Application of the Electronic Differential Analyzer to the Oscillation of Beams, Including Shear and Rotary Inertia

by

C. E. Howe
Professor of Physics, Oberlin College

R. M. Howe
Instructor in Aeronautical Engineering,
University of Michigan

L. L. Rauch
Associate Professor of Aeronautical Engineering,
University of Michigan

Aeronautical Research Center
Engineering Research Institute
University of Michigan

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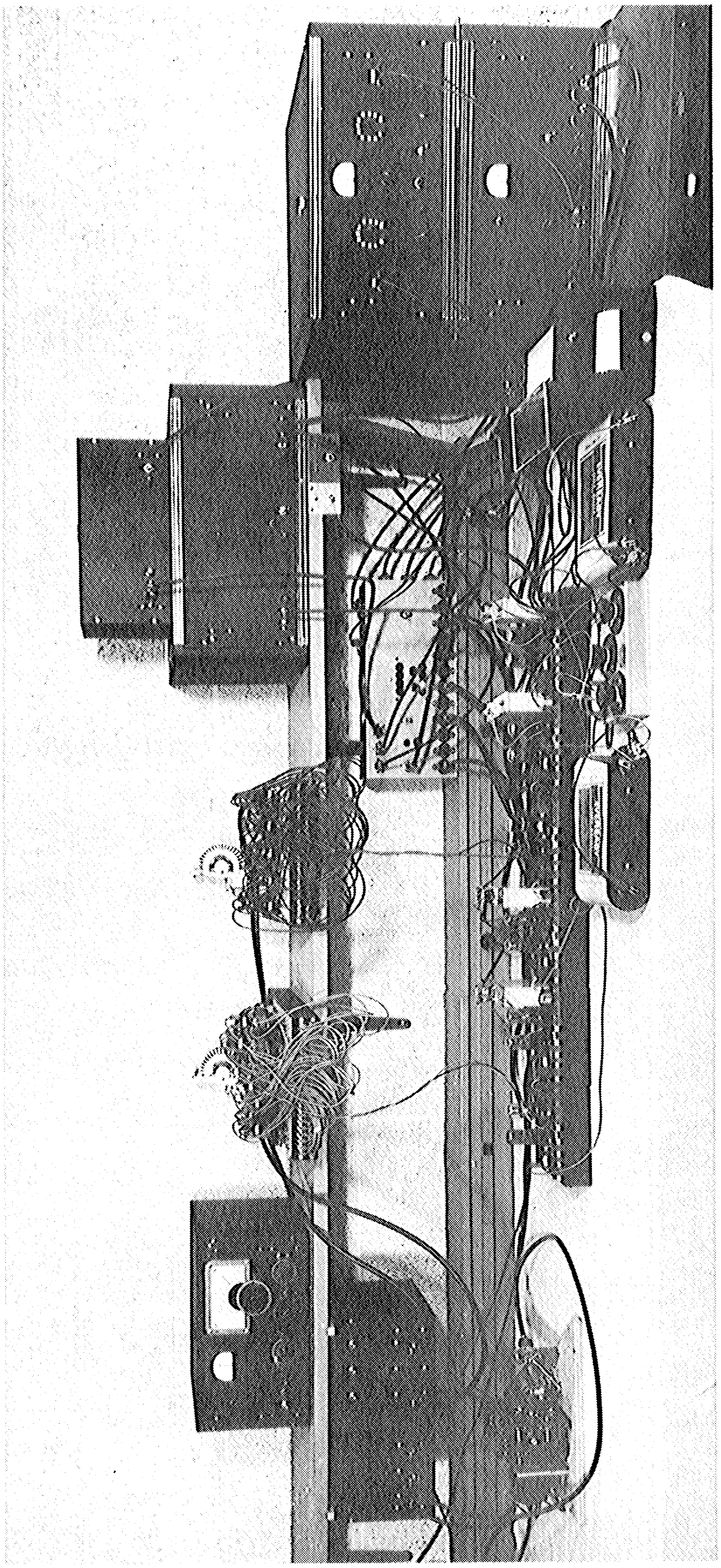

Electronic Differential Analyzer Set Up to Solve
for Normal Modes of Vibration of a Non-Uniform Beam

PREFACE

This report on the vibrating beam problem represents a continuation of work presented in previous reports[9,10]. The problem as first proposed for solution by the electronic differential analyzer was presented as one involving the solution of a fourth order differential equation subject to boundary conditions imposed by the restraints, or lack of them, placed on the ends of the beam.

In this communication it is demonstrated that, by using the basic first order differential equations from which the fourth order differential equation was derived, the electronic differential analyzer serves to determine the eigen-frequencies and mode shapes for vibrations of both uniform and non-uniform beams with the effects of rotary inertia and of transverse shear force included.

Chapters 1 and 2 are devoted to a theoretical analysis of the vibrating beam and the setting up of equations suitable for solution by the computer. Chapters 3 and 4 explain the use of the electronic differential analyzer for obtaining solutions for uniform and non-uniform beams. Chapter 5 describes some experimental techniques involved in the use of the computer. Chapter 6 outlines some proposed work evolving from that which has already been done. In the appendix are given tables and curves for the results obtained for the vibrations of uniform free-free beams. In addition there is a description of a drift-stabilized dc amplifier, along with a highly stabilized power supply.

While a very brief discussion of the principles of operation of the electronic differential analyzer are given in Chapters 3 and 4, it is assumed that anyone who wishes to do experimental work in this field will thoroughly indoctrinate himself in the construction and use of the computer[9,10].

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INTRODUCTION

The study of flexural vibrations of beams, with particular reference to the determination of eigen-frequencies and mode shapes, is a problem in which considerable interest has been developed. It is the purpose of this report to give a brief theoretical analysis of the problem and to demonstrate the usefulness of an electronic differential analyzer in determining the desired solutions for both uniform and non-uniform beams.

The ordinary textbook on vibrations presents a very simplified analysis of the vibrating beam problem. Generally it is assumed that the vibration of the beam takes place in one of the principal planes of flexure of the beam and that the length of the beam is large in comparison with its transverse dimensions. This latter assumption permits the effects of vertical shear force and of rotary inertia to be neglected without much loss in accuracy. However, if the beam being studied no longer has a length large compared with its cross-sectional dimensions, these simple solutions are not sufficiently accurate.

Lord Rayleigh[1] derived a correction factor to include the effect of rotary inertia. In 1921 Timoshenko[2] set up a group of differential equations which included the effect of transverse shear force as well as that of rotary inertia. From these equations he derived, for the special case of a beam hinged at both ends, a correction factor[2,3] which could be used to determine an approximately correct frequency. This method of approximation is not readily applicable to beams with other types of end fastenings, as was pointed out by Goens[4]. Goens derives exact expressions for the free-free beam with the effects of rotary inertia and vertical shear force included. The roots of these expressions yield the frequencies of vibration if the constants of the beam are known. Actually Goens uses the relations to determine the modulus of elasticity of the material of the bar from experimentally determined eigen-frequencies.

More recently Ormondroyd, et al.,[5,6] have presented results of theoretical research on the dynamics of a ship's structure with a view toward the determination of the eigen-frequencies and "normal" mode shapes of both uniform and non-uniform beams. In addition studies were made of forced damped vibrations.

In this work the effect of rotary inertia is neglected, and for this case orthogonality of the modes of vibration is demonstrated.

Hess[7] investigates the general problem of transverse vibrations in both uniform and non-uniform beams. After developing the basic equations for the beam and deriving differential equations for mode shape, bending moment, shear force, etc., the author obtains frequency equations and shape factors for beams with different kinds of end restraints. In his study of non-uniform beams the effect of rotary inertia is omitted. For this case it is shown that the modes of vibration are orthogonal. When rotary inertia is not neglected, Dolph[13] has shown that the solutions are not orthogonal, except for a certain type of end fastening.

Kruszewski[8] considers the effects of transverse shear and rotary inertia. The following summary of the work is given by the author: "A theoretical analysis of the effect of transverse shear and rotary inertia on the natural frequencies of a uniform beam is presented. Frequency equations are derived for the cases of the cantilever beam, the symmetrically vibrating free-free beam, and the antisymmetrically vibrating free-free beam. Numerical results are given in the form of curves giving the frequencies of the first three modes of the cantilever beam and the first six modes, three symmetrical and three antisymmetrical, of the free-free beam."

In all of the above work the solution of the basic equations led to a fourth order differential equation for the mode shape. This equation was solved, subject to the boundary conditions at the ends of the beam, the solutions being equations from which the eigen-frequencies were determined. In most cases no attention was given to mode shapes.

Hagelbarger, et al.,[9] demonstrated that an electronic differential analyzer can be used for solving the fourth order differential equation with the effects of rotary inertia and transverse shear force included. The eigen-frequencies and mode shapes are determined simultaneously. For higher modes of oscillation of relatively short beams the mode shapes obtained corresponded to oscillations of the center of gravity of the beam. This result was explained by the fact that incorrect boundary conditions had been used.

In subsequent work Howe[10] showed that satisfactory solutions can be obtained by the electronic differential analyzer, using the correct end conditions for a free-free beam, when the effects of transverse shear force and rotary inertia are included.

This particular method of attack does not permit the electronic differential analyzer to operate on the fourth order differential equation to obtain solutions for non-uniform beams. Such a procedure would neglect many derivatives which are important.

It is shown in subsequent analysis that satisfactory solutions can be obtained for both uniform and non-uniform beams by having the computer operate on the original basic differential equations.

CHAPTER 1

THEORETICAL ANALYSIS OF THE VIBRATING BEAM

1.1 Basic Equations

By applying the laws of dynamics and of elementary strength of materials to a free body diagram, one obtains the five basic equations,

$$\bar{\rho}(x) \frac{\partial^2 y(x,t)}{\partial t^2} + \frac{\partial \bar{V}(x,t)}{\partial x} = 0 , \qquad (1\text{-}1)$$

$$- \frac{\partial \bar{M}(x,t)}{\partial x} + \bar{I}_r(x) \frac{\partial^2 \bar{\beta}(x,t)}{\partial t^2} + \bar{V}(x,t) = 0 , \qquad (1\text{-}2)$$

$$\bar{V}(x,t) = - \overline{kAG}(x) \cdot \bar{\alpha}(x,t) , \qquad (1\text{-}3)$$

$$\frac{\partial \bar{\beta}(x,t)}{\partial x} = \frac{\bar{M}(x,t)}{\overline{EI}(x)} , \qquad (1\text{-}4)$$

$$\frac{\partial \bar{y}(x,t)}{\partial x} = \bar{\alpha}(x,t) + \bar{\beta}(x,t) . \qquad (1\text{-}5)$$

These equations, with changes of notation, are taken from Ormondroyd, et al.,[5]. Equation (1-1) equates to zero the sum of the transverse forces. Equation (1-2) is the sum of the moments acting on a small element of the beam and includes the effect of rotary inertia. Equations (1-3), (1-4), and (1-5) relate the neutral axis slope to shear and bending. The equivalent of the first two of these equations can be found in textbooks on strength of materials. Equation (1-5) states that the neutral axis slope is the sum of the slopes due to shear and bending and holds true for small deflections only. The derivation of equations (1-1) and (1-2) can be found in Timoshenko's original paper[2].

1.2 Notation

x = horizontal distance from left end of beam

ℓ = length of beam

t = time

$\bar{y}(x,t)$ = transverse deflection of beam at any instant

$y(x)$ = maximum transverse deflection of beam, at x

$\bar{M}(x,t)$ = bending moment at any instant

$M(x)$ = maximum bending moment, at x

$\bar{V}(x,t)$ = transverse shear force at any instant

$V(x)$ = maximum transverse shear force, at x

I = area moment of inertia

E = modulus of elasticity, or Young's modulus

$\overline{EI}(x)$ = flexural rigidity, at x

EI = a constant value of $\overline{EI}(x)$

A = cross-sectional area

G = modulus of shear, or rigidity modulus

k = ratio of average shear stress to stress at neutral axis

$\overline{kAG}(x)$ = shear rigidity, at x

kAG = a constant value of $\overline{kAG}(x)$

$\bar{\rho}(x)$ = mass per unit length of beam, at x

ρ = a constant value of $\bar{\rho}(x)$

$\bar{I}_r(x)$ = mass moment of inertia per unit length of beam, at x

I_r = a constant value of $\bar{I}_r(x)$

$\bar{\alpha}(x,t)$ = neutral axis slope due to shear at any instant

$\bar{\alpha}(x)$ = maximum neutral axis slope due to shear, at x

$\bar{\beta}(x,t)$ = neutral axis slope due to bending at any instant

$\bar{\beta}(x)$ = maximum neutral axis slope due to bending, at x

$\left(\overline{\frac{kG}{E}}\right)(x)$ = a dimensionless parameter, expressing the effect of rotary inertia

$N = \frac{kG}{E}$ = a constant value of $\left(\overline{\frac{kG}{E}}\right)(x)$

$\Phi_d(x)$ = a dimensionless variable, reflecting the variation of $\bar{\rho}(x)$ with x

$\phi_r(x)$ = a dimensionless variable, reflecting the variation of $\bar{I}_r(x)$ with x

$\phi_s(x)$ = a dimensionless variable, reflecting the variation of $\overline{kAG}(x)$ with x

$\phi_f(x)$ = a dimensionless variable, reflecting the variation of $\overline{EI}(x)$ with x

$\phi_n(x)$ = a dimensionless variable, reflecting the variation of $\overline{\left(\frac{kG}{E}\right)}(x)$ with x

$S = \frac{EI}{kAG\,\ell^2}$, a dimensionless parameter

$\lambda = \omega \ell^2 \sqrt{\frac{\rho}{EI}}$, a dimensionless parameter

ξ = a dimensionless independent variable, $x = \ell\xi$

Υ = the independent variable for the computer, $x = \frac{\ell}{L}\Upsilon$

μ = a root of equation (1-23)

m = a parameter, see equation (1-28)

n = a parameter, see equation (1-29)

1.3 Separation of Variables

In equations (1-1) to (1-5) there are two independent variables, x and t. The quantities $\bar{y}(x,t)$, $\bar{M}(x,t)$, $\bar{V}(x,t)$, $\bar{\alpha}(x,t)$ and $\bar{\beta}(x,t)$ are functions of both of these variables. $\overline{EI}(x)$, $\overline{kAG}(x)$, $\bar{I}_r(x)$ and $\bar{\rho}(x)$ are constants for a uniform beam and functions of x for a non-uniform beam. To separate the two independent variables x and t it is assumed that the time dependent variables are sinusoidal functions of time. This assumption represents one possible solution with respect to time, though not necessarily the only one. Let

$$\bar{y}(x,t) = y(x)\ e^{j\omega t}\ ,$$

$$\bar{M}(x,t) = M(x)\ e^{j\omega t}\ ,$$

$$\bar{V}(x,t) = V(x)\ e^{j\omega t}\ ,$$

$$\bar{\alpha}(x,t) = \bar{\alpha}(x)\ e^{j\omega t}\ ,$$

and

$$\bar{\beta}(x,t) = \bar{\beta}(x)\ e^{j\omega t}\ .$$

Equations (1-1) to (1-5) then become

$$- \bar{\rho}(x)\,\omega^2 y(x) + \frac{dV(x)}{dx} = 0 \,, \tag{1-6}$$

$$- \frac{dM(x)}{dx} - \bar{I}_r(x)\omega^2\bar{\beta}(x) + V(x) = 0 \,, \tag{1-7}$$

$$V(x) = - \overline{kAG}(x) \cdot \bar{\alpha}(x) \,, \tag{1-8}$$

$$\frac{d\bar{\beta}(x)}{dx} = \frac{M(x)}{\overline{EI}(x)} \,, \tag{1-9}$$

$$\frac{dy(x)}{dx} = \bar{\alpha}(x) + \bar{\beta}(x) \,. \tag{1-10}$$

Since the above set of equations are to be applied in general to the study of a non-uniform beam in which the physical characteristics change along the length of the beam, it will be convenient to use the following relations:

$$\bar{\rho}(x) = \rho\phi_d(x) \,,$$

$$\bar{I}_r(x) = I_r\phi_r(x) \,,$$

$$\overline{kAG}(x) = kAG\,\phi_s(x) \,,$$

and

$$\overline{EI}(x) = EI\,\phi_f(x) \,,$$

where ρ, I_r, kAG and EI are now constants, independent of distance along the beam, and where the variations of the physical properties of the non-uniform beam are reflected in the ϕ's.

Using the above relations there are obtained from equations (1-6) to (1-10),

$$- \rho\phi_d(x)\omega^2 y(x) + \frac{dV(x)}{dx} = 0 \,, \tag{1-11}$$

$$-\frac{dM(x)}{dx} - I_r\phi_r(x)\omega^2\bar{\beta}(x) + V(x) = 0 , \tag{1-12}$$

$$V(x) = - kAG\phi_s(x)\,\bar{\alpha}(x) , \tag{1-13}$$

$$\frac{d\bar{\beta}(x)}{dx} = \frac{M(x)}{EI\phi_f(x)} , \tag{1-14}$$

and

$$\frac{dy(x)}{dx} = \bar{\alpha}(x) + \bar{\beta}(x) . \tag{1-15}$$

In case equations (1-11) to (1-15) are to be used for a uniform beam the variables ϕ_d, ϕ_r, ϕ_s and ϕ_f may be set equal to unity.

1.4 Boundary Conditions

These equations are to be solved subject to the end conditions, or restraints, if any, imposed on the two ends of the beam to be studied. The end of a beam may be in any one of three states; free, hinged, or built-in. The boundary conditions to be met are given in Table I.

TABLE I

BOUNDARY CONDITIONS

End of Beam	Boundary Conditions	
Free	$M = 0$;	$V = 0$
Hinged	$y = 0$;	$M = 0$
Built-In	$y = 0$;	$dy/dx = 0$,
	or $y = 0$;	$\beta' = 0$

In the case of the built-in end there is some difference of opinion as to whether the slope of the neutral axis is zero or whether it has a finite value due to transverse shear force alone.

The solution of the differential equations given above for a freely vibrating beam may be accomplished in either of two ways; by a theoretical analysis or by a differential analyzer. Both methods are presented here, the theoretical analysis being limited to that for a uniform beam.

1.5 Theoretical Analysis of the Vibrations of a Uniform Beam

For a uniform beam ϕ_d, ϕ_r, ϕ_s and ϕ_f in equations (1-11) to (1-15) are set equal to unity. From these modified equations there may be obtained,

$$\frac{EI}{\rho\omega^2}\frac{d^4y(x)}{dx^4} + \left(\frac{EI}{kAG} + \frac{I_r}{\rho}\right)\frac{d^2y(x)}{dx^2} - \left(1 - \frac{I_r\omega^2}{kAG}\right)y(x) = 0\ , \qquad (1\text{-}16)$$

$$M(x) = \rho\omega^2\left[\frac{EI}{\rho\omega^2}\frac{d^2y(x)}{dx^2} + \frac{EI}{kAG}\,y(x)\right]\ , \qquad (1\text{-}17)$$

$$V(x) = \frac{\rho\omega^2}{1 - \frac{I_r\omega^2}{kAG}}\left[\frac{EI}{\rho\omega^2}\frac{d^3y(x)}{dx^3} + \left(\frac{EI}{kAG} + \frac{I_r}{\rho}\right)\frac{dy(x)}{dx}\right]\ , \qquad (1\text{-}18)$$

$$\beta(x) = \frac{dy}{dx} + \frac{1}{kAG}\,\frac{\rho\omega^2}{1 - \frac{I_r\omega^2}{kAG}}\left[\frac{EI}{\rho\omega^2}\frac{d^3y(x)}{dx^3} + \left(\frac{EI}{kAG} + \frac{I_r}{\rho}\right)\frac{dy(x)}{dx}\right]\ , \qquad (1\text{-}19)$$

where the bar over the β is omitted because it is of no significance in this part of the work.

In the above equations we shall replace the independent variable $x\ (-\frac{\ell}{2} \leq x \leq \frac{\ell}{2})$ by $\xi(-\frac{1}{2} \leq \xi \leq \frac{1}{2})$ using the relation $x = \ell\xi$, where ℓ is the length of the beam. Recognizing that

$$\frac{I_r}{\rho} = \frac{I}{A}\ ,$$

and letting

$$\lambda^2 = \frac{\rho\omega^2\ell^4}{EI}\ , \qquad (1\text{-}20)$$

$$S = \frac{EI}{kAG\ell^2}, \tag{1-21}$$

and

$$N = \frac{kG}{E}, \tag{1-22}$$

equations (1-16) to (1-19) may be written,

$$\frac{1}{\lambda^2}\frac{d^4y}{d\xi^4} + S(1 + N)\frac{d^2y}{d\xi} - (1 - \lambda^2S^2N)y = 0 , \tag{1-23}$$

$$M = \rho\omega^2\ell^2\left[\frac{1}{\lambda^2}\frac{d^2y}{d\xi^2} + Sy\right] , \tag{1-24}$$

$$V = \frac{\rho\omega^2\ell}{1 - \lambda^2S^2N}\left[\frac{1}{\lambda^2}\frac{d^3y}{d\xi^3} + S(1 + N)\frac{dy}{d\xi}\right] , \tag{1-25}$$

and

$$\beta = \frac{1}{\ell(1 - \lambda^2S^2N)}\left[S\frac{d^3y}{d\xi^3} + (1 + \lambda^2S^2)\frac{dy}{d\xi}\right] . \tag{1-26}$$

The fourth order differential equation (1-23) is to be solved, subject to the boundary conditions on both ends of the beam as given in Table I.

If in equation (1-23) we let $y = C\, e^{\mu\xi}$ we have

$$\frac{1}{\lambda^2}\mu^4 + S(1 + N)\mu^2 - (1 - \lambda^2S^2N) = 0 , \tag{1-27}$$

from which

$$\mu^2 = \lambda^2\,\frac{-S(1 + N) \pm \sqrt{S^2(1 + N)^2 + \frac{4}{\lambda^2} - 4S^2N}}{2} .$$

Since the last term on the left hand side of equation (1-27) is negative for all practicable beams, one of the roots of the equation must be positive and the other one negative. Hence

$$\mu_1 = m\lambda , \quad \mu_2 = - m\lambda ,$$

where

$$m = \sqrt{\frac{- S(1 + N) + \sqrt{S^2(1 - N)^2 + \frac{4}{\lambda^2}}}{2}} , \tag{1-28}$$

and

$$\mu_3 = jn\lambda , \quad \mu_4 = - jn\lambda ,$$

where

$$n = \sqrt{\frac{S(1 + N) + \sqrt{S^2(1 - N)^2 + \frac{4}{\lambda^2}}}{2}} . \tag{1-29}$$

In relations (1-28) and (1-29) only the positive values of the radicals are to be taken. The solution of equation (1-23) may then be written as

$$y = C_1 \cosh m\lambda\xi + C_2 \sinh m\lambda\xi + C_3 \cos n\lambda\xi + C_4 \sin n\lambda\xi . \tag{1-30}$$

In order to obtain the eigen-frequencies for a particular beam it is necessary to substitute the value of y, as given by equation (1-30), and the derivatives of y, into the boundary condition relations for both ends of the beam as given in Table I. This process of substitution results in a set of four homogeneous linear equations in C_1, C_2, C_3 and C_4. For solutions other than zero to exist the determinant of the coefficients of C_1, C_2, C_3 and C_4 must be equal to zero. The expression obtained by setting this determinant equal to zero is an equation the roots of which give all of the eigen-frequencies.

1.6 Solution for a Free-Free Uniform Beam

The boundary conditions for a free-free uniform beam are, M = 0 and V = 0 at both ends of the beam. From equations (1-24) and (1-25) we have

$$\frac{1}{\lambda^2}\frac{d^2y}{d\xi^2} + Sy = 0 , \text{ for } \xi = -\frac{1}{2} , \frac{1}{2} , \tag{1-31}$$

$$\frac{1}{\lambda^2}\frac{d^3y}{d\xi^3} + S(1 + N)\frac{dy}{d\xi} = 0 \text{ , for } \xi = -\frac{1}{2}, \frac{1}{2} . \tag{1-32}$$

Substitution of y and its derivatives, as obtained from equation (1-23), into relations (1-31) and (1-32) gives four homogeneous linear equations in C_1, C_2, C_3 and C_4. By setting equal to zero the determinant of the coefficients of the C's, there is obtained,

$$\left[\frac{\tanh\frac{m\lambda}{2}}{\tan\frac{n\lambda}{2}} - \frac{n(n^2 - S)}{m(m^2 + S)}\right]\left[\frac{\tanh\frac{m\lambda}{2}}{\tan\frac{n\lambda}{2}} + \frac{m(m^2 + S)}{n(n^2 - S)}\right] = 0 . \tag{1-33}$$

The roots of equation (1-33) determine eigen-values of λ. From equation (1-20) and the known characteristics of the beam an eigen-frequency, ω, can be determined for each eigen-value of λ.

A very useful family of curves can be made by computing and plotting the eigen-values of λ as a function of 1/S for various values of N. Such a set of curves for a free-free uniform beam, obtained from data given by an electronic differential analyzer, is shown in Figure 3-10.

CHAPTER 2

EQUATIONS OF THE VIBRATING BEAM FOR SOLUTION BY AN ELECTRONIC DIFFERENTIAL ANALYZER

2.1 Inexpedience of Using the Fourth Order Differential Equation

The fourth order differential equation (1-23), derived above, can be solved by an electronic differential analyzer[9,10]. For a uniform free-free beam the results obtained are very satisfactory. It was proposed that non-uniform beams be investigated in the same manner, but by representing a non-uniform beam as composed of a discrete number of short uniform beams, each of these short uniform beams having appropriate properties. The coefficients of equation (1-23) and of the corresponding equations which impose the end conditions were to have been changed at appropriate intervals in accordance with the properties of the short uniform beams.

In order to investigate the validity of this method, equations similar to (1-16), (1-17) and (1-18) were derived for the bending moment, shear force, and mode shape, taking cognizance of the fact that for a non-uniform beam the coefficients are functions of x. The relations obtained are,

$$V = \frac{EI\frac{d^3y}{dx^3} + \frac{d(EI)}{dx}\frac{d^2y}{dx^2} + \left(\rho\omega^2\frac{EI}{kAG} + I_r\omega^2\right)\frac{dy}{dx} + \left[\frac{d}{dx}\left(\rho\omega^2\frac{EI}{kAG}\right) + \rho\omega^2 EI\frac{d}{dx}\left(\frac{1}{kAG}\right)\right]y}{1 - \frac{I_r\omega^2}{kAG} - EI\frac{d^2}{dx^2}\left(\frac{1}{kAG}\right) - \frac{d(EI)}{dx}\frac{d}{dx}\left(\frac{1}{kAG}\right)}, \qquad (2\text{-}1)$$

$$M = EI\frac{d^2y}{dx^2} + \rho\omega^2\frac{EI}{kAG} + EI\frac{d}{dx}\left(\frac{1}{kAG}\right)V, \qquad (2\text{-}2)$$

$$y = \frac{1}{\rho\omega^2}\frac{dV}{dx}. \qquad (2\text{-}3)$$

Examination of the equation for M shows that there should be no difficulty in applying the end condition, M = 0, at either end of a free-free non-uniform beam, since V is also equal to zero at the same positions. However, the equation for V is sufficiently complicated as to make its use in fulfilling boundary conditions exceedingly involved. An insurmountable difficulty arises when one attempts to obtain an explicit equation for the mode shape, y. To obtain this expression it is necessary to take the derivative, with respect to x, of equation (2-1) for V. This process results in an expression involving over a hundred terms, of which very few combine or cancel out. It would be hopeless to attempt to set up a computer to handle this equation.

On the other hand, all of these additional terms are neglected, or ignored, in any attempt to apply equation (1-23) to non-uniform beams.

2.2 Use of the Basic First Order Differential Equations

Fortunately the actual problem consists not of obtaining the solution of a single fourth order differential equation but of a set of first order differential equations. When these equations are set up on the electronic differential analyzer a number of simplifications result. The end conditions are very readily applied, the technique of obtaining solutions is simplified, and non-uniform beams can be investigated without neglecting any derivatives.

