Asymptotic Theory of Diffraction by Smooth Convex Surfaces of Variable Curvature

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by

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FOREWORD

This report was prepared by the Radiation Laboratory of the Department of Electrical Engineering of The University of Michigan under the direction of Dr. Raymond F. Goodrich, Principal Investigator, and Burton A. Harrison, Contract Manager. The work was performed under Contract AF 04(694)-834 "Investigation of Re-entry Vehicle Surface Fields (SURF)". The work was administered under the direction of the Air Force Ballistic Systems Division, Norton Air Force Base, California 92409 by Major A. Aharonian BSYDF and was monitored by Mr. Henry J. Katzman of the Aerospace Corporation.

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ABSTRACT

A general method is presented for obtaining successive terms in short-wavelength asymptotic expansions of the diffracted field produced by plane acoustic and electromagnetic waves incident on an arbitrary smooth convex surface. By introducing the geodesic coordinate system on arbitrary surfaces of non-constant curvature, both scalar and vector integral equations governing the surface fields are solved directly. The expressions for leading and second order terms in the asymptotic expansion of the diffracted fields are obtained explicitly and the differences between acoustic and electromagnetic creeping waves are shown.

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INTRODUCTION

A shadow is formed when a wave is incident upon a smooth convex body which is large compared to the incident wavelength. In the neighborhood of the shadow boundary the surface field does not abruptly vanish and a penumbra region exists. Some waves penetrate into the shadow region and account for the non-zero fields there. These phenomena are due to diffraction of the incident wave by the object.

The mathematical problem of analyzing diffraction of waves involves finding the short-wavelength asymptotic form of a solution of the wave equation satisfying an appropriate boundary condition of the diffracting surface and the radiation condition at infinity. Detailed studies of the surface field on a circular cylinder and a sphere (Franz, 1954) for which the exact solutions are available, indicate that the incident wave is diffracted near the shadow boundary and the diffracted waves proceed along the geodesic into the shadow region, spilling off energy as they travel. Their phases are determined primarily by the distance traveled from the shadow boundary. The waves diffracted by a smooth convex surface are frequently called creeping waves.

In obtaining a description of the waves diffracted by an arbitrary smooth convex surface of variable curvature, two techniques can be used:

(1) finding the asymptotic form of an exact solution for a canonical body and generalizing the results;
(2) solving the boundary value problem directly by an asymptotic method for a general surface but in restricted regions.

The difficulty with the first method is that very few canonical problems can be solved exactly. Thus, in the well-known geometrical theory of diffraction (Levy and Keller, 1959), a locally cylindrical body is chosen as the canonical body in analyzing diffraction of waves by arbitrary smooth surfaces. While this theory gives the correct leading term in the asymptotic expansion of the diffracted fields, it does not yield higher-order terms.

The purpose of this paper is to discuss an integral equation approach based on the second technique. It can yield not only the leading term but also higher-order terms in the asymptotic expansion of the fields diffracted by an arbitrary shape with a smooth convex surface. The method to be used is the following. The geodesic coordinate system is introduced to describe the geometry of the diffracting surface (Sect. 2). In terms of this coordinate system, the short-wavelength asymptotic form of the integral equation governing the surface fields is derived (Sect. 3.1) for the acoustic case, and its solutions are derived for the penumbra (Sect. 3.2) and shadow (Sect. 3.3) regions. The same procedure is repeated for the electromagnetic case (Chap. 4).

II

THE GEODESIC COORDINATE SYSTEM

From the analysis of the sphere solution (Franz, 1954), it is observed that the creeping waves propagate along the geodesic. Thus, we propose to use the geodesic coordinate system to describe the diffracting surface. An important advantage of this coordinate system is that it can be defined on any smooth surface. For the sake of simplicity, it is assumed that the diffracting surface is symmetric with respect to the shadow boundary and that the torsion of the geodesic is zero.

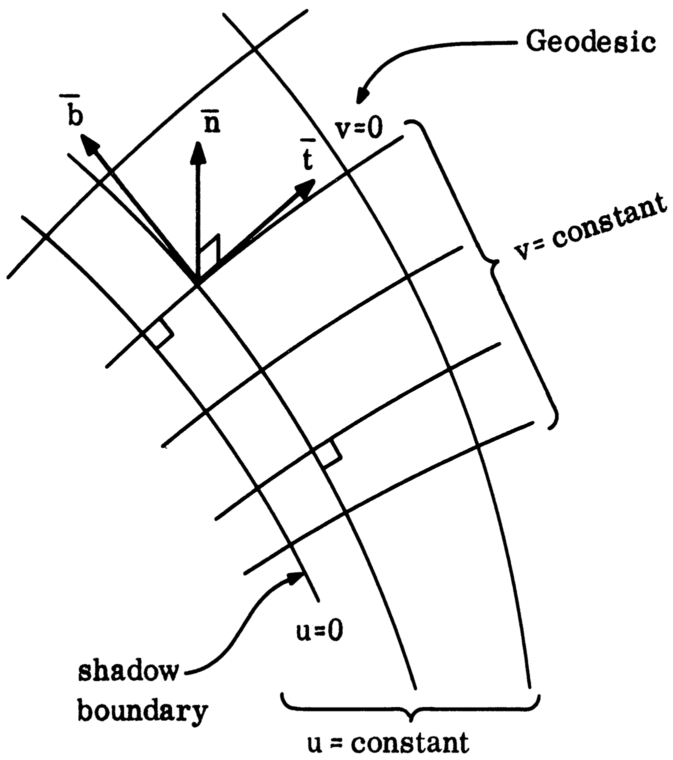

FIG. 2-1: GEODESIC COORDINATE SYSTEM

Let us define the geodesic coordinate system as follows. The curve u=0 is taken to represent the shadow boundary with v denoting arc length along it. At each point of u=0 the incident wave is tangent in a given direction, and this defines a geodesic through each point of u=0; these geodesics are taken as the coordinate curves v=const.; with u taken as arc length along the geodesic measured positively from the shadow boundary. The geodesic coordinate system is orthogonal and the linear element is given as:

$$ds^2 = du^2 + G\,dv^2 \quad \text{with} \quad G(u=0) \equiv 1\,. \tag{2.1}$$

Because of the assumption that the geodesics are planar, G is independent of v.

The diffracting surface may be described by the Gauss-Weingarten equations (Struik, 1950):

$$\begin{aligned}
\frac{\partial \bar{t}}{\partial u} &= -\kappa_g \bar{n} \\
\frac{\partial \bar{t}}{\partial v} &= \frac{\partial \bar{b}}{\partial u} = \kappa_{tt} \bar{b} \\
\frac{\partial \bar{b}}{\partial v} &= -G\left[\kappa_{tt}\bar{t} + \kappa_{tn}\bar{n}\right] \qquad \text{with } \kappa_t = \sqrt{\kappa_{tt}^2 + \kappa_{tn}^2} \\
\frac{\partial \bar{n}}{\partial u} &= \kappa_g \bar{t} \quad , \quad \frac{\partial \bar{n}}{\partial v} = \kappa_{tn} \bar{b}
\end{aligned} \tag{2.2}$$

where

$$\frac{\partial \bar{r}}{\partial u} = \bar{t}\,, \qquad \frac{\partial \bar{r}}{\partial v} = \bar{b} \qquad \text{with } \bar{r} = \text{the position vector.} \tag{2 3}$$

Here $\bar{n}$, $\bar{t}$ and $\frac{\bar{b}}{\sqrt{G}}$ are unit normal, tangent and binormal vectors along the geodesic, respectively. κ_g is the curvature of the geodesic. κ_{tt} and κ_{tn} are respectively, the tangential and the normal components of the curvature of the u = const. curves. Thus, the two principal curvatures are κ_g and κ_{tn} and their product is

$$\kappa_g \kappa_{tn} = - \frac{1}{\sqrt{G}} \frac{\partial^2 \sqrt{G}}{\partial u^2} \quad . \tag{2.4}$$

while κ_{tt} is related to the function G by

$$\kappa_{tt} = \frac{\frac{\partial G}{\partial u}}{2G} \quad . \tag{2.5}$$

In addition to Eq. (2.2), the Codazzi equation must be satisfied:

$$\begin{aligned} \frac{\partial \kappa_{tn}}{\partial u} &= \kappa_{tt}(\kappa_g - \kappa_{tn}) \\ \frac{\partial \kappa_g}{\partial v} &= 0 \quad . \end{aligned} \tag{2.6}$$

A more detailed analysis of the geodesic coordinate system can be found in most books on differential geometry (e.g. Struik, 1950).

