

Asymptotic stabilization of the hanging equilibrium manifold of the 3D pendulum

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SUMMARY

The 3D pendulum consists of a rigid body, supported at a fixed pivot, with three rotational degrees of freedom; it is acted on by gravity and it is fully actuated by control forces. The 3D pendulum has two disjoint equilibrium manifolds, namely a hanging equilibrium manifold and an inverted equilibrium manifold. This paper shows that a controller based on angular velocity feedback can be used to asymptotically stabilize the hanging equilibrium manifold of the 3D pendulum. Lyapunov analysis and nonlinear geometric methods are used to assess the global closed-loop properties. We explicitly construct compact sets that lie in the domain of attraction of the hanging equilibrium of the closed-loop. Finally, this controller is shown to achieve almost global asymptotic stability of the hanging equilibrium manifold. An invariant manifold of the closed-loop that converges to the inverted equilibrium manifold is identified. Copyright © 2007 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Pendulum models have provided a rich source of examples in nonlinear dynamics and, in recent decades, in nonlinear control. The most common rigid pendulum model consists of a mass particle that is attached to one end of a massless, rigid link; the other end of the link is fixed to a pivot point that provides a rotational joint for the link and mass particle. If the link and mass particle are constrained to move within a fixed plane, the system is referred to as a planar 1D pendulum. If the link and mass particle are unconstrained, the system is referred to as a

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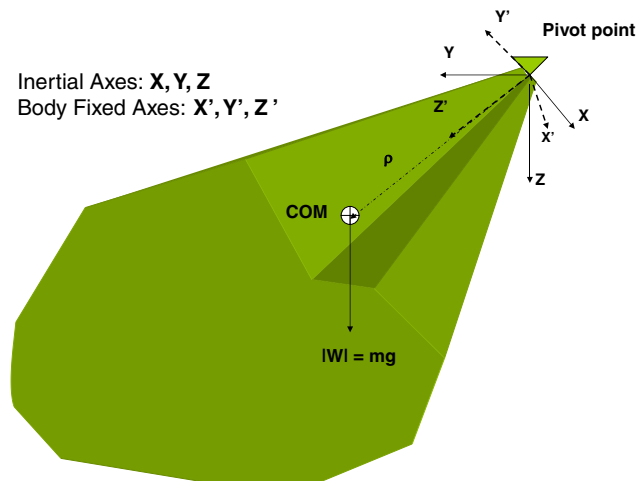


Figure 1. Schematic diagram of the 3D pendulum.

spherical 2D pendulum. Control problems for planar and spherical pendulum models have been studied in [1–6].

Pendulum models are useful for both pedagogical and research reasons. They represent simplified versions of mechanical systems that arise in robotics and spacecraft. In addition to their role in teaching the foundations of nonlinear dynamics and control, pendulum models have motivated research in nonlinear dynamics and nonlinear control.

The 3D pendulum is a non-trivial generalization of the planar pendulum and the spherical pendulum that has been studied only in recent publications of the authors and their colleagues in [7–11]. The 3D pendulum is a rigid body, supported at a fixed pivot, with three rotational degrees of freedom; it is acted on by a uniform gravity force and by control forces as shown in Figure 1.

The 3D pendulum has two equilibrium manifolds, namely the hanging and the inverted equilibrium manifolds. This paper studies one interesting stabilization problem for the 3D pendulum. In particular, we show that angular velocity feedback can be used to asymptotically stabilize the hanging equilibrium manifold of the 3D pendulum. The hanging equilibrium manifold and the inverted equilibrium manifold of the 3D pendulum are carefully introduced. The fact that angular velocity feedback provides local asymptotic stability of the hanging equilibrium follows from a simple passivity argument.

The paper formalizes this intuitive result. In addition, methods of nonlinear analysis are used to study the global closed-loop dynamics of the 3D pendulum. We develop explicit results that describe the domain of attraction of the hanging equilibrium manifold as an open and dense invariant set, thus resulting in *almost global* asymptotic stabilization of the hanging equilibrium manifold. We also identify an invariant manifold that characterizes solutions that converge to the inverted equilibrium manifold of the closed-loop 3D pendulum. These closed-loop dynamics are subtle due to the compactness of the configuration space, so that formal developments are required to describe the geometric features of the closed-loop dynamics.

This paper arose out of our continuing research on a laboratory facility, referred to as the Triaxial Attitude Control Testbed (TACT). The TACT provides a testbed for physical experiments on attitude dynamics and attitude control. The TACT has been described in several conference publications [12, 13]. Issues of nonlinear dynamics for the TACT have been treated in [13, 14] and stability issues have been treated in [11]. The present paper is partly motivated by the realization that the TACT is, in fact, a physical implementation of a 3D pendulum.

2. MATHEMATICAL MODELS FOR THE 3D PENDULUM

The 3D pendulum is a rigid body supported by a fixed, frictionless pivot, acted on by constant uniform gravity as well as control forces. A schematic diagram of a 3D pendulum is shown in Figure 1. Two co-ordinate frames are introduced. An inertial co-ordinate frame has its origin at the pivot; the first two co-ordinate axes lie in the horizontal plane while the third co-ordinate axis is vertical in the direction of gravity. A body-fixed co-ordinate frame with origin at the pivot point is fixed to the pendulum body. In this body-fixed frame, the moment of inertia of the pendulum is constant.

Rotation matrices are used to describe the attitude of the 3D pendulum. In this paper, we follow the convention in which a rotation matrix maps representations of vectors expressed in the body-fixed frame to representations expressed in the inertial frame. Rotation matrices provide global representations of the attitude of the pendulum, which is why they are utilized here. Although attitude representations, such as exponential co-ordinates, quaternions, and Euler angles, can also be used, each of these representations has a disadvantage of introducing an ambiguity or singularity. The attitude of the 3D pendulum is a rotation matrix R , viewed as an element of the special orthogonal group $SO(3)$. The associated angular velocity, expressed in the body-fixed frame, is denoted by ω in \mathbb{R}^3 .

The equations of motion of a 3D pendulum are now presented. The constant inertia matrix, in the body-fixed frame, is denoted by J . The vector from the pivot to the centre of mass of the 3D pendulum, resolved in the body-fixed frame, is denoted by ρ . The symbol g denotes the constant acceleration due to gravity. These are the data on which the equations of motion are based.

Standard techniques yield the equations of motion for the 3D pendulum. The dynamics are given by the Euler–Poincaré equation that includes the moment due to gravity and a body-fixed control moment $u \in \mathbb{R}^3$

$$J\dot{\omega} = J\omega \times \omega + mg\rho \times R^T e_3 + u \quad (1)$$

where $e_3 = [0 \ 0 \ 1]^T$. The rotational kinematics equations are

$$\dot{R} = R\hat{\omega} \quad (2)$$

where $R \in SO(3)$, $\omega \in \mathbb{R}^3$ and the operator $\hat{\cdot}: \mathbb{R}^3 \rightarrow \mathfrak{so}(3)$ denotes the isomorphism from \mathbb{R}^3 to the Lie algebra $\mathfrak{so}(3)$. Represented in co-ordinates, $\hat{\cdot}: \mathbb{R}^3 \rightarrow \mathbb{R}^{3 \times 3}$ is the usual skew-symmetric operator. The cross-product notation $a \times b$ for vectors a and b in \mathbb{R}^3 is

$$a \times b = [a_2 b_3 - a_3 b_2, \ a_3 b_1 - a_1 b_3, \ a_1 b_2 - a_2 b_1] = \hat{a}b \quad (3)$$

where, the skew-symmetric matrix \hat{a} is defined as

$$\hat{a} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \quad (4)$$

In (1), since $e_3 = [0 \ 0 \ 1]^T$ denotes the unit vector in the direction of gravity in an inertial frame, $R^T e_3$ denotes the unit vector of gravity in the body-fixed frame.

A special case occurs when the centre of mass of the 3D pendulum is located at the pivot. In this case $\rho = 0$, so that (1) is given by Euler–Poincaré equations with no gravity terms. This is the controlled free rigid body and in the context of the 3D pendulum, this is referred to as the balanced case. Since there is a large literature on the control of the free rigid body, that is the balanced case, we subsequently study the more general unbalanced case, where $\rho \neq 0$.

The equations of motion (1) and (2) for the 3D pendulum are viewed as a model where its dynamics evolve on the tangent bundle $T\text{SO}(3)$ [15]; these are referred to as the equations of motion of the 3D pendulum. It is possible to obtain a lower-dimensional reduced model for the pendulum. This reduction is made apparent by noting that the dynamics and kinematics equations can be written in terms of the reduced attitude vector $\Gamma = R^T e_3 \in S^2$, which is the unit vector that expresses the direction of gravity in the body-fixed co-ordinate frame.