2.3 Change of Independent Variable

In order to solve these equations, (1-11) to (1-15), with the differential analyzer it is necessary to change the independent variable. The length of the beam is ℓ so that the range for the independent variable, x, is $0 \leq x \leq \ell$. The independent variable of the computer is time. In solving the equations the independent variable, x, is proportional to the time in seconds which has elapsed since the start of the solution by the computer. If the solution obtained by the computer is completed in L seconds, then the independent variable, x, should be changed to a new independent variable, τ, $(0 \leq \tau \leq L)$, according to the relations,

$$x = \frac{\ell}{L}\tau \ , \quad \text{and} \quad \frac{d}{dx} = \frac{L}{\ell}\frac{d}{d\tau} .$$

Equations (1-11) to (1-15) then become,

$$-\rho\phi_d(\tau)\,\omega^2 y(\tau) + \frac{L}{\ell}\frac{dV(\tau)}{d\tau} = 0 \ , \tag{2-4}$$

$$-\frac{L}{\ell}\frac{dM(\Upsilon)}{d\Upsilon} - I_r\phi_r(\Upsilon)\,\omega^2\bar{\beta}(\Upsilon) + V(\Upsilon) = 0 \,, \tag{2-5}$$

$$V(\Upsilon) = -\,kAG\phi_s(\Upsilon)\,\bar{\alpha}(\Upsilon) \,, \tag{2-6}$$

$$\frac{L}{\ell}\frac{d\bar{\beta}(\Upsilon)}{d\Upsilon} = \frac{M(\Upsilon)}{EI\phi_f(\Upsilon)} \,, \tag{2-7}$$

$$\frac{L}{\ell}\frac{dy(\Upsilon)}{d\Upsilon} = \bar{\alpha}(\Upsilon) + \bar{\beta}(\Upsilon) \,. \tag{2-8}$$

In order to place these equations in more suitable form, assume

$$\bar{\alpha}(\Upsilon) = \frac{L}{\ell}\,\alpha(\Upsilon) \,, \tag{2-9}$$

and

$$\bar{\beta}(\Upsilon) = \frac{L}{\ell}\,\beta(\Upsilon) \,. \tag{2-10}$$

Equations (2-4) to (2-8) may then be written,

$$-\rho\phi_d\,\omega^2 y + \frac{L}{\ell}\frac{dV}{d\Upsilon} = 0 \,, \tag{2-11}$$

$$-\frac{L}{\ell}\frac{dM}{d\Upsilon} - I_r\omega^2\phi_r\frac{L}{\ell}\beta + V = 0 \,, \tag{2-12}$$

$$V = -\,kAG\phi_s\frac{L}{\ell}\alpha \,, \tag{2-13}$$

$$\frac{L^2}{\ell^2}\frac{d\beta}{d\Upsilon} = \frac{M}{EI\phi_s} \,, \tag{2-14}$$

and

$$\frac{dy}{d\Upsilon} = \alpha + \beta \,, \tag{2-15}$$

where y, M, V, α, β, ϕ_d, ϕ_r, ϕ_s, and ϕ_f are functions of Υ.

2.4 Combination and Simplification of Coefficients

The use of the differential analyzer in solving these equations will be made much easier if their coefficients are combined and simplified. To this end we substitute equation (2-13) into equation (2-11) and obtain, after dividing through by kAG/ℓ^2,

$$\frac{\rho\omega^2\ell^2}{kAG}\phi_d y + L^2 \frac{d}{d\Upsilon}(\phi_s\alpha) = 0 \qquad (2\text{-}16)$$

This may be written as,

$$\frac{EI}{kAG\ell^2}\cdot\frac{\rho\omega^2\ell^4}{EI}\phi_d y + L^2\frac{d}{d\Upsilon}(\phi_s\alpha) = 0 ,$$

or

$$y + \frac{1}{S}\frac{L^2}{\lambda^2}\frac{1}{\phi_d}\frac{d}{d\Upsilon}(\phi_s\alpha) = 0 , \qquad (2\text{-}17)$$

where

$$S = \frac{EI}{kAG\ell^2} , \qquad (2\text{-}18)$$

and

$$\lambda^2 = \frac{\rho\omega^2\ell^4}{EI} . \qquad (2\text{-}19)$$

Substitution of equations (2-14) and (2-13) into equation (2-12), after division by $kAGL/\ell$, results in

$$\frac{EI}{kAG\ell^2}L^2\frac{d}{d\Upsilon}\left(\phi_f\frac{d\beta}{d\Upsilon}\right) + \frac{I_r\omega^2}{kAG}\phi_r\beta + \phi_s\alpha = 0 . \qquad (2\text{-}20)$$

Now

$$\bar{I}_r = \frac{\bar{\rho}\bar{I}}{\bar{A}} = \bar{\rho}\,\frac{\overline{EI}}{\overline{kAG}}\cdot\left(\frac{kG}{E}\right) ,$$

$$= \rho \phi_d \frac{EI\phi_f}{kAG\phi_s} \frac{kG}{E} \phi_n \ ,$$

where

$$\left(\overline{\frac{kG}{E}}\right) = \frac{kG}{E} \phi_n = N\phi_n \ , \qquad (2\text{-}21)$$

Hence

$$I_r\phi_r \frac{\omega^2}{kAG} = \frac{EI}{kAG\ell^2} \frac{EI}{kAG\ell^2} \frac{\rho\omega^2\ell^4}{EI} \frac{kG}{E} \frac{\phi_d\phi_f\phi_n}{\phi_s} \ ,$$

or

$$I_r\phi_r \frac{\omega^2}{kAG} = S^2\lambda^2 N\phi_r \ , \qquad (2\text{-}22)$$

where

$$\phi_r = \frac{\phi_d\phi_f\phi_n}{\phi_s} \ . \qquad (2\text{-}23)$$

Substitution of relation (2-22) into equation (2-20) gives, after division by SL^2,

$$\frac{d}{d\tau}\left(\phi_f \frac{d\beta}{d\tau}\right) + SN\frac{\lambda^2}{L^2}\phi_r\beta + \frac{1}{SL^2}\phi_s\alpha = 0 \ . \qquad (2\text{-}24)$$

2.5 The Equations to be Solved by the Computer

The equations to be solved by the computer are then,

$$\frac{dy}{d\tau} = \alpha + \beta \ , \qquad (2\text{-}15)$$

$$y + \frac{1}{S}\frac{L^2}{\lambda^2}\frac{1}{\phi_d}\frac{d}{d\tau}(\phi_s\alpha) = 0 \ , \qquad (2\text{-}17)$$

$$\frac{d}{d\tau}\left(\phi_f \frac{d\beta}{d\tau}\right) + SN\frac{\lambda^2}{L^2}\phi_r\beta + \frac{1}{SL^2}\phi_s\alpha = 0 \ . \qquad (2\text{-}24)$$

2.6 Boundary Conditions

Reference to Table I, Section 1.4, indicates that some of the boundary conditions require that $M = 0$ and $V = 0$. Now M and V do not appear explicitly in the equations, (2-15), (2-17) and (2-24), to be solved by the computer. However, by reference to equation (2-14) it is seen that $M = 0$ when $d\beta/d\Upsilon = 0$, and equation (2-13) shows that $V = 0$ when $\alpha = 0$. To meet the boundary condition $dy/dx = 0$, we may let $dy/d\Upsilon = 0$ or $\alpha + \beta = 0$. The boundary conditions for use with the electronic differential analyzer are given in Table II.

TABLE II

BOUNDARY CONDITIONS FOR COMPUTER

End of Beam	Boundary Conditions
Free	$d\beta/d\Upsilon = 0$; $\alpha = 0$
Hinged	$y = 0$; $d\beta/d\Upsilon = 0$
Built-In	$y = 0$; $dy/d\Upsilon = \alpha + \beta = 0$,
	or $y = 0$; $\beta = 0$.

CHAPTER 3

APPLICATION OF THE ELECTRONIC DIFFERENTIAL ANALYZER TO THE SOLUTION OF THE UNIFORM-BEAM PROBLEM

3.1 Introduction to Operational Amplifiers

Before proceeding with a description of how the vibrating-beam problem can be solved by means of the electronic differential analyzer, it would seem appropriate to review very briefly the principles behind the operation of such a computer.

The basic component of the electronic differential.analyzer is the operational amplifier, which is shown schematically in Figure 3-1. It consists of a dc voltage amplifier of high gain (usually about 40,000), an input impedance Z_i, and a feedback impedance Z_f (see Appendix 2 for the dc amplifier circuit).

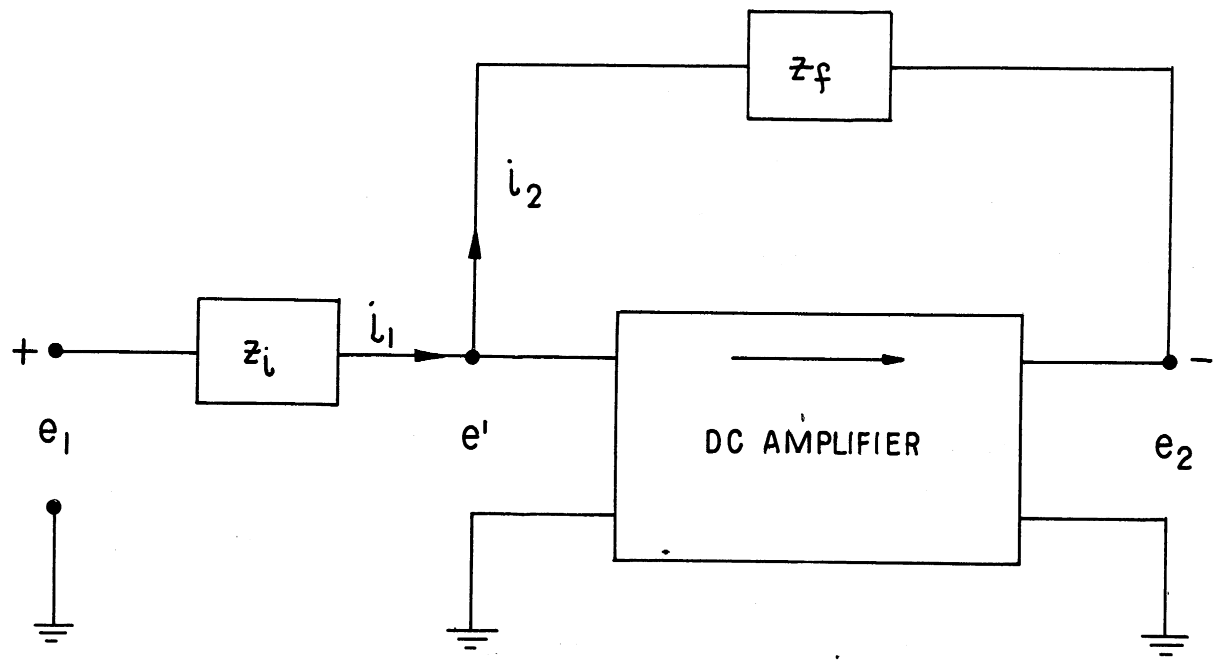


Figure 3-1. Operational Amplifier

If we eglect the current into the dc amplifier itself (i.e., neglect the current to the rid of the input tube), it follows that $i_1 = i_2$. Let us also neglect the voltage input e' to the dc amplifier in comparison with the output voltage e_2 or the input voltage e_1 to the operational amplifier ($e' = 1/40{,}000\ e_2$). We then have

$$i_1 = i_2$$

or

$$\frac{e_1}{Z_i} = -\frac{e_2}{Z_f}$$

from which

$$e_2 = -\frac{Z_f}{Z_i} e_1 , \qquad (3\text{-}1)$$

which is the fundamental equation governing the behavior of the operational amplifier. In general Z_f/Z_i is made the order of magnitude of unity. We shall now consider the scheme by which the amplifier can be used to perform three different types of operations.

(a) Multiplication by a constant.

If we wish to multiply a certain voltage e_1 by a constant factor k, we need only make $Z_f/Z_i = k$. From equation (3-1), then, the output voltage e_2 of the operational amplifier will be given by

$$e_2 = -k\ e_1 . \qquad (3\text{-}2)$$

Thus the required multiplication by a constant has been achieved, except for a reversal of sign. For example, if we wish k to be 10, we may let Z_i = 1 megohm resistance, Z_f = 10 megohms resistance. If we also desire the sign of e_2 to be the same as e_1, we need only feed e_2 through an additional operational amplifier with $Z_i = Z_f$ = 1 megohm. This second operational amplifier merely acts as a sign changer by multiplying any voltage by -1.

(b) Addition.

In order to add a number of voltages, say e_a, e_b, and e_c, the arrangement shown in Figure 3-2 is used. Here $i_a + i_b + i_c = i_2$,

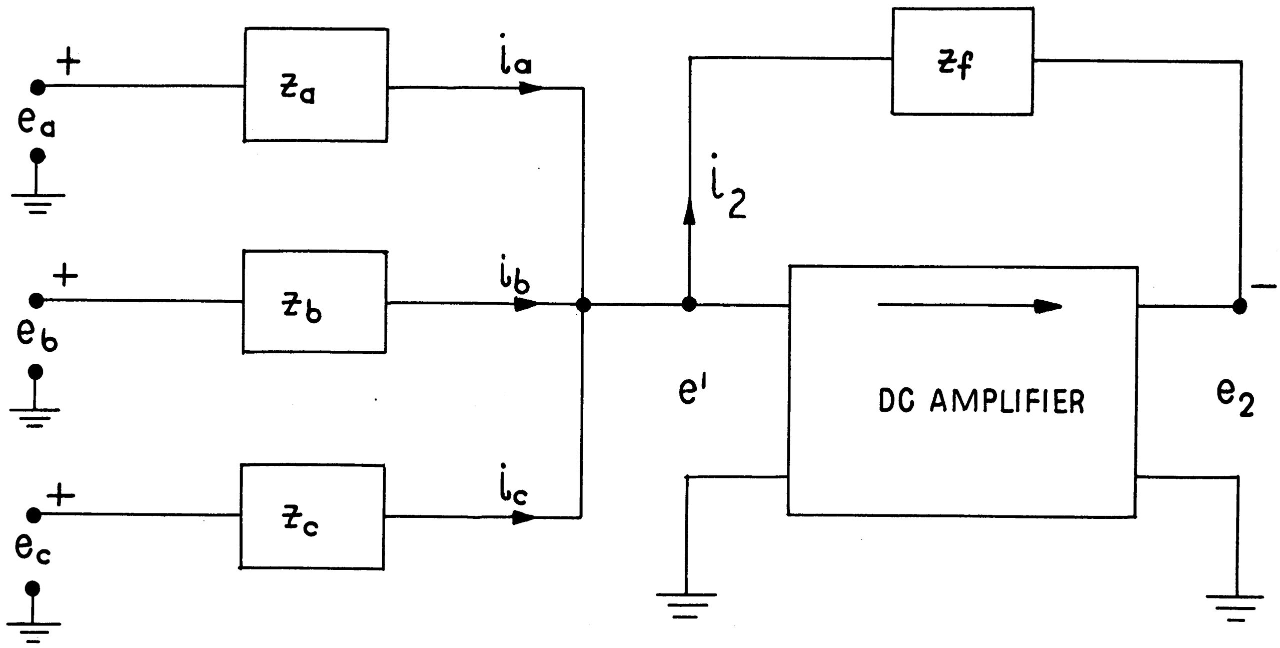


Figure 3-2. Operational Amplifier Used for Summation

and if we neglect e' as small compared with input or output voltages, we have

$$\frac{e_a}{Z_a} + \frac{e_b}{Z_b} + \frac{e_c}{Z_c} = - \frac{e_2}{Z_f}$$

or

$$e_2 = - \left(\frac{Z_f}{Z_a} e_a + \frac{Z_f}{Z_b} e_b + \frac{Z_f}{Z_c} e_c \right) . \qquad (3\text{-}3)$$

Thus the output voltage e_2 is the sum of the three input voltages, each multiplied respectively by a constant $- Z_f/Z_n$ (n = a, b, or c). The operational amplifier can, of course, be used in general to sum any number of input voltages.

(c) Integration.

If we make the input impedance Z_i a resistor and the feedback impedance Z_f a capacitor, then the operational amplifier serves as an integrator.

Referring to Figure 3-3, we see that if we neglect e' and let $i_1 = i_2$ as before, we have

$$e_2 = -\int \frac{i_1 d\tau}{C} \text{ and } i_1 = \frac{e_1}{R},$$

from which

$$e_2 = -\frac{1}{RC}\int e_1 d\tau. \tag{3-4}$$

The output voltage e_2 is then the integral with respect to time of the input voltage e_1 (multiplied by a constant factor - 1/RC).

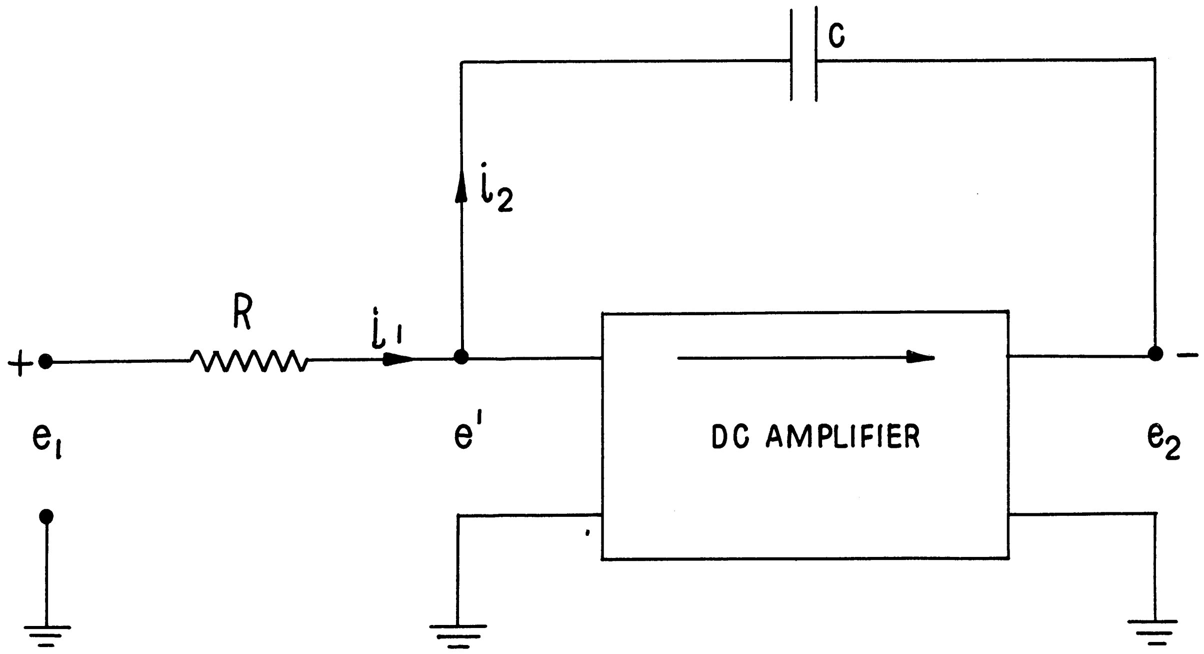


Figure 3-3. Operational Amplifier as an Integrator

In order to demonstrate how operational amplifiers performing the above three functions can be combined to solve ordinary linear differential equations, we will now set up the amplifier circuits required to solve the uniform beam problem.

3.2 Computer Arrangement for Solving the Uniform-Beam Problem

(a) The equations to be solved.

In chapter 2 we reduced the differential equations of the vibrating beam to the following three simultaneous equations:

$$\frac{dy}{d\Upsilon} = \alpha + \beta \;, \tag{2-15}$$

$$y + \frac{1}{S}\frac{L^2}{\lambda^2}\frac{1}{\phi_d}\frac{d}{d\Upsilon}(\phi_s \alpha) = 0 \;, \tag{2-17}$$

and

$$\frac{d}{d\Upsilon}\left(\phi_f \frac{d\beta}{d\Upsilon}\right) + SN\frac{\lambda^2}{L^2}\phi_r\beta + \frac{1}{SL^2}\phi_s\alpha = 0 \;. \tag{2-24}$$

We remember that y is the dependent variable (volts in the case of the computer), and that Υ is computer time (corresponding to distance along the beam). S and N are dimensionless parameters depending upon the physical characteristics of the beam, and λ is the characteristic root of the equations such that the appropriate boundary conditions are satisfied. Bending moment is proportional to $d\beta/d\Upsilon$ and shear force is proportional to α. The length of the computer solution is L.

For a uniform beam $\phi_d = \phi_f = \phi_r = \phi_s = 1$, so that equations (2-15), (2-16), and (2-24) reduce to

$$\frac{dy}{d\Upsilon} = \alpha + \beta \tag{2-15}$$

$$y + \frac{1}{S}\frac{L^2}{\lambda^2}\frac{d\alpha}{d\Upsilon} = 0 \tag{3-5}$$

$$\frac{d^2\beta}{d\Upsilon^2} + SN\frac{\lambda^2}{L^2}\beta + \frac{1}{SL^2}\alpha = 0 \tag{3-6}$$

These equations are of course subject to the end conditions given in Table II at the end of Chapter 2.

(b) The computer circuit.

The computer circuit for solving the above three equations is shown in Figure 3-4. The output functions of each of the operational units are clearly labeled so that the reader should have no trouble in tracing through the circuits to see how the three equations (2-15), (3-5), and (3-6) are set up. Note that the input to each of the integrating units is the derivative of the output of the same unit.

By combining several functions in one amplifier, one can simplify the circuit of Figure 3-4 to that shown in Figure 3-5. This arrangement could be further simplified by eliminating A_7 and feeding the output of A_6 $(-\beta)$ directly into A_4 through a resistor of value $L^2/SN\lambda$ megohms. Since $L^2/SN\lambda$ is generally large compared with 1 megohm, it is more convenient to distribute this factor through several resistors. The addition of A_7 also allows the feedback resistor λ^2/L^4 to be changed in gang with the feedback resistance of A_1.

3.3 Method of Obtaining the Correct Solution

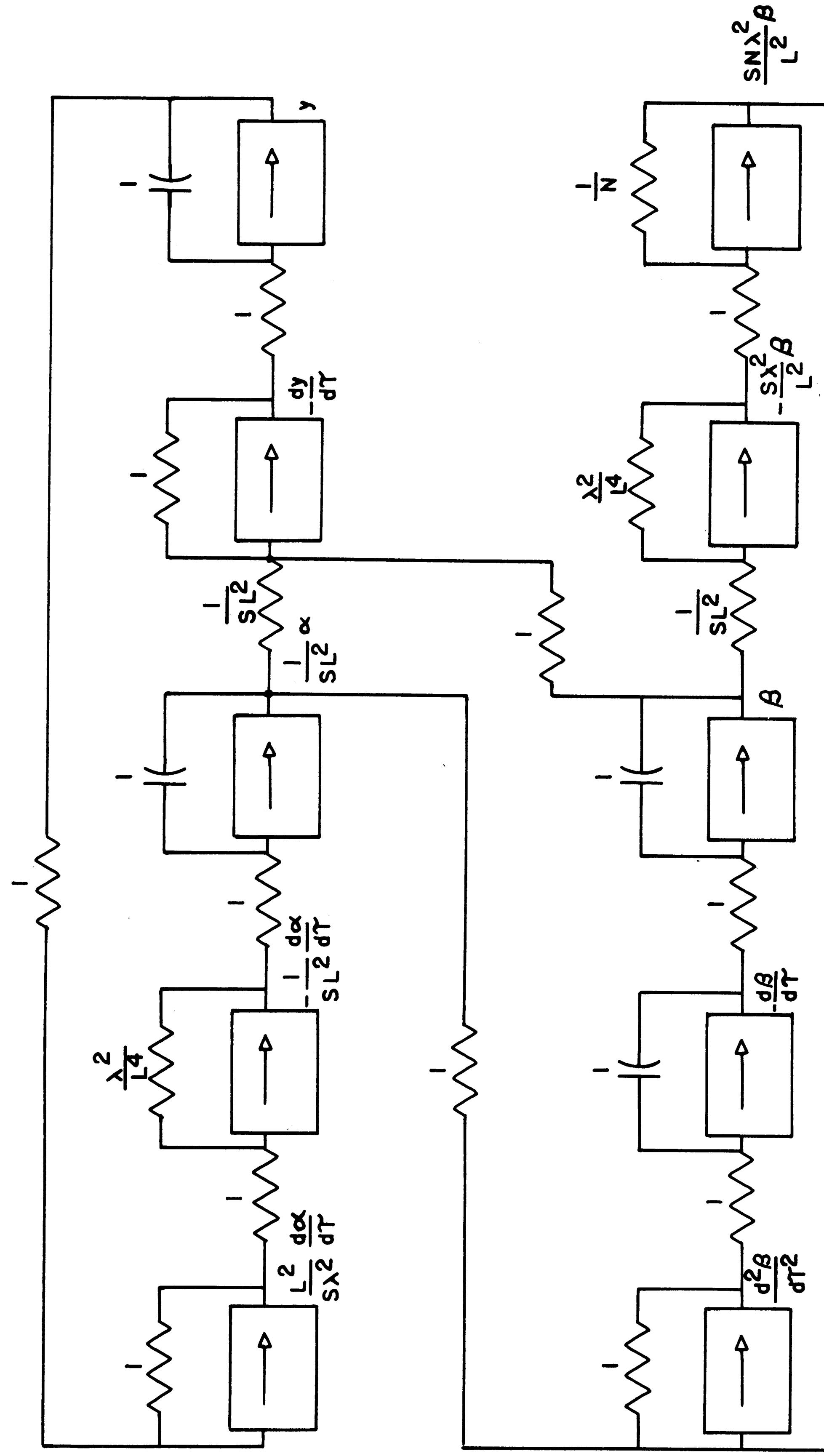
(a) Initial conditions.

The initial conditions utilized on the computer depend upon the type of end fastening for the beam in question. For a "free-free" beam we see from Table II at the end of Chapter 2 that the initial conditions (and also final conditions) are

$$\frac{d\beta}{d\Upsilon} = 0, \ \alpha = 0 \text{ at } \Upsilon = 0 \tag{3-7}$$

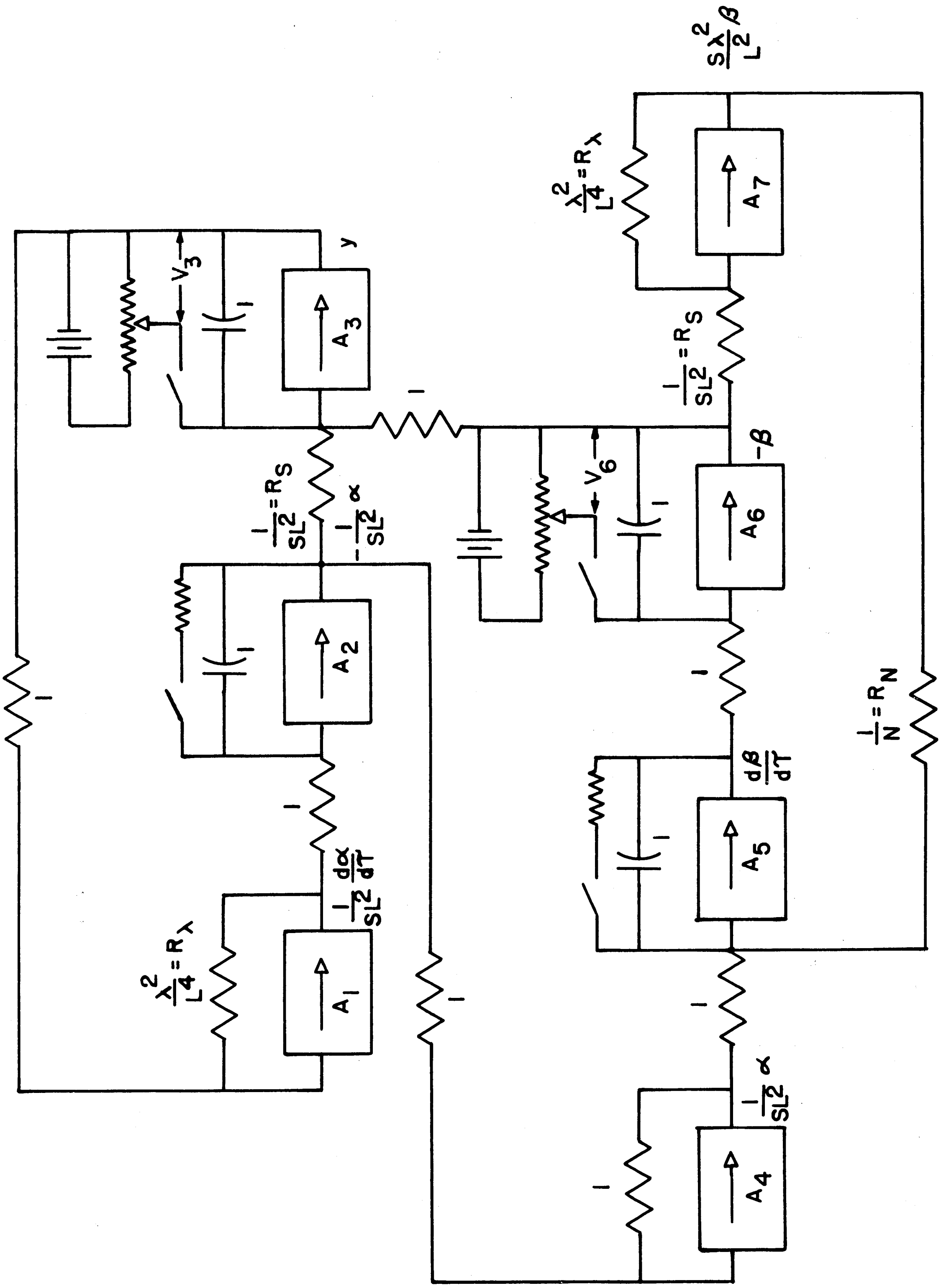
$$\frac{d\beta}{d\Upsilon} = 0, \ \alpha = 0 \text{ at } \Upsilon = L \tag{3-8}$$

The initial conditions given in (3-7) are imposed by short-circuiting the feedback capacitors of amplifiers A_2 and A_5 in Figure 3-5 through initial-condition relays. This is equivalent to making the vertical shear force and bending moment zero at the one end of the beam. At the same end the deflection and slope must be finite; these conditions are imposed initially on A_3 and A_6 through voltages V_3 and V_6. The solution of the problem is begun by releasing simultaneously the four initial condition relays, two of which were holding $d\beta/d\Upsilon = 0$ and $\alpha = 0$, and two of which were holding $y = V_3$ and $-\beta = V_6$.



All Resistor Units are Megohms
All Capacitors are I mfd
Ground Connections are Omitted for Clarity

Figure 3-4. Complete Computer Circuit for Solving the Uniform Beam



All Resistor Units are Megohms
All Capacitors are 1 mfd
Ground Connections are Omitted for Clarity

Figure 3-5. Simplified computer Circuit for Solving the Uniform Beam

(b) Trial and error method of obtaining the normal-mode solution.