III

DIFFRACTION OF A PLANE ACOUSTIC (SCALAR) WAVE

The first problem to be considered is the diffraction of a plane acoustic (scalar) wave by an acoustically hard surface, i. e a Neumann boundary condition is imposed.

3.1 Integral Equation Governing the Surface Field.

We suppose that a plane acoustic wave is incident upon a smooth convex surface and that the normal derivative of the total field on the surface vanishes. Then the integral equation governing the surface field can be easily derived by Green's theorem (Hönl et al, 1961):

$$U(\bar{r}) = 2U_{inc.}(\bar{r}) - \frac{1}{2\pi}\iint da' \, U(\bar{r}') \frac{1-ikR}{R^3} \left\{ \bar{n}(\bar{r}') \cdot \bar{R} \right\} e^{ikR} \tag{3.1}$$

where $\bar{R} = \bar{r}' - \bar{r}$, and $U_{inc.}$ is the incident field. Without loss of generality, we will consider the surface field on a geodesic which will be called the curve $v=0$. In terms of the geodesic coordinate system, the incident wave on the geodesic $v=0$ is

$$U_{inc.}(u, v=0) = e^{ik\bar{t}(u=0,\, v=0) \cdot \bar{r}(u,\, v=0)} \quad . \tag{3.2}$$

In the above two equations the time dependence factor $e^{-i\omega t}$ is omitted. As observed in the study of a circular cylinder and a sphere (Franz, 1954), the phase of the diffracted (creeping) wave is determined mainly by the distance traveled from the shadow boundary, thus we shall set

$$U(\bar{r}) = e^{iku} I(\bar{r}) \tag{3.3}$$

and for a large $k(=\frac{2\pi}{\lambda}$, the wave number) $I(\bar{r})$ is assumed to be slowly varying in comparison with e^{iku}. Substitution of this expression into Eq. (3.1) gives:

$$I(u,0)=2\exp.\left\{ik\bar{t}(0,0)\cdot\bar{r}(u,0)-iku\right\}-\frac{1}{2\pi}\iint\sqrt{G}(u')\,du'dv'\,I(u',v')\,\frac{1-ikR}{R^3}$$

$$\left\{\bar{n}(u',v')\cdot\bar{R}\right\}\exp.\left\{ikR-ik(u-u')\right\}\quad. \tag{3.4}$$

Since we are interested in the short-wavelength behavior of the solution, we will replace the second term in Eq. (3.4) by its asymptotic form. For large k, the integrand has a saddle point where the derivative of the function $R-(u-u')$ vanishes. The Taylor series expansion of the vector $\bar{R}$ near $\bar{r}=\bar{r}'$ is easily derived by means of Eq. (2.2) and is given by:

$$\bar{R}=\bar{r}'(u',v')-\bar{r}(u,0)\simeq(u'-u)\bar{t}(u)+v'\bar{b}(u)-\frac{1}{2}\Big[(u'-u)^2\kappa_g(u)\bar{n}(u)-2(u'-u)v'\kappa_{tt}(u)\bar{b}(u)$$

$$+v'^2G(u)\left\{\kappa_{tt}(u)\bar{t}(u)+\kappa_{tn}(u)\bar{n}(u)\right\}\Big]\;-\frac{1}{6}\Big[(u'-u)^3\left\{\dot{\kappa}_g(u)\bar{n}(u)+\kappa_g^2(u)\bar{t}(u)\right\}$$

$$+3(u'-u)^2v'\kappa_g(u)\kappa_{tn}(u)\bar{b}(u)+3(u'-u)v'^2G(u)\kappa_{tt}(u)\left\{\kappa_{tt}(u)\bar{t}(u)+\kappa_{tn}(u)\bar{n}(u)\right\}$$

$$+v'^3G(u)\kappa_t^2(u)\bar{b}(u)\Big]\;+----\;. \tag{3.5}$$

Above, and in following pages, the curvatures (κ_g etc.), $\bar{t}$, $\bar{b}$ and $\bar{n}$ without the argument for the v coordinate represents their values at v=0. The dots denote the derivative with respect to the argument of the function. Using the above expression, the solution of the equation

$$\frac{\partial}{\partial v'}\left[\sqrt{\bar{R}\cdot\bar{R}}\;-(u-u')\right]=\frac{v'}{R}G(u)\left[(u-u')\kappa_{tt}(u)+---\right]=0 \tag{3.6}$$

yields the saddle point at v' = 0 for the v' integration. Applying the method of steepest descents (Brekhovskikh, 1960) to the v' integration in Eq. (3. 4), we obtain an asymptotic expression of the integral equation for large k.

$$I(u,0) = 2\exp.\left\{ik\bar{t}(0,0)\cdot\bar{r}(u,0) - iku\right\} - \frac{e^{i\frac{\pi}{4}}}{\sqrt{2\pi}}\int_{-\infty}^{u} du' \; e^{ikR_o - ik(u-u')} \frac{\sqrt{G(u')}}{\sqrt{k\frac{\partial^2 R}{\partial v'^2}(v'=0)}}$$

$$\left[-ik\frac{\bar{n}(u',0)\cdot\bar{R}_o}{R_o^2} I(u',0) - \frac{\left\{\bar{n}(u',0)\cdot\bar{R}_o\right\}\frac{\partial^4 R}{\partial v'^4}(v'=0)}{8R_o^2\left\{\frac{\partial^2 R}{\partial v'^2}(v'=0)\right\}^2} I(u',0)\right.$$

$$\left. + \frac{\kappa_{tn}(u')}{2R_o} I(u',0) + \frac{\left\{\bar{n}(u',0)\cdot\bar{R}_o\right\}}{2R_o^2\left\{\frac{\partial^2 R}{\partial v'^2}(v'=0)\right\}} \frac{\partial^2 I}{\partial v'^2}(u',0)\right] + 0(k^{-3/2}) \qquad (3.7)$$

where

$$\bar{R}_o = \bar{r}'(u',0) - \bar{r}(u,0) \qquad (3.8)$$

$$R_o \simeq (u-u') - \frac{\kappa_g^2(u)}{24}(u-u')^3 + \frac{\kappa_g(u)\kappa_g'(u)}{24}(u-u')^4$$

$$+\left[\frac{\kappa_g^4(u)}{1920} - \frac{\kappa_g'^2(u)}{90} - \frac{\kappa_g(u)\kappa_g''(u)}{80}\right](u-u')^5 \qquad (3.9)$$

$$\frac{\partial^2 R}{\partial v'^2}(v'=0) \simeq \frac{G(u')}{R_o}\left[1-\bar{R}_o\left\{\kappa_{tt}(u')\bar{t}(u') + \kappa_{tn}(u')n(u')\right\}\right] \tag{3.10}$$

and

$$\frac{\partial^4 R}{\partial v'^4}(v'=0) = -\frac{3G(u')}{R_o^2}\left[1-\bar{R}_o\cdot\left\{\kappa_{tt}(u')\bar{t}(u')+\kappa_{tn}(u')\bar{n}(u')\right\}\right.$$

$$\left. + \frac{R_o^2\kappa_t^2(u')}{3}\right]\frac{\partial^2 R}{\partial v'^2}(v'=0) \quad . \tag{3.11}$$

In Eq. (3.7), the contribution from the u' integration between u and infinity is neglected. The reason is the following: From Eq. (3.5) it can be shown that R_o near $u=u'$ is

$$R_o \simeq (u-u') + \ldots \qquad \text{for } u > u'$$

$$\simeq (u'-u) + \ldots \qquad \text{for } u < u' ,$$

thus, exp. $ik\left\{R_o-(u-u')\right\}$ in Eq. (3.7) has a saddle point at $u=u'$ only if $u > u'$; therefore, by integrating by parts, one can show that the contribution from the region $u < u' \leqslant \infty$ is asymptotically negligible for large k.

Now the integral equation governing the surface field is thus reduced to a one dimensional Volterra equation.