Specifically, let Π_ψ denote the S^1 group action $\Pi_\psi : \text{SO}(3) \rightarrow \text{SO}(3)$ as $\Pi_\psi(R) = Re^{\hat{\Gamma}\psi}$, where $\Gamma = R^T e_3$. Then, the orbit space $\text{SO}(3)/S^1$ is the equivalent set of rotations

$$[R] \triangleq \{R' \in \text{SO}(3) : R' = Re^{\hat{\Gamma}\psi}, \Gamma = R^T e_3, \psi \in \mathbb{R}\} \quad (5)$$

For the equivalence relation in (5), it is easy to see that $R_1 \sim R_2$ if and only if $R_1^T e_3 = R_2^T e_3$ and hence, the equivalence relation in (5) can alternatively be expressed as

$$[R] \triangleq \{R_s \in \text{SO}(3) : R_s^T e_3 = R^T e_3\} \quad (6)$$

Thus, for each $R \in \text{SO}(3)$, $[R]$ can be identified with $\Gamma = R^T e_3 \in S^2$ and hence $\text{SO}(3)/S^1 \cong S^2$. Now, since $T\text{SO}(3) \cong \text{SO}(3) \times \mathbb{R}^3$, Π_ψ induces a projection $\pi : T\text{SO}(3) \rightarrow T\text{SO}(3)/S^1$ given by $\pi : (R, \omega) \mapsto ([R], \omega)$, where $[R]$ is as given in (6).

Proposition 1

The dynamics of the 3D pendulum given by (1) and (2) induce a flow on the quotient space $T\text{SO}(3)/S^1$ through the projection $\pi : T\text{SO}(3) \rightarrow T\text{SO}(3)/S^1$, given by the dynamics

$$J\dot{\omega} = J\omega \times \omega + mg\rho \times \Gamma + u \quad (7)$$

and the kinematics for the reduced attitude

$$\dot{\Gamma} = \Gamma \times \omega \quad (8)$$

Furthermore, $T\text{SO}(3)/S^1 \cong S^2 \times \mathbb{R}^3$.

Proof

Let $\pi : T\text{SO}(3) \rightarrow T\text{SO}(3)/S^1$ be the quotient map defined as $\pi : (R, \omega) \mapsto ([R], \omega)$ where $[R]$ is defined as in (6). Since, $[R] \in \text{SO}(3)/S^1$ is identified with $\Gamma \in S^2$, substituting $\Gamma = R^T e_3$ in (1)

$$\begin{array}{ccc}
 TSO(3) \cong SO(3) \times \mathbb{R}^3 & & \\
 \pi \downarrow & \searrow p & \\
 TSO(3)/S^1 & \xrightarrow{\quad} & S^2 \times \mathbb{R}^3 \\
 & f = p \circ \pi^{-1} &
 \end{array}$$

Figure 2. Quotient map.

yields (7). Furthermore,

$$\dot{\Gamma} = \dot{R}^T e_3 = -\widehat{\omega} R^T e_3 = \Gamma \times \omega$$

yields (8). Thus, (1) and (2) induce a flow on the quotient space $TSO(3)/S^1$ given by (7) and (8) via the projection π .

To see that $TSO(3)/S^1 \cong S^2 \times \mathbb{R}^3$, consider the quotient map $p : SO(3) \times \mathbb{R}^3 \rightarrow S^2 \times \mathbb{R}^3$ given by $p(R, \omega) = (\Gamma, \omega)$ as shown in Figure 2. From (6), it is clear that p is constant on $\pi^{-1}(y) \subset TSO(3)$ for each $y \in TSO(3)/S^1$. Therefore, p induces a quotient map $p \circ \pi^{-1} : TSO(3)/S^1 \rightarrow S^2 \times \mathbb{R}^3$. Since $[R]$ is identified with $\Gamma \in S^2$, $p \circ \pi^{-1}$ is a bijection. Hence, $p \circ \pi^{-1}$ is a homeomorphism. Thus, $TSO(3)/S^1 \cong S^2 \times \mathbb{R}^3$. \square

Equations (7) and (8) are expressed in a non-canonical form; they are referred to as the reduced attitude dynamics of the 3D pendulum on $TSO(3)/S^1$. We subsequently show that the reduced attitude dynamics are fundamental in stabilization of the hanging equilibrium manifold of (1) and (2).

3. EQUILIBRIUM STRUCTURE OF THE 3D PENDULUM

We first study the uncontrolled dynamics of the 3D pendulum. We obtain two integrals of motion for the 3D pendulum. These integrals expose the free dynamics of the 3D pendulum and can be used to construct control-Lyapunov functions.

There are two conserved quantities for the 3D pendulum. First, the total energy, which is the sum of the rotational kinetic energy and the gravitational potential energy, is conserved. The other conserved quantity is the component of angular momentum about the vertical axis through the pivot.

Proposition 2 ([7, 14])

Let $u = 0$ in (1). The total energy, $E = \frac{1}{2} \omega^T J \omega - mg\rho^T R^T e_3 = \frac{1}{2} \omega^T J \omega - mg\rho^T \Gamma$, and the component of the angular momentum vector about the vertical axis through the pivot, $h = \omega^T J R^T e_3 = \omega^T J \Gamma$, are each constant along motions of the 3D pendulum given by (1) and (2).

To further understand the dynamics of the 3D pendulum, we study the equilibria of (1) and (2). Equating the RHS of (1) and (2) to zero for $u = 0$ yields

$$J\omega \times \omega + mg\rho \times R^T e_3 = 0 \quad (9)$$

$$R\widehat{\omega} = 0 \quad (10)$$

Since $R \in \text{SO}(3)$ is non-singular and $\widehat{\cdot}: \mathbb{R}^3 \rightarrow \mathfrak{so}(3)$ is an isomorphism, $R\widehat{\omega} = 0$ if and only if $\omega = 0$. Substituting $\omega = 0$ in (9), we obtain

$$\rho \times R^T e_3 = 0 \quad (11)$$

Since ρ denotes the centre of mass vector resolved in the body-fixed frame, an equilibrium of the 3D pendulum corresponds to the case where ρ is collinear with the gravity direction in the body-fixed frame given by $R^T e_3$. This implies that either the centre of mass vector ρ points in the direction of the gravity vector or the centre of mass vector ρ points in the direction opposite to the gravity vector.

Let R_e denote an attitude rotation matrix that satisfies (11) and define $\Gamma_e = R_e^T e_3$. Then, every attitude in the configuration manifold given by

$$\{R \in \text{SO}(3) : R = R_e e^{\widehat{\Gamma_e \bar{\psi}}}, \bar{\psi} \in \mathbb{R}\} \quad (12)$$

satisfies (11) and defines an equilibrium attitude corresponding to $\omega = 0$. We can use Rodrigues's formula to write

$$e^{\widehat{\Gamma_e \bar{\psi}}} = I_3 + \sin \bar{\psi} \widehat{\Gamma_e} + (1 - \cos \bar{\psi}) \widehat{\Gamma_e}^2$$

Thus, if the attitude R_e satisfies (11), then a rotation of the 3D pendulum about the gravity vector by an arbitrary angle is also an equilibrium. Consequently, there are two disjoint equilibrium manifolds of the 3D pendulum. The manifold corresponding to the case where the centre of mass is below the pivot for each attitude in the manifold is referred to as the *hanging* equilibrium manifold, and the manifold corresponding to the case where the centre of mass is above the pivot for each attitude in the manifold is referred to as the *inverted* equilibrium manifold.

This paper focuses on stabilization of the hanging equilibrium manifold of (1) and (2). To study the stabilization problem, it is advantageous to consider the equations of the 3D pendulum in terms of the reduced attitude as in (7) and (8). If R_e satisfies (11), then $(0, \Gamma_e)$ is an equilibrium of (7) and (8). Thus, corresponding to the hanging equilibrium manifold and the inverted equilibrium manifold of (1) and (2), there exist two isolated equilibrium solutions of the reduced attitude equations (7) and (8). These are given by the hanging equilibrium $(0, \Gamma_h)$ and the inverted equilibrium $(0, \Gamma_i)$, where

$$\Gamma_h = \frac{\rho}{\|\rho\|} \quad \text{and} \quad \Gamma_i = -\frac{\rho}{\|\rho\|}$$

Proposition 3

The hanging equilibrium manifold and the inverted equilibrium manifold of the 3D pendulum given by (1) and (2) are identified with the hanging equilibrium $(0, \Gamma_h)$ and the inverted equilibrium $(0, \Gamma_i)$ of the reduced attitude equations given by (7) and (8).