The first-trial settings of V_3 and V_6 were arbitrary, and in general a correct solution will not result. For a correct solution $d\beta/d\tau$ must go through zero at precisely the same instant as α goes through zero. The time elapsed between the start of the solution and the instant when $d\beta/d\tau = \alpha = 0$ is then the length L of the computer solution. A typical first-trial solution is shown in Figure 3-6a, in which the condition $d\beta/d\tau(L) = \alpha(L) = 0$ is obviously not met. The small pips on the $d\beta/d\tau$ curve indicate when α has passed through zero (see Chapter 5 - Section 4). By varying V_3 and holding V_6 constant (or vice versa), the ratio V_3/V_6 can be varied until an exact solution is obtained. The trial solutions leading to an exact first mode solution are shown in Figure 3-6. Usually about half a dozen trial solutions are needed until an exact solution is reached. Higher modes are obtained in exactly the same manner. Second and third mode solutions are shown in Figure 3-7 and Figure 3-8.

If we are looking for the solution of a given uniform beam, then we must have chosen originally a suitable computer-solution length L. But the values for the feedback resistor R_λ in A_1 and A_7 (Figure 3-5) had to be selected arbitrarily, so that in general the observed computer length L for a correct solution will be different from the L originally chosen. It is then necessary to select a new R_7 and rerun the solution, obtaining a new L. By repeating this process several times and by interpolating, we can arrive at the value of R_7 which gives a correct solution of the desired length L. The frequency parameter λ is then given by

$$\lambda = L^2 \sqrt{R_\lambda} \ . \tag{3-9}$$

On the other hand, if we are interested in solving the problem for a whole family of beams characterized by different values of the parameter S (i.e. having various thickness to length ratios), we may keep R_λ fixed, and calculate λ from equation (3-9) and S from the formula

$$\frac{1}{S} = L^2 R_s \ . \tag{3-10}$$

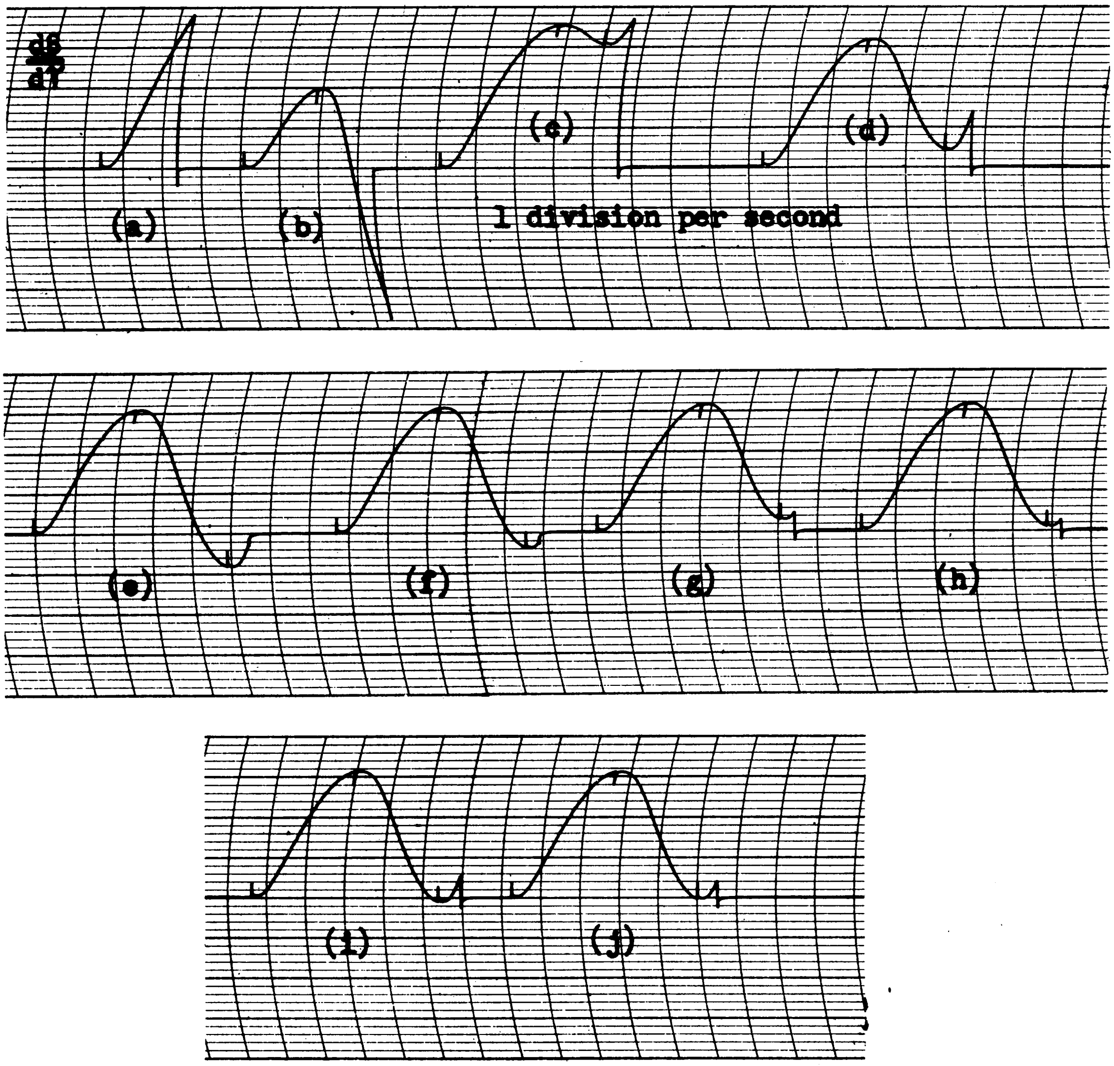


Figure 3-6. Trial and Error Method for Obtaining a Correct Solution

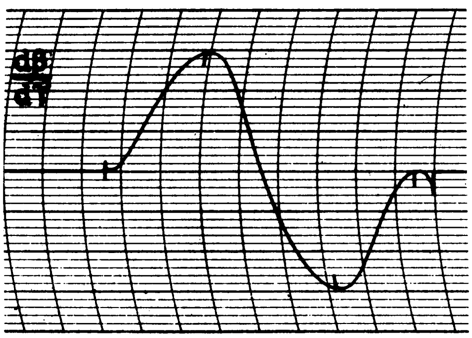


Figure 3-7. Second-Mode Solution for a Uniform, Free-Free Beam

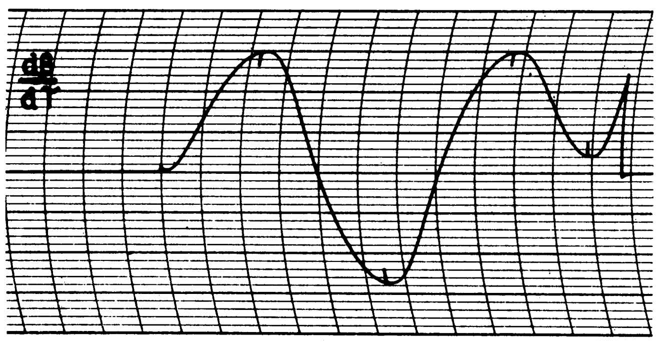


Figure 3-8. Near-Correct Third-Mode Solution for a Uniform, Free-Free Beam

In this procedure every correct solution of length L represents a certain beam, but we cannot choose the exact 1/S value for that beam beforehand.

(c) Method of recording computer solutions.

Computer output voltages are recorded by means of a Brush, Model BL-202, two-channel magnetic oscillograph. Since the pen motors require more current than the operational amplifiers can furnish, computer voltages are first fed through Brush, Model BL-913, dc amplifiers, which in turn drive the pens of the magnetic oscillograph. The Brush dc amplifiers were modified as described in a previous report[10] in order to reduce zero-drift to a negligible value. The frequency response of the oscillograph and modified amplifier together is flat out to 20 cps, which is entirely adequate for the vibrating-beam solutions, except where sharp pulses are also recorded (see Sections 5.4 and 5.5).

3.4 Computer Solutions for a Free-Free Beam, Infinitely Long

In order to check the accuracy of the computer it is instructive to solve the vibrating-beam problem for the case in which the length of the beam is very long compared with the thickness. This means that rotary inertia and shear forces are negligible ($N = S = 0$); such an idealized beam is often termed "infinitely long". The frequency parameter λ for the first five modes of an infinitely long free-free beam are given by Timoshenko[3] and can be obtained from the roots of equation (1-33).

(a) Comparison of computer solutions with theoretical solutions.

In order to solve the problem of the infinitely long beam the connection between A_2 and A_3 in Figure 3-5 was broken, along with the connection between A_7 and A_5. R_λ was set equal to unity (1 megohm), so that $\lambda = L^2$ or $\sqrt{\lambda} = L$. By using the experimental techniques described in Chapter 5, we obtained the values for $\sqrt{\lambda}$ shown in Table III.

Evidentally the computer solutions are accurate to hundredths of a percent. It is felt that one of the chief limitations on accuracy attainable was the inability to measure the length L of the solution to better than a few milliseconds. In Chapter 5 Section 6 an electronic clock is proposed which would eliminate this inaccuracy in measurement.

TABLE III

COMPARISON OF COMPUTER AND THEORETICAL SOLUTIONS

FOR A "FREE-FREE" BEAM, INFINITELY LONG

Mode	$\sqrt{\lambda} = L$ (Computer)	$\sqrt{\lambda} = L$ (Theoretical)
1	4.735 sec	4.730
2	7.852	7.853
3	10.997	10.996
4	14.136	14.137

(b) Method of interpolation for higher mode solutions.

When obtaining solutions of higher modes one finds that the ratio of V_3/V_6 in Figure 3-5 becomes extremely critical. A change in these initial voltages of one part in five thousand causes considerable deviation from one solution to the next when a third or fourth mode is being sought. As a result the solutions for the third and fourth modes do not repeat from run to run, and it is extremely fortunate ever to get an exact third or fourth mode solution.

Let us define the length L' of an inexact solutions as the time elapsed from the start of the solution to the instant α goes through zero near the "end" of the inexact solution. If the value of $d\beta/d\tau$ at $\alpha = 0$ is plotted as a function of L', we obtain a curve similar to the one shown in Figure 3-9. From this curve the length L of an exact solution can readily be obtained by interpolation. The L values for the third and fourth modes in Table III were obtained in this manner.

It turns out that when one includes shear and rotary inertia effects (N and S finite), the initial conditions for obtaining higher modes become much less critical and solutions for these modes are more easily obtained.

The cause for the lack of repetition of solutions for higher modes is believed to be the polarization of the dielectric in the capacitors of the integrators. Thus a solution begun after the capacitors were charged in one direction before being returned to their initial voltages will differ from a solution begun after the capacitors were charged in the opposite direction before being returned to their initial voltages.

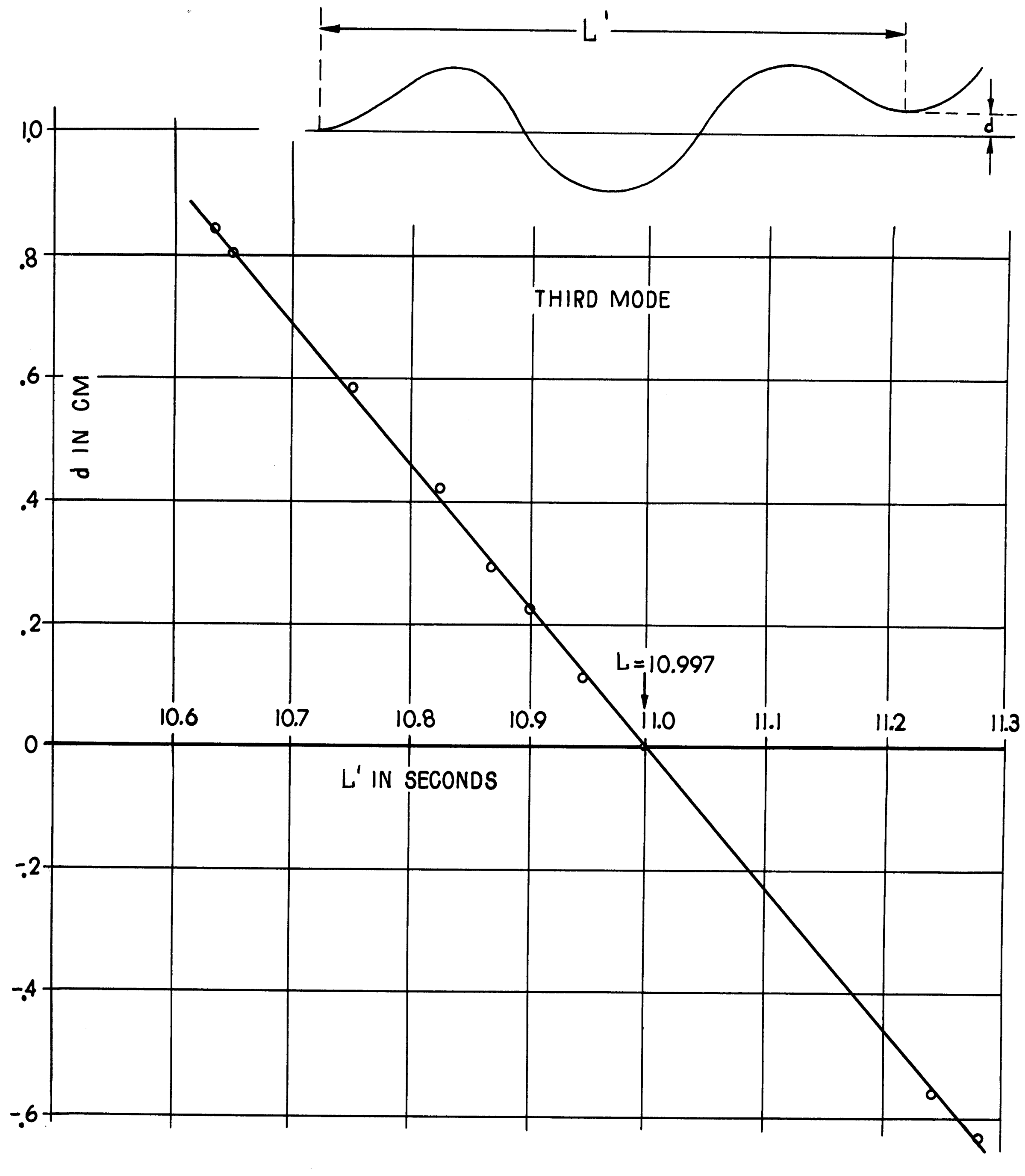


Figure 3-9. Method for Obtaining Exact Solution Length L from Inexact Solutions

3.5 Computer Solutions for a Free-Free Beam of Finite Length

The differential equations presented in Chapter 1 for the vibrating beam include the effects of shear and rotary inertia forces, and hence are representative of a beam whose thickness may be appreciable in comparison with its length. The relative effects of shear force and rotary inertia present in the vibrating beam are represented respectively by the values of the dimensionless parameters S and N (see equations (1-21) and (1-22)). For a long, thin beam, S will be small; for a short, thick beam, S will be large. N, on the other hand, depends only on the ratio of shear modulus G to Young's modulus E for a given shape of beam cross-section.

(a) Method of varying parameters.

The computer was used to find the frequency parameter λ for a considerable range of S and N values in the case of a free-free beam. The circuit shown in Figure 3-5 was used; the resistor R_λ was set equal to unity. The value of R_s was chosen arbitrarily, and N was made 0, 0.1, 0.2, and 0.3 by setting $R_n = \infty$, 10, 5, and 3 1/3 megohms. For each value of R_n and R_s the length L of the solution of the first three modes was determined. The parameter S for each of these solutions is given by equation (3-10), while $\lambda = L^2$ since $R_\lambda = 1$.

(b) Results of the computer solutions.

Figure 3-10 shows the family of curves obtained when λ is plotted as a function of 1/S for the four values of N. The actual data obtained, along with more accurate curves, can be seen in Appendix 1.

The values chosen for N(0, 0.1, 0.2, 0.3) seemed appropriate for the various types of beams which might be encountered. Variation of λ as a function of N is almost linear, so a much wider range in N could be obtained by interpolation from the above values of N without too much loss in accuracy.

The parameter 1/S ranges from very high values down to about 14, 35, and 72 in the case of the first, second, and third modes respectively. These lower limits on 1/S correspond to length-thickness ratios of about 2, 3.5, and 5 for a rectangular steel beam; i.e., the length to thickness ratios for the beams represented in Figure 3-10 go down to the region where the wave-length of

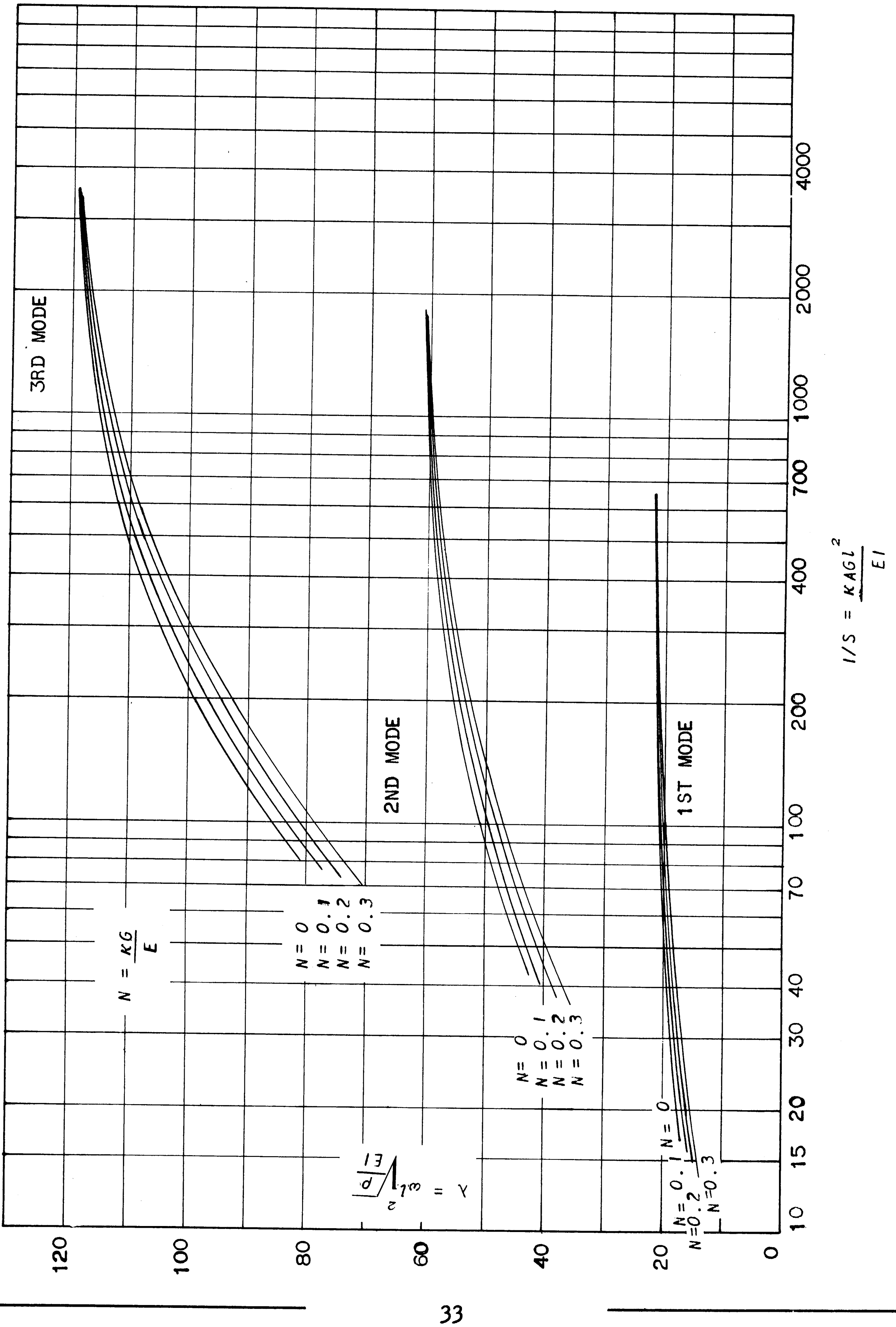


Figure 3-10. Frequency Parameter λ as a Function of N and S for the First Three Modes of a Free-Free Beam

vibration is almost the same as the thickness of the beam itself. This is the region where one would expect the assumptions involved in deriving the original beam equations (1-1) to (1-5) to break down.

(c) Orthogonality of the modes.

Dolph[13] has shown that when rotary inertia forces are included, the normal mode solutions are not in general orthogonal. The solutions are orthogonal when rotary inertia effects are neglected. Therefore it is of considerable interest to compare the mode shapes obtained with and without rotary inertia effects included. This has been done in Figure 3-11, where three modes are shown, both for $N = 0$ (no rotary inertia) and $N = 0.3$. Evidentally there is no great difference in mode shapes. It therefore seems safe to say that a Fourier series of the modes for $N = 0$ would not be too bad a representation of an arbitrary function, particularly if we only worry about the first few modes, even though an appreciable rotary inertia effect is present in the beam.

(d) Comparison of computer results with theoretical results.

The roots of equation (1-33) determine the frequency parameter λ. Since this equation is transcendental, the only method of determining the roots is by trial substitution of values of λ. However, the computer actually solves for the roots λ of equation (1-33), so to check the accuracy of the computer solution it is only necessary to substitute the value of λ obtained by the computer into equation (1-33). In general, due to inaccuracies in the computer, the equation will not be satisfied exactly (i.e., the bracketed term yielding the root in question will not be quite zero). By substituting two additional λ's which are slightly different than the original λ, one can easily interpolate to find the λ for which one of the bracketed terms in equation (1-33) does indeed vanish. This theoretical λ is compared with the computer λ for a third mode solution in Table IV.

Even though we know beforehand almost the exact value of λ, the task of computing λ from equation (1-33) requires a good many man-hours of work plus the use of a calculating machine. The accuracy of the computer, as exhibited in Table IV, is not as good as that exhibited in Table III. The percentage error in Table IV, however, is doubled since we are comparing λ directly and not $\sqrt{\lambda}$,

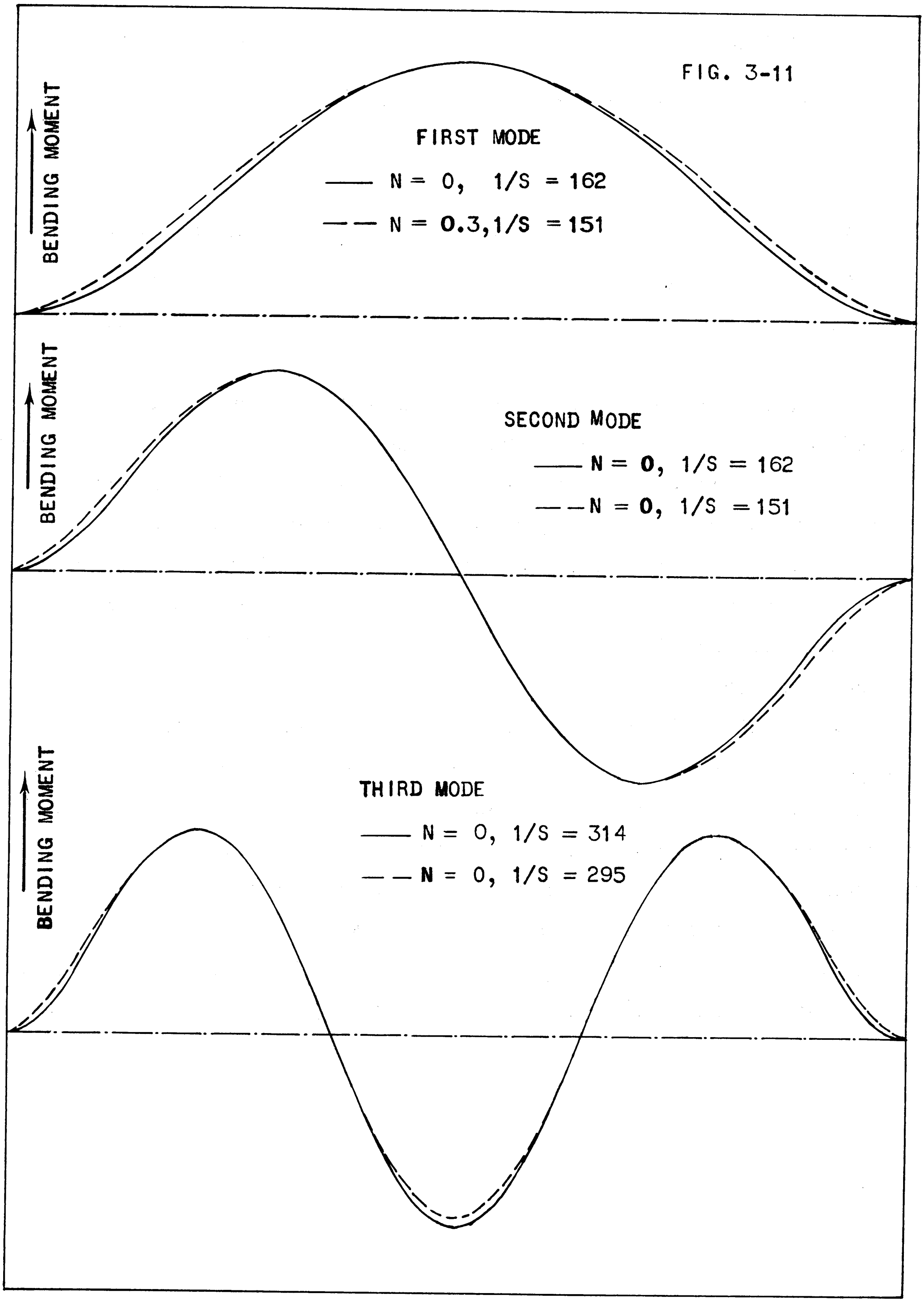


Figure 3-11. Comparison of Mode Shapes with and without Rotary Inertia Included

TABLE IV

COMPARISON OF COMPUTER AND THEORETICAL SOLUTIONS FOR A FREE-FREE BEAM, INCLUDING SHEAR AND ROTARY INERTIA FORCES

Third Mode

1/S	N	λ(Computer)	λ(Theoretical)
551	0	111.43	111.32
537	0.2	108.56	108.32

which is actually what the computer finds ($L = \sqrt{\lambda}$). The error can probably be attributed to a small increase in the time-constant of the integrating units between the time they were calibrated and the time the solution was run off. Additional errors are involved in measuring the length L of the exact solution.

It is probably safe to say that the λ's of Figure 3-10 are good to 0.2%. By incorporating better measuring and calibration techniques, we feel that this accuracy could be improved by a factor of 10 (see Chapter 5, in particular Section 5.6).

Accuracies of 0.01% or even 0.1% are obviously of no great engineering significance in this vibrating-beam work. Usually the physical quantities describing the beams themselves are not known to this accuracy, and most engineers would be very happy with 1% accuracy in normal-mode frequency predictions. The purpose behind these measurements of high accuracy is really twofold; (1) to check the accuracies attainable with the electronic differential analyzer, and (2) to stimulate further investigation of lateral vibrations of actual beams in the laboratory in order to check the theory given in Chapter 1.

CHAPTER 4

APPLICATION OF THE ELECTRONIC DIFFERENTIAL ANALYZER TO THE SOLUTION OF NON-UNIFORM BEAM PROBLEMS

4.1 The Equation to be Solved

In Chapter 2 we found, after separation of variables, the following equations for the non-uniform beam:

$$\frac{dy}{d\Upsilon} = \alpha + \beta \tag{2-15}$$

$$y + \frac{1}{S}\frac{L^2}{\lambda^2}\frac{1}{\phi_d}\frac{d}{d\Upsilon}(\phi_s\alpha) = 0 \tag{2-17}$$

$$\frac{d}{d\Upsilon}\left(\phi_f \frac{d\beta}{d\Upsilon}\right) + SN\frac{\lambda^2}{L^2}\phi_r\beta + \frac{1}{SL^2}\phi_s\alpha = 0 \tag{2-24}$$

where the notation has been defined in Section 1.2. These equations are subject to the boundary conditions given in Table II. Note that ϕ_d, ϕ_s, ϕ_f, and ϕ_r are all functions of the independent variable Υ. Since α, β, and $d\beta/d\Upsilon$ are dependent variables, it is necessary to vary somehow the gain of the operational amplifiers as a function of Υ (computer time) in order to obtain $\phi_s\alpha$, $1/\phi_d\ d/d\Upsilon\ (\phi_s\alpha)$, $\phi_f\ d\beta/d\Upsilon$, and $\phi_r\beta$ respectively. In other words, we have the problem of solving differential equations with coefficients which are varying functions of the independent variable. Several of the methods for accomplishing this are described in the next section.

4.2 Methods for Varying the Gain of an Operational Amplifier as a Function of Time

(a) Two methods of approach.