3.2 The Surface Field in the Penumbra Region.

In this section, the asymptotic integral equation (3.7) governing the surface field is solved for the penumbra region. It is assumed that the curvatures are slowly varying and that $\frac{\rho_{tt}}{\rho_{tn}}$ is of order one or less. In order to obtain an appropriate form of Eq. (3.7) in the neighborhood of the shadow boundary, we shall set

$$M_o = \left[k\rho_g(0,0)\right]^{1/3} \qquad \left(\rho_g = \frac{1}{\kappa_g}\right)$$

$$ku = M_o^2 \xi$$

$$ku' = M_o^2 \tau \quad , \tag{3.12}$$

and further assume that $\left[k\rho_g(u,v)\right]^{1/3} \gg 1$. Near the shadow boundary (u=0), the phase function, $\bar{t}(0,0)\cdot\bar{r}(u,0)-u$, of the incident wave term in Eq. (3.7) can be expanded in Taylor series by means of Eq. (2.2) :

$$\bar{t}(0,0)\cdot\bar{r}(u,0)-u \simeq -\frac{u^3}{6}\kappa_g^2(0,0) + \frac{u^5}{120}\kappa_g^4(0,0)\left\{1+4\rho_g(0,0)\ddot{\rho}_g(0,0)\right\} \tag{3.13}$$

$\left(\dot{\rho}_g(0,0)=0\right.$ by assumption of symmetry of the diffracting surface with respect to the shadow boundary$\left.\right)$.

Substitution of Eq. (3.12) into the above expression yields the asymptotic form of the incident wave:

$$e^{ik\bar{t}(0,0)\cdot\bar{r}(u,0)-iku} = e^{-i\frac{\xi^3}{6}}\left[1+i\frac{\left\{1+4\rho_g(0,0)\ddot{\rho}_g(0,0)\right\}}{120\, M_o^2}\xi^5\right] + 0(M_o^{-3}) \tag{3.14}$$

Similarly, an appropriate asymptotic form of the second term in the right-hand side of Eq. (3.7) can be easily derived by expanding the integrand near u=u', and by substituting the relationships of (3.12) along with

$$\rho_g(u,0) \simeq \rho_g(0,0) + \frac{u^2}{2}\ddot{\rho}_g(0,0) \qquad \text{near } u=0$$

The high frequency asymptotic form of the integral equation governing the surface field in the penumbra region is then

$$I(\xi,0)=2e^{-i\frac{\xi^3}{6}}\left[1+i\frac{\left\{1+4\rho_g(0,0)\ddot{\rho}_g(0,0)\right\}}{120\,M_o^2}\,\xi^5\right]$$

$$-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\xi} d\tau\, I(\tau,0)\, e^{-i\frac{(\xi-\tau)^3}{24}}\left[e^{-i\frac{\pi}{4}}\frac{(\xi-\tau)^{1/2}}{2}+\frac{K_2(\xi,\tau)}{M_o^2}\right]+0(M_o^{-3}) \qquad (3.15)$$

where

$$K_2(\xi,\tau)= e^{-i\frac{\pi}{4}}(\xi-\tau)^{1/2}\left[-\frac{(\xi-\tau)^2}{96}+i\frac{(\xi-\tau)^5}{20}\left\{\frac{1}{192}+\frac{\rho_g(0,0)\ddot{\rho}_g(0,0)}{8}\right\}+\right.$$

$$+\rho_g(0,0)\ddot{\rho}_g(0,0)\left\{\frac{\xi^2}{12}-\frac{\xi\tau}{3}-\frac{(\xi-\tau)^2}{8}+i\frac{\xi\tau(\xi-\tau)^3}{48}\right\}+\frac{\rho_g(0,0)}{8\rho_{tn}(0,0)}(\xi^2-\tau^2)$$

$$\left.+i\,\frac{\left\{\frac{3}{16}+\frac{\rho_g(0,0)}{2\rho_{tn}(0,0)}\right\}}{(\xi-\tau)}\right] \qquad (3.16)$$

Since there is no term of order M_o^{-1} in the above equation, we shall take the asymptotic expansion of I as

$$I(\xi,0)=I_o(\xi,0)+\frac{I_1(\xi,0)}{M_o^2}+0(M_o^{-3}) \qquad (3.17)$$

Substitution of this expression into Eq. (3.15) simplifies the integral equation and the following equations for I_o and I_1 are obtained:

$$I_o(\xi,0) = 2\,e^{-i\frac{\xi^3}{6}} - \frac{e^{-i\frac{\pi}{4}}}{4}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\xi} d\tau\, I_o(\tau,0)(\xi-\tau)^{1/2}\, e^{-i\frac{(\xi-\tau)^3}{24}} \tag{3.18}$$

and

$$I_1(\xi,0) = i\,\frac{\left\{1+4\rho_g(0,0)\ddot{\rho}_g(0,0)\right\}}{60}\,\xi^5\, e^{-i\frac{\xi^3}{6}} - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\xi} d\tau\, I_o(\tau,0)\,\kappa_2(\xi,\tau)\, e^{-i\frac{(\xi-\tau)^3}{24}}$$

$$- \frac{e^{-i\frac{\pi}{4}}}{4}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\xi} d\tau\, I_1(\tau,0)(\xi-\tau)^{1/2}\, e^{-i\frac{(\xi-\tau)^3}{24}} \quad . \tag{3.19}$$

We observe, from the above two equations, that the kernel functions are the same, and that substitution of the solution for the leading term I_o yields the solution for the second order term. Similarly, integral equations governing higher order terms in the high frequency expansion of the field can be derived by including further terms in the asymptotic expansion of Eqs. (3.7) and (3.15).

Since Eq. (3.18) is a Volterra type and its kernel is a function of $\xi-\tau$ only, the use of Fourier transform is suggested. We shall set

$$\tilde{I}_o(t) = \int_{-\infty}^{\infty} I_o(\xi,0)\, e^{-i\xi t}\, dt \quad . \tag{3.20}$$

Application of the Fourier transform to both sides of Eq. (3.18) and use of the convolution theorem yields

$$\tilde{I}_0(t) = 2\int_{-\infty}^{\infty} d\xi\, e^{-i\xi t - i\frac{\xi^3}{6}} \Big/ \left[1 + \sqrt{\frac{2}{\pi}}\frac{e^{-i\frac{\pi}{4}}}{4}\int_0^{\infty} d\xi\, \xi^{1/2} \cdot e^{-i\xi t - i\frac{\xi^3}{24}}\right]. \tag{3.21}$$

The numerator of the above equation is an Airy function (Miller, 1946)

$$\int_{-\infty}^{\infty} d\xi\, e^{-i\xi t - i\frac{\xi^3}{6}} = 2^{4/3}\, Ai(t2^{1/3}) \quad . \tag{3.22}$$

The denominator and other integrals for $\tilde{I}_1$ can be evaluated by means of the functions

$$F_n(p) = \int_0^{\infty} dx\, x^{n-\frac{1}{2}}\, e^{-i(12)^{1/3}px - ix^3} \tag{3.23}$$

for various n (Weston, 1960). In particular,

$$F_0 = \pi^{3/2}\, 2^{2/3}\, 3^{-1/6}\, e^{i\frac{\pi}{4}}\, Ai(p)\left[Ai(p) - iBi(p)\right] \tag{3.24}$$

$$F_1 = \pi^{3/2}\, 3^{-1/2}\, e^{i\frac{3}{4}\pi}\left[2Ai(p)\left\{\dot{A}i(p) - i\dot{B}i(p)\right\} + \frac{i}{\pi}\right] \tag{3.25}$$

$$F_2 = 2\pi^{3/2}\, 3^{-1/2}\, (12)^{-1/3}\, e^{i\frac{5\pi}{4}}\left[\left\{\dot{A}i(p)\right\}^2 + p\left\{Ai(p)\right\}^2 - i\left\{pAi(p)Bi(p) + \dot{A}i(p)\dot{B}i(p)\right\}\right] \quad . \tag{3.26}$$

The remaining F_n may be expressed in terms of the above three functions by the relation,

$$F_n(p) = C_n(12)^{-\frac{n}{3}} i^n F_o(p) + D_n(12)^{-\frac{(n-1)}{3}} i^{(n-1)} F_1(p) + E_n(12)^{-\frac{(n-2)}{3}} i^{(n-2)} F_2(p) \tag{3.27}$$

where C_n, D_n and E_n are given in Table I.

n	C_n	D_n	E_n
3	2	4p	0
4	0	6	4p
5	8p	$16p^2$	10
6	28	80p	$16p^2$
7	$32p^2$	$108+64p^3$	112p
8	288p	$672p^2$	$220+64p^3$
9	$2912+512p^3$	$10048p+1024p^4$	$3456p^2$