Proof

Let $(0, R_e)$ be an equilibrium of (1) and (2). Note that each of the hanging and the inverted equilibrium manifolds can be expressed as $(0, [R_e])$ where $[R_e]$ is as given in (6). Further, from Proposition 1 we know that for all $R \in [R_e]$, $\pi(R, 0) = ([R_e], \omega) \cong (\Gamma_e, \omega)$, where Γ_e is Γ_h for the hanging equilibrium manifold and Γ_i for the inverted equilibrium manifold. \square

In the ensuing sections, we show that angular velocity feedback asymptotically stabilizes the hanging equilibrium manifold. Based on Proposition 3, this is demonstrated by showing that angular velocity feedback asymptotically stabilizes the hanging equilibrium solution of the reduced attitude equations (7) and (8).

4. CLOSED-LOOP EQUATIONS FOR THE 3D PENDULUM

In this section, simple angular velocity feedback controllers are developed that asymptotically stabilize the hanging equilibrium manifold of (1) and (2). This development is based on the observation that the reduced attitude dynamics given by (7) and (8) are locally input–output passive if angular velocity is taken as the output. The total energy is the storage function. Since the total energy, $\frac{1}{2}\omega^T J\omega - mg\rho^T\Gamma$, has a minimum at the hanging equilibrium $(0, \Gamma_h)$, a control law based on angular velocity feedback is suggested.

Let $\Psi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a smooth function such that

$$\begin{cases} \Psi'(0)^T = \Psi'(0), & \Psi'(0) > 0 \\ \mathcal{P}(x) \leq x^T \Psi(x) \leq \alpha(\|x\|) & \forall x \in \mathbb{R}^3 \end{cases} \quad (13)$$

where $\mathcal{P} : \mathbb{R}^3 \rightarrow \mathbb{R}$ is a positive definite function, and $\alpha(\cdot)$ is a class- \mathcal{K} function. Thus, we propose controllers of the form

$$u = -\Psi(\omega) \quad (14)$$

where $\Psi(\cdot)$ satisfies (13). With the above control law, the closed-loop attitude dynamics on $TSO(3)$ are

$$\begin{cases} J\dot{\omega} = J\omega \times \omega + mg\rho \times R^T e_3 - \Psi(\omega) \\ \dot{R} = R\hat{\omega} \end{cases} \quad (15)$$

and the closed-loop *reduced* attitude dynamics on $TSO(3)/S^1$ are

$$\begin{cases} J\dot{\omega} = J\omega \times \omega + mg\rho \times \Gamma - \Psi(\omega) \\ \dot{\Gamma} = \Gamma \times \omega \end{cases} \quad (16)$$

Equating the RHS of (15) to zero, (ω, R) is an equilibrium if $\omega = 0$ and R_e satisfies (11). Hence, (15) has two equilibrium manifolds, namely the hanging and the inverted equilibrium manifolds. Similarly, (16) has exactly two equilibrium solutions, namely, the hanging equilibrium and the inverted equilibrium.

The existence of multiple equilibria of (16) is expected since there exists a topological obstruction [16] to the existence of a continuous time-invariant controller that globally stabilizes any equilibrium of the reduced attitude dynamics of the 3D pendulum.

5. LOCAL ANALYSIS OF THE CLOSED-LOOP 3D PENDULUM

We now study the linearization of the closed-loop system (15) about an arbitrary equilibrium in one of the two equilibrium manifolds. Consider an equilibrium $(0, R_e) \in TSO(3)$ where $R_e \in SO(3)$ satisfies (11). To linearize the 3D pendulum model (1) and (2), consider a small

perturbation of $(\Delta\omega, R)$ from the equilibrium $(0, R_e)$. We denote the perturbation in the rotation matrix using exponential co-ordinates $\Delta\Theta$ as $R = R_e e^{\widehat{\Delta\Theta}}$, where $\Delta\Theta \in \mathbb{R}^3$ [17] and $\Delta\omega \in \mathbb{R}^3$.

Expanding Rodrigues's formula [17],

$$R = R_e [I + \widehat{\Delta\Theta}] + O(\Delta\Theta^2)$$

Next, from $\dot{R} = R_e e^{\widehat{\Delta\Theta}} \widehat{\Delta\dot{\Theta}}$ and (2) it follows that

$$\dot{R} = R \widehat{\Delta\omega} = R_e e^{\widehat{\Delta\Theta}} \widehat{\Delta\omega} = R_e e^{\widehat{\Delta\Theta}} \widehat{\Delta\dot{\Theta}}$$

Hence, $\Delta\dot{\Theta} = \Delta\omega$. Therefore, $J\omega \times \omega = O(\Delta\omega^2) = O(\Delta\dot{\Theta}^2)$. Hence, expressing (15) in terms of $\Delta\Theta$, we obtain

$$J\Delta\ddot{\Theta} + \Psi'(0)\Delta\dot{\Theta} - k \frac{mg}{\|\rho\|} \widehat{\rho}^2 \Delta\Theta + O(\Delta\Theta^2, \Delta\dot{\Theta}^2) = 0 \tag{17}$$

where $k = 1$ if $(0, R_e)$ is a hanging equilibrium and $k = -1$ if $(0, R_e)$ is an inverted equilibrium. If the higher-order terms are ignored, we obtain the linearization of (17) at $(0, R_e)$ as

$$J\Delta\ddot{\Theta} + \Psi'(0)\Delta\dot{\Theta} - k \frac{mg}{\|\rho\|} \widehat{\rho}^2 \Delta\Theta = 0 \tag{18}$$

Now note that $\widehat{\rho}^2$ is a rank 2, symmetric, negative-semidefinite matrix. Thus, it follows from [18, 19] that one can simultaneously diagonalize J and $\widehat{\rho}^2$. Thus, there exists a non-singular matrix M such that $J = MM^T$ and $-(mg/\|\rho\|)\widehat{\rho}^2 = M\Lambda M^T$, where Λ is a diagonal matrix. Denote $\Lambda = \text{diag}(mgl_1, mgl_2, 0)$, where l_1 and l_2 are positive. Define $x \triangleq M^T \Delta\Theta$ and denote $D = M^{-1} \Psi'(0) M^{-T}$. Since $\Psi'(0)$ is symmetric and positive definite, $D^T = D$ and D is positive definite.

Thus, the linearization of (17) can be expressed as

$$\ddot{x} + D\dot{x} + k \Lambda x = 0 \tag{19}$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Equation (19) consists of three coupled second-order linear differential equations.

Note that in the linearization of the closed-loop, the damping in (19) represented by the matrix D arises from the angular velocity feedback.

We next study linearization of (16) about an equilibrium $(0, R_e^T e_3)$, where $(0, R_e)$ is an equilibrium of (15). Since $\dim[TSO(3)/S^1] = 5$, the linearization of (16) evolves on \mathbb{R}^5 .

Proposition 4

The linearization of the reduced attitude dynamics of the 3D pendulum, about the equilibrium $(0, R_e^T e_3)$ described by equations (16) can be expressed using $(x_1, x_2, \dot{x}_1, \dot{x}_2, \dot{x}_3) \in \mathbb{R}^5$ according to (19).