In Section 3.1 we saw that the output voltage e_2 of an operational amplifier is given by

$$e_2 = - \frac{Z_f}{Z_i} e_1 , \tag{3-1}$$

where e_1 is the input voltage, and Z_f and Z_i are feedback and input impedances respectively. Our problem is to find an arrangement whereby

$$e_2 = - f(\Upsilon) \; e_1 , \tag{4-1}$$

where $f(\Upsilon)$ is the variable coefficient. Evidentally equation (4-2) will be realized providing we can make

$$\frac{Z_f}{Z_i} = f(\Upsilon) . \tag{4-2}$$

But this can be accomplished if we let Z_i be a fixed resistor R_i and make Z_f a resistor R_f which varies with time according to the relation

$$R_f(\Upsilon) = R_i f(\Upsilon) . \tag{4-3}$$

Likewise, if we want $e_2 = - 1/f(\Upsilon) \; e_1$, we may let Z_f be a fixed resistor R_f and Z_i be a resistance R_i which varies with time in accordance with

$$R_i(\Upsilon) = R_f f(\Upsilon) . \tag{4-4}$$

Thus one scheme of changing the operational amplifier gain as a function of time Υ is to vary the feedback or input resistance as a function of time.

A second scheme for varying the gain of the operational amplifier is indicated in Figure 4-1. Here the impedances Z_i and Z_f are fixed; the resistance r between the tap on the potentiometer R and ground is made to vary with time in accordance with $\bar{f}(\Upsilon) = r/R$, where $\bar{f}(\Upsilon)$ is the function $f(\Upsilon)$ multiplied by a constant factor K so that the maximum value of $\bar{f}(\Upsilon)$ is unity. Thus we let

$$\bar{f}(\Upsilon) = K \; f(\Upsilon) , \tag{4-5}$$

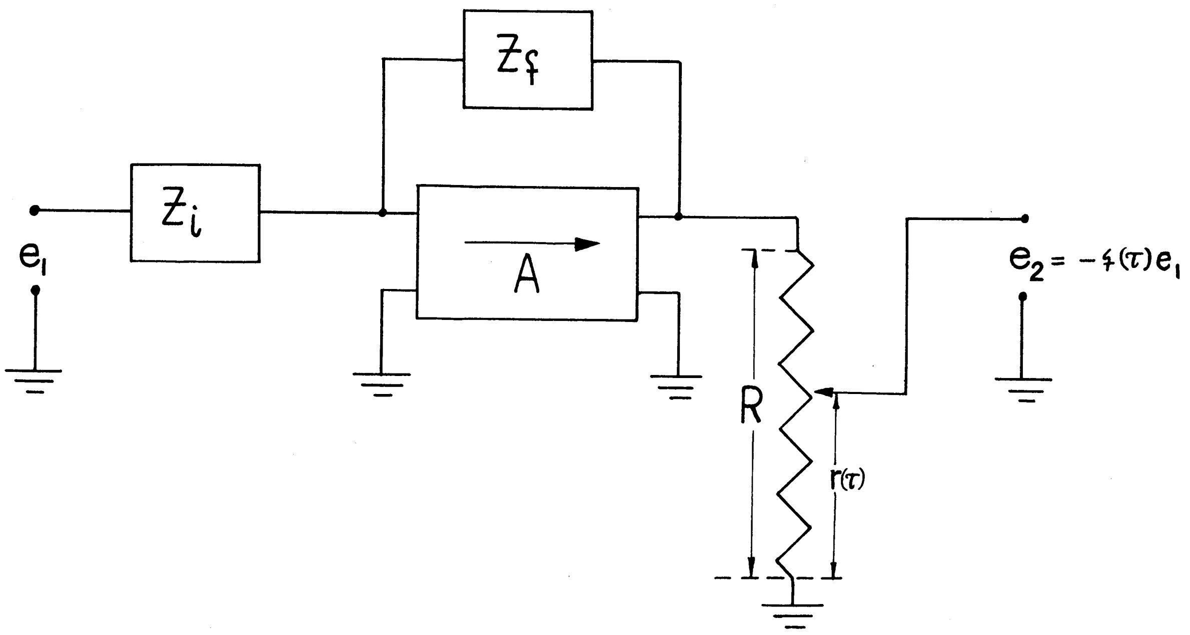


Figure 4-1. A Method for Obtaining Variable Gain

and if we set $K = Z_i/Z_f$, then $e_2 = -f(\Upsilon)\, e_1$, which is the relationship desired.

In general the voltage e_2 in Figure 4-1 will be fed into an additional operational amplifier having input resistance R_i'. Unless R is very much smaller than R_i', an appreciable error in e_2 results. In order to eliminate this error we must arrange $r(\Upsilon)$ such that

$$\frac{r\,R_i'}{R\,(r + R_i')} = \bar{f}(\Upsilon)\ . \tag{4-6}$$

Thus we see that there are two general methods for varying the gain of an operational amplifier; (1) by varying feedback or input resistors, and (2) by varying the tap position of a potential divider across the output. (1) has the advantage of allowing division as well as multiplication by a function $f(\Upsilon)$. (2) has the disadvantage of introducing an error unless the correction factor given in equation (4-6) is incorporated.

Let us consider several methods for varying resistors as a function of time.

(b) Cam operated variable resistances.

Angular rotation of an output shaft can be made any desired function of the angle of rotation of an input shaft by driving the output shaft through a properly-shaped cam turned by the input shaft. The output shaft is then connected to the sliding-contact shaft of a linear potentiometer, so that the resistance of the potentiometer will vary with input shaft angle by the desired functional relationship. If the input shaft is turned at constant speed by a synchronous motor, the resistance of the potentiometer will vary properly with time.

This scheme can be used to vary the gain of an operational amplifier as a function of time by either of the two methods described in (a). Its accuracy is limited by the precision of the cam and connecting linkage as well as the linearity of the potentiometer. The proper cutting of cams may also involve a costly or time-consuming procedure.

(c) Non-linear potentiometers.

It is possible to obtain a potentiometer wound so that the resistance is any desired function of the angle of rotation of the sliding-contact shaft. If this shaft is driven at constant speed, then the resistance of the non-linear potentiometer will vary with the desired function of time. Again cost and lack of flexibility may be serious disadvantages to this method.

(d) Stepping-relay method.

One method for varying resistance as a function of time which has been used with considerable success by the authors of this report is the so-called stepping-relay method. In this scheme the resistance, instead of being varied continuously, is changed in discrete steps after equal time intervals. For example, a linear function of resistance with time would be replaced by the staircase function shown in Figure 4-2. Note that at the end of each step the integral of the step-function (the area under the curve) is the same as the integral of the continuous function. The circuit for obtaining the resistance steps of Figure 4-2 is shown in Figure 4-3.

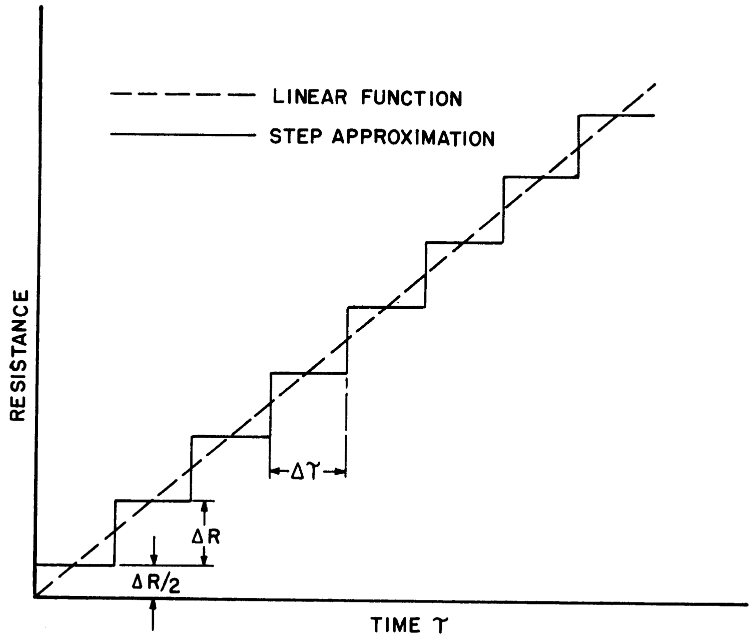


Figure 4-2. Step-Method of Approximating a Linear Function

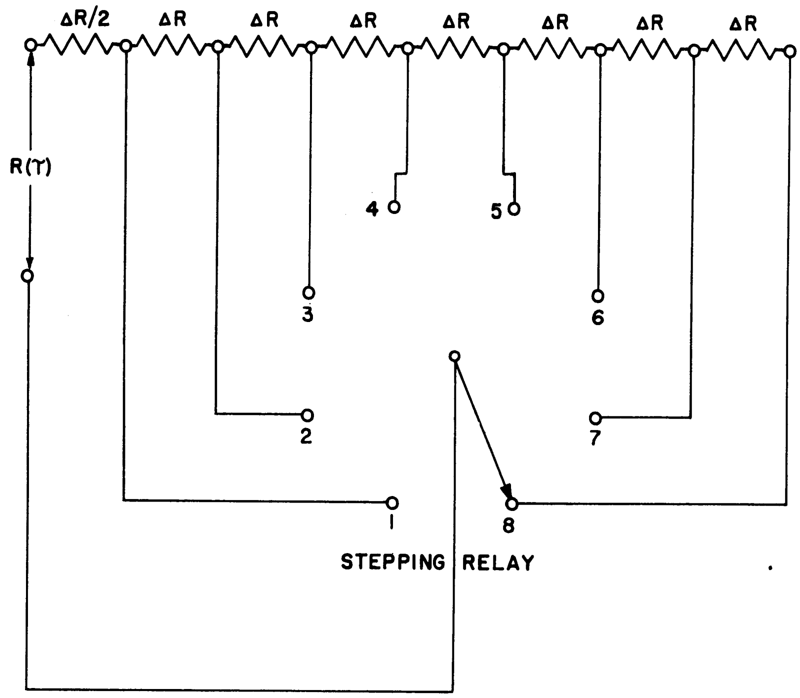


Figure 4-3. Circuit for Obtaining Staircase Resistance Function

At $\tau = 0$ the contact is in the position shown in the figure. After time $\Delta\tau$ has elapsed the contact moves to position (2), after additional time $\Delta\tau$ to position (3), etc. The contact is moved around by the stepping relay mechanism, which is energized every $\Delta\tau$ seconds once a problem solution has begun. The $\Delta\tau$ pulses to drive the stepping relay are obtained through a synchronous contactor driven by a synchronous motor. For complete details of the entire stepping-relay mechanism, the reader is referred to a previous report by the authors.[9]

The step-method is most effective when used to vary directly the input or feedback resistors of the operational amplifier, as described in Section 4.1a. The input or feedback circuit is never opened during the switching operations because bridging-type relay contacts are employed.

The extension of this step-approximation scheme to arbitrary continuous functions is obvious. The whole scheme has proved highly satisfactory, and surprising accuracy is obtained even when the steps are made comparatively large. Bessel's equation and Legendie's equation have both been solved with good results on the electronic differential analyzer using the stepping-relay method of varying coefficients.[9] A total of forty steps were employed in this work.

Several advantages are apparent for the stepping-relay scheme. First of all, it is easy to build up any desired function of time. One need only plug appropriate resistors into a panel provided for this purpose. Then too, accuracy of resistance is limited not by the percentage accuracy of full-scale resistance, but only by the percentage accuracy of the resistor in question. Thus 0.1% accuracy can be maintained in going from an operational amplifier gain of 1/50 to 50. This cannot be realized with the methods of continuously varying the resistance described in Sections 4.1b or 4.1c. A third advantage to the stepping-relay scheme is that the total resistance utilized can be of the order of megohms, which is convenient when that resistance is being used as input or feedback impedance in the operational units.

Many times the data for a given engineering problem involving variable coefficients is presented in the form of discontinuous or stepped functions. The stepping relay method would seem ideally suited for transforming this type of data into the computer circuit.

One disadvantage to the stepping-relay method is that it introduces discontinuous functions. However, in most computer circuits one or more integrators follow the amplifiers employing the stepping relays, so that the discontinuous functions get smoothed out fairly well. If one desires to synthesize an input or forcing function using the step-method, a smoothly-varying function can be obtained by setting up the derivative of the desired function with resistance steps and by then running the resulting stepped function through an integrating unit.

The stepping-relay method of varying amplifier gain is utilized in the computer to solve the non-uniform beam problem.

(e) Binary digital method.

An alternative circuit arrangement for varying resistance by steps is shown in Figure 4-4. Here any desired resistance is obtained by selecting the proper series combination of the resistors at the top of the figure. A particular resistor is added to the total series resistance R if the relay across this resistor is open. When the relay is closed, this resistor is short-circuited and makes no contribution to R. If we desired R to be 41 ohms, we would close relays R_5, R_3 and R_2 and leave relays R_6, R_4, and R_1 open. R would then be equal to $1 \times 32 + 0 \times 16 + 1 \times 8 + 0 \times 4 + 0 \times 2 + 1 \times 1 = 41$ ohms. In the binary system which is employed here the number 41 is then represented as 101001. The position which each of the relays R_1 to R_6 will assume is controlled by the position of the corresponding toggle switches in one of the rows of toggle switches shown in Figure 4-4. Thus for Step 1 the switches are set in positions up, down, up, down, down, up, going from left to right. This represents the number 41 and makes R = 41 ohms.

The position of the 6-gang stepping relay determines which row of toggle switches is connected to the resistor-controlling relays R_1 to R_6. During the first step, the top row of toggle switches controls the total resistance R; during the second step, the second row of toggle switches controls R, etc. Any step function of resistance varying with time can be set up merely by positioning the toggle switches properly to represent the function at each step. The same set of resistors and resistor-controlling relays is used over and over again.

If we wish 0.01% accuracy of maximum R, 13 resistors instead of 6 will be required ($2^{13} = 8192 = 10^4$). This means of course that we need 13 resistor-controlling relays, 13 toggle switches in each row, and a 13 gang stepping relay.

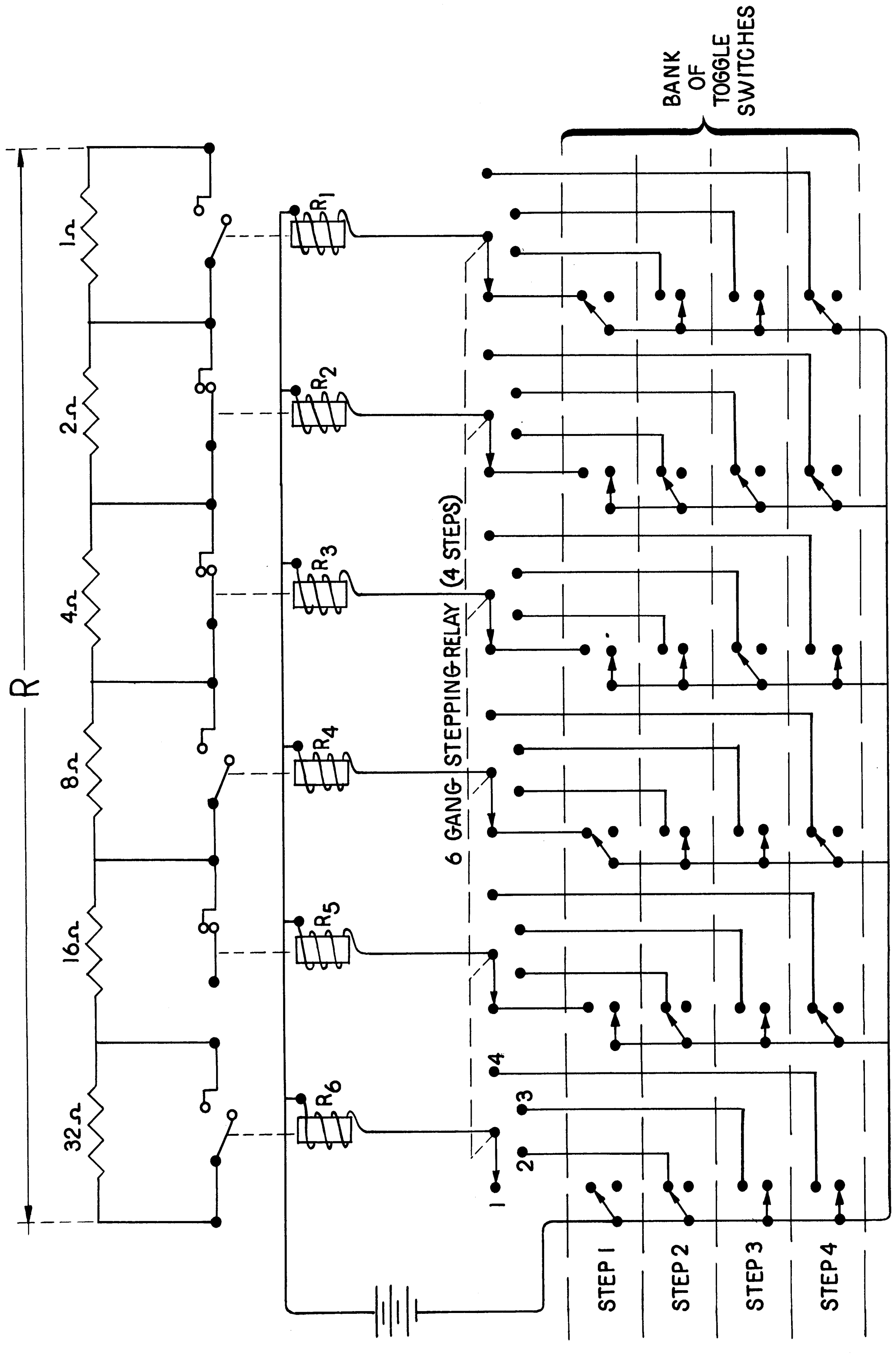


Figure 4-4. Binary Digital Scheme for Varying Coefficients

The total number of steps possible is determined merely by the number of steps available on the stepping relay. Naturally there must also be as many rows of toggle switches as there are steps.

It would probably be desirable to add an additional digit to change the sign of a function. This could be done by adding an extra gang and accompanying relay. A sign-changing operational amplifier would be connected into or out of the circuit, depending on the sign of the variable coefficient.

The initial cost of this binary digital circuit would probably be somewhat higher than the stepping-relay circuit used at present (Figure 4-3), but the increased versatility would soon pay off in decreased man hours of operation. Any function could be set up in a matter of a very few minutes - merely the time required to position the toggle switches. Furthermore, the digital scheme requires only one set of 13 precision resistors for 0.01% accuracy, whereas the arrangement of Figure 4-3 necessitates a vast supply of different resistors which could in themselves involve considerable expense.

4.3 Computer Circuit for Solving the Non-Uniform Beam Problem

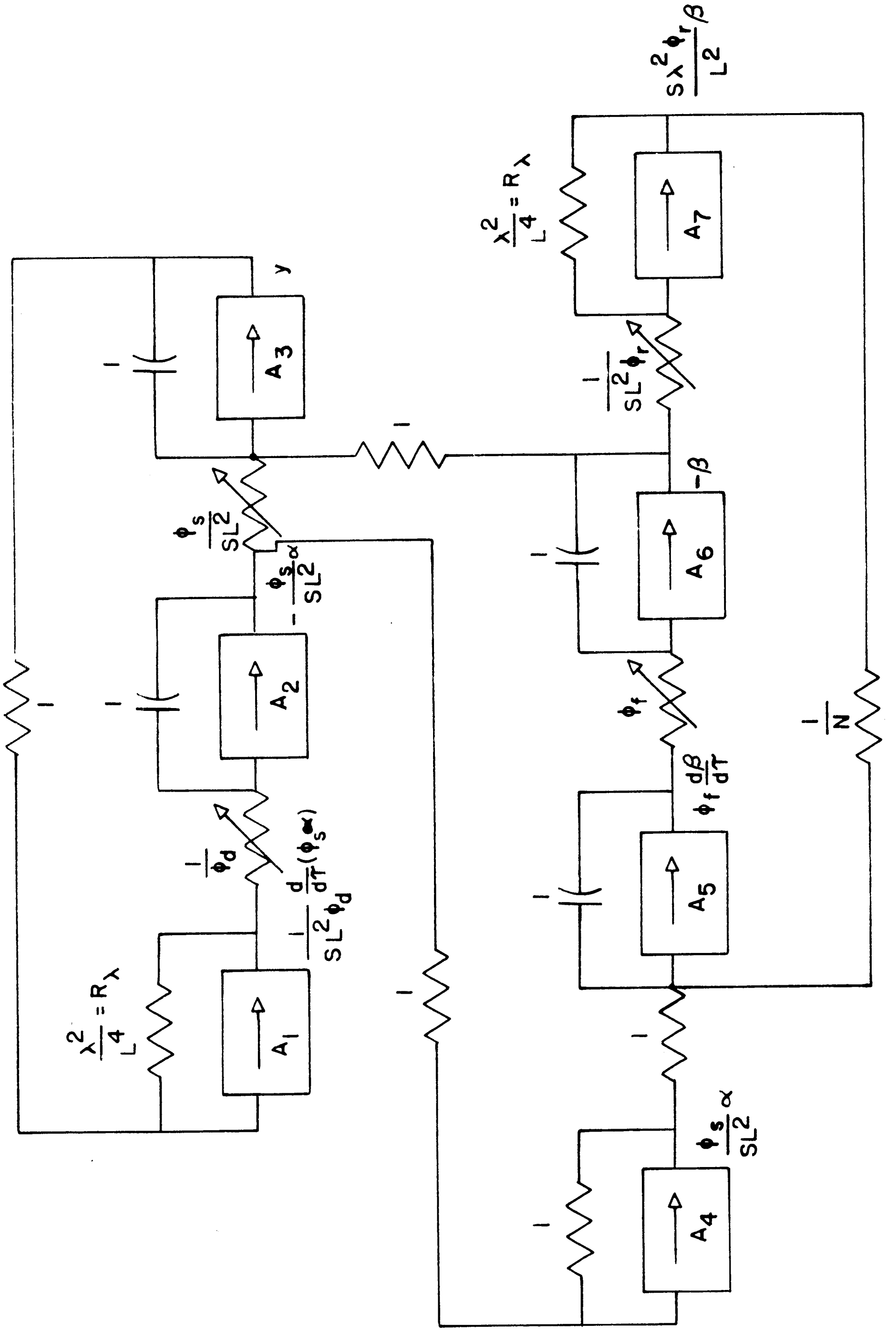
The equations to be solved for the non-uniform beam are repeated again for convenience.

$$\frac{dy}{d\tau} = \alpha + \beta \qquad (2\text{-}15)$$

$$y + \frac{1}{S}\frac{L^2}{\lambda^2}\frac{1}{\phi_d}\frac{d}{d\tau}(\phi_s\alpha) = 0 \qquad (2\text{-}17)$$

$$\frac{d}{d\tau}\left(\phi_f\frac{d\beta}{d\tau}\right) + SN\frac{\lambda^2}{L^2}\phi_r\beta + \frac{1}{SL^2}\phi_s\alpha = 0 \qquad (2\text{-}24)$$

The computer circuit for solving the above equations is shown in Figure 4-5. The variable functions ϕ_d, ϕ_s, ϕ_f and ϕ_r are approximated by the step-method described in the previous section. No initial condition circuits have been indicated. These will be dictated by the type of end-conditions appropriate for the beam in question (see Table II at the end of Chapter 2).



All Resistor Units are Megohms
All Capacitors are 1 mfd
Ground Connections are Omitted for Clarity

Figure 4-5. Computer Circuit for Solving the Non-Uniform Beam

The techniques for solving the uniform beam are again used to obtain the solution of the non-uniform beam. When a starting button is pressed, the initial condition relays are released and the stepping relays begin changing the appropriate resistors. One of the initial voltages is varied until a solution with the correct end conditions is obtained. In general this solution will have a length different from the assumed length L for the computer solution. R_λ must then be varied until a correct solution of length L is obtained (or a length close enough to L to allow interpolation of R_λ). The frequency parameter λ is then given by

$$\lambda = L^2 \sqrt{R_\lambda} \tag{3-9}$$

where R_λ is the exact feedback resistance for A_1 and A_7 which gives a correct solution of exact length L.

For beams of reasonable length to width ratios the value of the feedback resistor of A_1 in Figure 4-4 will be much more critical than that of A_7. Hence R_λ in equation (3-9) should be the value of the feedback resistor of A_1, although ideally this resistor should be equal to the feedback resistor of A_7.

For solving the non-uniform beam problem it is quite important to have a good standard-frequency source available for driving the stepping relays (see Section 5.6). This is necessary not only to insure good accuracy but also to allow solutions to repeat when solving for the critical higher modes. The time scale of the computer is fixed; so, therefore, must be the time scale of the stepping-relays.

As was the case for the uniform-beam computer circuit, here again care should be taken to maintain the gains of amplifiers in the main loop (A_1-A_2-A_4-A_5-A_6-A_3-A_1) close to unity. This means that one should make an intelligent guess for λ beforehand, and choose L so that R_λ is about 1 megohm.

The electronic differential analyzer has been used to solve several non-uniform vibrating beam problems to date with good success,[9,11] although shear and rotary inertia forces were not included properly in either case. It is in the solution of the non-uniform beam problem that the computer should have its greatest utility, for here is a realm in which mathematical solutions that include shear and rotary inertia forces are virtually impossible by any hand methods.

CHAPTER 5

COMPONENTS AND EXPERIMENTAL TECHNIQUES

5.1 Components

(a) Power supply.

The dc power supply for the direct current amplifiers furnishes the following voltages: +300, -190, and -350. The absolute values of these voltages are extremely close to the values given and are held constant to ± 1.5 millivolts by continuous automatic checking against a standard cell. The ac ripple is approximately 1 millivolt. The details of the power supply are given in Appendix 4.

(b) Direct current amplifiers.

The dc amplifiers used in this work are those described previously[9,10], modified for use with the power supply voltages given above (see Appendix 2). Improved amplifiers with a continuous automatic balancing feature are in the process of construction. These are described in Appendix 3.

(c) Resistors, feedback and input.

For most of the work the feedback and input resistors were Continental "Nobeloy X" type. This type of resistor has a low voltage coefficient and a low noise characteristic. The temperature coefficient is less than 0.05 percent per degree Centigrade negative. There is some change of resistance with age so that for very accurate work frequent calibration is necessary. The humidity characteristics are fair, but not perfect. A ten megohm resistor is temporarily changed in value about ten percent when breathed upon by a good slow "ah" with the open mouth very close to the resistor. A coat of ceresin wax removes most of this difficulty.

For the most accurate work it is suggested that precision wire-wound resistors be used. These can be obtained with a temperature coefficient as low as ± 0.002 percent per degree Centigrade.

There would be considerable advantage in housing the computing equipment in a constant temperature and constant humidity room. However, this procedure would probably be very expensive, since considerable power is dissipated.

One exception to the use of low temperature coefficient resistors occurs in the case of resistors associated with feedback capacitors.

(d) Capacitors, feedback.

The feedback capacitors used in the integrating circuits are Western Electric D161270 Condensers, 1 microfarad, with polystyrene dielectric. The characteristics of these capacitors are not known by us but it is assumed that they are not far different from those of the Plasticon laboratory-grade condensers with polyethylene dielectric manufactured by the Condenser Products Company. These Plasticon condensers have a negative temperature coefficient of 0.04-0.05 percent per degree Centigrade.

When an amplifier is used as an integrator it is important that the feedback capacitance, C, and the associated input resistance, R, be such that the product RC remains constant. If the temperature coefficients of both R and C are negative the product RC will be temperature sensitive. Such was found to be the case. Fairly satisfactory results were obtained by using Akrohm precision wire-wound resistors with a positive temperature coefficient of approximately 0.017 percent per degree Centigrade. Using this type of resistor with a Western Electric condenser gave an RC product that remained constant to within 0.05 percent for a temperature change of $5^{\circ}C$.

Ideally R and C should have equal and opposite temperature coefficients, as well as the other properties demanded by accurate computing. The Condenser Products Company make condensers with temperature coefficients from +800 to -800 parts per million per degree Centigrade. The suitability of these condensers should be investigated.

5.2 Measurement of Resistances

Most of the resistors used have a tolerance of ± 1 percent and must be calibrated if better than one percent accuracy is desired. The ordinary box bridge, or test set, was not found suitable, particularly for measuring resistances of one megohm and higher. The bridge arrangement shown in Figure 5-1 proved to be satisfactory both in accuracy and in speed of measurement.

All of the equipment, with the exception of the galvanometer and recorder, is mounted on a small polystyrene panel (1/4" thick). The bridge proper consists of two "ratio" resistors, R_1 and R_2, the unknown resistor, R_x, and a decade box, R_s. General Radio (or equivalent) jacks, with standard 3/4" separation, are provided for plugging in the resistors and the galvanometer and recorder connections.

The resistors R_1 and R_2 are General Radio Type 500 with plug type terminals. Minimum requirements are two 10,000 ohm standards and one each of 1,000, 100 and 10 ohms. A decade box similar to General Radio Type 602-L (111,100 ohms total, in steps of 10 ohms) makes a suitable variable resistor at R_s. B_2 is a battery of suitable voltage (45 volts) and B is a shorting plug which may be removed to insert either a protective resistance or an additional battery in series with B_2.

The bridge may be balanced either by the galvanometer G or by a Speedomax recorder (0 - 10 mv), the recorder being used merely as an indicator. The dry cell B_1 and the associated network permits a five millivolt bias voltage to be put on the recorder to place its zero point for bridge balancing at the middle of the scale. The use of the recorder as a null indicator gives speed to the measurements and good sensitivity. With R_1 = 10,000 ohms and R_2 = 100 ohms a sensitivity of well over one-tenth of one percent exists in measuring a ten megohm resistor. Resistances as high as 30 megohms have been measured. For most measurements an accuracy of the order of one-tenth of one percent is to be expected.