TABLE I: THE FUNCTION F_n

Using the F_1 function, Eq. (3.21) reduces to

$$\tilde{I}_o(t) = \frac{2^{4/3}\sqrt{\pi}}{\dot{w}_1(t2^{1/3})} \quad , \tag{3.28}$$

where

$$w_1 = i\sqrt{\pi}\left[Ai - iBi\right] \quad .$$

The inverse Fourier transform of Eq. (3.28) yields the solution for the leading term:

$$I_o(\xi, v=0) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dp \frac{e^{ip\xi 2^{-1/3}}}{\dot{w}_1(p)} \qquad (p = t\,2^{1/3}) \tag{3.29}$$

Similarly, the application of the Fourier transform to the integral equation governing the second order term (3.19) gives

$$\tilde{I}_1(t) = \frac{\tilde{N}}{2\sqrt{\pi}\, Ai(p)\, \dot{w}_1(p)} \qquad (p = t2^{1/3}) \tag{3.30}$$

where

$$\tilde{N} = \int_{-\infty}^{\infty} d\xi\, e^{-it\xi} \left[i \frac{\left\{1 + 4\rho_g(0,0)\rho''_g(0,0)\right\}}{60} \xi^5 e^{-i\frac{\xi^3}{6}} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\xi} d\tau\, I_o(\tau, 0) \kappa_2(\xi, \tau) e^{-i\frac{(\xi-\tau)^3}{24}} \right] \tag{3.31}$$

Using the convolution theorem, the functions F_n of Table I and Eq. (3.28), one can show that:

$$\frac{\tilde{N}}{4\pi Ai(p)} = p\left[-\frac{2}{15} + \rho_g(0,0)\rho''_g(0,0)\frac{17}{15} + \frac{\rho_g(0,0)}{\rho_{tn}(0,0)}\right] + \frac{w_1(p)}{\dot{w}_1(p)}\left[-\frac{1}{5} + \frac{\beta^3}{30} + \left(\frac{8}{15} - \frac{6}{5}\beta^3\right)\rho_g(0,0)\rho''_g(0,0)\right]$$

$$-\left[\frac{p w_1(p)}{\dot{w}_1(p)}\right]^2 \left[\frac{7}{3}\rho_g(0,0)\rho''_g(0,0) + \frac{\rho_g(0,0)}{\rho_{tn}(0,0)}\right] + \rho_g(0,0)\rho''_g(0,0)\frac{1}{3}\left[\frac{w_1(p)}{\dot{w}_1(p)}\right]^3 p^4 \tag{3.32}$$

Substitution of this expression into Eq. (3.30) and inverse Fourier transformation gives the solution for I_1. Combining I_1 with Eqs. (3.3) and (3.29), we obtain the desired expression for the surface field in the penumbra region:

$$U(u,0) \simeq e^{iku} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dp \, \frac{e^{ip\frac{u}{d}}}{\dot{w}_1(p)} + \frac{e^{iku}}{\left[\frac{k\rho_g(0,0)}{2}\right]^{2/3} \sqrt{\pi}} \int_{-\infty}^{\infty} dp \, e^{ip\frac{u}{d}} \left[\frac{p}{\dot{w}_1(p)} \left\{ -\frac{2}{15} + \right.\right.$$

$$\left. + \rho_g(0,0) \frac{d^2\rho_g}{du^2}(0,0) \frac{17}{15} + \frac{\rho_g(0,0)}{\rho_{tn}(0,0)} \right\} + \frac{w_1(p)}{\left\{\dot{w}_1(p)\right\}^2} \left\{ -\frac{1}{5} \right.$$

$$\left. + \frac{p^3}{30} + \rho_g(0,0) \frac{d^2\rho_g}{du^2}(0,0) \left(\frac{8}{15} - \frac{6}{5}p^3\right) \right\}$$

$$- \left\{ \frac{p w_1(p)}{\dot{w}_1(p)} \right\}^2 \left\{ \frac{7}{3} \rho_g(0,0) \frac{d^2\rho_g}{du^2}(0,0) + \frac{\rho_g(0,0)}{\rho_{tn}(0,0)} \right\}$$

$$\left. + \frac{4 \, p \left\{ p w_1(p) \right\}^3 \rho_g(0,0) \frac{d^2\rho_g}{du^2}(0,0)}{3 \left\{ \dot{w}_1(p) \right\}^4} \right] \tag{3.33}$$

where

$$d = \left[\frac{\lambda \rho_g^2(0,0)}{\pi} \right]^{1/3}$$ and $\lambda =$ the incident wavelength.

When $\frac{u}{d}$ is positive and sufficiently large (far away from the shadow boundary into the shadow region) Eq. (3.33) can be expressed as a rapidly convergent series in terms of the residues at the poles $\dot{w}_1(p)=0$. This residue series represents the creeping waves. When $\frac{u}{d}$ goes to negative infinity (illuminated region), Eq. (3.33) reduces to $2e^{ik\bar{t}(0,0)\cdot\bar{r}(u,0)}$ (Logan, 1959) which is the geometrical optics term. The width of the penumbra region is of order d.

3.3 The Surface Field in the Shadow Region.

The incident plane wave cannot reach the shadow region directly (otherwise the shadow does not exist), and only the waves diffracted near the shadow boundary proceed into the shadow region. An expression for the surface field in this region may be obtained by following two steps (Hönl et al, 1961):

1) obtain the initial values of the diffracted (creeping) waves from the solution for the penumbra region at the shadow boundary; and
2) solve the homogeneous integral equation (without the plane wave term in Eq. 3.7 and the limit of the integration only over the surface in the shadow region) and match the initial values at the shadow boundary.

The initial values of the diffracted waves may be obtained from Eq. (3.33). When u is positive, the integrals of this equation can be expressed in terms of the residues at the poles $\dot{w}_1(p)=0$, and each residue represents a creeping wave (Goodrich, 1959). The values of these residues at the shadow boundary (u=0) yield the necessary initial values of the creeping waves. The residue series of Eq. (3.33) at the shadow boundary is

$$U(0,0) \sim 2\sqrt{\pi}\, i \sum_{\ell=1}^{\infty} \frac{1}{p_\ell w_1(p_\ell)} \left[1 + \frac{1}{2\left[\frac{k\rho_g(0,0)}{2}\right]^{1/3}} \left\{ p_\ell \left(-\frac{1}{15} + \frac{\rho_g(0,0)}{2\rho_{tn}(0,0)} - \frac{1}{90}\rho_g(0,0)\frac{d^2\rho_g}{du^2}(0,0) \right) + \frac{1}{p_\ell^2} \left(\frac{1}{5} + \frac{\rho_g(0,0)}{2\rho_{tn}(0,0)} - \frac{1}{30}\rho_g(0,0)\frac{d^2\rho_g}{du^2}(0,0) \right) \right\} \right] \qquad (3.34)$$

where p_ℓ is the ℓ^{th} root of $\dot{w}_1(p_\ell)=0$. In the above expression, each term in the series represents the initial value (birth weight) of the ℓ^{th} mode of the creeping waves.

The short-wavelength asymptotic form of the homogeneous integral equation governing the surface field in the shadow region is the same as Eq. (3.7), except for the incident field term which vanishes now. Before attempting to solve the integral equation, we can observe that the common factor

$$\sqrt{G(u')}\left[\frac{\partial^2 R}{\partial v'^2}\right]^{-1/2}$$

in the kernel of Eq. (3.7) behaves near the saddle point u=u' as follows

$$\sqrt{G(u')}\left[R_o\frac{\partial^2 R}{\partial v'^2}(v'=0)\right]^{-1/2} = \sqrt{G(u')}\left[1-\bar{R}_o\cdot\left\{\kappa_{t\kappa}(u')\bar{t}(u')\right.\right.$$

$$\left.\left.+\kappa_{tn}(u')\bar{n}(u')\right\}\right]^{-1/2} \simeq \left[\frac{G(u')}{G(u)}\right]^{1/4} \qquad (3.35)$$