Proof

Consider a perturbation in Γ and ω in terms of $\Delta\Theta$ and $\Delta\omega$, where $\Delta\Theta \in \mathbb{R}^3$ are the exponential co-ordinates as before and $\Delta\omega \in \mathbb{R}^3$. Then, since $\Gamma = R^T e_3$, it follows from Rodrigues's formula that $\Gamma = [I - \widehat{\Delta\Theta}] \Gamma_e + O(\Delta\Theta^2)$, where $\Gamma_e = R_e^T e_3$. Denoting $\Delta\Gamma \triangleq \widehat{\Gamma}_e \Delta\Theta \in T_{\Gamma_e} S^2$ yields

$$\Gamma - \Gamma_e = \Delta\Gamma + O(\Delta\Theta^2) \tag{20}$$

Note that $\Gamma_e = \rho/||\rho||$ if R_e is in the hanging equilibrium manifold and $\Gamma_e = -\rho/||\rho||$ if R_e is in the inverted equilibrium manifold. Hence, (20) yields

$$\rho \times \Gamma = \rho \times (\Gamma_e + \Delta\Gamma + O(\Delta\Theta^2)) = \hat{\rho} \Delta\Gamma + O(\Delta\Theta^2)$$

Since $J\Delta\omega \times \Delta\omega = O(\Delta\omega^2)$ and $\Delta\dot{\Theta} = \Delta\omega$, (16) can be expressed as

$$J\Delta\dot{\omega} = -\Psi'(0)\Delta\omega + mg\hat{\rho} \Delta\Gamma + O(\Delta\Theta^2, \Delta\omega^2) \quad (21)$$

$$\Delta\dot{\Gamma} = k \frac{\hat{\rho}}{||\rho||} \Delta\omega \quad (22)$$

where if R_e is a hanging equilibrium then $k = 1$, and if R_e is an inverted equilibrium then $k = -1$. Next, neglecting the higher-order terms of $\Delta\Theta$ and $\Delta\omega$, we obtain the linearization of (21) and (22) as

$$J\Delta\dot{\omega} = -\Psi'(0)\Delta\omega + mg\hat{\rho}\Delta\Gamma \quad (23)$$

$$\Delta\dot{\Gamma} = k \frac{\hat{\rho}}{||\rho||} \Delta\omega \quad (24)$$

We can express (23) and (24) in terms of (x, \dot{x}) . Specifically, we show that $(\Delta\Gamma, \Delta\omega)$ can be expressed using $(x_1, x_2, \dot{x}_1, \dot{x}_2, \dot{x}_3) \in \mathbb{R}^5$.

Since $x = M^T\Delta\Theta$ and M is non-singular, $\Delta\omega = M^{-T}\dot{x}$ and $\Delta\Gamma = k(\hat{\rho}/||\rho||)M^{-T}x$. We now give an orthogonal decomposition of the vector $\Delta\Theta = M^{-T}x$ into a component along the vector ρ and a component normal to the vector ρ . This decomposition is

$$M^{-T}x = -\frac{\hat{\rho}^2}{||\rho||^2}(M^{-T}x) + \frac{1}{||\rho||^2}[\rho^T(M^{-T}x)]\rho$$

where

$$\frac{1}{||\rho||^2}[\rho^T(M^{-T}x)]\rho \in \text{span}\{\rho\} \quad \text{and} \quad -\frac{\hat{\rho}^2}{||\rho||^2}(M^{-T}x) \in \text{span}\{\rho\}^\perp$$

Thus,

$$\Delta\Gamma = k \frac{\hat{\rho}}{||\rho||} \Delta\Theta = k \frac{\hat{\rho}}{||\rho||} M^{-T}x = \frac{k}{mg||\rho||^2} \hat{\rho} M \Lambda x$$

does not depend on x_3 since $\Lambda = \text{diag}(mgl_1, mgl_2, 0)$. Thus, we can express the linearization of (21)–(22) and hence, the linearization of (16) at $(0, R_e^T e_3)$ in terms of the state variables $(x_1, x_2, \dot{x}_1, \dot{x}_2, \dot{x}_3)$ according to (19). \square

Remark 1

Since the matrix Λ has a zero eigenvalue, (19) is not asymptotically stable for $k = 1$. This is due to the fact that the hanging equilibrium manifold and the inverted equilibrium manifold are non-trivial 1D centre submanifolds in $T\text{SO}(3)$. However, due to our careful choice of variables, one can discard x_3 from (19) to study the stability property of the equilibrium manifold.

Thus, intuitively, x_3 corresponds to a component of the perturbation in the attitude that is tangential to the equilibrium manifold. The following lemmas are needed.

Lemma 1

Consider the linear model (19), representing linearization of (15) at a hanging equilibrium $(0, R_e)$ expressed in first-order form as

$$\frac{d}{dt} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & I \\ -\Lambda & -D \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \triangleq A_h \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \tag{25}$$

where $x = (x_1, x_2, x_3)$. Then, A_h has one zero eigenvalue and all other eigenvalues have negative real part and at least one of them is negative real.

Proof

In the prior notation, $k = 1$ in (19) for a hanging equilibrium. Let $v = [v_1^T \ v_2^T]^T$, $v_1, v_2 \in \mathbb{C}^3$ be an eigenvector of A_h corresponding to the eigenvalue λ . Then, $A_h v = \lambda v$ yields $v_2 = \lambda v_1$ and $Dv_2 + \Lambda v_1 = -\lambda v_2$. Combining these equations yields $\lambda^2 v_1 + D\lambda v_1 + \Lambda v_1 = 0$. Thus, every eigenvalue–eigenvector pair $(\lambda, [v_1^T \ v_2^T]^T)$ of A_h satisfies $v_2 = \lambda v_1$ and $\lambda^2 v_1 + \lambda Dv_1 + \Lambda v_1 = 0$.

Next, taking the inner product of the above equation with respect to the complex conjugate of v_1 yields

$$a\lambda^2 + b\lambda + c = 0 \tag{26}$$

where $a = \bar{v}_1^T v_1$, $b = \bar{v}_1^T Dv_1$ and $c = \bar{v}_1^T \Lambda v_1$. Since D is symmetric and positive definite, Λ is diagonal and positive semidefinite, and $v_1 \neq 0$, it follows that $a, b, c \in \mathbb{R}$ satisfy $a > 0$, $b > 0$ and $c \geq 0$. Furthermore, since $\Lambda = \text{diag}(mgl_1, mgl_2, 0)$, $c = 0$ if and only if $v_1 = \beta e_3$, where $e_3 = [0 \ 0 \ 1]^T$ and $\beta \in \mathbb{C} \setminus \{0\}$.

Now, since (26) has two solutions and $v = [v_1^T \ v_2^T]^T = [v_1^T \ \lambda v_1^T]^T$, it is clear that the eigenvalue–eigenvector pair (λ, v) of A_h can be written as

$$\left\{ \left(\lambda_i, \begin{bmatrix} v_{1i} \\ \lambda_i v_{1i} \end{bmatrix} \right), \left(\lambda_i^*, \begin{bmatrix} v_{1i} \\ \lambda_i^* v_{1i} \end{bmatrix} \right) \right\}, \quad i \in \{1, 2, 3\} \tag{27}$$

where λ_i and λ_i^* are the two solutions to the quadratic equation (26) corresponding to $a = a_i = \bar{v}_{1i}^T v_{1i}$, $b = b_i = \bar{v}_{1i}^T Dv_{1i}$ and $c = c_i = \bar{v}_{1i}^T \Lambda v_{1i}$, $i \in \{1, 2, 3\}$.

Now choose $v_{11} = \beta e_3$. Then, $a_1 = \beta\beta = \|\beta\|^2$, $b_1 = \|\beta\|^2 e_3^T D e_3 > 0$ and $c_1 = 0$. Therefore, the roots of (26) are given by $\lambda_1 = 0$ and $\lambda_1^* = -e_3^T D e_3$. Hence, 0 and $-e_3^T D e_3$ are two of the six eigenvalues of A_h in (25). Thus, A_h has a zero and a negative real eigenvalue.

Now for each of v_{12} and v_{13} , we obtain a corresponding quadratic equation as given in (26). First, note that since $v_{11} = \beta e_3$ yields a zero eigenvalue, neither v_{12} nor v_{13} is equal to βe_3 . This follows since if not, then there is a repeated zero eigenvalue which implies that A_h has rank less than or equal to four. However, it is easy to see that all columns of A_h except the third column, which is identically zero, are linearly independent. Since both v_{12} and v_{13} are not equal to βe_3 , it follows that a_i, b_i and c_i , $i \in \{2, 3\}$ are positive. Then, the corresponding roots of (26) are given by

$$\lambda_i = -\frac{b_i}{2a_i} + \frac{\sqrt{b_i^2 - 4a_i c_i}}{2a_i} \quad \text{and} \quad \lambda_i^* = -\frac{b_i}{2a_i} - \frac{\sqrt{b_i^2 - 4a_i c_i}}{2a_i}$$

where $i \in \{2, 3\}$. Thus, since $4a_i c_i > 0$, it follows that if $b_i^2 - 4a_i c_i < 0$, then λ_i and λ_i^* are complex with negative real part given by $-b_i/2a_i$, and if $b_i^2 - 4a_i c_i \geq 0$, then λ_i and λ_i^* are real negative since $b_i^2 > b_i^2 - 4a_i c_i$. Thus, the real part of λ_i and λ_i^* is negative for $i \in \{2, 3\}$. \square

Lemma 2

Consider the reduced attitude dynamics of the 3D pendulum given by (7) and (8). Let $\Psi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a smooth function satisfying (13) and choose the controller (14). Then, the hanging equilibrium of the closed-loop reduced attitude dynamics (16) is asymptotically stable and the convergence is locally exponential.