In using the bridge care must be taken not exceed the current rating of any of the resistors. If R_1 were made permanently 10,000 ohms and a fixed 10,000 ohm resistor were connected in series with R_s (and added to the value of R_s) the bridge would be protected against overload up to B_2 = 90 volts.

5.3 Measurement of Capacitance

A very suitable method for determining the capacity of condensers involves measurements of the period of oscillation of the computer circuit shown in Figure 5-2. At least three capacitors are needed. In the circuit shown C_1 and C_2 are two of the 1 mfd condensers to be measured. R_1 and R_2 are one megohm

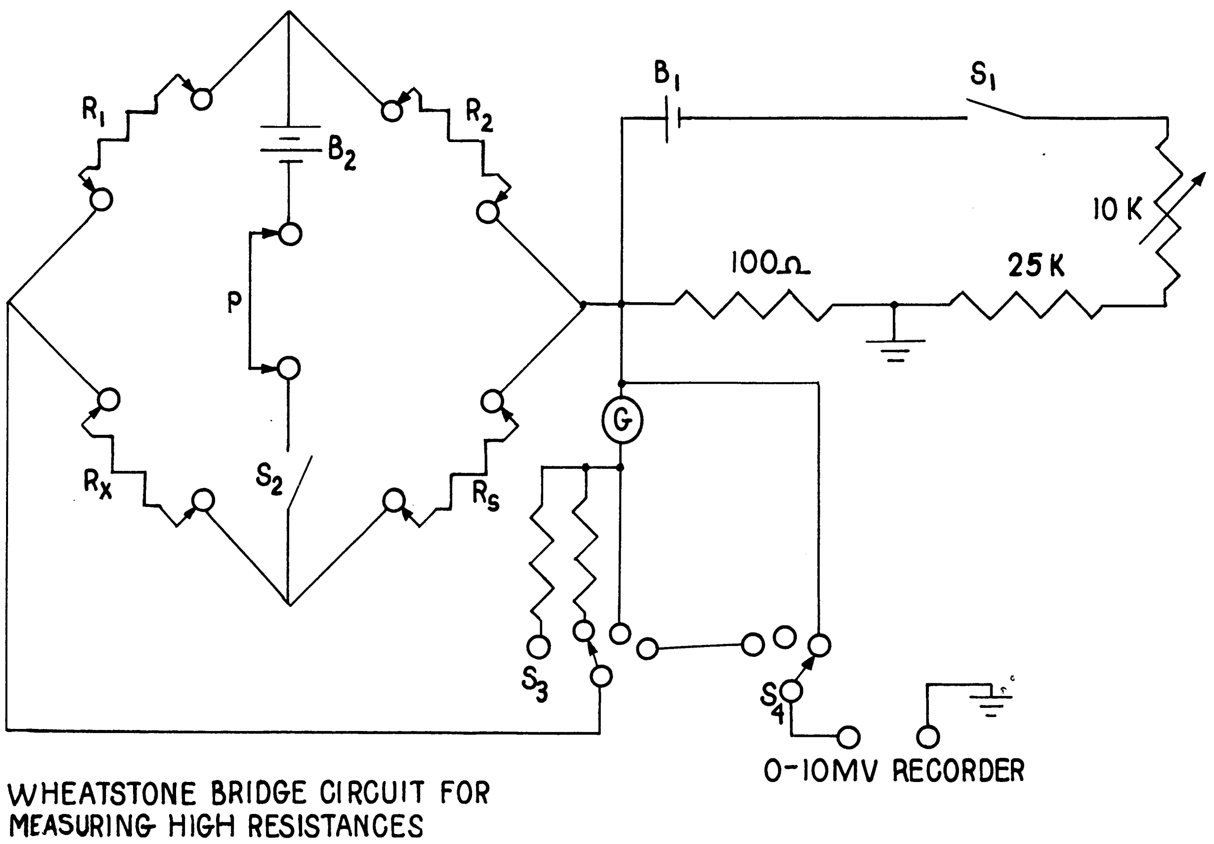


Figure 5-1.

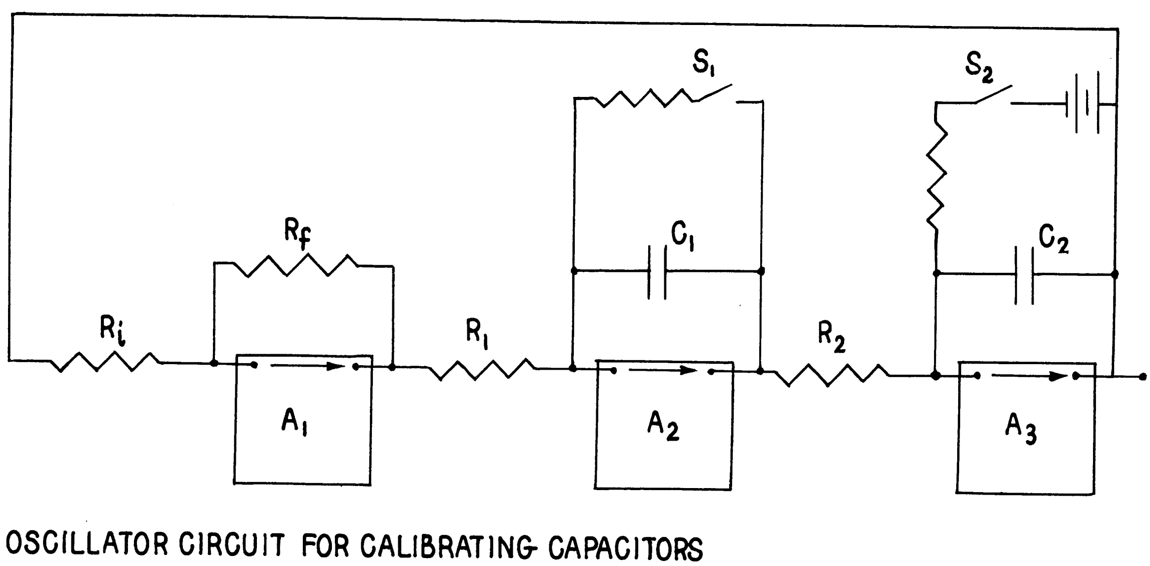


Figure 5-2

resistors which have been carefully calibrated. R_i and R_f are a pair of accurately matched resistors of equal value.

The circuit is an undamped harmonic oscillator representing the equation

$$y + R_1R_2C_1C_2\ddot{y} = 0 ,$$

for which the frequency is given by

$$\omega^2 = \frac{1}{R_1R_2C_1C_2} .$$

The circuit is put into operation by simultaneously opening switches S_1 and S_2, and the output is recorded on a Brush oscillograph. Timing pulses from an accurate source are recorded on the same oscillograph. The timing pulses should be accurate to at least 0.01 percent. In the work associated with this report seconds pulses from WWV were used. From the oscillograph records the period of oscillation is determined. This procedure is repeated for condensers C_1 and C_3, and then for C_2 and C_3.

On the assumption that resistors R_1 and R_2 differ but very little from one megohm and that the capacitors differ but little from one microfarad, we write

$$R_1 = 1 + r_1 , \qquad C_1 = 1 + c_1 ,$$
$$R_2 = 1 + r_2 , \qquad C_2 = 1 + c_2 ,$$
$$C_3 = 1 + c_3 .$$

Also let

$$\tau = 1 + \epsilon = \frac{T}{2\pi} = \frac{1}{\omega} .$$

Then for the first oscillator set up, using C_1 and C_2,

$$\tau_{12} = 1 + \epsilon_{12} = \sqrt{R_1R_2C_1C_2}$$

$$= \sqrt{(1 + r_1)(1 + r_2)(1 + c_1)(1 + c_2)}$$

$$\doteq \sqrt{1 + r_1 + r_2 + c_1 + c_2}$$

$$\doteq 1 + \frac{r_1 + r_2 + c_1 + c_2}{2} .$$

Hence

$$\epsilon_{12} \doteq \frac{r_1 + r_2 + c_1 + c_2}{2} .$$

Similarly,

$$\epsilon_{13} \doteq \frac{r_1 + r_2 + c_1 + c_3}{2} ,$$

and

$$\epsilon_{23} \doteq \frac{r_1 + r_2 + c_2 + c_3}{2} .$$

Solving these last three equations for c_1, c_2, and c_3, we obtain

$$c_1 = \epsilon_{12} + \epsilon_{13} - \epsilon_{23} - \frac{r_1 + r_2}{2} ,$$

$$c_2 = \epsilon_{12} + \epsilon_{23} - \epsilon_{13} - \frac{r_1 + r_2}{2} ,$$

$$c_3 = \epsilon_{13} + \epsilon_{23} - \epsilon_{12} - \frac{r_1 + r_2}{2} .$$

From the observed deviations, ϵ_{12}, ϵ_{13} and ϵ_{23}, of Υ from unity and the known values of r_1 and r_2, the values of c_1, c_2, and c_3 and hence C_1, C_2 and C_3 can be determined.

Having measured the capacitances in this manner it is possible to select resistors to obtain a desired RC product to within 0.01 percent. In selecting these resistors cognizance must be made of the fact that the temperature at the computer may be several degrees higher than room temperature.

5.4 Pulse-Forming Circuit for Checking End Conditions

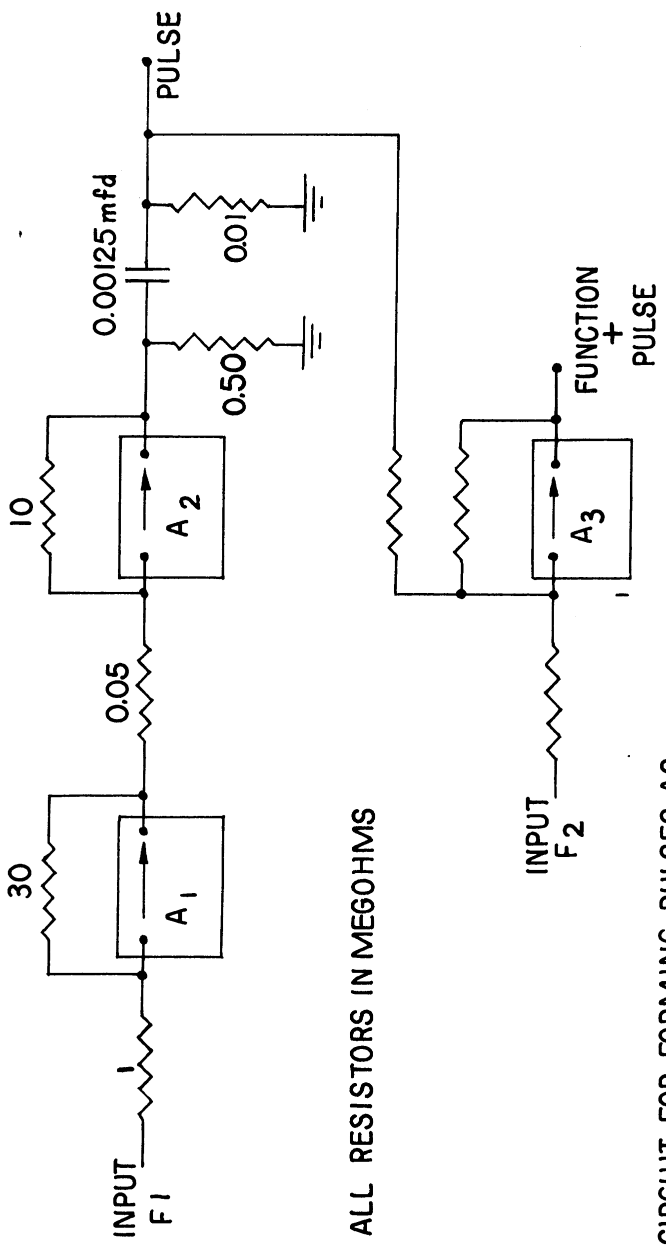
Frequently occasions arise which make it necessary to determine accurately when a function becomes zero, or when two functions become zero simultaneously. Ordinarily a precise observation is difficult because of such effects as recorder pen lag, error in determining the exact point of crossing the zero axis, etc. A method which aids in making more precise measurements is shown in Figure 5-3.

Assume that a function F_1 crosses the zero axis with a finite slope. Amplifier A_1 increases the amplitude by a factor of 30. The 0.05 megohm input resistor to amplifier A_2 acts as a load on amplifier A_1, preventing, due to saturation effects, a large voltage from being developed across the output of A_1. However, the rapid passage through zero of the output voltage is a faithful reproduction of the function F_1 as it passes through zero.

An additional gain of 200 is obtained through amplifier A_2, the output of which is a saturation voltage at all times except for the extremely abrupt passage through zero. The differentiating CR circuit produces a sharp pulse at the exact instant of the passage through zero of the output voltage. This pulse can be recorded directly, or mixed in amplifier A_3 with another function F_2. Using the output of A_3 it is possible to observe on the same record the passage through zero of each of the two functions, F_1 and F_2.

Figure 5-4 shows a sinusoidal input function, F_1. The output of amplifier A_1 is shown in Figure 5-5 and that of A_2 in Figure 5-6. Figures 5-7 and 5-8 show the pulse recorded at paper speeds of 1 division per second and 25 divisions per second, respectively.

Figure 5-9 shows the output of amplifier A_3 when the same function is fed into amplifiers A_1 and A_3. It can be seen that the pulse occurs slightly before the function, as recorded, crosses the zero axis. This effect is caused by pen lag in recording the sinusoidal function. On the same record are shown seconds time pulses as received from WWV. In this particular recording of the



CIRCUIT FOR FORMING PULSES AS FUNCTION GOES THROUGH ZERO

Figure 5-3

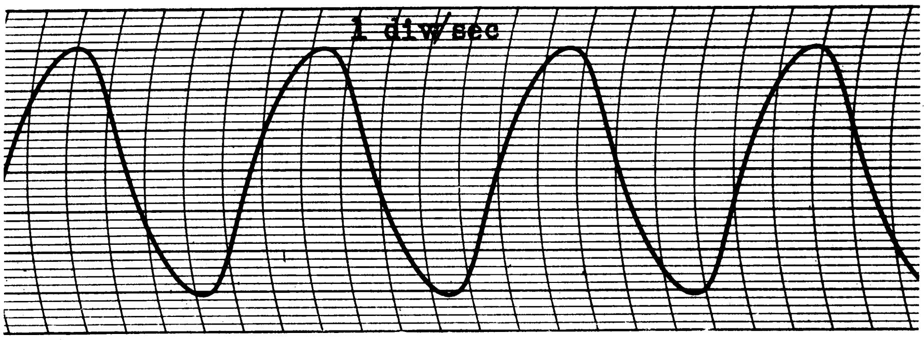


Figure 5-4. Sinusoidal Input Function, F_1

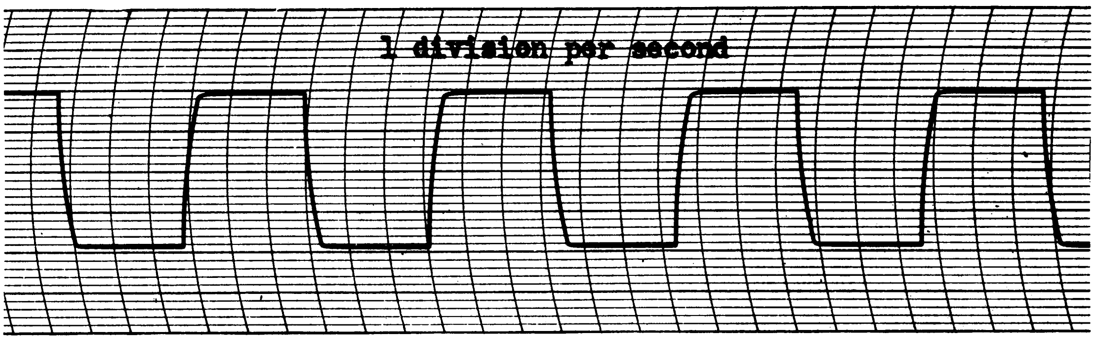


Figure 5-5. Output of Amplifier A_1

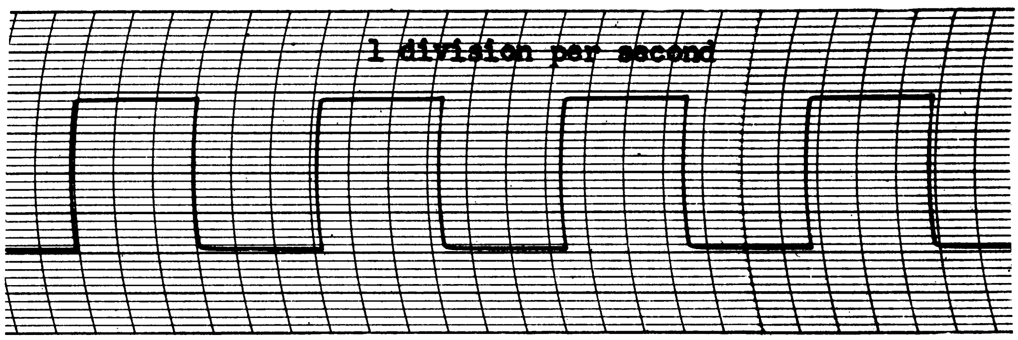


Figure 5-6. Output of Amplifier A_2

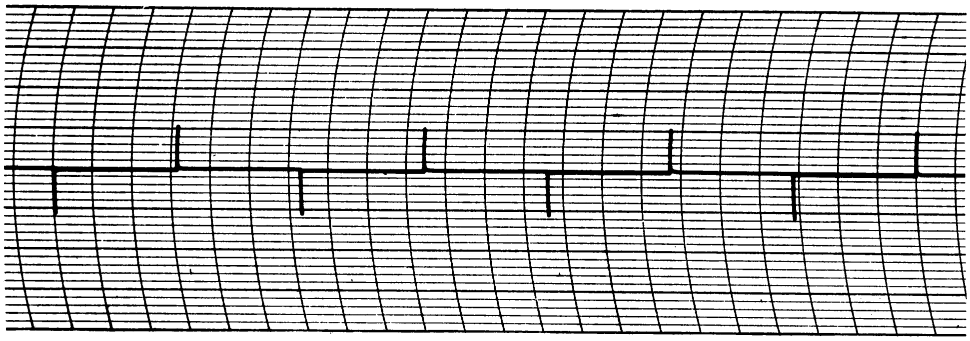


Figure 5-7. Pulse Recorded at a Speed of 5 div/sec

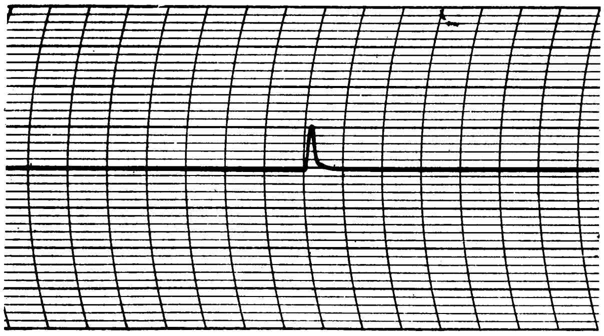


Figure 5-8. Pulse Recorded at a Speed of 25 div/sec

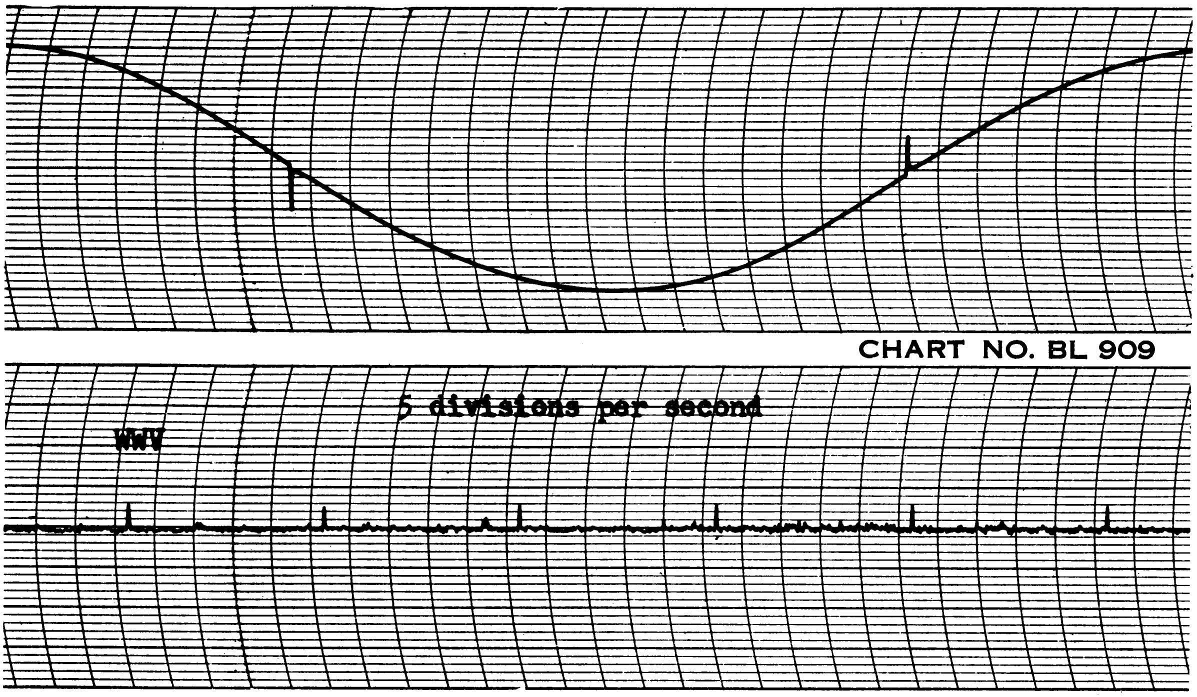


Figure 5-9. Sinesoidal Function with Superimposed "Zero" Pulses, and Seconds Pulses from WWV

time signals the noise level was rather high, but the time pulses are readily distinguishable.

5.5 Use of Time Signals from Radio Station WWV

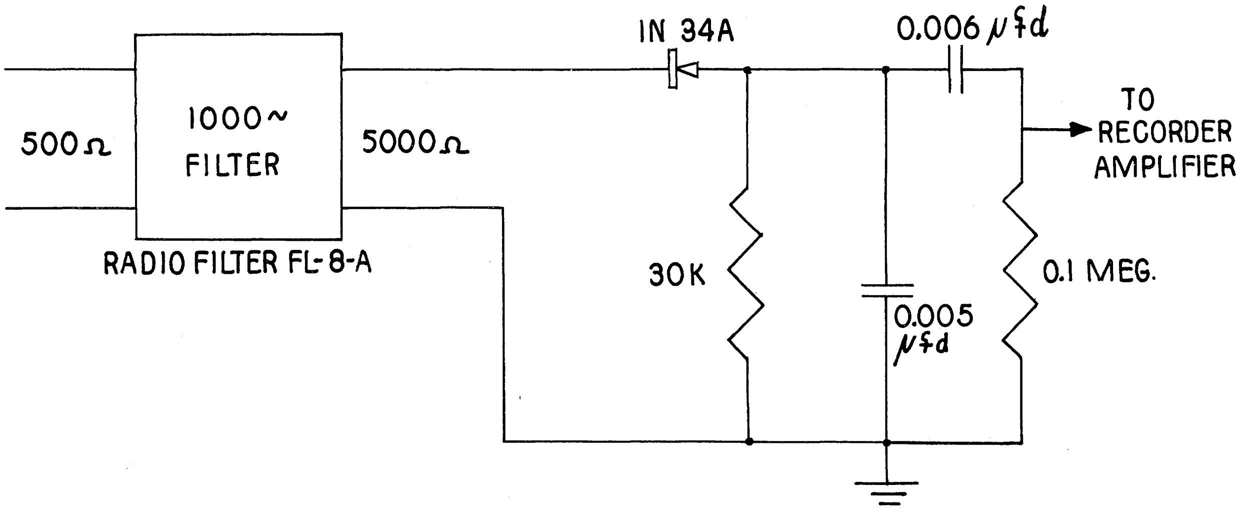
Very accurate timing pulses are needed for several reasons: (1) to aid in the measurement of capacitances, (2) to determine L, the exact time taken by the computer to complete its solution, and (3) to operate stepping relays for changing computer resistors when studying non-uniform beams. By far the most satisfactory method of obtaining accurate timing pulses is to use an oscillating quartz crystal frequency standard or equivalent, as described in a previous report[10]. Such a standard should have an accuracy of at least 0.01 percent under all operating conditions.

Since a standard frequency source was not available for the present work, seconds time signals from Radio Station WWV were used instead. WWV transmits continuously, day and night, standard radio frequencies of 2.5, 5, 10, 15, 20, 25, 30 and 35 megacycles per second with an accuracy of one part in 50,000,000. Each of these carrier frequencies is modulated by three audio frequencies, 1, 440 and 600 cycles per second. A 1,000 cycles per second pulse of 0.005 second duration may be heard as a faint tick every second, except the 59th second of each minute. The time interval marked by the pulse every second is accurate to one microsecond.

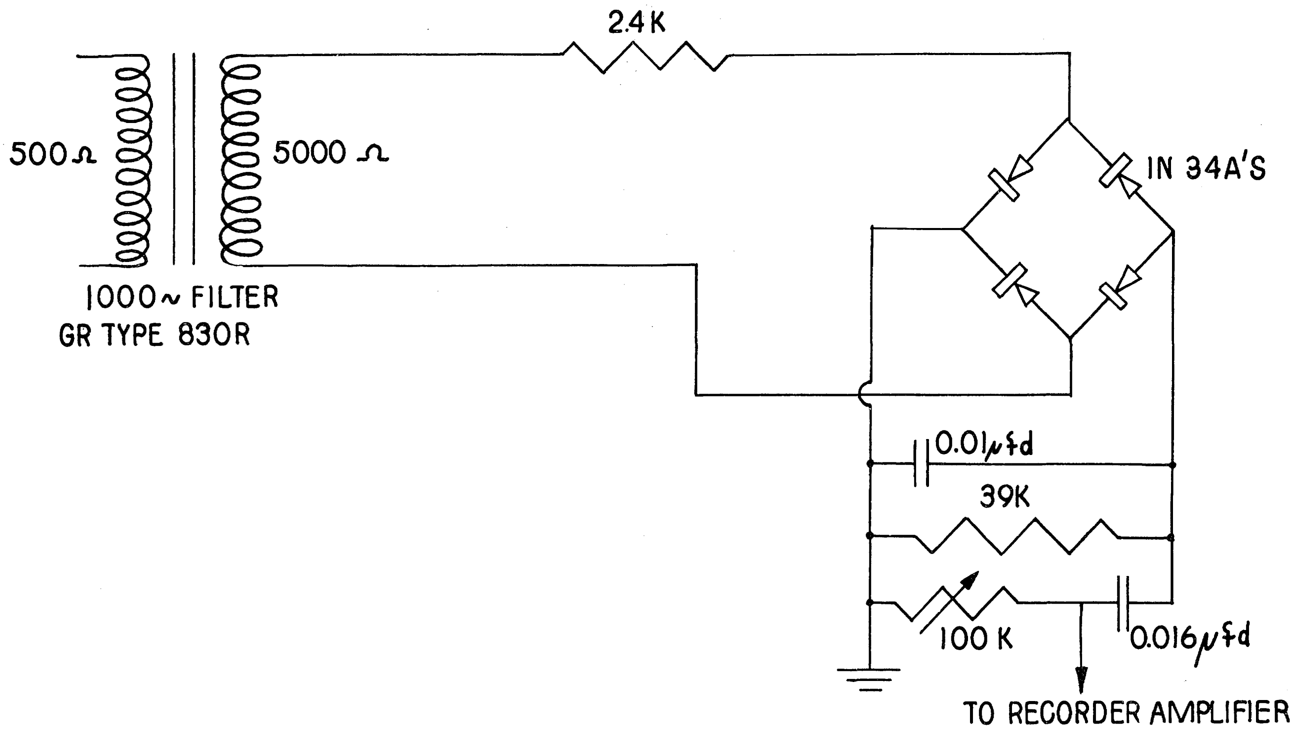
The two audio frequencies of 440 and 600 cycles per second are transmitted alternately for four minutes out of each five minute interval. The audio frequency is interrupted at precisely one minute before each hour and each five minutes thereafter (59th minute; 4 minutes past the hour; 9 minutes past the hour, etc.); the alternate audio frequency is resumed in each case after an interval of precisely one minute. This one minute interval is used to announce Eastern Standard Time and to give certain other information.

In order to receive the seconds-pulses, exclusive of other modulation frequencies, the output of a radio receiver tuned to one of WWV's carrier signals is passed through a 1,000 cycle filter. The output of the filter is rectified and the resulting DC pulse used to place time markers on the oscillograph record.

Figures 5-10a and 5-10b show the circuit arrangements used. In Figure 5-10a a General Radio band pass filter Type 830R is employed. This filter permits



(a)



(b)

Figure 5-10. Circuits for Obtaining Seconds Pulses from WWV the use of a full wave bridge rectifier as shown. In Figure 5-10b the filter is a surplus property Radio Filter FL-8-A. Since this filter has a terminal common to both input and output, a half wave rectifier is used. The networks associated with the filter outputs were not scientifically designed, but merely selected by trial and error to give satisfactory pulses. The time pulses obtained are shown in Figure 5-9.

These time pulses were obtained and used during the latter part of August and the first part of September (1950). At night and during the forenoon suitable signals could usually be picked up on 5 or 10 megacycles; sometimes one carrier frequency was better than the other. After about 2:30 P.M. it was difficult to receive satisfactory signals. For this and other reasons it would be highly desirable to have a local standard frequency source.

