{yy}}$$

Out of this the moduli of elasticity may be determined, and indeed we find for the modulus of elasticity in the x direction:

$$E_{xx}^{xx} = E_g{}'(1 = \phi) \frac{1 + \frac{n_1}{1 - \nu_g{}^2}(1 + \cos^2 \omega - \nu_g \sin^2 \omega + 4n_2 \cos^2 \omega) + 2n_2(1 + \cos^2 \omega + 2n_2 \cos^2 \omega)}{1 + \frac{n_1}{2}(1 - \cos \omega)^2 + n_2(1 + \cos^2 \omega + \nu_g \sin^2 \omega)}$$

And in the y direction:

$$E_{yy}^{xx} = E{}'(1 - \phi) \frac{1 + \frac{n_1}{1 - \nu_g{}^2}(1 + \cos^2 \omega - \nu_g \sin^2 \omega + 4n_2 \cos^2 \omega) + 2n_2(1 + \cos^2 \omega + 2n_2 \cos^2 \omega)}{1 + \frac{n_1}{2}(1 + \cos \omega)^2 + n_2(1 + \cos^2 \omega + \nu_g \sin^2 \omega)}$$

Modulus of cross contraction:

$$= E_{yy}^{xx} = - E_g{}'(1 - \phi) \frac{1 + \frac{n_1}{1 - \nu_g{}^2}(1 + \cos^2 \omega - \nu_g \sin^2 \omega + 4n_2 \cos^2 \omega) + 2n_2(1 + \cos^2 \omega + 2n_2 \cos^2 \omega)}{\nu_g + \frac{n_1}{2} \sin^2 \omega + n_2(\sin^2 \omega + \nu_g + \nu_g \cos^2 \omega)}$$

$$\tag{10}$$

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With the help of Eqs. (10) the influence of the elastic constants of the individual materials and the effect of the arrangement of the textile reinforcements on the stiffness of the composite body may be determined.

Because of the anisotropy of the composite element, the following special cases will now be studied:

(a) The influence of the protective rubber layer should be negligibly small, so that if $n_2 = 0$, Eqs. (10) becomes:

$$E_{xx}^{xx} = E_g'(1 - \emptyset) \cdot \frac{1 + \dfrac{n_1}{1 - v_g^2}(1 + \cos^2 \omega - v_g \sin^2 \omega)}{1 + \dfrac{n_1}{2}(1 - \cos \omega)^2}$$

$$E_{yy}^{yy} = E_g'(1 - \emptyset) \cdot \frac{1 + \dfrac{n_1}{1 - v_g^2}(1 + \cos^2 \omega - v_g \sin^2 \omega)}{1 + \dfrac{n_1}{2}(1 + \cos \omega)^2}$$

$$E_{xx}^{yy} = E_{yy}^{xx} = -E_g'(1 - \emptyset) \cdot \frac{1 + \dfrac{n_1}{1 - v_g^2}(1 + \cos^2 \omega - v_g \sin^2 \omega)}{v_g + \dfrac{n_1}{2}\sin^2 \omega}$$

These are the same relationships which F. Martin derived in his investigations into the state of stress of pneumatic aircraft tires.

(b) The textile reinforcement lies parallel to the x-axis, that is, $\omega = 0$ and therefore

$$E_{xx}^{xx} = E_g'(1 - \emptyset) \frac{1 + \dfrac{2n_1}{1 - v_g^2}(1 + 2n_2) + 4n_2(1 + n_2)}{1 + 2n_2}$$

$$E_{yy}^{yy} = E_g'(1 = \emptyset) \frac{1 + \dfrac{2n_1}{1 - v_g^2}(1 + 2n_2) + 4n_2(1 + n_2)}{1 + 2n_1 + 2n_2}$$

$$E_{xx}^{yy} = E_{yy}^{xx} = -E_g'(1 - \emptyset) \frac{1 + \dfrac{2n_1}{1 - v_g^2}(1 + 2n_2) + 4n_2(1 + n_2)}{v_g(1 + 2n_2)}$$

If the textile reinforcement lies parallel to the y-axis, that is, $\omega = \pi$, then we obtain the same value for $E_{xx}^{yy} = E_{yy}^{xx}$ while the E moduli are interchanged in the x- and y-direction.