Proof

Consider the linearization of the closed-loop system (16) about the hanging equilibrium, given by (19) written in terms of the state variable $z = (x_1, x_2, \dot{x}_1, \dot{x}_2, \dot{x}_3) \in \mathbb{R}^5$. Writing (19) in terms of z yields $\dot{z} = \bar{A}_h z$, where \bar{A}_h is obtained by deleting the third row and third column from A_h given in (25). Let $\text{Spec}(M)$ denote the eigenvalues of the matrix M . Since, the third column of A_h is identically zero, it can be shown that $\text{Spec}(\bar{A}_h) = \text{Spec}(A_h) \setminus \{0\}$. Then, from Lemma 1, it follows that all eigenvalues of \bar{A}_h have negative real parts and at least one eigenvalue is negative real.

Hence, it follows that all eigenvalues of the linearization of the closed-loop system (16) about the hanging equilibrium have negative real parts. Thus, the hanging equilibrium of the nonlinear system (16) is asymptotically stable with locally exponentially fast convergence. \square

Remark 2

Let $(0, R_e)$ be a hanging equilibrium. Suppose $D = \text{diag}(d_1, d_2, d_3)$ is diagonal and $d_i > 0$, $i \in \{1, 2, 3\}$. Then, the eigenvalues of the linearized closed-loop reduced attitude dynamics at $(0, R_e^T e_3)$ given by (16) are the roots of the polynomial

$$(s^2 + d_1 s + mgl_1)(s^2 + d_2 s + mgl_2)(s + d_3) = 0$$

Next, we study the linearization of the closed-loop about the inverted equilibrium. This yields the local structure of closed-loop trajectories near the inverted equilibrium.

Lemma 3

Consider the linear model (19), representing linearization of (15) at an inverted equilibrium $(0, R_e)$ expressed in first-order form as

$$\frac{d}{dt} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & I \\ \Lambda & -D \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \triangleq A_{\text{inv}} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \quad (28)$$

where $x = (x_1, x_2, x_3)$. Then, A_{inv} has one zero, three negative and two positive real eigenvalues.

Proof

In the prior notation $k = -1$ in (19) for an inverted equilibrium. Let $v = [v_1^T \ v_2^T]^T$, $v_1, v_2 \in \mathbb{C}^3$ be an eigenvector corresponding to the eigenvalue λ of A_{inv} . Then, as in Lemma 1, one can show that all eigenvalues of A_{inv} satisfy

$$a\lambda^2 + b\lambda - c = 0$$

where $a = \bar{v}_1^T v_1$, $b = \bar{v}_1^T D v_1$ and $c = \bar{v}_1^T \Lambda v_1$. Arguing as in Lemma 1, one can show that $\lambda_1 = 0$ and $\lambda_1^* = -e_3^T D e_3$ are two of the six eigenvalues of A_{inv} in (28) and the other four eigenvalues are of the form

$$\lambda_i = -\frac{b_i}{2a_i} + \frac{\sqrt{b_i^2 + 4a_i c_i}}{2a_i} \quad \text{and} \quad \lambda_i^* = -\frac{b_i}{2a_i} - \frac{\sqrt{b_i^2 + 4a_i c_i}}{2a_i}$$

where a_i, b_i and $c_i, i \in \{2, 3\}$ are positive. Thus, since $4a_i c_i > 0$, it follows that λ_i is positive and λ_i^* is negative for $i \in \{2, 3\}$. Hence, A_{inv} has one zero eigenvalue, three negative eigenvalues, and two positive eigenvalues. \square

Lemma 4

Consider the reduced attitude dynamics of the 3D pendulum given by (7) and (8). Let $\Psi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a smooth function satisfying (13) and choose the controller (14). Then, the inverted equilibrium of the closed-loop reduced attitude dynamics (16) is unstable. Furthermore, the set of closed-loop trajectories that converge to the inverted equilibrium is a 3D invariant manifold \mathcal{M}_i .

Proof

Consider the linearization of the closed-loop system (16) about the inverted equilibrium, given by (19) written in terms of the state variable $z = (x_1, x_2, \dot{x}_1, \dot{x}_2, \dot{x}_3) \in \mathbb{R}^5$. Writing (19) in terms of z yields $\dot{z} = \bar{A}_{\text{inv}} z$, where \bar{A}_{inv} is obtained by deleting the third row and third column from A_{inv} given in (28). Since, the third column of A_{inv} is identically zero, it can be shown that $\text{Spec}(\bar{A}_{\text{inv}}) = \text{Spec}(A_{\text{inv}}) \setminus \{0\}$. Then, from Lemma 3, it follows that \bar{A}_{inv} has three negative and two positive eigenvalues. Hence, the inverted equilibrium of (16) is unstable.

Furthermore, there exists a 3D stable invariant manifold \mathcal{M}_i of the closed-loop (16) such that all solutions that start in \mathcal{M}_i converge to the inverted equilibrium [20]. The tangent space to this manifold at the inverted equilibrium is the stable eigenspace corresponding to the negative eigenvalues. Also, since there are no eigenvalues on the imaginary axis, the closed-loop (16) has no centre manifold and every closed-loop trajectory that converges to the inverted equilibrium lies in the stable manifold \mathcal{M}_i . \square

Remark 3

Let $(0, R_e)$ be an inverted equilibrium. Suppose $D = \text{diag}(d_1, d_2, d_3)$ is diagonal and $d_i > 0, i \in \{1, 2, 3\}$. Then, the eigenvalues of the linearized closed-loop reduced attitude dynamics at $(0, R_e)$ are the roots of the polynomial

$$(s^2 + d_1 s - mgl_1)(s^2 + d_2 s - mgl_2)(s + d_3) = 0$$

In summary, we have shown that the hanging equilibrium of the closed-loop (16) is locally exponentially stable and the inverted equilibrium of (16) is unstable. Furthermore, the set of all closed-loop trajectories that converge to the inverted equilibrium constitutes a 3D, invariant manifold \mathcal{M}_i in $\text{TSO}(3)/S^1$.

6. GLOBAL ANALYSIS OF THE CLOSED-LOOP 3D PENDULUM

In this section, we construct Lyapunov functions and use the local results proven in the last section to obtain global properties of closed-loop trajectories of the 3D pendulum.

Lemma 5

Consider the reduced attitude dynamics of the 3D pendulum given by (7) and (8). Let $\Psi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a smooth function satisfying (13) and choose the controller (14). Then, for

every $\epsilon \in (0, 2mg\|\rho\|)$, all solutions of the closed-loop given by (16), such that $(\omega(0), \Gamma(0)) \in \mathcal{H}_\epsilon$, where

$$\mathcal{H}_\epsilon = \{(\omega, \Gamma) \in \mathbb{R}^3 \times S^2 : \frac{1}{2}\omega^T J \omega + \frac{1}{2}mg\|\rho\| \|\Gamma - \Gamma_h\|^2 \leq 2mg\|\rho\| - \epsilon\} \quad (29)$$

satisfy $(\omega(t), \Gamma(t)) \in \mathcal{H}_\epsilon$, $t \geq 0$, and $\lim_{t \rightarrow \infty} \omega(t) = 0$ and $\lim_{t \rightarrow \infty} \Gamma(t) = \Gamma_h$.

Proof

Consider the closed-loop system given by (16). We propose the following candidate Lyapunov function:

$$V(\omega, \Gamma) = \frac{1}{2}[\omega^T J \omega + mg\|\rho\| \|\Gamma - \Gamma_h\|^2] \quad (30)$$

Note that the above Lyapunov function is positive definite on $T\text{SO}(3)/S^1 \cong \mathbb{R}^3 \times S^2$ and $V(0, \Gamma_h) = 0$. Furthermore, the derivative along a solution of the closed-loop (16) is

$$\begin{aligned} \dot{V}(\omega, \Gamma) &= \omega^T(J\dot{\omega}) + mg\|\rho\|(\Gamma - \Gamma_h)^T \dot{\Gamma} \\ &= \omega^T(J\omega \times \omega + mg\rho \times \Gamma - \Psi(\omega)) + mg\|\rho\|(\Gamma - \Gamma_h)^T(\Gamma \times \omega) \\ &= -\omega^T \Psi(\omega) + mg\omega^T(\rho \times \Gamma) - mg\rho^T(\Gamma \times \omega) = -\omega^T \Psi(\omega) \leq -\mathcal{P}(\omega) \end{aligned}$$

where the last inequality follows from (13). Thus, $V(\cdot)$ is positive definite and $\dot{V}(\cdot)$ is negative semidefinite on $\mathbb{R}^3 \times S^2$.