5.6 Proposed Electronic Clock

In many problems it is necessary to determine with a high degree of accuracy the length (in time) of a solution obtained by the computer. Thus far the procedure has been to record simultaneously the solution of the problem and accurate time pulses. The time of solution is then determined by measuring (in centimeters) the length of solution shown on the oscillogram and a nearly equal length (in centimeters) covered by an integral number of time pulses. From these two measurements the length (in time) is computed.

Considerable labor could be saved and better accuracy obtained by using an electronic clock for measuring the time it takes for the computer to obtain the solution. This electronic clock would consist of a local standard frequency source and an accurate electronic counting device. The standard frequency source should furnish pulses at the rate of 1,000 per second. The counting device would consist of a set of decade scaling units to indicate elapsed time to the nearest 0.001 second. The Decascale Unit SC-11 of Tracerlab Inc., would serve very well for the basic counting units.

The clock would be started by a pulse at the time the solution is begun and stopped by a pulse at the end of the solution. This latter pulse could be obtained in the manner described in Section 5.4 above. The clock would indicate the length (in time) of the solution without calculation or measurements of length. Figure 5-11 shows a block diagram of the proposed electronic clock.

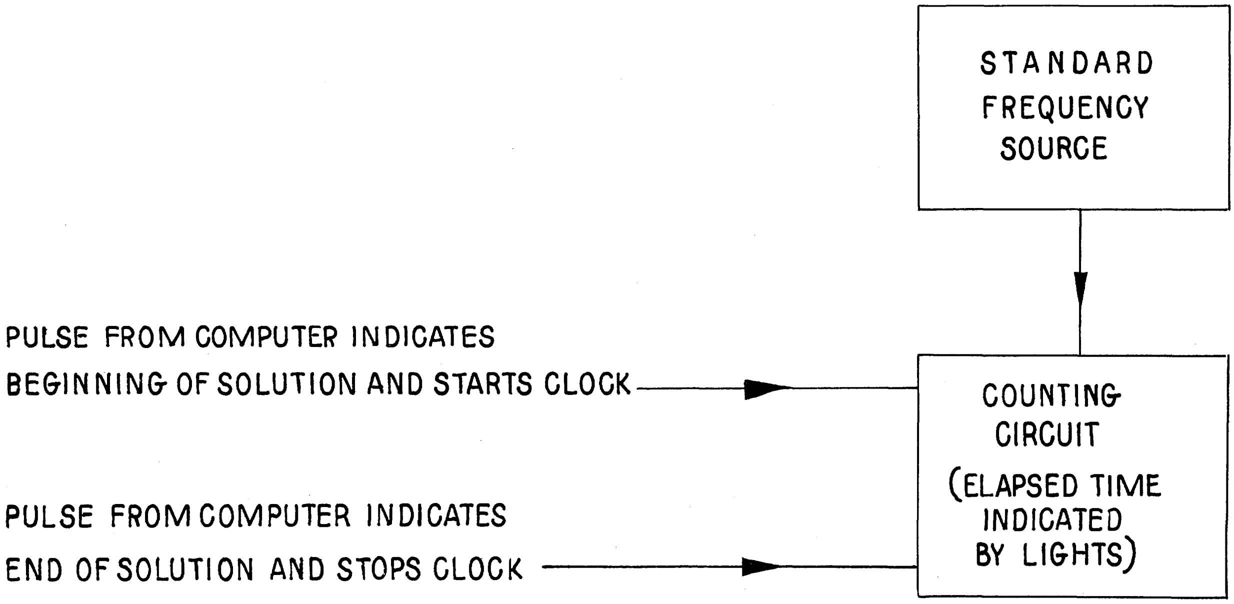


Figure 5-11. Block Diagram for Proposed Electronic Clock

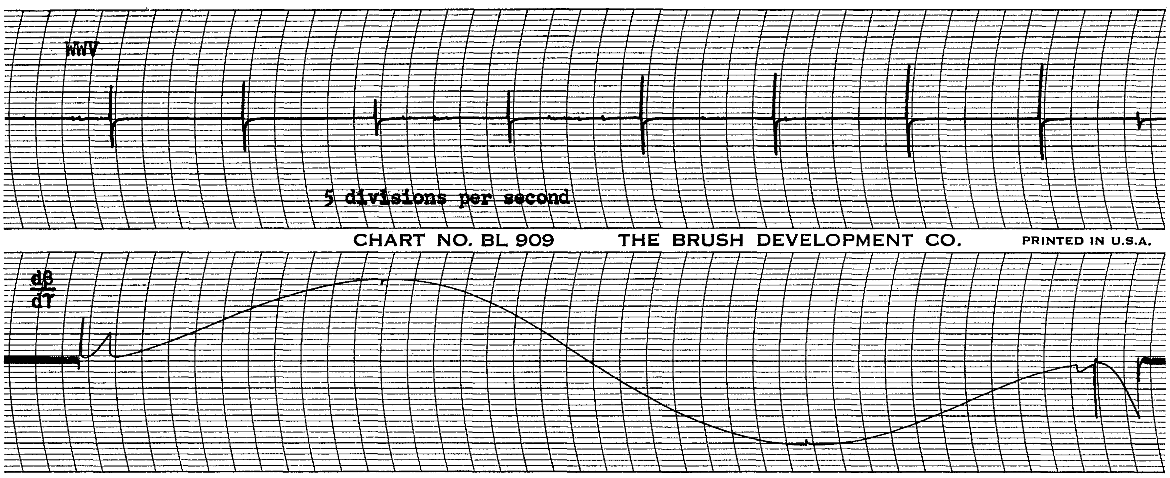


Figure 5-12. Computer Solution for Second Mode, Showing Superimposed "Zero" Pulses and Seconds Pulses from WWV

5.7 Technique for Measuring the Length of the Computer Solution

In order to obtain the frequency parameter λ of the vibrating beam it is necessary to measure the length L of the computer solution (see Sections 3.2 and 3.3). L is defined as the elapsed time from the beginning of the solution to the instant when the proper end conditions are met. For the free-free beam the initial and end (or final) conditions are that $d\beta/d\Upsilon = 0$ and that $\alpha = 0$ (i.e., that shear and bending moment are zero).

The function $d\beta/d\Upsilon$ is recorded from the computer, along with pulses occurring at $\alpha = 0$ (using the technique described in Section 5.4). One of the initial voltages is varied until a solution satisfying the proper end conditions is obtained. In order to determine more accurately when the correct end conditions have been realized, the $d\beta/d\Upsilon$ record is "blown-up" by a factor of ten in the region where it is near zero in value, i.e., at the end of a correct solution. The $d\beta/d\Upsilon$ record is also blown-up by a factor of ten at the beginning of a solution. A correct solution obtained in this manner for a second mode is shown in Figure 5-12. The pulses superimposed on the $d\beta/d\Upsilon$ curve occur at the instant α goes through zero. The length of the solution L is just the time between initial and final pulse. This time is obtained by comparison with the WWV seconds-pulses on the upper channel.

CHAPTER 6

PROPOSED WORK

In addition to continuing the investigation of the general utility of the electronic differential analyzer for obtaining solutions to theoretical and physical problems, there are a number of specific lines of investigation suggested by the work done thus far, including the following:

(1) the publication of a complete set of frequency and mode-shape curves for all types of uniform beams,

(2) an experimental study of the proper end conditions for a cantilever beam,

(3) the comparison of experimentally determined eigen-frequencies for vibrating uniform beams with the eigen-frequencies obtained with the computer,

(4) a check on the accuracy obtained in using the electronic computer to determine solutions for non-uniform beams,

(5) the degree of validity of the basic equations of Chapter 1 for eigen-vibrations having wave-lengths comparable to the transverse dimensions of the beam, and

(6) a complete and comprehensive bibliography of the work which has been done on the problem of vibrating beams.

6.1 A Complete Set of Frequency and Mode-Shape Curves for all Types of Uniform Beams

In this report we have included data and families of curves giving the relation between frequency parameter λ and shear and rotary inertia parameters S and N for the first three "normal modes" of vibration of free-free beams. A definite contribution would be made by obtaining similar data and curves for all types of uniform beams. Including the one already done, these would be:

(1) Free-Free Beam,

(2) Clamped-Free Beam,

(3) Hinged-Free Beam,

(4) Clamped-Clamped Beam,

(5) Clamped-Hinged Beam, and

(6) Hinged-Hinged Beam.

Although some of the above types of beams are more important than others, it is suggested, for the sake of completeness, that data and curves for all types be obtained.

In determining the data for the free-free-beam curves as given in this report the values for rotary inertia parameter N (0.0, 0.1, 0.2 and 0.3) were selected somewhat arbitrarily. It is desirable to determine values of N for beams of various materials and forms of cross-section. The cross-sectional forms should include cylinders, hollow cylinders, rectangles, I shapes, U shapes, etc. From the values of N found in this way one can determine the most suitable values to be included in the data for the families of frequency curves. Publication of these data and curves, together with typical mode shape curves, is desirable.

6.2 An Experimental Study of the Proper End Conditions for the Clamped End of a Beam

There seems to be some question concerning the correct expressions for the type of restraint placed on a beam end when it is "clamped". When the effects of rotary inertia and transverse shear force are neglected, the boundary conditions at the built-in end are assumed to be $y = 0$, and $dy/dx = 0$. When the effects of rotary inertia and transverse shear force are included, several authors[7,8] have given the boundary conditions as $y = 0$ and $\beta = 0$, where β is the neutral axis slope due to bending moment.

This latter condition means that dy/dx, the slope of the neutral axis, has a finite value at the built-in end of the beam. This condition seems to contradict the usual definition of a clamped beam. However, in general there will be distortion in the built-in portion of the beam so that the assumption made in deriving our basic equations of Chapter 1, namely that the planes of flexure remain parallel, will no longer be valid. In order to solve the problem rigorously we must consider carefully not only the elastic properties of the beam but also of the supporting wall. To apply the equations of Chapter 1 with either end condition ($y = \beta = 0$ or $y = dy/dx = 0$) is an oversimplification. The problem is not merely academic, since preliminary results with the electronic computer indicate a difference of about 20 percent in eigen frequencies of the first mode, for a l/S value of 11, depending upon which end condition is used.

One interesting sidelight to this problem is that Dolph has shown[13] the $\beta = 0$ end condition is the only one which leads to orthogonal solutions when the effect of rotary inertia is included.

A direct experimental approach to the problem is to construct a small uniform cantilever beam and experimentally to observe its eigen-frequencies. These values should then be compared with those obtained by calculations using the two proposed sets of end conditions.

To compute the eigen-frequencies from theoretical considerations it is necessary to know the values of the modulus of elasticity, E, and the modulus of rigidity, G. It is suggested that these moduli be determined dynamically, following the general procedure given by Goens[4]. Attention is called to the fact that static and dynamic determinations of G differ by approximately one percent.

Little difficulty should be experienced in exciting transverse and longitudinal vibrations in a steel bar by an audio oscillator with an electromagnetic drive. The excitation of torsional vibrations cannot be obtained as easily. One method worth considering is the use of a fluctuating, as opposed to a rotating, magnetic field. This fluctuating field could be obtained by means of a four pole stator, similar to that used in a small two phase motor. One pair of opposing poles should be excited by direct current, the other pair by alternating current. One would expect the round rod, acting as a rotor, to receive oscillatory torque impulses of the same frequency as the alternating current applied to the stator, and that relatively large torsional oscillations would be set up in the rod when the correct frequency is applied.

In setting up the experimental cantilever beam care must be taken to imbed the beam tightly in the "wall" at all points. The mass into which the beam is imbedded should be so large that no measurable vibration of the wall mass takes place.

A somewhat less direct method, but one which might possibly give more enlightenment on the conditions which exist in the built-in portion of the beam, involves the use of the electronic computer for solving non-uniform beams. Consider the cantilever beam as consisting of a uniform rod of relatively small diameter imbedded in another uniform beam of much larger diameter. By obtaining a series of solutions for greater and greater sizes of this large beam it is possible to approach the ideal case of having the smaller beam clamped in a rigid massive wall. From the observed curves of y, dy/dx, α, and β there should be some indication of the conditions which exist when the mass of the "wall" becomes infinite.

6.3 Comparison of Experimental and Computer Values of Eigen-Frequencies

Goens[4] has made some very precise experimental measurements on vibrations of free-free cylindrical steel bars and on an aluminum bar of rectangular cross-section. In most cases the necessary data is given to four significant figures. The comparison of his results with those obtained by the electronic differential analyzer would serve as a good check on the validity of the equations.

Alternately, the check could be made with vibrating beams set up and measured in the laboratory.

Dolph[13] has shown that when the effect of rotary inertia is included, the solutions are in general not orthogonal. Dolph further argues that when rotary inertia effects are included, equations (1-1) to (1-5) are fourth order in both time and displacement along the beam, resulting in four boundary and four initial conditions. But the standing wave hypothesis involved in separating the time variable from equations (1-1) to (1-5) allows only two initial conditions in time to be satisfied. Thus the validity of equations (1-1) to (1-5) might seriously be questioned. Good laboratory checks on beam frequencies, where rotary inertia effects are quite appreciable, ought to throw considerable light on this whole question.

6.4 Checking the Computer for the Solutions of Non-Uniform Beams

It is assumed that a non-uniform beam may be represented approximately by a finite number of equal length uniform beams having appropriate dimensions and properties. It is suggested that small non-uniform beams be constructed and that the experimentally determined eigen-frequencies be compared with those obtained by using the electronic computer. At least one of the beams should consist of a series of equal length uniform beam increments. In this case the computer solution would be made without approximations and a direct check could be obtained between frequencies determined by experiment and by the computer.

In addition a beam should be constructed of continuously varying size, preferably with symmetry about the center of the beam. For solution by the computer this beam would be approximated by a finite number of uniform beam increments. Frequency comparisons would indicate the accuracy with which the computer gives the frequency when operating on the approximated equations.

Another check on the computer would be to use the approximation data available for the ship APA87. This ship is 400 feet long and is considered divided into 20 equal-length sections for each of which are given the necessary properties and dimensions. This ship problem has been solved by several other methods.

6.5 Validity of Basic Equations for Eigen-Vibrations of Wave-Lengths Comparable to Transverse Dimensions of Beams

For higher and higher modes of vibration of a beam the corresponding wave-lengths become smaller and smaller. The question arises as to what happens when the wave-lengths become comparable in size to the transverse dimensions of the beam. Goens[4] attacks the problem analytically and shows that above a certain critical frequency limit a change in the type of oscillation should develop. He states that Giebe and Scheibe, working with prismatic quartz bars of a few millimeters thickness, obtained limiting frequencies which, from data unpublished at the time of writing his paper, seemed to be somewhat smaller than the above mentioned critical frequency. Goens raises the question as to whether this limiting frequency has any physical meaning, even under the assumption that the basic equations give at least a qualitative description of the vibration process under such extreme conditions.

It would be of interest to determine to what extent the basic equations used in the present approach to the beam problem are valid when the wave-lengths become comparable in size to the transverse dimensions of the beam.

6.6 The Need for a Complete and Comprehensive Bibliography

Originally there was no thought of making the study of vibrating beams a major project associated with the computer. The problem was presented to the group working with the electronic differential analyzer as the solution of a specific fourth order differential equation subject to certain boundary conditions for a free-free beam. The problem was welcomed as an opportunity to show what the computer could do.

The computer, itself, demonstrated that the assumed end conditions were in error. Furthermore it was soon discovered that the use of the fourth order equation for non-uniform beams neglected many derivatives which are important. However, after some investigation it was found that by using the basic relations from which the fourth order equation was derived the computer could obtain solutions without neglecting any derivatives.

If, before attempting to solve the fourth order differential equation by the computer, adequate reference had been made to easily available literature, it is probable that neither of these problems would have arisen. Subsequently, enough reference has been made to various articles to find that there is available a considerable amount of information and data which would be helpful in continuing work on the general problem of vibrating beams. Among the vast amount of work which has been done on vibrating quartz crystals there is considerable information directly applicable to the present problem.

If work is continued on vibrating beams, it is suggested that a complete bibliography, with abstracts, be compiled.

APPENDIX 1

SUMMARY OF NORMAL MODE FREQUENCIES FOR A FREE-FREE BEAM, INCLUDING TRANSVERSE SHEAR AND ROTARY INERTIA FORCES

Notation (see Chapter 1)

$$\lambda = \omega \ell^2 \sqrt{\frac{\rho}{EI}}$$

$$S = \frac{EI}{kAG\ell^2}$$

$$N = \frac{kG}{E}$$

$\lambda_o = \lambda$ for $S = N = 0$ (i.e., for an infinitely-long beam)

L = length of computer solution in seconds

FIRST MODE

"FREE-FREE" BEAM

$\lambda_o = 22.373$

$\frac{1}{\lambda_o} = 0.0446967$

$\frac{1}{SL^2}$	N	$\lambda = L^2$	$\frac{1}{S}$	$\frac{\lambda}{\lambda_o} \times 100$
29.55	0.0	22.203	656.1	99.240
	0.1	22.099	653.0	98.775
	0.2	22.034	651.1	98.484
	0.3	21.930	648.0	98.019
19.83	0.0	22.090	438.0	98.735
	0.1	21.977	435.8	98.229
	0.2	21.837	433.0	97.604
	0.3	21.697	430.3	96.978
14.92	0.0	21.977	327.9	98.229
	0.1	21.809	325.4	97.479
	0.2	21.678	323.4	96.893
	0.3	21.511	320.9	96.147
9.92	0.0	21.772	216.0	97.313
	0.1	21.520	213.5	96.187
	0.2	21.326	211.6	95.320
	0.3	21.114	209.5	94.372
6.958	0.0	21.548	149.9	96.312
	0.1	21.197	147.5	94.743
	0.2	20.867	145.2	93.268
	0.3	20.593	143.3	92.043
4.946	0.0	21.197	104.84	94.743
	0.1	20.739	102.58	92.696
	0.2	20.349	100.64	90.953
	0.3	19.927	98.56	89.067

$\frac{1}{SL^2}$	N	$\lambda = L^2$	$\frac{1}{S}$	$\frac{\lambda}{\lambda_o} \times 100$
3.982	0.0	20.867	83.09	93.268
	0.1	20.412	81.28	91.234
	0.2	19.909	79.28	88.986
	0.3	19.430	77.37	86.845
2.977	0.0	20.421	60.79	91.275
	0.1	19.803	58.95	88.512
	0.2	19.193	57.14	85.786
	0.3	18.602	55.38	83.144
1.990	0.0	19.483	38.77	87.082
	0.1	18.697	37.21	83.569
	0.2	17.893	35.61	79.975
	0.3	17.148	34.12	76.645
1.502	0.0	18.636	27.99	83.296
	0.1	17.707	26.60	79.144
	0.2	16.810	25.25	75.135
	0.3	15.928	23.92	71.192
0.9963	0.0	16.991	16.928	75.944
	0.1	15.952	15.893	71.300
	0.2	14.900	14.845	66.598
	0.3	13.883	13.832	62.052

SECOND MODE

"FREE-FREE" BEAM

$\lambda_o = 61.6696$

$\frac{1}{\lambda_o} = 0.01621544$

$\frac{1}{SL^2}$	N	$\lambda = L^2$	$\frac{1}{S}$	$\frac{\lambda}{\lambda_o} \times 100$
29.55	0.0	60.934	1800.6	98.807
	0.1	60.746	1795.0	98.502
	0.2	60.575	1790.0	98.225
	0.3	60.482	1787.2	98.074
19.83	0.0	60.528	1200.2	98.148
	0.1	60.187	1193.5	97.595
	0.2	60.063	1191.0	97.394
	0.3	59.939	1188.6	97.193
14.92	0.0	60.125	897.1	97.495
	0.1	59.815	892.4	96.992
	0.2	59.537	888.3	96.541
	0.3	59.213	883.5	96.016
9.92	0.0	59.367	588.9	96.266
	0.1	58.890	584.2	95.492
	0.2	58.446	579.8	94.772
	0.3	57.912	574.5	93.906
6.958	0.0	58.446	406.7	94.772
	0.1	57.821	402.3	93.759
	0.2	57.199	398.0	92.750
	0.3	56.475	393.0	91.576
4.946	0.0	57.184	282.8	92.726
	0.1	56.355	278.7	91.382
	0.2	55.413	274.1	89.854
	0.3	54.597	270.0	88.531

$\frac{1}{SL^2}$	N	$\lambda = L^2$	$\frac{1}{S}$	$\frac{\lambda}{\lambda_o} \times 100$
3.982	0.0	56.190	223.7	91.114
	0.1	55.145	219.6	89.420
	0.2	54.081	215.4	87.694
	0.3	52.984	211.0	85.915
2.977	0.0	54.361	161.83	88.148
	0.1	53.086	158.04	86.081
	0.2	51.926	154.58	84.200
	0.3	50.652	150.79	82.134
1.990	0.0	51.094	101.68	82.851
	0.1	49.421	98.35	80.138
	0.2	47.734	94.99	77.402
	0.3	46.050	91.64	74.672
1.502	0.0	48.066	72.20	77.941
	0.1	46.158	69.33	74.847
	0.2	44.209	66.40	71.686
	0.3	42.250	63.46	68.510
0.9963	0.0	42.576	42.42	69.038
	0.1	40.284	40.13	65.322
	0.2	37.896	37.76	61.450
	0.3	35.355	35.22	57.329

THIRD MODE

"FREE-FREE" BEAM

$\lambda_o = 120.890$

$\frac{1}{\lambda_o} = 0.008271983$

$\frac{1}{SL^2}$	N	$\lambda = L^2$	$\frac{1}{S}$	$\frac{\lambda}{\lambda_o} \times 100$
29.55	0.0	119.40	3528	98.767
	0.1	119.18	3522	98.585
	0.2	118.81	3511	98.279
	0.3	118.61	3505	98.113
19.83	0.0	118.77	2355	98.246
	0.1	118.16	2343	97.741
	0.2	117.81	2336	97.452
	0.3	117.57	2331	97.253
14.92	0.0	117.74	1756.7	97.394
	0.1	117.25	1749.4	96.989
	0.2	116.62	1740.0	96.467
	0.3	116.25	1734.4	96.161
9.92	0.0	116.23	1153.0	96.145
	0.1	115.30	1143.8	95.375
	0.2	114.40	1134.8	94.631
	0.3	113.53	1126.2	93.911
6.958	0.0	114.08	793.8	94.366
	0.1	113.06	786.7	93.523
	0.2	111.98	779.2	92.629
	0.3	110.97	772.1	91.794
4.946	0.0	111.43	551.1	92.174
	0.1	109.96	543.9	90.958
	0.2	108.56	536.9	89.800
	0.3	107.02	529.3	88.526

$\frac{1}{SL^2}$	N	$\lambda = L^2$	$\frac{1}{S}$	$\frac{\lambda}{\lambda_o} \times 100$
3.982	0.0	109.29	435.2	90.404
	0.1	107.45	427.9	88.882
	0.2	105.78	421.2	87.501
	0.3	104.04	414.3	86.061
2.977	0.0	105.33	313.6	87.128
	0.1	103.33	307.6	85.474
	0.2	101.24	301.4	83.475
	0.3	99.14	295.1	82.008
1.977	0.0	98.66	195.05	81.611
	0.1	95.90	189.59	79.328
	0.2	93.20	184.26	77.094
	0.3	90.44	178.80	74.811
1.502	0.0	92.37	138.74	76.408
	0.1	89.36	134.22	73.918
	0.2	86.27	129.58	71.362
	0.3	83.14	124.88	68.773
0.9963	0.0	81.07	80.77	67.060
	0.1	77.67	77.38	64.248
	0.2	74.11	73.84	61.303
	0.3	70.34	70.08	58.185

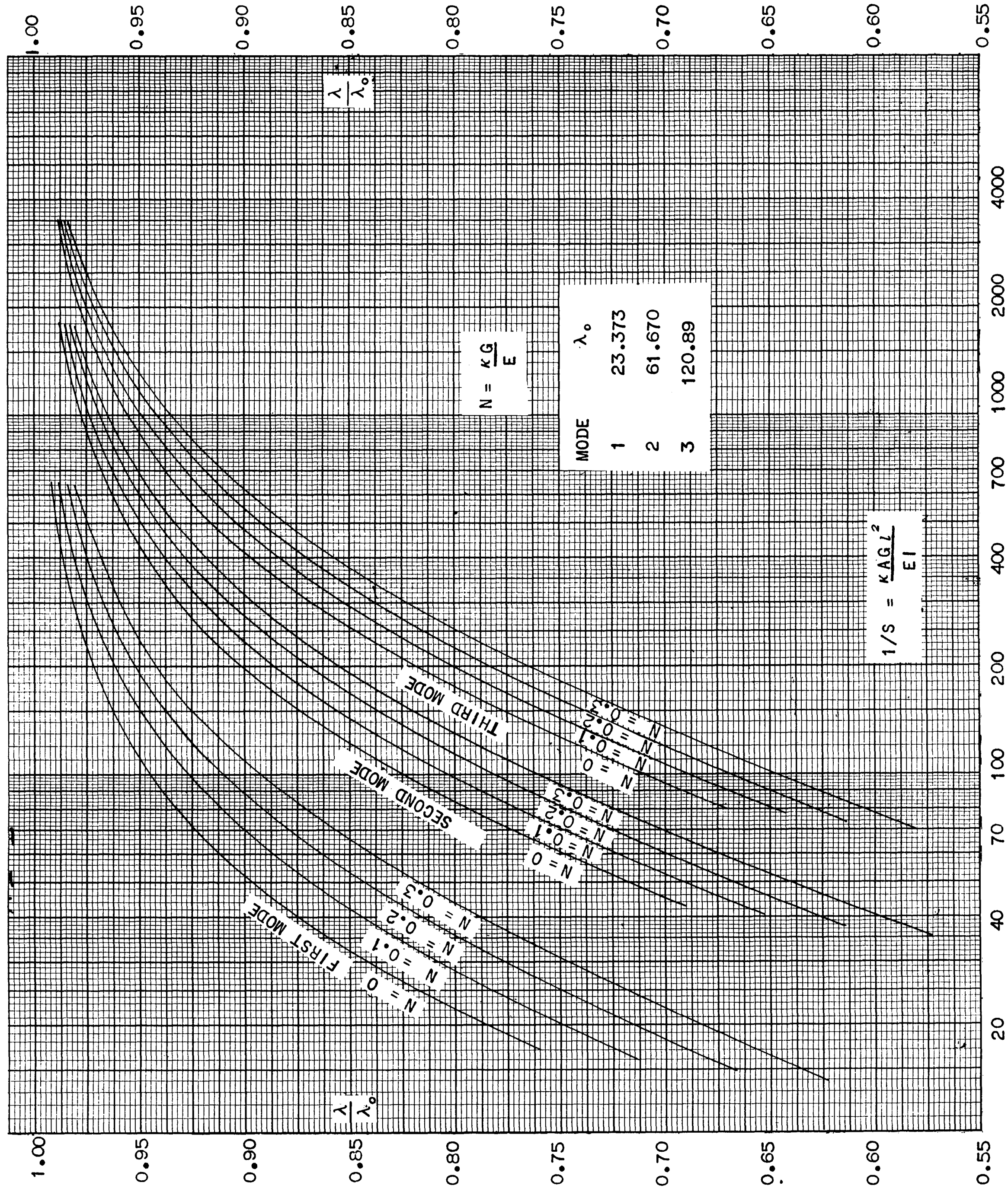


Figure A1-1. λ/λ_o for the First Three Modes of a Free-Free Beam

APPENDIX 2

DESCRIPTION OF THE OLD DC AMPLIFIER

The work covered in this report was carried on using the dc amplifier shown in Figure A2-1. This circuit is similar to one presented by Ragazzini et al[12] and has been used with very satisfactory results in the electronic differential analyzer.[9,10] It has an overall dc gain of about 40,000, which is completely adequate for the problems solved thus far. This amplifier is liable to drift off of balance over extended periods of time.