(Refer to Eqs. (2.2), (2.4), (2.5) and (3.10).) The above relation indicates that the solution of the homogeneous integral equation has a factor $\left[G(u)\right]^{-1/4}$. In view of the phase factor for the solution in the penumbra region, Eq. (3.33), we shall set the solution of the homogeneous integral equation in the form

$$I(u,0) = A\left[G(u)\right]^{-1/4} \exp\left[ik\int_0^u ds\frac{2^{-1/3}}{M^2(s)}\left\{\gamma_o(s)+\frac{\gamma_1(s)}{M(s)}+\frac{\gamma_2(s)}{M^2(s)}+\ldots\right\}\right], \qquad (3.36)$$

where the constant A is the initial value, and

$$M(u) = \left[k\rho_g(u,0)\right]^{1/3} \quad .$$

The propagation factors γ_o, γ_1 and γ_2 are yet to be determined. On setting

$$k(u-u') = M^2(u)\tau \qquad (3.37)$$

we obtain the following expression under the assumption that the curvatures are slowly varying:

$$k\int_{u'}^{u} \frac{ds}{M^2(s)} \left\{\gamma_o(s) + \frac{\gamma_1(s)}{M(s)} + \frac{\gamma_2(s)}{M^2(s)} + \ldots\right\}$$

$$\simeq \gamma_o(u)\tau + \frac{1}{M(u)}\left[\gamma_1(u)\tau - \frac{\tau^2}{2}\left\{\rho_g(u,0)\dot{\rho}_o(u) - \frac{2}{3}\dot{\rho}_g(u,0)\gamma_o(u)\right\}\right]$$

$$+ \frac{1}{M^2(u)}\left[\gamma_2(u)\tau + \frac{\tau^3}{6}\left\{\rho_g^2(u,0)\ddot{\gamma}_o(u) - \frac{2}{3}\rho_g(u,0)\ddot{\rho}_g(u,0)\gamma_o(u)\right.\right.$$

$$\left. - \frac{4}{3}\rho_g(u,0)\dot{\rho}_g(u,0)\dot{\gamma}_o(u) + \frac{10}{9}\dot{\rho}_g^2(u,0)\gamma_o(u)\right\}$$

$$\left. - \frac{\tau^2}{2}\left\{\rho_g(u,0)\dot{\gamma}_1(u) - \dot{\rho}_g(u,0)\gamma_1(u)\right\}\right] + 0(M^{-3}) \quad . \qquad (3.38)$$

Now combine Eq. (3.7), without the incident plane wave term, with Eq. (3.36) and expand the integrand near the saddle point u=u' by Taylor series (using Eq. 2.2).

After these algebraic manipulations and making use of Eq. (3.37) and (3.38), we can obtain the following asymptotic homogeneous equation; for the propagation factors γ_i:

$$1 = -\frac{e^{-i\frac{\pi}{4}}}{2\sqrt{2\pi}} \int_0^\infty d\tau\ \tau^{1/2} \left[1 - \frac{\dot{\rho}_g(u,0)}{M(u)} \left\{-\frac{2}{3}\tau\right.\right.$$

$$\left. + i\frac{\tau^3}{24}\right\} + \frac{\tau^2}{M^2(u)} \left\{-\frac{1}{48} + \frac{\dot{\rho}_g^2(u,0)}{2} - \frac{\rho_g(u,0)\ddot{\rho}_g(u,0)}{4}\right\}$$

$$+ i\frac{\tau^5}{M^2(u)} \left\{\frac{1}{1920} - \frac{23}{360}\dot{\rho}_g^2(u,0) + \frac{\rho_g(u,0)\ddot{\rho}_g(u,0)}{80}\right\} - \frac{\tau^8}{1152\, M^2(u)}\dot{\rho}_g^2(u,0)$$

$$\left. + i\frac{\left\{\frac{3}{8} + \frac{\rho_g(u,0)}{\rho_{tn}(u,0)}\right\}}{M^2(u)\tau}\right]\left[1 + \frac{i2^{-1/3}}{M^2(u)}\left\{\gamma_2(u)\tau + \frac{\tau^3}{6}\right.\right.$$

$$\left(\rho_g^2(u,0)\ddot{\gamma}_o(u) - \frac{2}{3}\rho_g(u,0)\ddot{\rho}_g(u,0)\gamma_o(u) - \frac{4}{3}\rho_g(u,0)\dot{\rho}_g(u,0)\dot{\gamma}_o(u) + \frac{10}{9}\dot{\rho}_g^2(u,0)\gamma_o(u)\right)$$

$$\left.\left. - \frac{\tau^2}{2}\left(\rho_g(u,0)\dot{\gamma}_1(u) - \dot{\rho}_g(u,0)\gamma_1(u)\right)\right\}\right]$$

$$\exp.\left[-i\frac{\tau^3}{24} + i\gamma_o(u)\,2^{-1/3}\tau + \frac{i\,2^{-1/3}}{M(u)}\left\{\gamma_1(u)\tau - \frac{\tau^2}{2}\left(\rho_g(u,0)\dot{\gamma}_o(u,0)\right.\right.\right.$$

$$\left.\left.\left. + 0(M^{-3}) - \frac{2}{3}\dot{\rho}_g(u,0)\gamma_o(u)\right)\right\}\right] \qquad (3.39)$$

Upon comparing coefficients of the leading term in the expansion in powers of $\frac{1}{M}$, we obtain

$$1 = - \frac{e^{-i\frac{\pi}{4}}}{2\sqrt{2\pi}} \int_0^{\infty} d\tau \; \tau^{1/2} \; \exp\left\{\left[-i\frac{\tau^3}{24} - i\,\gamma_o(u)\,\tau\, 2^{-1/3}\right]\right\} \tag{3.40}$$

Comparing with Eq.(3.23), it can be shown that the right-hand side is related to the function F_1 of Eq. (3.25). Substitution of Eq. (3.23) and (3.25) into Eq. (3.40) yields the following:

$$Ai(\gamma_o)\, \dot{w}_1(\gamma_o) = 0 \tag{3.41}$$

The solution of this equation determines γ_o. To be consistent with the initial values (Eq. 3.34) of the creeping waves, the roots of $\dot{w}_1(\gamma_o)=0$ must be chosen. In terms of the definition

$$\gamma_{o\ell} = e^{i\frac{\pi}{3}} \beta_\ell \; ,$$

the various roots are given by Table II at the end of this section.

Since γ_o is constant, comparison of the coefficients of $\frac{1}{M}$ in Eq.(3.39) yields

$$0 = -\frac{e^{-i\frac{\pi}{4}}}{2\sqrt{2\pi}}\int_0^\infty d\tau\ \tau^{1/2}\left[\frac{2}{3}\dot{\rho}_g(u,0)\tau - i\frac{\dot{\rho}_g(u,0)}{24}\tau^4\right.$$

$$\left.-i\tau\left\{\gamma_1(u)+\frac{\gamma_0}{3}\dot{\rho}_g(u,0)\tau\right\}2^{-1/3}\right]\exp.\left\{-i\gamma_0 2^{-1/3}\tau - i\frac{\tau^3}{24}\right\} \quad (3.42)$$

Various integrals in Eq. (3.42) can be identified with Eq. (3.23). Thus, after substitution of F_n from Table I, we can evaluate γ_1 from Eq. (3.42) and the result is

$$\gamma_1(u) = i\,2^{1/3}\frac{\dot{\rho}_g(u,0)}{6} \quad . \quad (3.43)$$

From the coefficients of M^{-2} in Eq. (3.39), one finds:

$$\gamma_2(u)2^{-2/3}F_2(\gamma_0) = (12)^{-2/3}F_o(\gamma_o)\left[\frac{1}{5}+\frac{\rho_g(u,0)}{2\rho_{tn}(u,0)} - \frac{\rho_g(u,0)\ddot{\rho}_g(u,0)}{30}\right.$$

$$\left.+\frac{\dot{\rho}_g^2(u,0)}{45}\right] + \gamma_o^2\,F_2(\gamma_o)\left[\frac{1}{60}-\frac{2}{45}\rho_g(u,0)\ddot{\rho}_g(u,0)+\frac{4}{135}\dot{\rho}_g^2(u,0)\right] \quad (3.44)$$

and upon substituting the values of F_o and F_2 given by Eqs. (3.24) and (3.26), we obtain γ_2, namely

$$\gamma_2(u)2^{-2/3} = -\frac{1}{\gamma_0}\left[\frac{1}{10}+\frac{\rho_g(u,0)}{4\rho_{tn}(u,0)}-\frac{\rho_g(u,0)\ddot{\rho}_g(u,0)}{60}+\frac{\dot{\rho}_g^2(u,0)}{90}\right]$$

$$+\gamma_0^2\left[\frac{1}{60}-\frac{2}{45}\rho_g(u,0)\ddot{\rho}_g(u,0)+\frac{4}{135}\dot{\rho}_g^2(u,0)\right]. \qquad (3.45)$$