Next, consider the sublevel set $\mathcal{H}_\epsilon = \{(\omega, \Gamma) \in \mathbb{R}^3 \times S^2 : V(\omega, \Gamma) \leq 2mg\|\rho\| - \epsilon\}$. Note that the compact set \mathcal{H}_ϵ contains the hanging equilibrium $(0, \Gamma_h)$ but does not contain the inverted equilibrium $(0, \Gamma_i)$. Since, $\dot{V}(\omega, \Gamma) \leq 0$, all solutions such that $(\omega(0), \Gamma(0)) \in \mathcal{H}_\epsilon$ satisfy $(\omega(t), \Gamma(t)) \in \mathcal{H}_\epsilon$ for all $t \geq 0$. Thus, \mathcal{H}_ϵ is a compact, positively invariant set for (16).

Furthermore, from the invariant set theorem, we obtain that solutions satisfying $(\omega(0), \Gamma(0)) \in \mathcal{H}_\epsilon$ converge to the largest invariant set in $\{(\omega, \Gamma) \in \mathcal{H}_\epsilon : \omega = 0\}$, that is $\rho \times \Gamma = 0$. Therefore, either $\Gamma(t) \rightarrow \rho/\|\rho\| = \Gamma_h$ or $\Gamma(t) \rightarrow -\rho/\|\rho\| = \Gamma_i$ as $t \rightarrow \infty$. Since $(0, \Gamma_i) \notin \mathcal{H}_\epsilon$, it follows that $\Gamma(t) \rightarrow \Gamma_h$ as $t \rightarrow \infty$. \square

From Lemma 2, we know that $(0, \Gamma_h)$ is asymptotically stable. Thus, $(0, \Gamma_h)$ is an asymptotically stable equilibrium of the closed-loop reduced attitude dynamics given by (16), with a domain of attraction that contains \mathcal{H}_ϵ .

Lemma 6

Consider the reduced attitude dynamics of the 3D pendulum given by (7) and (8). Let $\Psi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a smooth function satisfying (13) and choose the controller (14). Define the level set

$$\mathcal{A} = \{(\omega, \Gamma) \in \mathbb{R}^3 \times S^2 : \frac{1}{2}\omega^T J \omega + \frac{1}{2}mg\|\rho\| \|\Gamma - \Gamma_h\|^2 = 2mg\|\rho\|\} \quad (31)$$

Then, all solutions of the closed-loop given by (16), such that $\omega(0) \neq 0$ and $(\omega(0), \Gamma(0)) \in \mathcal{A}$ satisfy $\lim_{t \rightarrow \infty} \omega(t) = 0$ and $\lim_{t \rightarrow \infty} \Gamma(t) = \Gamma_h$.

Proof

Consider the closed-loop given by (16) and the Lyapunov function given in (30). As already shown in Lemma 5, $\dot{V}(\omega, \Gamma) = -\omega^T \Psi(\omega)$. Thus,

$$\begin{aligned} \dot{V} &= -\dot{\omega}^T \Psi(\omega) - \omega^T \dot{\Psi}(\omega) \\ &= -\dot{\omega}^T \left(\Psi(\omega) + \left[\frac{\partial \Psi}{\partial \omega} \right]^T \omega \right) \end{aligned} \tag{32}$$

since $\Psi(\cdot)$ is smooth. Furthermore, for all $(\omega, \Gamma) \in \mathcal{H}_0$, ω is bounded and

$$\dot{\omega} = J^{-1}(J\omega \times \omega + mg\rho \times \Gamma - \Psi(\omega))$$

is also bounded. Define

$$N = \frac{1}{2} \sup_{(\omega, \Gamma) \in \mathcal{H}_0} \left\{ \left\| \dot{\omega}^T \left(\Psi(\omega) + \left[\frac{\partial \Psi}{\partial \omega} \right]^T \omega \right) \right\| \right\} < \infty$$

Next, since $(\omega(0), \Gamma(0)) \in \mathcal{A}$, $V(\omega(0), \Gamma(0)) = 2mg\|\rho\|$ and $\|\dot{V}(\omega(t), \Gamma(t))\| \leq 2N, t \geq 0$. Expanding $V(\omega(t), \Gamma(t))$ in a Taylor series expansion, we obtain

$$\begin{aligned} V(\omega(t), \Gamma(t)) &= 2mg\|\rho\| - \omega(0)^T \Psi(\omega(0))t + R(t) \\ &\leq 2mg\|\rho\| - \mathcal{P}(\omega(0))t + Nt^2 \end{aligned}$$

since the remainder necessarily satisfies $\|R(t)\| \leq Nt^2$. Since, $\omega(0) \neq 0$ and \mathcal{P} is a positive definite function, it follows that $\mathcal{P}(\omega(0)) > 0$. Denote $\gamma = \mathcal{P}(\omega(0))$ and define

$$\bar{\epsilon} = \min \left(2mg\|\rho\|, \frac{3\gamma^2}{16N} \right)$$

It can be easily shown that for all $t \in [T_1, T_2]$,

$$t^2 - \frac{\gamma}{N}t + \frac{\bar{\epsilon}}{N} \leq 0$$

where

$$T_1 = \frac{\gamma}{2N} \left(1 - \sqrt{1 - \frac{4\bar{\epsilon}N}{\gamma^2}} \right) > 0$$

and $T_2 = T_1 + \gamma/(2N)$. Choose an $\epsilon \in (0, \bar{\epsilon})$. Then, for all $t \in [T_1, T_2]$, $V(\omega(t), \Gamma(t)) \leq 2mg\|\rho\| - \bar{\epsilon} < 2mg\|\rho\| - \epsilon$ and hence, $(\omega(t), \Gamma(t)) \in \mathcal{H}_\epsilon$, where \mathcal{H}_ϵ is a positively invariant set given in Lemma 5. Thus, from Lemma 5, we obtain the result that $\omega(t) \rightarrow 0$ and $\Gamma(t) \rightarrow \Gamma_h$, as $t \rightarrow \infty$. □

Theorem 1

Consider the reduced attitude dynamics of the 3D pendulum given by (7) and (8). Let $\Psi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a smooth function satisfying (13) and choose the controller (14). Then, all solutions of the closed-loop given by (16), such that $(\omega(0), \Gamma(0)) \in \mathcal{N} \setminus \{(0, \Gamma_i)\}$, where

$$\mathcal{N} = \{(\omega, \Gamma) \in \mathbb{R}^3 \times S^2 : \frac{1}{2}\omega^T J\omega + \frac{1}{2}mg\|\rho\| \|\Gamma - \Gamma_h\|^2 \leq 2mg\|\rho\|\} \tag{33}$$

satisfy $(\omega(t), \Gamma(t)) \in \mathcal{N}, t \geq 0$, and $\lim_{t \rightarrow \infty} \omega(t) = 0$ and $\lim_{t \rightarrow \infty} \Gamma(t) = \Gamma_h$.

Proof

From Lemmas 5 and 6, we obtain the result that for every $\epsilon \in (0, 2mg\|\rho\|)$ and $(\omega(0), \Gamma(0)) \in \mathcal{H}_\epsilon \cup \mathcal{A} \setminus \{(0, \Gamma_i)\}$, where \mathcal{H}_ϵ and \mathcal{A} are as defined in Lemmas 5 and 6, $\omega(t) \rightarrow 0$ and $\Gamma(t) \rightarrow \Gamma_h$ as $t \rightarrow \infty$. Since, \mathcal{N} can be written as

$$\mathcal{N} = \bigcup_{\epsilon \in (0, 2mg\|\rho\|)} (\mathcal{H}_\epsilon \cup \mathcal{A})$$

the result follows. \square

It is important to note that $\mathcal{N} \setminus \{(0, \Gamma_i)\}$ is a positively invariant set of the closed-loop (16), and hence it is contained in the maximal domain of attraction of the hanging equilibrium. Any trajectory of (16), that intersects $\mathcal{A} \setminus \{(0, \Gamma_i)\}$ transversely, must at some time lie outside $\mathcal{N} \setminus \{(0, \Gamma_i)\}$, but it eventually enters $\mathcal{N} \setminus \{(0, \Gamma_i)\}$. Hence, the maximal domain of attraction of the hanging equilibrium is a proper superset of the invariant set $\mathcal{N} \setminus \{(0, \Gamma_i)\}$.

