

DEPARTMENT OF AERONAUTICAL ENGINEERING
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A TABLETOP ELECTRONIC
DIFFERENTIAL ANALYZER

by

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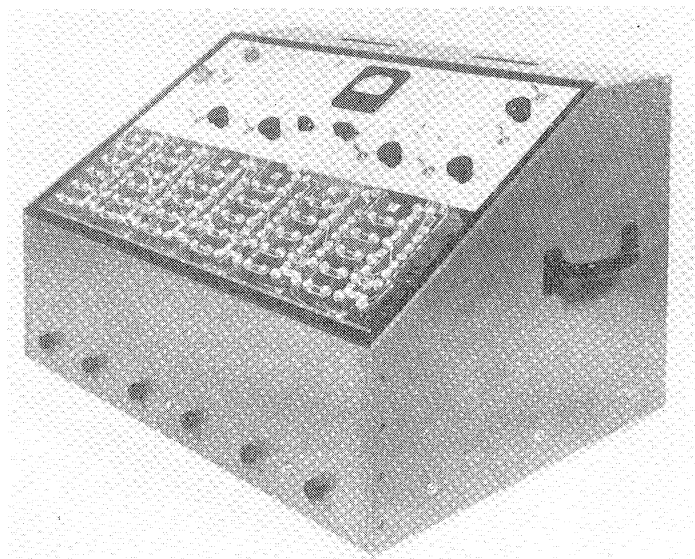
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TABLETOP ELECTRONIC DIFFERENTIAL ANALYZER

Preface

This report describes the theory, design, and operation of a six-amplifier electronic differential analyzer, capable of solving up to fourth-order ordinary linear differential equations. The analyzer has complete, self-contained power supplies, the only input required being the 115 volt 60 cycle line power. Accuracy capabilities are as high as 0.1 percent, depending on the precision with which computing resistors and capacitors are calibrated.

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1. Introduction

1.1 Utility of Electronic Differential Analyzers

The electronic differential analyzer has become an extremely valuable tool for the modern engineer. It is widely used in the field of automatic control as an aid in design, in simulation, in testing, and as a component of the control system. There are many other fields in which the electronic differential analyzer has wide applicability. Some of the more elementary problems which it can solve are discussed in Section 4 at the end of this report. The reader is directed to other references¹⁻⁷ for more detailed discussion of the many applications.

The electronic differential analyzer is limited to the solution of ordinary differential equations, both linear and non-linear. The size of the computer installation required depends on the order and complexity of the equations being solved. Non-linear differential equations often require the use of multipliers, which are usually more expensive and elaborate than the basic linear components. However, the authors have found that for many elementary problems, for demonstration, and for student use a small, self-contained electronic differential analyzer capable of solving only elementary differential equations has great utility. This report describes the theory and operation of such a unit.

1.2 Introduction to operational Amplifiers

The basic computing element of the electronic differential analyzer is the operational amplifier. It consists of a high-gain dc amplifier along with input impedance Z_i and feedback impedance Z_f as shown in Figure 1. If the current into the dc amplifier proper is negligible (this amounts to neglecting the grid current in the first stage of vacuum-tube amplification) the current i_1 through the input impedance is equal to the current i_2 through the feedback impedance. Thus

$$i_1 = i_2 \quad (1)$$

and from ohms law

$$\frac{e_1 - e'}{Z_i} = \frac{e' - e_2}{Z_f} \quad (2)$$

where e_1 = input voltage and e_2 = output voltage of the operational amplifier, and where e' is the input voltage to the dc amplifier proper.