Combining Eqs. (3.36), (3.43) and (3.45) and matching the initial values given by Eq. (3.34) by letting $u=0$, we obtain the desired solution for the surface field in the shadow region:

$$U(u,0) = \left[\frac{G(0)}{G(u)}\right]^{1/4}\left[\frac{\rho_g(0,0)}{\rho_g(u,0)}\right]^{1/6} e^{iku}\sum_{\ell=1}^{\infty}\frac{1}{\beta_\ell Ai(-\beta_\ell)}$$

$$\left[1+\frac{e^{i\frac{\pi}{3}}}{2^{-2/3}M^2(0)}\left\{\beta_\ell\left(-\frac{1}{30}+\frac{\rho_g(0,0)}{4\rho_{tn}(0,0)}+\frac{1}{180}\rho_g(0,0)\frac{d^2\rho_g}{du^2}(0,0)\right)\right.\right.$$

$$\left.\left.-\frac{1}{\beta_\ell^2}\left(\frac{1}{10}+\frac{\rho_g(0,0)}{4\rho_{tn}(0,0)}-\frac{1}{60}\rho_g(0,0)\frac{d^2\rho_g}{du^2}(0,0)\right)\right\}\right]$$

$$\exp.\left[-e^{-i\frac{\pi}{6}}\beta_\ell\int_0^u\frac{ds}{\rho_g(s,0)}M(s)2^{-1/3}-e^{i\frac{\pi}{6}}\int_0^u\frac{ds}{\rho_g(s,0)}\frac{2^{1/3}}{M(s)}\right.$$

$$\left\{\left[\frac{1}{\beta_\ell}\left(\frac{1}{10}+\frac{\rho_g(s,0)}{4\rho_{tn}(s,0)}-\frac{\rho_g(s,0)\ddot{\rho}_g(s,0)}{60}+\frac{\dot{\rho}_g^2(s,0)}{90}\right)\right.\right.$$

$$\left.\left.+\beta_\ell^2\left(\frac{1}{60}-\frac{2}{45}\rho_g(s,0)\ddot{\rho}_g(s,0)+\frac{4}{135}\dot{\rho}_g^2(s,0)\right)\right\}\right] \qquad (3.46)$$

where $M(u)=\left[k\rho_g(u,0)\right]^{1/3}$ and various values of β_ℓ and $A_i(-\beta_\ell)$ are given in Table II.

ℓ	β_ℓ	$Ai(-\beta_\ell)$
1	1.01879	+0.53566
2	3.24820	-0.41902
3	4.82010	+0.38041

TABLE II: THE VALUES OF β_ℓ and $Ai(-\beta_\ell)$

In deriving the Eq. (3.46) the following relationships are used:

$$\gamma_{o\ell}=e^{i\frac{\pi}{3}}\beta_\ell \quad \text{and} \quad w_1(p)=e^{i\frac{\pi}{6}}\,2\sqrt{\pi}\,Ai(p\,e^{i\frac{2\pi}{3}}) \quad . \qquad (3.47)$$

IV

DIFFRACTION OF A PLANE ELECTROMAGNETIC (VECTOR) WAVE

The second problem to be investigated is the diffraction of a plane electromagnetic wave by a perfectly conducting smooth convex surface of nonconstant curvature. Since much of the analysis is similar to that which we have already discussed for the acoustic case, the details will be omitted wherever possible.

4.1 Integral Equation Governing the Induced Currents on the Conducting Surface.

If a plane electromagnetic wave is incident upon a smooth convex conducting surface, the integral equation governing the induced currents on the conductor is (Hönl et al, 1961)

$$\bar{J}(\bar{r}) = 2\bar{n}(\bar{r}) \times \bar{H}^{inc.}(\bar{r}) - \frac{1}{2\pi}\bar{n}(\bar{r}) \times \iint da' \frac{1-ikR}{R^3}\left\{\bar{J}(\bar{r}') \times \bar{R} \quad e^{ikR}\right\} . \tag{4.1}$$

Here $\bar{H}^{inc.}$ is the incident field and the time dependence factor $e^{-i\omega t}$ is omitted. Again, without loss of generality, we shall consider the induced current along the geodesic $v=0$. The expression for the incident field is

$$\bar{H}^{inc.}(u,0) = \left[-\cos\theta_o \bar{b}(0,0) + \sin\theta_o \bar{n}(0,0)\right] e^{ik\bar{t}(0,0)\cdot\bar{r}(u,0)} \tag{4.2}$$

where $\theta_o = \sin^{-1}\left\{\frac{\bar{n}(0,0) \times \bar{H}_o}{|\bar{H}_o|}\right\}$ is the polarization angle of the incident wave.

With the substitution

$$\bar{J}(\bar{r}) = \left[\bar{t}(\bar{r})\, I_t(\bar{r}) + \bar{b}(\bar{r})\, I_b(\bar{r})\right] e^{iku} \tag{4.3}$$

the vector integral Eq. (4.1) is reduced to two coupled scalar equations:

$$I_t(u,0) = 2\,I_t^{inc.}(u,0) - \frac{1}{2\pi}\iint du'dv' \sqrt{G(u')}\,\frac{1-ikR}{R^3}$$

$$\bar{t}(u)\cdot\Big[\bar{n}(u)\,x\,\Big\{I_t(u',v')\,\bar{t}(u',v')\,x\bar{R} + I_b(u',v')\,\bar{b}(u',v')\,x\bar{R}\Big\}\Big]$$

$$e^{ikR - ik(u-u')} \tag{4.4}$$

and

$$I_b(u,0) = 2\,I_b^{inc.}(u,0) - \frac{1}{2\pi}\iint du'dv' \sqrt{G(u')}\,\frac{1-ikR}{R^3}$$

$$\bar{b}(u)\cdot\Big[\bar{n}(u)\,x\Big\{I_t(u',v')\,\bar{t}(u',v')\,x\bar{R} + I_b(u',v')\,\bar{b}(u',v')\,x\bar{R}\Big\}\Big]$$

$$e^{ikR - ik(u-u')} \quad . \tag{4.5}$$

The above two equations are similar to that of acoustic case, Eq. (3.4). They also have saddle points at $v'=0$ for the v' integration and at $u=u'$ for the u' integration along the $v'=0$ curve. Performing the v' integration by the method of steepest descents, we obtain

$$I_t(u,0) = 2\, I_t^{\text{inc.}}(u,0) - \frac{1}{2\pi}\int_{-\infty}^{u} du' \sqrt{G(u')}\sqrt{\frac{2\pi i}{k}}\frac{1}{\left[\frac{\partial^2 R}{\partial v'^2}(v'=0)\right]^{1/2}}$$

$$\Bigg[\frac{-ik\, I_t(u',0)}{R_o^2}\,\bar{t}(u)\cdot\left\{\bar{n}(u)\times\left(\bar{t}(u')\times\bar{R}_o\right)\right\}$$

$$-\frac{\frac{\partial^4 R}{\partial v'^4}(v'=0)}{8R_o^2\left\{\left[\frac{\partial^2 R}{\partial v'^2}(v'=0)\right]\right\}^2}\,\bar{t}(u)\cdot\left\{\bar{n}(u)\times\left(\bar{t}(u')\times\bar{R}_o\right)\right\} I_t(u',0)$$

$$-\frac{I_t(u',0)}{2R_o^2}\,\frac{G(u')}{\left\{\left[\frac{\partial^2 R}{\partial v'^2}(v'=0)\right]\right\}}\,\bar{t}(u)\cdot\left\{\left(\bar{n}(u)\cdot\bar{R}_o\,\kappa_{tt}(u') - \bar{n}(u)\cdot\bar{t}(u')\right)\right.$$

$$\left.\kappa_t(u')\bar{N}(u') + \left(\bar{t}(u') - \kappa_{tt}(u')\bar{R}_o\right)\kappa_t(u')\,\bar{n}(u)\cdot\bar{N}(u')\right\}$$

$$+\frac{\frac{\partial^2 I_t}{\partial v'^2}(u',0)}{2R_o^2\left\{\left[\frac{\partial^2 R}{\partial v'^2}(v'=0)\right]\right\}}\,\bar{t}(u)\cdot\left\{\bar{n}(u)\times\left(\bar{t}(u')\times\bar{R}_o\right)\right\}$$