Theorem 2

Consider the reduced attitude dynamics of the 3D pendulum model given by (7) and (8). Let $\Psi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a smooth function satisfying (13) and choose the controller (14). Let \mathcal{M}_i denote the 3D invariant manifold as in Lemma 4. Then all solutions of the closed-loop given by (16), such that $(\omega(0), \Gamma(0)) \in (TSO(3)/S^1) \setminus \mathcal{M}_i$, satisfy $\lim_{t \rightarrow \infty} \omega(t) = 0$ and $\lim_{t \rightarrow \infty} \Gamma(t) = \Gamma_h$. Furthermore, all solutions of the closed-loop given by (16), such that $(\omega(0), \Gamma(0)) \in \mathcal{M}_i$, satisfy $\lim_{t \rightarrow \infty} \omega(t) = 0$ and $\lim_{t \rightarrow \infty} \Gamma(t) = \Gamma_i$.

Proof

Consider the closed-loop system given by (16) and the Lyapunov function given in (30). Since S^2 is compact and the Lyapunov function $V(\omega, \Gamma)$ is quadratic in ω , each sublevel set of $V(\omega, \Gamma)$ is compact. Furthermore, since $\dot{V}(\omega, \Gamma) = -\omega^T \Psi(\omega) \leq 0$, by the invariant set theorem, all solutions converge to the largest invariant set in $\{(\omega, \Gamma) \in \mathbb{R}^3 \times S^2 : \dot{V}(\omega, \Gamma) = 0\}$.

Substituting $\omega \equiv 0$ in (16), it is easily shown that the largest such invariant set is given by $\{(0, \Gamma_h)\} \cup \{(0, \Gamma_i)\}$. However, from Lemma 4, we know that all trajectories that converge to the inverted equilibrium are contained in the 3D manifold \mathcal{M}_i . Therefore, all solutions of the closed-loop given by (16), such that $(\omega(0), \Gamma(0)) \in (TSO(3)/S^1) \setminus \mathcal{M}_i$, converge to the hanging equilibrium and satisfy $\lim_{t \rightarrow \infty} \omega(t) = 0$ and $\lim_{t \rightarrow \infty} \Gamma(t) = \Gamma_h$. \square

We briefly compare the conclusions in Theorems 1 and 2. Theorem 1 provides an explicit description of a domain of attraction of the hanging equilibrium in $TSO(3)/S^1$; however, that domain of attraction is not maximal. In contrast, Theorem 2 shows that the maximal domain of attraction consists of all points in $TSO(3)/S^1$ that are not in the stable manifold \mathcal{M}_i of the inverted equilibrium. The geometry of the stable manifold \mathcal{M}_i of the inverted equilibrium may be complicated; it depends on the model parameters and the specific controller.

Remark 4

Since \mathcal{M}_i is a 3D submanifold, its complement in $TSO(3)/S^1$ is open and dense. Thus, from Theorem 2 it follows that the domain of attraction of the hanging equilibrium for the closed-loop (16) is *almost global*.

Remark 5

According to Theorem 1, $\mathcal{N} \setminus \{(0, \Gamma_i)\}$ is a subset of the domain of attraction; it is clear that \mathcal{M}_i and $\mathcal{N} \setminus \{(0, \Gamma_i)\}$ are disjoint. Indeed, $\mathcal{M}_i \cap \mathcal{N} = \{(0, \Gamma_i)\}$.

The results presented in Theorems 1 and 2 apply to the solutions of the closed-loop reduced attitude dynamics of the 3D pendulum given by (16). Thus, the compact sets and the almost global results hold with respect to $TSO(3)/S^1$. We now study the implication for the 3D pendulum dynamics given by (15). Specifically, we show that the controller (14) almost globally asymptotically stabilizes the hanging equilibrium manifold. The following Lemma is needed.

Lemma 7

Let $\pi : TSO(3) \rightarrow TSO(3)/S^1$ denote the projection, where $TSO(3)/S^1$ is endowed with the quotient topology. Let $\mathcal{U} \subseteq TSO(3)/S^1$ be a set whose complement is open and dense in $TSO(3)/S^1$. Then, $\pi^{-1}(\mathcal{U}) \subseteq TSO(3)$ is a set whose complement is open and dense in $TSO(3)$.

Proof

We show that $\pi^{-1}(\mathcal{U})$ is closed and nowhere dense, and hence, its complement is open and dense in $TSO(3)$. Since, π is continuous and \mathcal{U} is closed, $\pi^{-1}(\mathcal{U})$ is closed. Suppose that $\pi^{-1}(\mathcal{U})$ is not nowhere dense. Then, there exists a non-trivial open set \mathcal{O} in $TSO(3)$ such that $\mathcal{O} \subseteq \pi^{-1}(\mathcal{U}) = \pi^{-1}(\mathcal{U})$. Then, since π is a quotient map, it follows that $\pi(\mathcal{O}) \subseteq \mathcal{U}$ is an open set. Further, since π is a surjection, \mathcal{O} is non-trivial and hence \mathcal{U} is not nowhere dense; this contradicts the assumption that the complement of \mathcal{U} is open and dense. \square

We now present the main result for stabilization of the hanging equilibrium manifold of the 3D pendulum given by (1) and (2). This result shows that controller (14) achieves almost global stabilization of the hanging equilibrium manifold.

Theorem 3

Consider the dynamics of the 3D pendulum model given by (1) and (2). Let $\Psi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a smooth function satisfying (13) and choose the controller (14). Then, the hanging equilibrium manifold is asymptotically stable with local exponential convergence. Furthermore, there exists an invariant set $M_i \subset TSO(3)$ such that $TSO(3) \setminus M_i$ is open and dense, and all solutions of the closed-loop given by (1), (2) and (14), such that $(\omega(0), R(0)) \in TSO(3) \setminus M_i$, converge to the hanging equilibrium manifold. All solutions of the closed-loop, such that $(\omega(0), R(0)) \in M_i$, converge to the inverted equilibrium manifold.

Proof

Asymptotic stability of the hanging equilibrium manifold and local exponential convergence of closed-loop trajectories follows immediately from Lemma 2 and Proposition 3. We next show almost global convergence of the closed-loop trajectories.

Consider the projection $\pi : TSO(3) \rightarrow TSO(3)/S^1$ which yields the reduced attitude dynamics as given in Proposition 1. Let \mathcal{M}_i be the invariant submanifold as in Theorem 2 and denote $M_i \triangleq \pi^{-1}(\mathcal{M}_i) \subset TSO(3)$. It follows from Lemma 7 that the complement of M_i in $TSO(3)$ is open and dense. Since \mathcal{M}_i is invariant for the closed-loop (16), it is easy to see that $M_i = \pi^{-1}(\mathcal{M}_i)$ is invariant for the closed-loop (15). Furthermore, from Propositions 1 and 3 and Theorem 2, it follows that for the closed-loop (1), (2) and (14) given by (15), all trajectories contained in

$TSO(3) \setminus M_i$ converge to the hanging equilibrium manifold and all trajectories in the set M_i converge to the inverted equilibrium manifold. \square

Remark 6

Since $TSO(3) \setminus M_i$ is open and dense, it follows from Theorem 3 that the domain of attraction of the hanging equilibrium manifold for the closed-loop (1), (2) and (14) is *almost global*.

Remark 7

Theorem 1 guarantees that $\mathcal{N} \setminus \{(0, \Gamma_i)\}$ is a subset of the domain of attraction of the hanging equilibrium. It is clear that M_i and $\pi^{-1}(\mathcal{N}) \setminus \mathbb{I}$ are disjoint, where \mathbb{I} is the inverted equilibrium manifold. Indeed, $M_i \cap \pi^{-1}(\mathcal{N}) = \mathbb{I}$.

7. SIMULATION RESULTS

In this section, we present results to show that controller (14) can stabilize the hanging equilibrium manifold with any arbitrary level of saturation.

Assume

$$J = \begin{bmatrix} 30 & 20 & 7 \\ 20 & 40 & 0 \\ 7 & 0 & 50 \end{bmatrix} \text{ kg m}^2$$

$m = 100 \text{ kg}$, $\rho = [0 \ 0 \ 0.2]^T \text{ m}$ and $g = 10 \text{ m s}^{-2}$. For the given values, $\Gamma_h = [0 \ 0 \ 1]^T$ and $\Gamma_i = [0 \ 0 \ -1]^T$. Suppose that the actuators for the 3D pendulum satisfy the control constraint $\|u\|_2 \leq k$ for some positive k . We next choose the function Ψ in (14) such that the controller does not violate the given constraint.