The amplifier chassis is pictured in Figure A2-2. Balance potentiometer controls are provided, along with jacks for plugging in feedback or input impedances. A more complete description of these amplifiers is given in a previous report.[9]

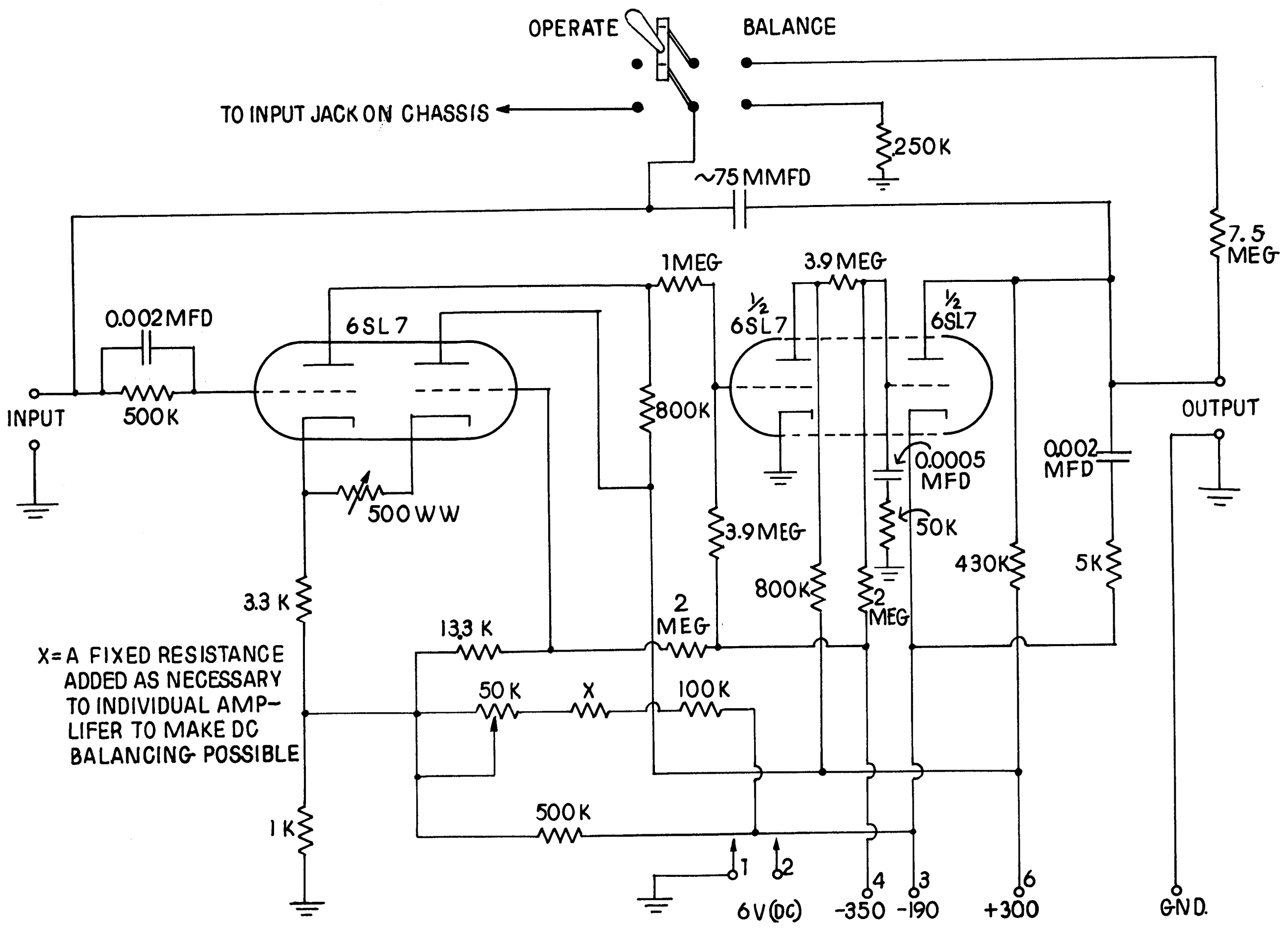


Figure A2-1. Old DC - Amplifier Circuit

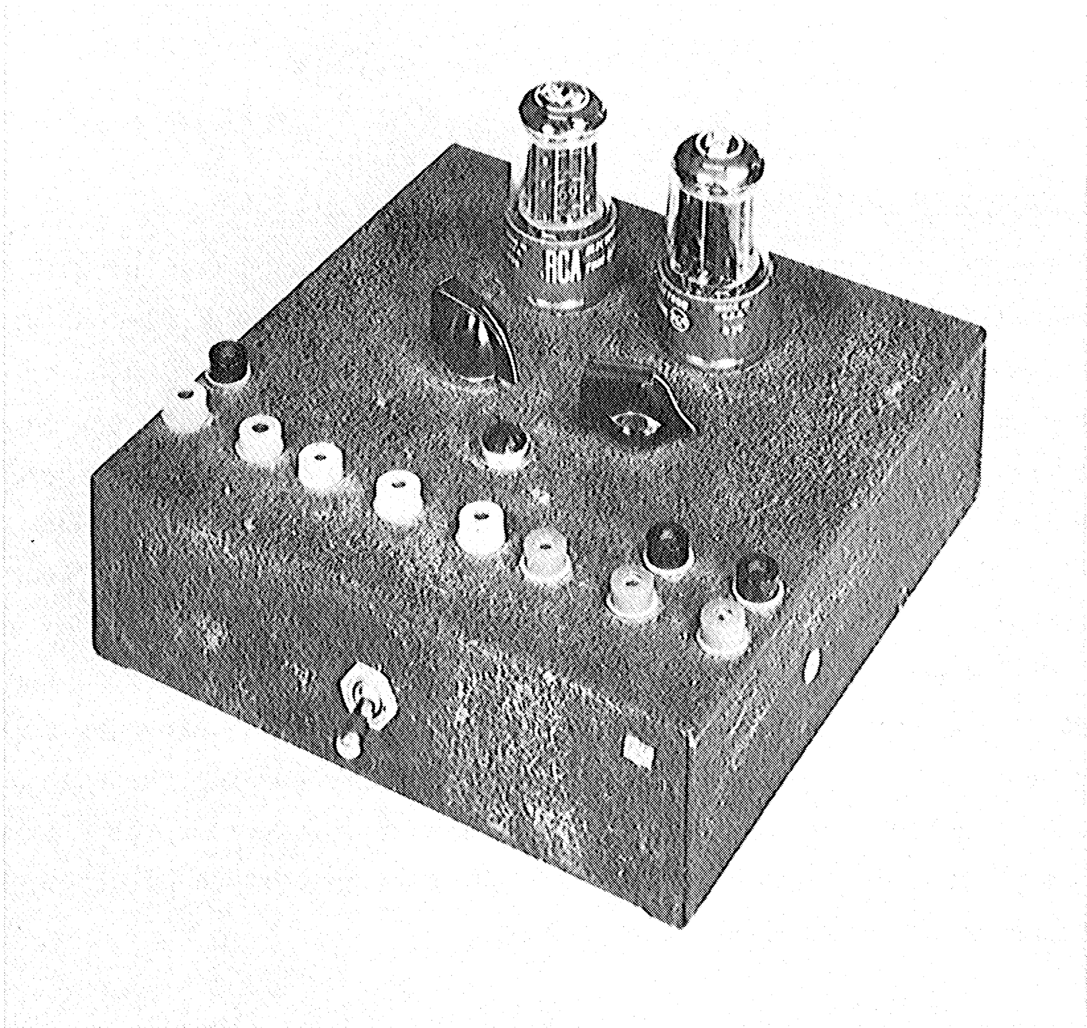


Figure A2-2. Operational Amplifier Chassis

APPENDIX 3

THE DRIFT-STABILIZED DC AMPLIFIER

A3-1 Importance of Low-Drift Operation

The dc amplifier described in Appendix 2 and used for the work covered in this report is liable to drift off of balance over extended periods of time. This drift may be due to very slight changes in power supply voltages or in amplifier components. In general the drift is quite small during the several seconds required for a computer solution, particularly when the feedback is enough so that the operational amplifier is operated at a gain of about unity. However, it only takes a zero-drift of the order of a millivolt to affect noticeably the behavior of higher-mode solutions (see Section 3.4b).

For many types of problems it is desirable to run solutions for several minutes at a time (this may be necessary, for example, if a recorder of low frequency response is used). It may even be necessary to stop the solution midway through a problem, perhaps to read the computer voltage outputs with a potentiometer. Low-drift dc amplifier operation is an obvious requirement if either of the above procedures are employed.

A3-2 General Description of the Drift-Stabilized DC Amplifier

An ingenious method for practically eliminating dc drift was worked out by RCA and Leeds and Northrup, and is shown schematically in Figure A3-1. The arrangement consists of a conventional dc amplifier of gain A and having input and feedback impedances Z_i and Z_f respectively. The dc amplifier actually has two isolated inputs, at P and P'. The net input voltage is the sum of the two voltages appearing at these points. The potential at the point P is fed through a 60 cycle vibrator, chopped into ac, amplified, and reconverted to dc. This amplified dc signal is fed into the dc amplifier at P'. From here it gets amplified through the dc amplifier and fed back through Z_f to P, which is driven back to zero potential (relative to ground).

The function of the additional drift-free loop, therefore, is to maintain zero potential at the junction point P. If ideally met, this condition

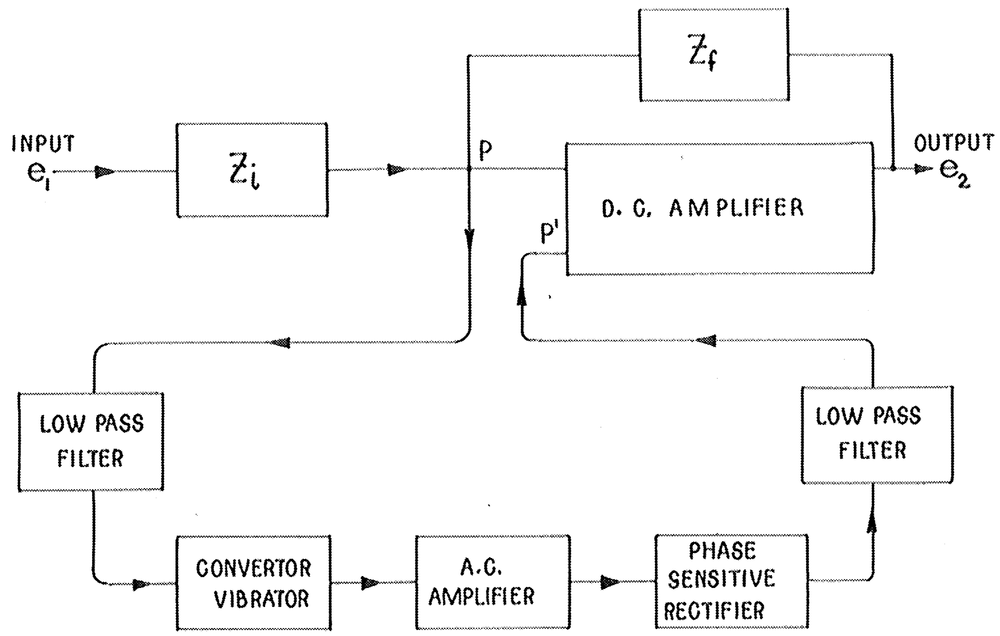


Figure A3-1. Drift-Stabilized DC Amplifier

guarantees that $e_2 = - Z_f/Z_i\ e_1$, where e_1 is the input voltage, e_2 is the output voltage. Thus the amplifier is kept in balance, since if $e_1 = 0$, then $e_2 = 0$.

It can be shown that the effective gain of the dc amplifier and ac loop system is A + AG, where A is the gain of the dc amplifier and G is the gain of the auxiliary, drift-free loop. The frequency response of the auxiliary loop begins to decrease at around 0.1 cps, so that this drift-stabiling loop can only correct for slow drifts. At higher frequencies for which $G \rightarrow 0$, the net gain of the system is still A, while for low frequencies, where $G >> 1$, the net gain is $A(G + 1)$. Thus for the actual amplifier described here, where A = 40,000 and G = 2000, the effective gain of the system at low frequencies (<0.1 cps) is about 80,000,000.

A3-3 Circuit Description of the Drift-Stabilized DC Amplifier

The actual circuit employed for the drift-stabilized dc amplifier is shown in Figure A3-2. This circuit is essentially similar to one developed by the Rand Corporation for their electronic differential analyzer.

The input proper and error-signal input are fed into the dc amplifier through cathode followers (each half of a 6SU7). Three stages of amplification follow using a 5691, 12SJ7, and 1631 tube respectively. The 6SU7 and 5691 tubes

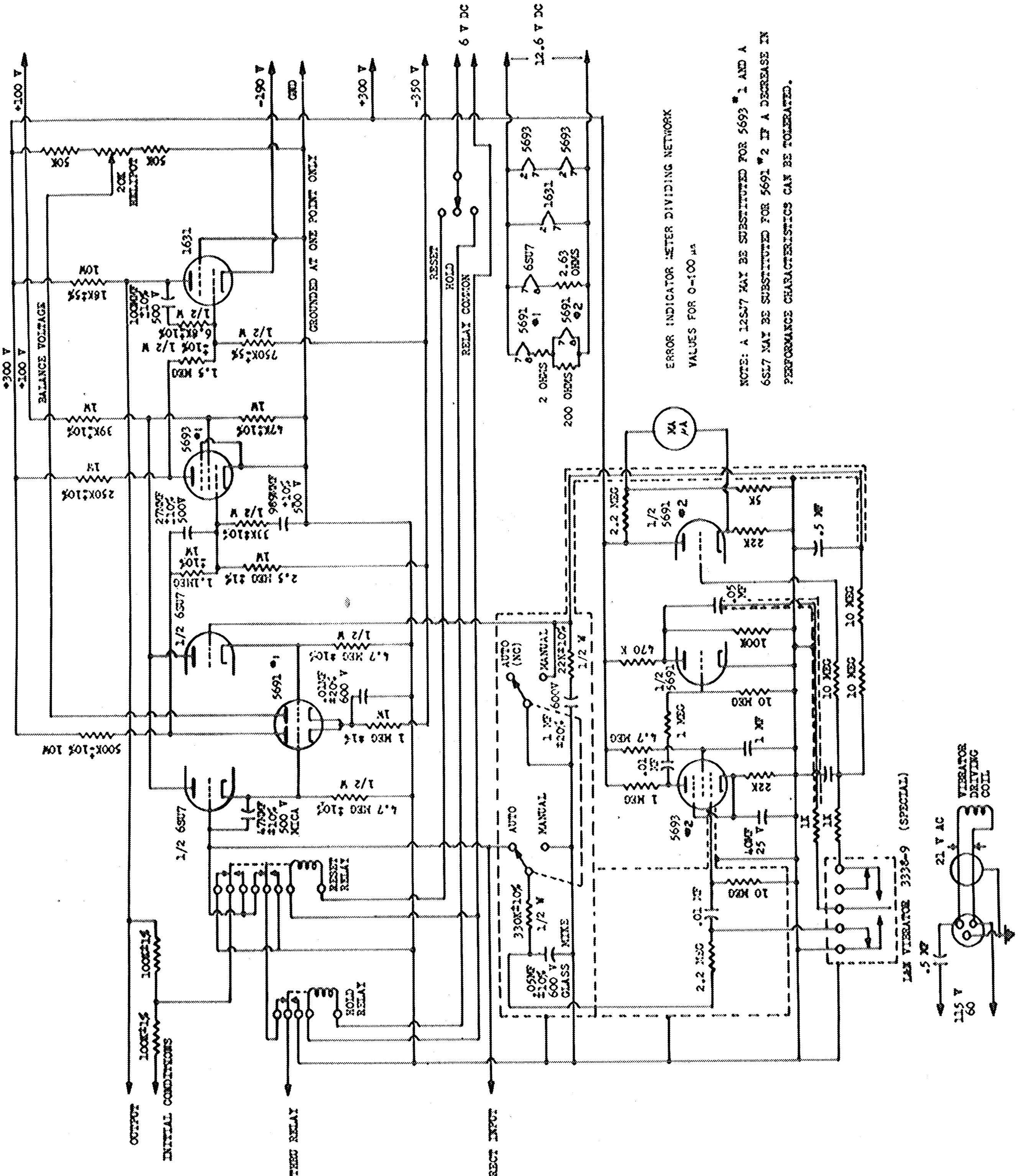


Figure A3-2. Circuit for Drift-Stabilized DC Amplifier

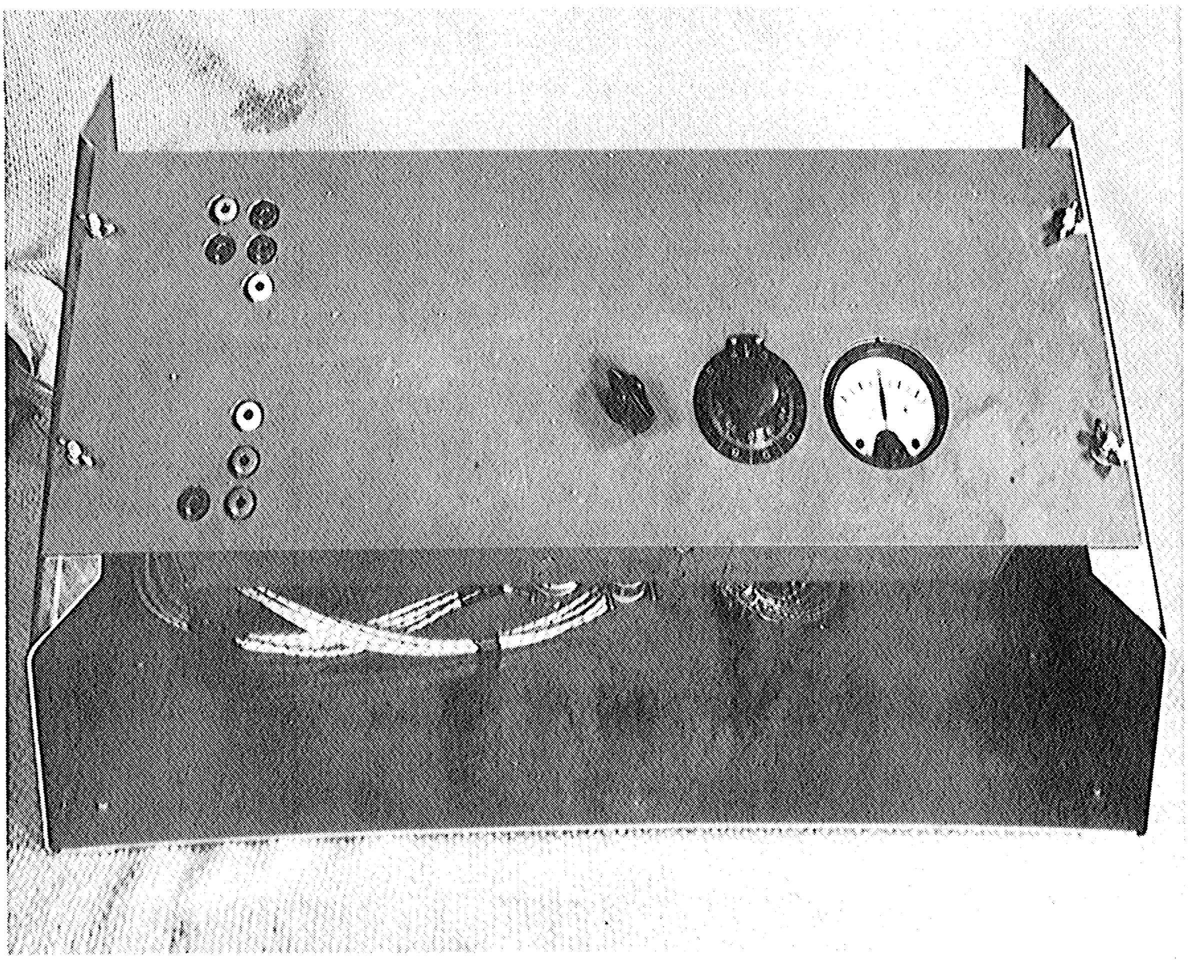


Figure A3-3. Panel for Drift-Stabilized DC Amplifier are operated at reduced filament voltages of 4.2 and 5.25 volts respectively. This prevents the flow of appreciable current to the grids.

A Leeds and Northrup Model 3338-9 special vibrator is used to chop the error signal into ac. A 5693 tube and a 6SL7 tube are employed in the ac amplifier, and the same L & N vibrator reconverts the ac signal to the amplified dc error signal. The error signal itself can be read directly off the error-indicating meter. It is desirable in normal operation to have the amplifier balanced with the manual control so that a minimum of error-signal is required to keep the system in balance.

Relays are provided in order that initial conditions can be imposed on the operational amplifier; a solution can also be stopped and held fixed at any time.

The amplifier panel is shown in Figure A3-3. Jacks are provided on the face of the panel to receive General Radio plugs with standard 3/4" separation, so that input or feedback resistors can readily be plugged into the unit. Space is available back of the chassis to plug in a feedback capacitor when the unit is to be used as an integrator. A switch is provided to select manual or automatic balance operation. The manual balance control operates a ten-turn heliopot. Error-signal can be read directly off of the meter on the panel provided for this purpose.

The signal-ground connections for all amplifiers should be connected to the power ground at just one point. This is necessary to prevent circulating ground currents from introducing voltage errors in the operational amplifiers.

The +300, +100, -190, and -350 volt dc power for the amplifier is supplied by the highly-stabilized power supply described in Appendix 4. All filaments are supplied by direct current. This is necessary to reduce 60 cycle ripple level, particularly in the ac loop, where any 60 cycle pickup is amplified and reconverted to dc error signal.

In the final dc amplifiers we plan to replace all 12SJ7 and 6SL7 tubes by 5693 and 5691 tubes respectively in order to prolong tube life.

A3-4 Performance Data on the DC Amplifier

The drift-stabilizing loop can work only when a feedback impedance is present across the dc amplifier. Therefore it is necessary to measure the

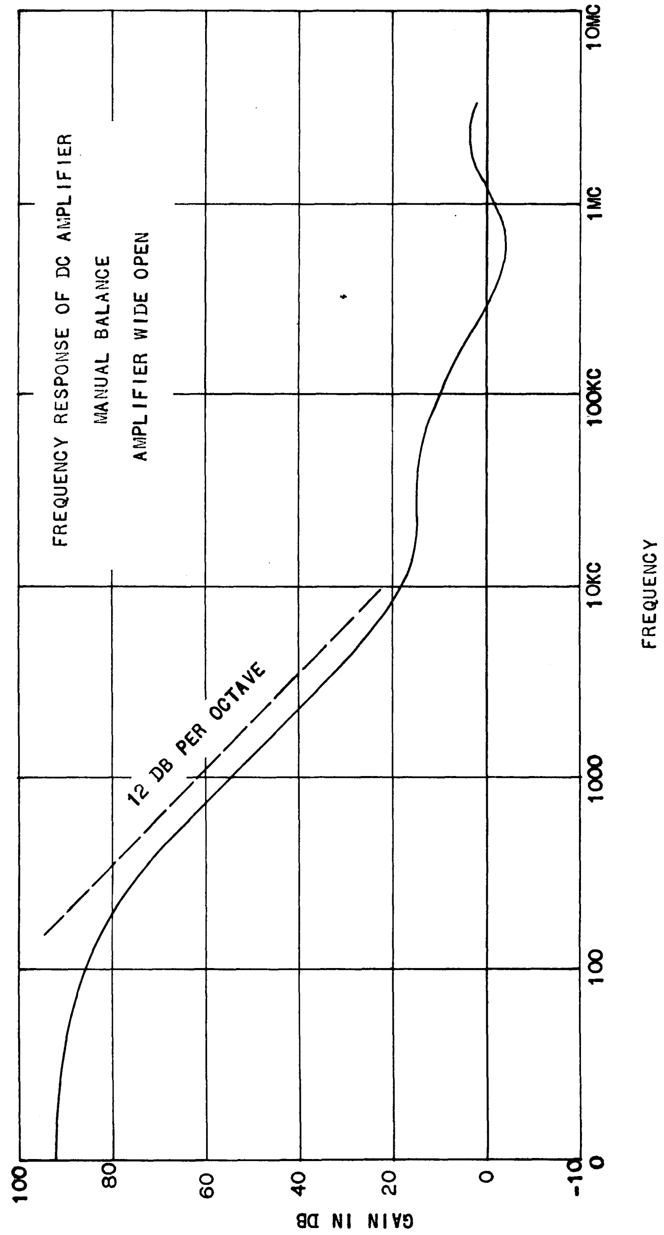


Figure A3-4. Frequency Response Curve for the Amplifier

wide-open gain of the amplifier with the balance control on manual, i.e., with the stabilizing loop disconnected. Amplifier gain as a function of frequency is shown in Figure A3-4. Note that the Bode criteria for stability is met; the gain does not fall off faster than 12 db per octave. No combination of input or feedback impedance can cause the amplifier to go into oscillation. Output saturation voltage as a function of load resistance is shown in Figure A3-5.

The automatic balance feature works extremely well; the stabilizing loop holds the output voltage within 50 microvolts of balance for an operational gain of unity. When the amplifier was set up as an integrator with a time constant of one second and with the input short-circuited, an initial output of 100 volts held constant to better than 0.1% for over one hour.

Because of the rapid decrease in the stabilizing-loop gain for frequencies above about 0.1 cps, recovery of the amplifier from a saturation condition is very slow when operated on automatic balance. For this reason the stabilizing loop cannot be used when the amplifier is employed in the zero-pulsing circuit described in Section 5.4.

At the time of writing of this report only one of the drift-stabilized amplifiers had been constructed. Performance was deemed highly satisfactory, and nine more are presently under construction. Eventually we hope to have twenty or even thirty of these amplifiers available for the electronic differential analyzer.

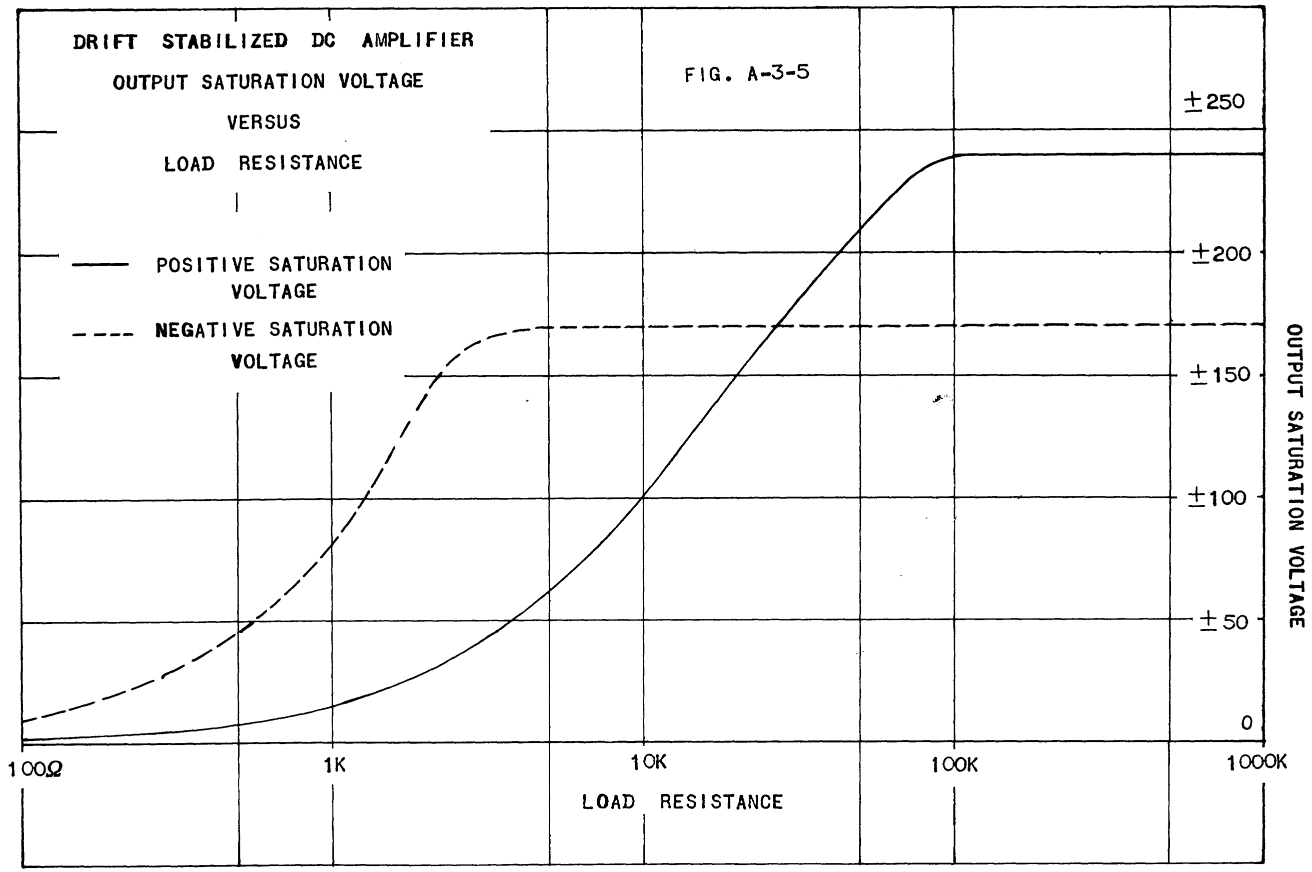


Figure A3-5. Saturation Voltage as a Function of Load

APPENDIX 4

DESCRIPTION OF THE HIGHLY-STABILIZED POWER SUPPLY

A4-1 Power Supply Requirements for the Computer

The high-gain dc amplifiers used in the electronic differential analyzer require well regulated dc power supplies, since the output balance of these amplifiers will be highly sensitive to any changes in the plate-voltage supplies of the vacuum tubes. Even with the considerable feedback employed in the operational amplifiers themselves, the output balance is still sensitive to fluctuations in the power-supply voltage. The solution of vibrating-beam problems depends in a very critical manner on the initial starting voltages and hence on the initial balance of the amplifiers. Amplifier balance is also very sensitive when the pulsing techniques described in Section 5.4 are employed (where the overall operational amplifier gain used is 6000). Thus the importance of well-regulated power supplies is apparent.

The dc amplifiers used for the work described in this report (see Appendix 2) require power supply voltages of +300, -190, and -350 volts. The new amplifiers now being put into operation (see Appendix 3) require +100 volts in addition to the above voltages.

A4-2 General Description of the Highly-Stabilized Power Supply

A well known scheme for obtaining a highly-regulated dc power-supply voltage is shown in the block diagram of Figure A4-1. The output voltage from the rectifier and filter is regulated in the usual way by means of loop (a), in which a VR tube is used as the reference voltage. This loop is able to provide regulation over a wide range of frequencies (for example it eliminates most of the 60 cycle ripple) but in general will be subject to slow drifts. Loop (b) provides an additional feedback which takes care of any slow drifts. The output voltage in this loop is compared with a standard cell reference. Any dc error signal is chopped into ac by means of a vibrator, amplified, and reconverted to dc with the phase sensitive rectifier. This amplified dc error signal is finally fed back into the regulator amplifier.