$$+\frac{\frac{\partial I_b}{\partial v'}(u',0)}{\left\{\left[\frac{\partial^2 R}{\partial v'^2}(v'=0)\right]\right\}R_o^2}\,G(u')\,\bar{t}(u)\cdot\left\{\bar{R}_o\left(\bar{n}(u)\cdot\bar{N}(u')\right)\kappa_t(u')\right.$$

$$\left.\left.-\kappa_t(u')\bar{N}(u')\left(\bar{n}(u)\cdot\bar{R}_o\right)\right\}\right] + 0(k^{-3/2}) \qquad (4.6)$$

and

$$I_b(u,0) = 2\, I_b^{inc.}(u, v=0) - \frac{1}{2\pi}\int_{-\infty}^{u} du' \sqrt{G(u')}\sqrt{\frac{2\pi i}{k}}\frac{1}{\left[\frac{\partial^2 R}{\partial v'^2}(v'=0)\right]^{1/2}}$$

$$\left[-ik\frac{\left\{\bar{n}(u)\cdot\bar{R}_o\right\}\left\{\bar{b}(u)\cdot\bar{b}(u')\right\}}{R_o^2} - \frac{\left\{\frac{\partial^4 R}{\partial v'^4}(v'=0)\right\}\left\{\bar{n}(u)\cdot\bar{R}_o\right\}\left\{\bar{b}(u)\cdot\bar{b}(u')\right\}}{8R_o^2\left\{\frac{\partial^2 R}{\partial v'^2}(v'=0)\right\}^2}\right.$$

$$I_b(u',0) - \frac{G(u')\, I_b(u',0)}{\left\{\frac{\partial^2 R}{\partial v'^2}(v'=0)\right\}2R_o^2}\left\{\bar{b}(u)\cdot\bar{b}(u')\right\}\left\{\kappa_t^2(u')\bar{n}(u)\cdot\bar{R}_o\right.$$

$$\left. - \kappa_t(u')\bar{n}(u)\cdot\bar{N}(u')\right\} + \frac{\frac{\partial^2 I_b}{\partial v'^2}(u',0)}{2R_o^2\left\{\frac{\partial^2 R}{\partial v'^2}(v'=0)\right\}}\left\{\bar{b}(u)\cdot\bar{b}(u')\right\}\left\{\bar{n}(u)\cdot\bar{R}_o\right\}$$

$$\left. + \frac{\partial I_t}{\partial v'}(v'=0, u')\frac{\left\{\bar{b}(u)\cdot\bar{b}(u')\right\}}{R_o^2\left\{\frac{\partial^2 R}{\partial v'^2}(v'=0)\right\}}\left\{\kappa_{tt}(u')\bar{n}(u)\cdot\bar{R}_o - \bar{n}(u)\cdot\bar{t}(u')\right\}\right]$$

$$+ 0\,(k^{-3/2}) \qquad (4.7)$$

where $\kappa_t(u)\bar{N}(u) \equiv \kappa_{tt}(u)\bar{t}(u) + \kappa_{tn}(u)\bar{n}(u)$.

4.2 The Induced Currents in the Penumbra and the Shadow Regions.

In this section, Eqs. (4.6) and (4.7) are solved by the same technique used in the acoustic case. In the penumbra region, substitution of Eq. (3.12) into Eq. (4.2) gives the asymptotic form of the incident field:

$$I_t^{inc.}(u,v=0) = \bar{t}(u) \cdot \left[\bar{n}(u) \times \bar{r}'_{inc.}\right] e^{-iku}$$

$$= \cos\theta_o \, e^{-i\frac{\xi^3}{6}} \left[1+i \frac{\left\{1+4\rho_g(0)\ddot{\rho}_g(0)\right\}}{120 M_o^2} \xi^5\right] + 0(M_o^{-3}) \tag{4.8}$$

and

$$I_b^{inc.}(u,v=0) = \bar{b}(u) \cdot \left[\bar{n}(u) \times \bar{H}_{inc.}\right] e^{-iku}$$

$$= -\sin\theta_o \frac{\xi}{M_o} e^{-i\frac{\xi^3}{6}} + 0(M_o^{-3}) \quad . \tag{4.9}$$

Combining Eqs. (3.12), (4.6) and (4.8), we have:

$$I_t(\xi,0) = 2\cos\theta_o \, e^{-i\frac{\xi^3}{6}} \left[1+i \frac{\left\{1+4\rho_g(0)\ddot{\rho}_g(0)\right\}}{120 M_o^2} \xi^5\right]$$

$$- \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\xi} d\tau \, I_t(\xi,0) e^{-i\frac{(\xi-\tau)^3}{24}} \left\{e^{-i\frac{\pi}{4}} \frac{(\xi-\tau)^2}{2} + \frac{K_{2t}(\xi,\tau)}{M_o^2}\right\} + 0(M_o^{-3}) \tag{4.10}$$

where

$$K_{2t}(\xi,\tau) = e^{-i\frac{\pi}{4}}(\xi-\tau)^{1/2}\left[-\frac{(\xi-\tau)^2}{96} + i\frac{(\xi-\tau)^5}{20}\left\{\frac{1}{192} + \frac{\rho_g(u=0)\ddot{\rho}_g(0)}{8}\right\}\right.$$

$$+\rho_g(0)\ddot{\rho}_g(0)\left\{\frac{\xi^2}{12} - \frac{\xi\tau}{3} - \frac{(\xi-\tau)^2}{8} + \frac{i}{48}(\xi-\tau)^3\,\xi\tau\right\}$$

$$\left. + \frac{1}{8}\frac{\rho_g(0)}{\rho_{tn}(0)}(\xi^2-\tau^2) + i\frac{\left(\frac{3}{16} - \frac{\rho_g(0)}{2\rho_{tn}(0)}\right)}{(\xi-\tau)}\right] \tag{4.11}$$

Now the above asymptotic form of the integral equation governing I_t is independent of I_b, and thus the original coupled vector integral Eq. (4.1) is decoupled in the asymptotic sense for large k.

Upon comparing Eq. (4.10) with the acoustic Eq. (3.15), we can easily observe that the only difference between the two equations is the sign of the term

$\dfrac{\rho_g(0,0)}{2\rho_{tn}(0,0)(\xi-\tau)}$ in K_{2t} and K_2. Thus, we can immediately obtain the solution for I_t from the acoustic solution given by Eq. (3.33).

The asymptotic form of the integral equation governing I_b in the penumbra region is

$$I_b(\xi,0) = -2\sin\theta_o\,\frac{\xi}{M_o}\,e^{-i\frac{\xi^3}{6}} + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\xi} d\tau\, I_b(\tau,0)\,e^{-i\frac{(\xi-\tau)^3}{24}}$$

$$\left[\frac{e^{-i\frac{\pi}{4}}}{2}(\xi-\tau)^{1/2} + 0(M_o^{-2})\right] + 0(M_o^{-3}) \ . \tag{4.12}$$

On setting

$$I_b(\xi, 0) = \frac{I_{0b}(\xi, 0)}{M_o} + 0(M_o^{-3}) \quad , \qquad (4.13)$$

application of the Fourier transform and the F_n function (Eq. 3.23) yields:

$$\tilde{I}_{bo} = -i \sin\theta_o 4\, 2^{2/3} \pi \dot{A}i\,(p) + \tilde{I}_{bo} \left[-1 + i 2\pi Ai(p) \left\{ \dot{A}i(p) - i\dot{B}i(p) \right\} \right] . \qquad (4.14)$$

Substituting the Wronskian relation

$$Ai\dot{B}i - Bi\dot{A}i = \frac{1}{\pi}$$

into Eq. (4.14), the inverse Fourier transform gives the solution for I_{bo} :

$$I_b(\xi, 0) = \frac{i \sin\theta_o}{2^{-1/3} M_o \sqrt{\pi}} \int_{-\infty}^{\infty} dp \frac{e^{i\xi 2^{-1/3} p}}{w_1(p)} \quad . \qquad (4.15)$$