Let $P \in \mathbb{R}^{3 \times 3}$ denote a positive definite symmetric matrix. Define the controller

$$u = -\Psi(\omega) \triangleq -k \frac{P\omega}{1 + \|P\omega\|_2} \quad (34)$$

It can be easily shown that Ψ defined as above satisfies (13) and that controller (34) satisfies $\|u\|_2 \leq k$. Choose the positive definite matrix $P = \text{diag}(5, 10, 15)$ and $k = 10$. Thus, $\|u\|_2 \leq 10 \text{ N m}$. Choose initial conditions for the simulation as

$$R(0) = \begin{bmatrix} 0 & -0.9999 & -0.0141 \\ -1 & 0 & 0 \\ 0 & 0.0141 & -0.9999 \end{bmatrix} \quad \text{and} \quad \omega(0) = 0 \text{ deg/s}$$

To plot the attitude error from the hanging equilibrium manifold, we plot the angle between the vector Γ and Γ_h . This is given by $\Theta(t) = \cos^{-1}(e_3^T R(t)e_3)$. It can be shown that $\Theta(t) \in [0, \pi]$, where $\Theta = 0$ and π correspond to the hanging and the inverted position of the 3D pendulum, respectively. Thus, computing $\Theta(0)$ for the chosen $R(0)$ yields $\Theta(0) = 179.2^\circ$. The initial condition of the 3D pendulum lies close to the inverted equilibrium manifold.

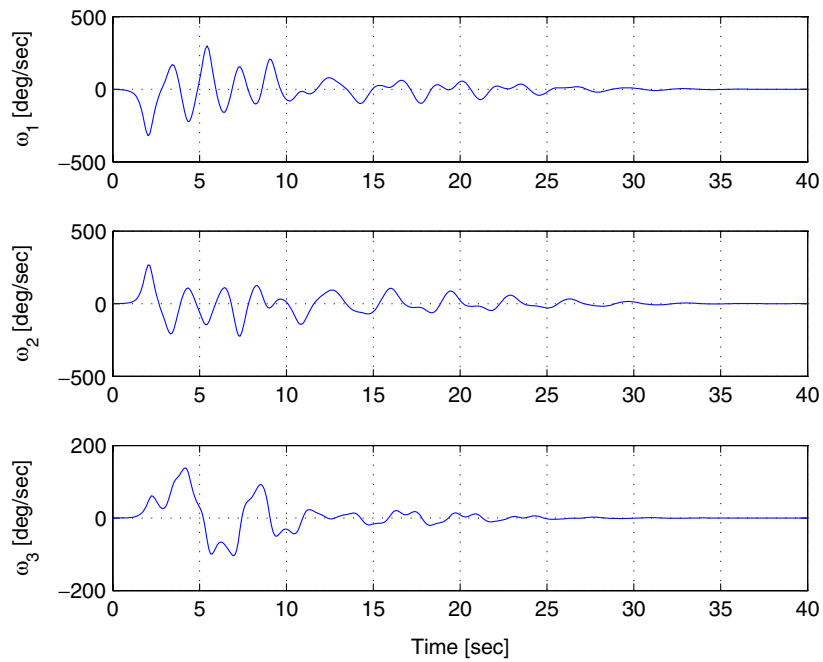


Figure 3. Angular velocity of the 3D pendulum.

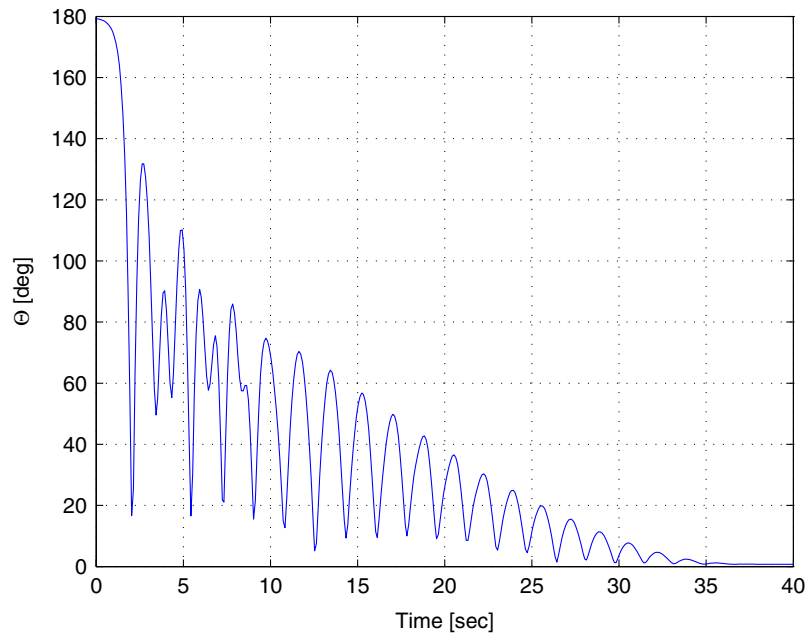


Figure 4. Error in attitude of the 3D pendulum.

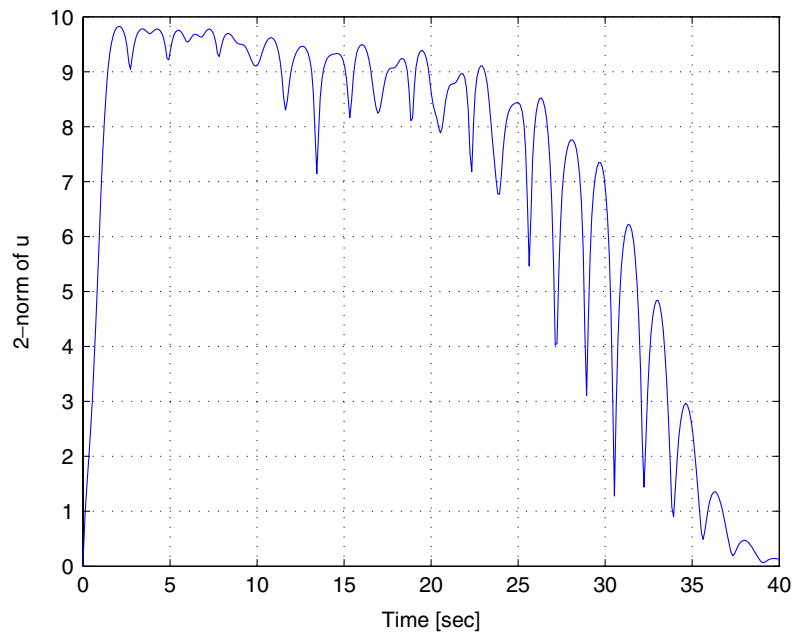


Figure 5. Magnitude of the control torque applied to 3D pendulum.

Figures 3–5 are simulation plots for the closed-loop 3D pendulum. Note from Figures 3 and 4 that both angular velocity and the error in attitude converge to zero. Furthermore, from Figure 5 it is clear that $\|u\|_2 \leq 10$ N m.

8. CONCLUSIONS

This paper has presented a complete analysis of the closed-loop dynamics of the 3D pendulum controlled by angular velocity feedback. The fact that the hanging equilibrium manifold is asymptotically stabilized is made formal, and the global closed-loop dynamics are studied. The hanging equilibrium manifold is not globally asymptotically stable and it cannot be globally stable due to a topological obstruction. The key contribution of this paper is the analysis of the global dynamics of the closed-loop system of the 3D pendulum and the dissipative controller on the compact configuration space $SO(3)$. In particular, it is shown that angular velocity feedback guarantees almost global asymptotic stabilization in the sense that the domain of attraction of the hanging equilibrium manifold is open and dense. Ideas developed in this paper have been used in [21] to study the global dynamics of the 3D pendulum for stabilization of the inverted equilibrium manifold.

This paper has focused on asymptotic stabilization of the hanging equilibrium manifold. In several other papers [8–10, 21] we studied the asymptotic stabilization of the inverted equilibrium manifold and in [21] we studied the asymptotic stabilization of a specific equilibrium in the inverted equilibrium manifold. These stabilization problems for the inverted 3D pendulum are fundamentally different from the problem considered in this paper although

some of the methods for analysing closed-loop dynamics are similar. In this paper, only angular velocity feedback is required to asymptotically stabilize the hanging equilibrium manifold. In order to stabilize the inverted equilibrium manifold, or an inverted equilibrium, it is necessary to feed back both the attitude of the 3D pendulum as well as the angular velocity of the 3D pendulum.

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