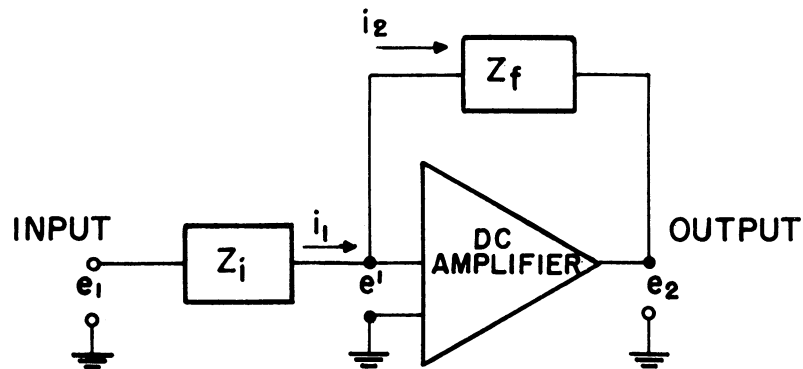


Figure 1. Operational Amplifier

If μ is the gain of the dc amplifier, we have

$$e_2 = - \mu e' \quad (3)$$

Eliminating e' from Equations (2) and (3) and solving for e_2 , we obtain

$$e_2 = - \frac{Z_f}{Z_i} \frac{1}{1 + \frac{1}{\mu} \left(1 + \frac{Z_f}{Z_i}\right)} e_1 \quad (4)$$

If the amplifier gain μ is very much larger than $1 + \frac{Z_f}{Z_i}$, then

$$e_2 \cong - \frac{Z_f}{Z_i} e_1, \mu \gg 1 + \frac{Z_f}{Z_i} \quad (5)$$

which is the fundamental equation governing the behavior of operational amplifiers. It states that the output voltage e_2 is equal to the ratio of feedback to input impedance times the input voltage e_1 , with a sign reversal. Thus a voltage e_1 can be multiplied by a constant K using resistors for impedances and by letting the ratio of feedback to input resistance be equal to K .

Next consider the operational amplifier shown in Figure 2. Here

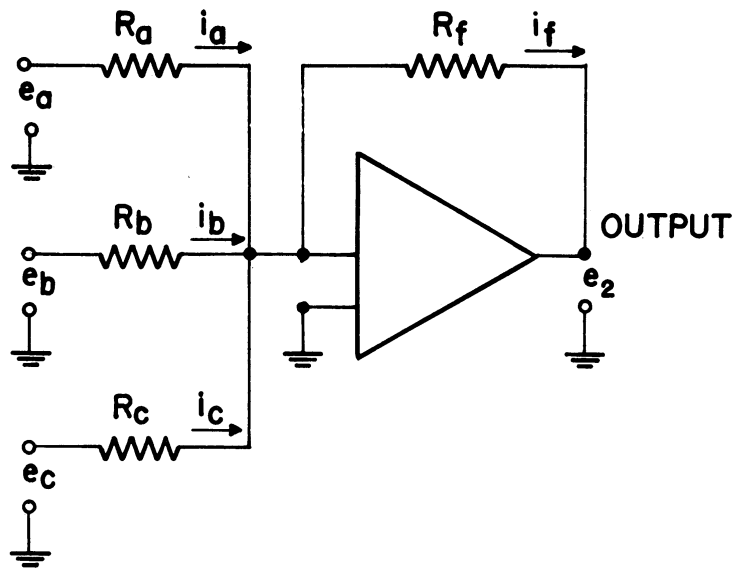


Figure 2. Operational Amplifier for Summation

there are several input resistors R_a , R_b , and R_c , each with a respective input voltage e_a , e_b , and e_c . Again neglecting input current to the dc amplifier proper, it follows that

$$i_a + i_b + i_c = i_f \quad (6)$$

i. e., the sum of the input currents equals the feedback current. If we assume as before that the amplifier gain μ is much greater than one plus the ratio of feedback resistance R_f to respective input resistance, we have

$$e_2 = -\left(\frac{R_f}{R_a} e_a + \frac{R_f}{R_b} e_b + \frac{R_f}{R_c} e_c\right) \quad (7)$$

Thus by employing several input resistors we can sum input voltages.