Thus, combining solutions for I_t and I_b, we obtain the expression for the induced currents in the penumbra region as

$$\bar{J}(u,0) = \bar{t}(u,0)\cos\theta_o\, e^{iku} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dp\, \frac{e^{ip\frac{u}{d}}}{\dot{w}_1(p)}$$

$$+ \bar{t}(u,0)\cos\theta_o\, e^{iku} \frac{1}{\sqrt{\pi}\, 2^{-2/3} M_o^2} \int_{-\infty}^{\infty} dp\; e^{ip\frac{u}{d}} \Bigg[\frac{p}{\dot{w}_1(p)}$$

$$\left\{ -\frac{2}{15} + \rho_g(0,0) \frac{d^2\rho_g}{du^2}(0,0) \frac{17}{15} + \frac{\rho_g(0,0)}{\rho_{tn}(0,0)} \right\}$$

$$+ \frac{w_1(p)}{\left\{\dot{w}_1(p)\right\}^2} \left\{ -\frac{1}{5} + \frac{p^3}{30} \frac{\rho_g(0,0)}{\rho_{tn}(0,0)} + \rho_g(0,0) \frac{d^2\rho_g}{du^2}(0,0) \left(\frac{8}{15} - \frac{6}{5} p^3 \right) \right\}$$

$$- \frac{\left\{p w_1(p)\right\}^2}{\left\{\dot{w}_1(p)\right\}^3} \left\{ \frac{7}{3}\, \rho_g(0,0) \frac{d^2\rho_g}{du^2}(0,0) + \frac{\rho_g(0,0)}{\rho_{tn}(0,0)} \right\}$$

$$+ \frac{4p \left\{p w_1(p)\right\}^3 \rho_g(0,0) \frac{d^2\rho_g}{du^2}(0,0)}{3 \left\{\dot{w}_1(p)\right\}^4} \Bigg]$$

$$+ \bar{b}(u,0) \frac{i \sin\theta_o}{2^{-1/3} M_o \sqrt{\pi}} \int_{\pm\infty}^{\infty} dp\, \frac{e^{ip\frac{u}{d}}}{w_1(p)} + O(M_o^{-3}) \qquad (4.16)$$

where

$$d = \left[\frac{\lambda \rho_g^2(0,0)}{\pi} \right]^{1/3} .$$

Due to the similarity of asymptotic forms of electromagnetic and acoustic integral equations, solutions for the shadow region can be obtained by the same method used in the acoustic case. The induced current in the shadow region is:

$$\bar{J}(u,0) = \bar{t}(u,0)\cos\theta_o \left[\frac{G(0)}{G(u)}\right]^{1/4}\left[\frac{\rho_g(0,0)}{\rho_g(u,0)}\right]^{1/6} e^{iku}\sum_{\ell=1}^{\infty}\frac{1}{\beta_\ell \mathrm{Ai}(-\beta_\ell)}\left[1+\frac{e^{i\frac{\pi}{3}}}{2^{-2/3}M^2(0)}\right.$$

$$\left\{\beta_\ell\left(-\frac{1}{30}+\frac{\rho_g(0,0)}{4\rho_{tn}(0,0)}+\frac{1}{180}\rho_g(0,0)\frac{d^2\rho_g}{du^2}(0,0)\right)-\frac{1}{\beta_\ell^2}\left(\frac{1}{10}-\frac{\rho_g(0,0)}{4\rho_{tn}(0,0)}-\frac{1}{60}\rho_g(0,0)\frac{d^2\rho_g}{du^2}(0,0)\right)\right\}\Bigg]$$

$$\exp.\left[-e^{-i\frac{\pi}{6}}\beta_\ell\int_0^u\frac{ds}{\rho_g(s,0)}2^{-1/3}M(s)-e^{i\frac{\pi}{6}}\int_0^u\frac{ds}{\rho_g(s,0)2^{-1/3}M(s,0)}\right.$$

$$\left\{\frac{1}{\beta_\ell}\left(\frac{1}{10}-\frac{\rho_g(s,0)}{4\rho_{tn}(s,0)}-\frac{\rho_g(s,0)\frac{d^2\rho_g}{ds^2}(s,0)}{60}+\frac{\left(\frac{d\rho_g}{ds}(s,0)\right)^2}{90}\right)\right.$$

$$\left.\left.+\beta_\ell^2\left(\frac{1}{60}-\frac{2}{45}\rho_g(s,0)\frac{d^2\rho_g}{ds^2}(s,0)+\frac{4}{135}\left(\frac{d\rho_g}{ds}(s,0)\right)^2\right)\right\}\right]$$

$$+\bar{b}(u,0)\sin\theta_o\left[\frac{G(0)}{G(u)}\right]^{1/4}\left[\frac{\rho_g(0,0)}{\rho_g(u,0)}\right]^{1/6}\frac{e^{iku+i\frac{\pi}{6}}}{2^{-1/3}M(0)}\sum_{\ell=1}^{\infty}\frac{1}{\mathrm{Ai}(-\alpha_\ell)}$$

$$\exp.\left[-e^{-i\frac{\pi}{6}}\alpha_\ell\int_0^u\frac{ds}{\rho_g(s,0)}2^{-1/3}M(s)\right]+0(M^{-3}) \quad . \qquad (4.17)$$

Here β_ℓ and $Ai(-\beta_\ell)$ are given in Table 2, and α_ℓ is the ℓ^{th} root of $Ai(-\alpha_\ell)=0$:

ℓ	α_ℓ	$\dot{A}i(-\alpha_\ell)$
1	2.33811	+0.70121
2	4.08795	-0.80311
3	5.52056	+0.86520

TABLE 3: THE VALUES OF α_ℓ AND $\dot{A}i(-\alpha_\ell)$

V

DISCUSSION

In both the acoustic and the electromagnetic diffraction problems considered, the short-wavelength asymptotic expressions for the surface fields have been obtained for the penumbra and the shadow regions. The second order terms in the asymptotic expansion of the surface fields are new results. The leading terms are the same as those of Fock (1946) and Levy and Keller (1959).

In the solutions for the shadow region, the factor $\left[\frac{G(0)}{G(u)}\right]^{1/4}$ is of interest. By definition of the function G (Eq. 2.1), $\sqrt{G}\,dv$ represents the width between the two adjacent geodesics. Thus, referring to the geometrical theory of diffraction (Levy and Keller, 1959), $\left[\frac{G(0)}{G(u)}\right]^{1/4}$ represents the so-called ray convergence factor for the creeping waves. In the geometrical theory of diffraction, this factor was obtained by physical reasoning (conservation of energy), and in the present paper, this factor is justified mathematically. The leading term for the acoustic and electromagnetic creeping waves is the same as that predicted by the geometrical theory of diffraction. This leading term, except the factor $\left[G(0)/G(u)\right]^{1/4}$, is independent of curvature in the direction transverse to the geodesic.

In the solutions of electromagnetic diffraction problems, it is shown that up to the terms of order $\left[k\rho_g\right]^{-2/3}$ in the asymptotic expansion, there is no coupling between the tangential and binormal components of the creeping waves. However, identity between the acoustic creeping waves under Neumann boundary condition and the tangential component of the electromagnetic creeping waves is true only in the leading term. The transverse curvature appears in the second order term. The effect of transverse curvature on the electromagnetic creeping waves differs from that on the acoustic creeping waves. This is one of the new results of the present investigation.

When the radius of curvature ρ_t in the transverse direction is infinite, the diffracting surface becomes cylindrical. In this case, the propagation factors of the creeping waves in the shadow region agree with those obtained by Franz and Klante (1959), and by Keller and Levy (1959). When the principal radii of curvature (ρ_g and ρ_{tn}) are the same and constant, the diffracting surface is spherical. In this case, the solutions of the creeping waves reduce to the results of Senior (1966), who obtained the creeping wave solution (including the second order terms) for the sphere by means of a Watson transformation of the Mie series (exact) solution.

The solutions for the shadow regions are not valid near a caustic where the radius of curvature (ρ_t) in the direction transverse to the geodesic is no longer large compared to the incident wavelength. The author feels that the integral equation method used here will be still applicable in investigating the surface fields near the caustic, provided that the saddle point integration for the v' coordinate (Sect. 3.1 and 4.1) is modified by some suitable means.

VI

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VII

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13. ABSTRACT

A general method is presented for obtaining successive terms in short-wavelength asymptotic expansions of the diffracted field produced by plane acoustic and electromagnetic waves incident on an arbitrary smooth convex surface. By introducing the geodesic coordinate system on arbitrary surfaces of non-constant curvature, both scalar and vector integral equations governing the surface fields are solved directly. The expressions for leading and second order terms in the asymptotic expansion of the diffracted fields are obtained explicitly and the differences between acoustic and electromagnetic creeping waves are shown.

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