(c) The textile reinforcement is arranged orthogonally, making an angle of 45° with the x and y axes, that is, $\omega = \pi/2$:

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$$E^{xx}_{xx} = E_g'(1 - \phi) \; \frac{1 + \dfrac{n_1}{1 + \nu_g} + 2n_2}{1 + \dfrac{n_1}{2} + n_2(1 + \nu_g)}$$

$$E^{yy}_{yy} = E_g'(1 - \phi) \; \frac{1 + \dfrac{n_1}{1 + \nu_g} + 2n_2}{1 + \dfrac{n_1}{2} + n_2(1 + \nu_g)} \qquad\qquad (11)$$

$$E^{yy}_{xx} = E^{xx}_{yy} = -E_g'(1 - \phi) \; \frac{1 + \dfrac{n_1}{1 + \nu_g} + 2n_2}{\nu_g + \dfrac{n_1}{2} + n_2(1 + \nu_g)}$$

In this case $E^{xx}_{xx} = E^{yy}_{yy}$. That means that the assumption of isotropy of the compound body is best approximated with orthogonal textile reinforcements symmetrical to the x and y axes. In this case, relationships derived using the classical theory of elasticity yield exact results. The greater the anisotropy, the less exact is the simplified calculation for the composite rubber-textile body.

The dependence of the anisotropy on the elastic properties and textile reinforcement is shown in two examples—with simple numerical values—and the moduli of elasticity of the composite body is plotted against the included angle of the textile reinforcement for:

$$E_g' = 100 \text{ kg/cm}^2; \; E_t = 50{,}000 \text{ kg/cm}^2; \; E_g'' = 50 \text{ kg/cm}^2$$

and

$$\nu_g = 0.5; \; \phi = 0.5; \; \psi = 0.5 \quad \text{in Fig. 5.}$$

In Fig. 6 the constants are chosen as

$$E_g' = E_g'' - 100 \text{ kg/cm}^2; \; E_t = 2{,}000{,}000 \text{ kg/cm}^2$$

and

$$\nu_g = 0.5; \; \phi = 0.5; \; \psi = 0.5.$$

IV. SUMMARY

Based on linear elasticity, a simplified calculation is presented for the magnitude of stress and strain in the composite rubber-textile body using the theory of elasticity. Under the assumption that the state of stress is free
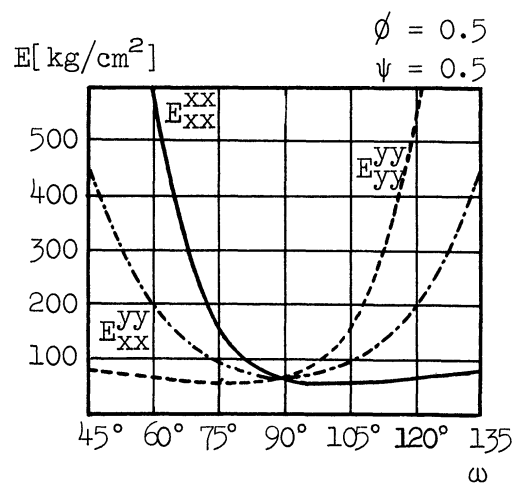
Fig. 5. Dependence of the modulii of elasticity of the composite body on the included angle ω for $E_g' = 100$ kg/cm^2; $E_t = 50,000$ kg/cm^2; $E_g'' = 50$ kg/cm^2.
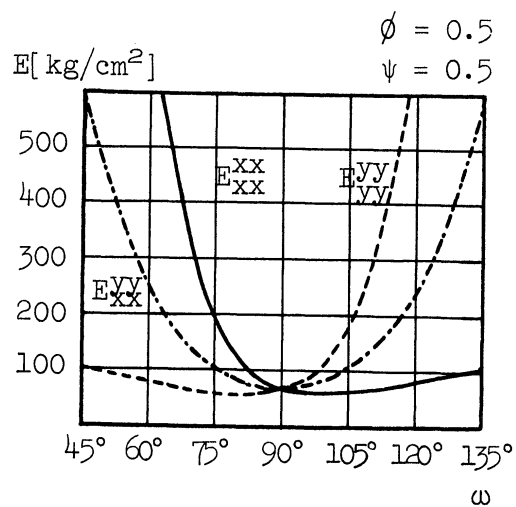

Fig. 6. Dependence of the modulii of elasticity of the composite body on the included angle ω for $E_g' = 100$ kg/cm^2; $E_t = 2,000,000$ kg/cm^2; $E_g'' = 100$ kg/cm^2.

from bending and that no shearing stresses are present, a linear relationship between the stresses and strains may be derived for the simple composite body whose walls consist of two crossed rubberized textile layers and a protective layer of rubber. The simplified relationships which have been derived predict the influence of elastic constants of the constituent materials and the construction geometry on the overall stiffness of the composite body. As an indication of the anisotropy, the moduli of elasticity of the composite body are determined and their dependence on the angle of the textile reinforcements is investigated for special cases. Two examples of it are calculated.

It was found that the assumption of isotropy for the composite rubber textile body is best approximated with orthogonal textile reinforcements symmetrical to the x and y axes.