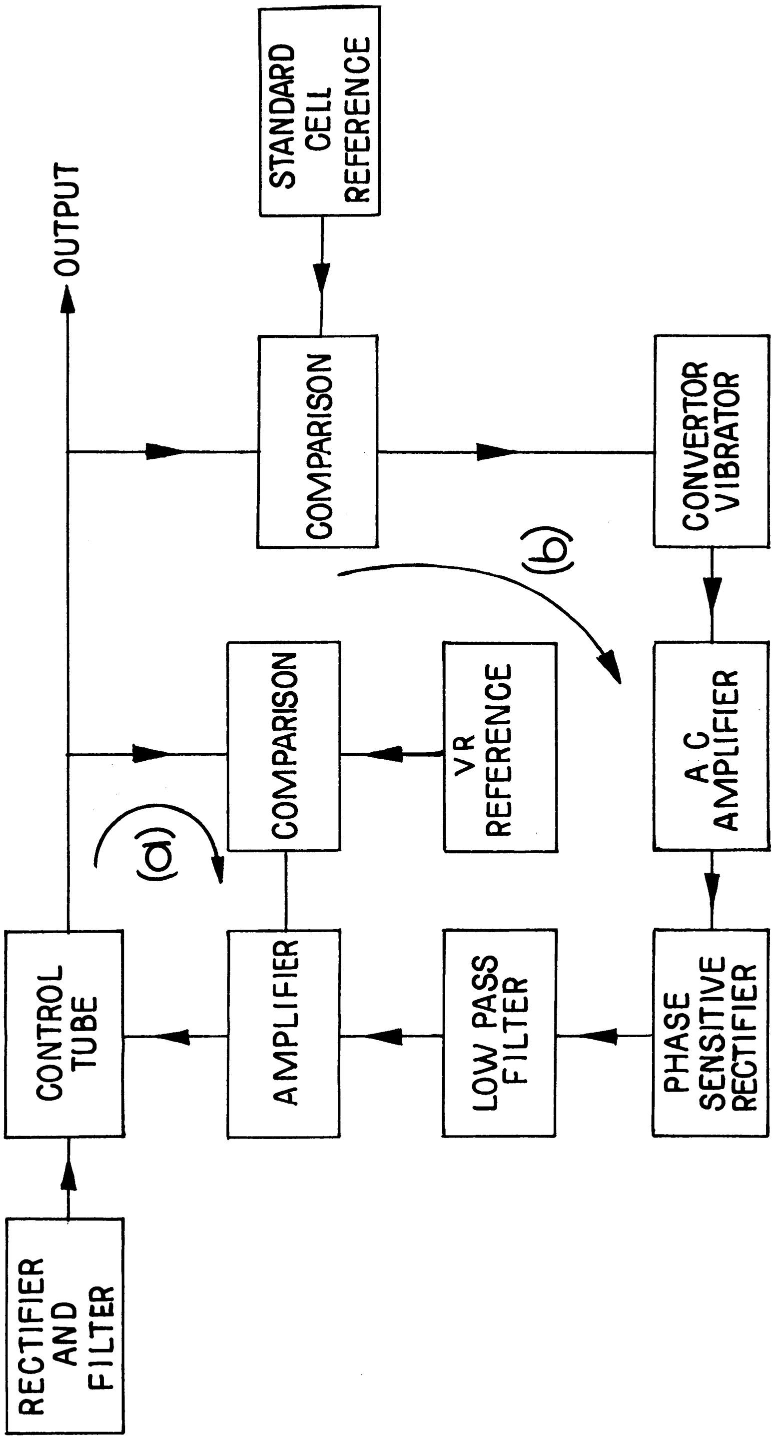


Figure A4-1. Block Diagram of Power Supply

The (a) loop therefore serves the purpose of stabilizing the output voltage against fast or sudden fluctuations while the (b) loop prevents the output voltage from drifting away from the voltage set by the standard cell reference. The frequency response of the (b) loop is such that it cannot compensate for fluctuations faster than about 0.1 cps.

A4-3 Components and Circuits Used in the Power Supply

The 60 cycle 115 volt input to the power supply is regulated by a Sorenson Model 2000S Regulator. The rectifier and filter circuits are shown in Figures A4-2, A4-3, A4-4, and A4-5, while the regulator circuits are given in Figures A4-6 and A4-7. The current capacities of the various voltage supplies are sufficient to operate 20 computer amplifiers under the most adverse load conditions and 30 computer amplifiers under normal load conditions.

About 1/350th of the -350 volt output is tapped off with a precision potential divider and compared with the standard cell voltage (see Figure A4-6). The other voltage outputs (-190, +100, and +300) are compared with the -350 volt supply by means of potential divider arrangements (Figures A4-6 and A4-7). Since only a small fraction of the -350 volt output is used for comparison with the standard cell, the gain of the ac amplifier in the -350 volt loop needs to be considerably higher than that of the ac amplifiers for the -190, +100, and +300 volt loops, where a considerable fraction of the voltage is compared with the -350 volt supply.

The 12AU7 and 6AU6 tubes in Figures A4-6 and A4-7 are part of the ac amplifiers. The 6AL5 tubes are used in the phase sensitive rectifier, the output voltage of which can be observed directly by means of an error-indicating meter. Two Leeds and Northrup Model 3881 standard vibrators are used to chop the dc error signal.

The panel housing for the highly-stabilized power supply is shown in Figure A4-8. Current and voltage outputs for each of the four supplies can be read directly off of panel meters. The error signal of the (b) loop for each of the supplies can also be read directly from panel meters. Full scale on the meters represents an error signal of 25 volts.

If it becomes desirable to add current capacity to the power supplies, it is only necessary to increase the capacity of the rectifier, filter, and control circuits. The regulator circuits can remain unchanged.

The output voltages of this supply remain constant to the order of one millivolt under normal conditions of operation. 60 cycle ripple in the outputs is also the order of one millivolt.

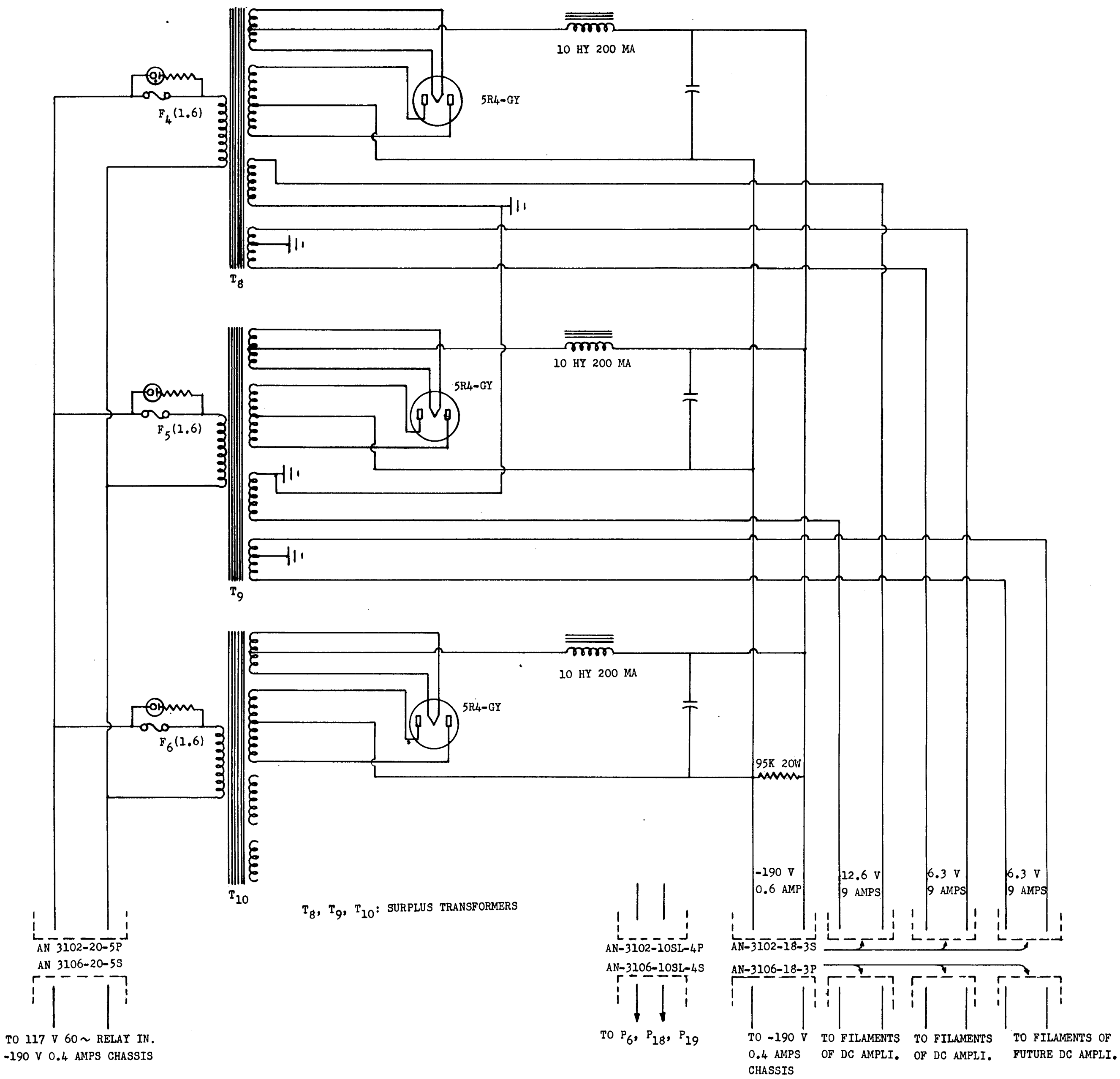


Figure A4-2. -190 Volt Power Supply and Filaments
Part I 0.6 Amps DC

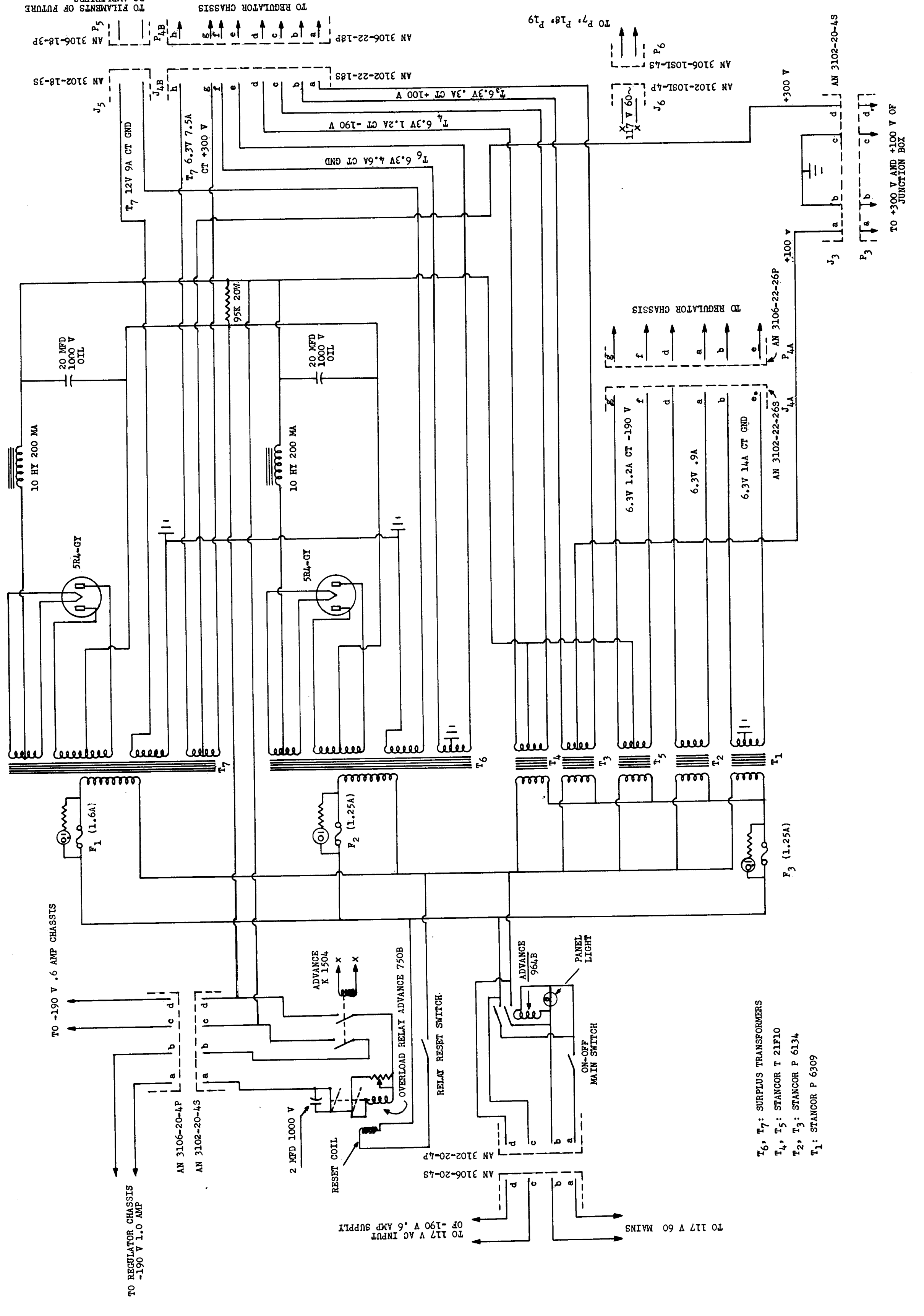


Figure A4-3. -190 Volt Power Supply and Filaments Part II 0.4 Amps DC

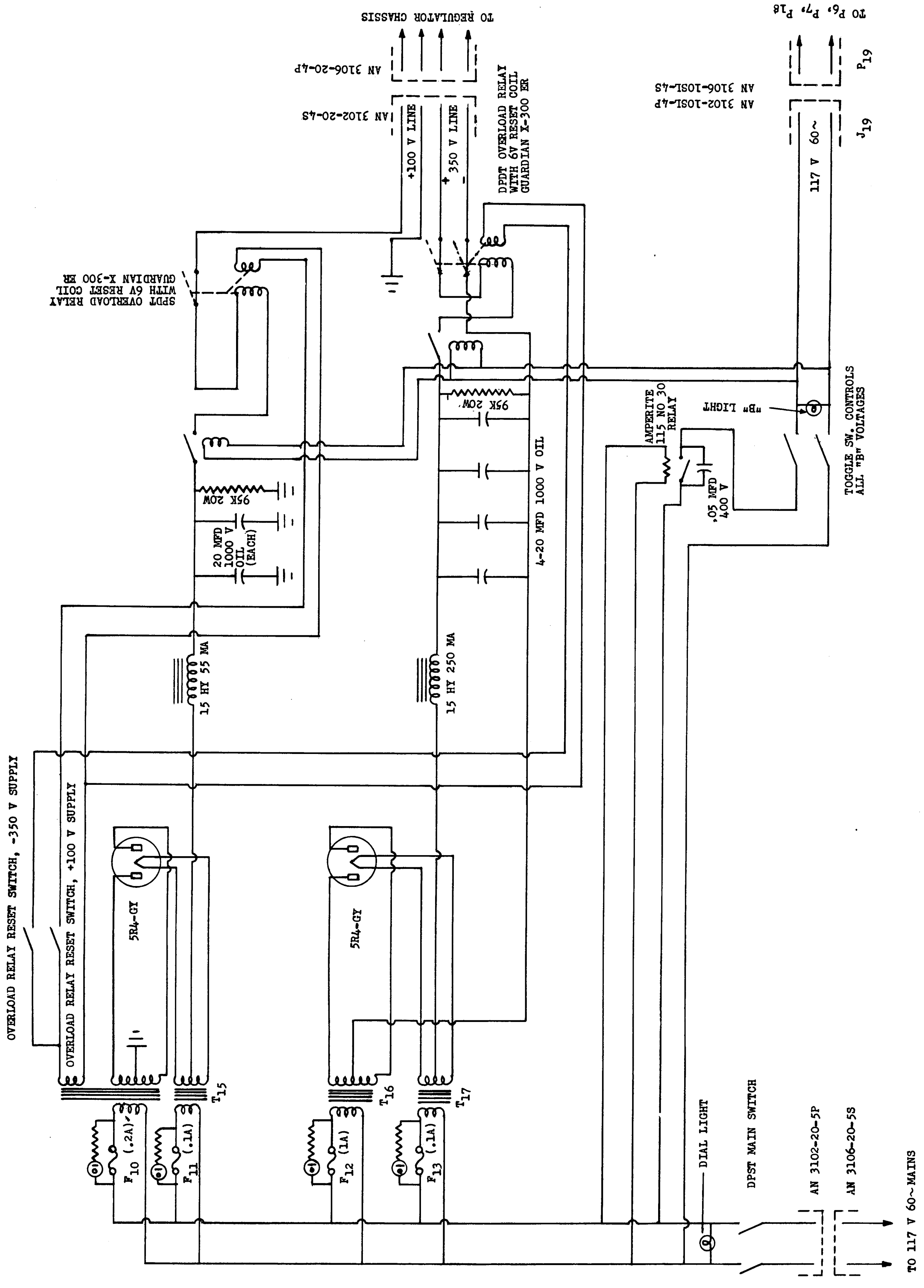


Figure A4-4. +100 Volt 30 Ma and -350 Volt 200 Ma Power Supplies

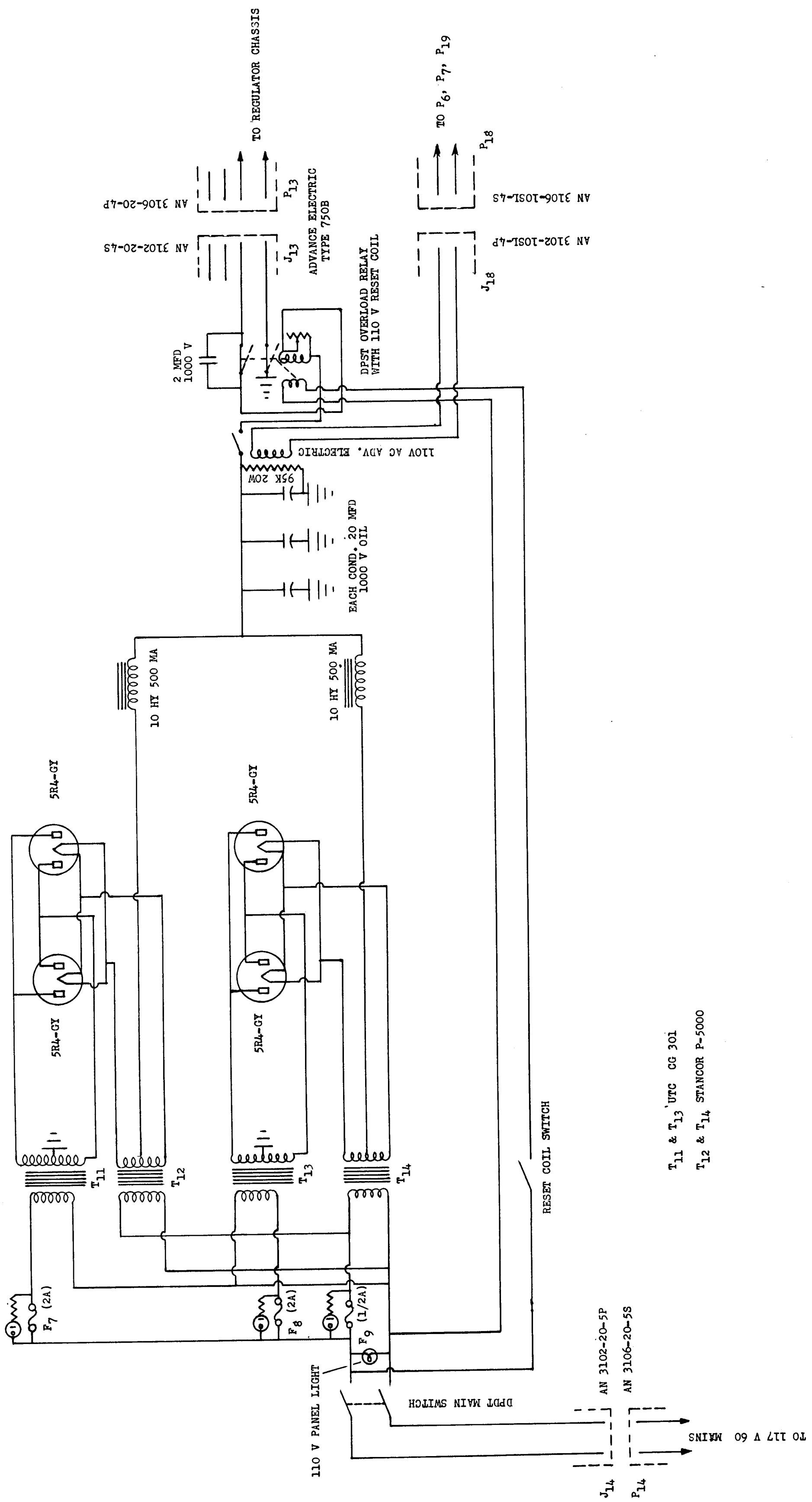


Figure A4-5. +300 Volt 780 Ma Power Supply

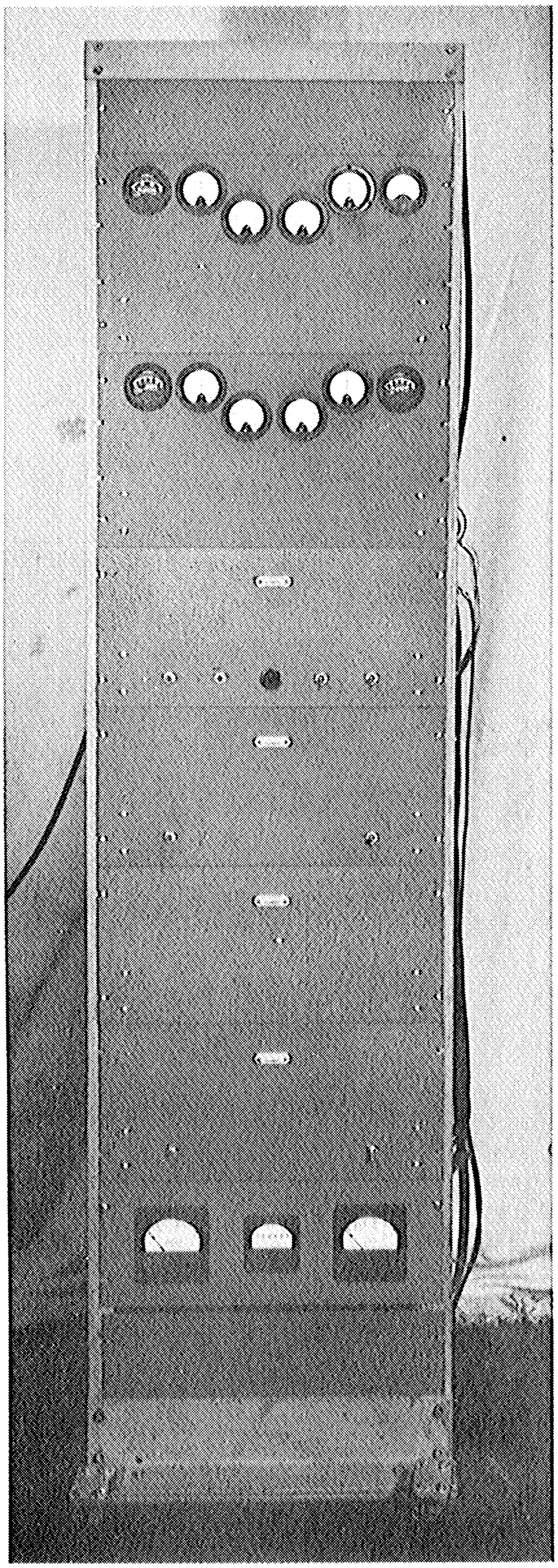


Figure A4-8. Panel Housing for the Highly-Stabilized Power Supply

APPENDIX 5

TIME CONSTANT OF THE ELECTRONIC INTEGRATOR

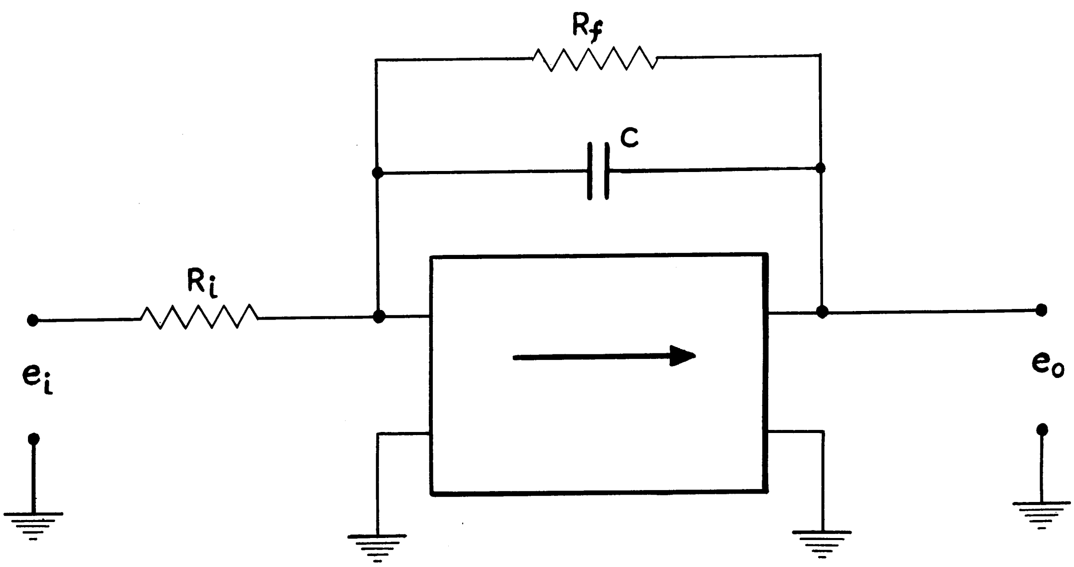


Figure A5-1.

The DC amplifier has a gain of $-\mu$ so that

$$e_o = -\mu e \; . \tag{A5-1}$$

The total current entering the input junction must be zero

$$\frac{e_o - e}{R_f} + C(\dot{e}_o - \dot{e}) + \frac{e_i - e}{R_i} = 0 \; . \tag{A5-2}$$

Using (A5-1) to eliminate e in (A5-2) gives

$$\frac{e_o}{R_f}\left(1 + \frac{1}{\mu}\right) + C\,\dot{e}_o\left(1 + \frac{1}{\mu}\right) + \frac{e_i}{R_i} + \frac{e_o}{\mu R_i} \tag{A5-3}$$

$$\dot{e}_o = -\frac{e_i}{R_i C\left(1 + \frac{1}{\mu}\right)} - \frac{e_o}{C}\left[\frac{1}{R_f} + \frac{1}{\mu R_i\left(1 + \frac{1}{\mu}\right)}\right] \tag{A5-4}$$

Integration furnishes the result

$$e_o = -\frac{1}{R_iC\,(1+\frac{1}{\mu})}\int_{t_o}^{t} e_i\,dt - \frac{1}{C}\left[\frac{1}{R_f} + \frac{1}{\mu R_i\,(1+\frac{1}{\mu})}\right]\int_{t_o}^{t} e_o\,dt\ . \qquad \text{(A5-5)}$$

It is seen that a finite value of μ reduces slightly the value of the coefficient before the integral of e_i as compared with the simplified derivation which assumes e = o. More important is the coefficient

$$-\frac{1}{\mu R_iC\,(1+\frac{1}{\mu})}$$

introduced before the integral of e_o. This second integral acts as a time constant of exponential decay for the integrator. This can be seen by assuming e_i = o for $t > t_1$ and $e_o = \bar{e}_o$ for $t > t_1$. Then equation (A5-4) can be solved to give

$$e_o = \bar{e}_o \exp\left(-\frac{t}{RC}\right) \qquad \text{(A5-6)}$$

where

$$\frac{1}{R} = \frac{1}{R_f} + \frac{1}{\mu R_i\,(1+\frac{1}{\mu})}\ . \qquad \text{(A5-7)}$$

Thus we see that when e_i = o the integrator time constant is equivalent to two leakage resistors paralleled across the condenser: the first is the actual resistance R_f and the second, due to the finite gain of the amplifier, is $\mu R_i\,(1 + 1/\mu)$. When $R_f = \infty$ and $R_iC = 1$ the time constant is closely equal to μ.

The general solution of (A5-4) for e_o as a linear time operation on e_i is

$$e_o = k \exp(-bt) - a \exp(-bt)\int_{o}^{t} \exp(bt)\, e_i\, dt \qquad \text{(A5-8)}$$

where

$$a = \frac{1}{R_i C \left(1 + \frac{1}{\mu}\right)} ,$$

$$b = \frac{1}{C}\left[\frac{1}{R_f} + \frac{1}{\mu R_i \left(1 + \frac{1}{\mu}\right)}\right] ,$$

and k is determined by the initial value of e_o when the integrator is started at $t = 0$.

Condensers with $C = 10^{-6}$ farads are available with $R_f > 10^{12}$ ohms.

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