In Figure 3 the operational amplifier circuit for integration is shown. Here an input resistance R and a feedback capacitance C are used. Neglecting the voltage e_1 as small compared with e_1 or e_2 (this is equivalent to the assumption that $\mu \gg 1 + Z_f/Z_i$), we have for the output voltage e_2

$$e_2 = -\frac{1}{C} \int i_2 dt \quad (8)$$

But if the input current to the dc amplifier proper is negligible, $i_2 = i_1$, and is given by

$$i_2 = i_1 = \frac{e_1}{R} \quad (9)$$

Eliminating i_2 from Equations (8) and (9) we have

$$e_2 = -\frac{1}{RC} \int e_1 dt \quad (10)$$

i. e., the output voltage e_2 is proportional to the integral with respect to time of the input voltage e_1 , the constant of proportionality being $1/RC$.

We have seen how operational amplifiers can be used for multiplication by a constant, sign inversion, summation, and integration. In the next section we will combine these operations to solve a linear differential equation with constant coefficients.

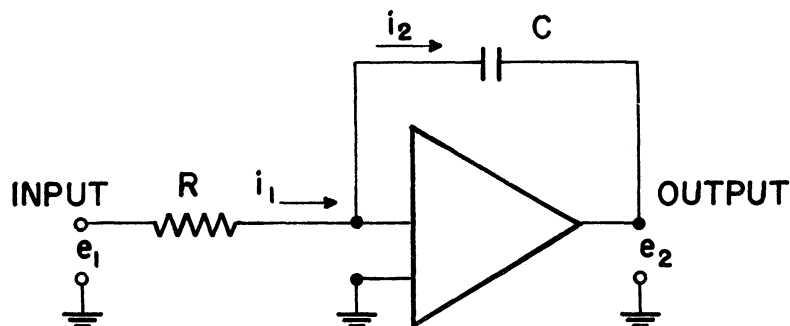


Figure 3. Operational Amplifier for Integration

1.3 Computer Circuit for Solving a Differential Equation

As a simple illustration of the solution of a physical problem, consider the mass-spring-damper shown in Figure 4.

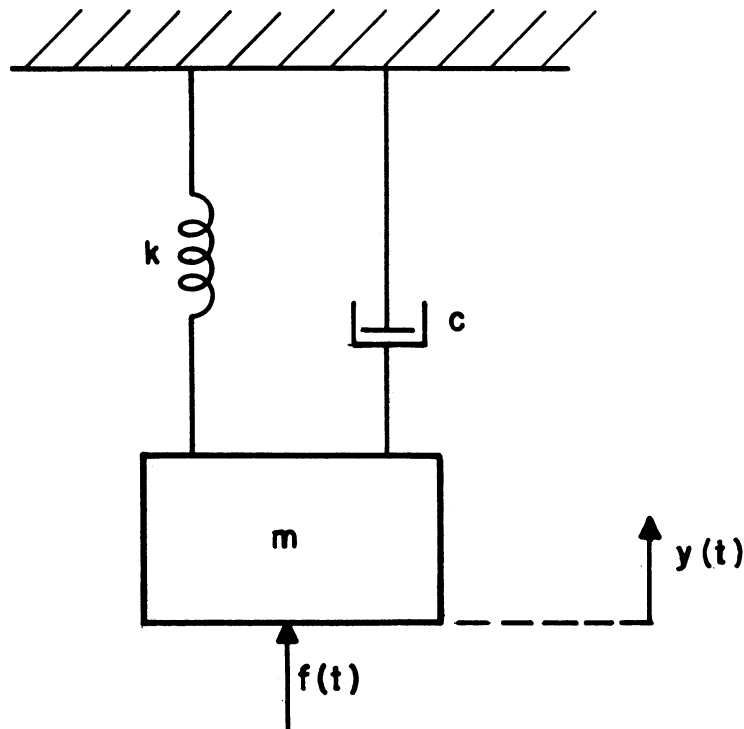


Figure 4. Mass-Spring-Damper System

Let the mass be m , the spring constant k , and viscous damping constant c , and consider only vertical displacement y of the mass. Summing all the forces acting on the mass, we have

$$m\ddot{y} + c\dot{y} + ky = f(t) \quad (11)$$

where $f(t)$ is the applied force. The problem is to find the differential analyzer circuit which will give an output voltage proportional to y for a given input voltage proportional to $f(t)$. To do this it is more convenient to rewrite Equation (11) as

$$m\ddot{y} = -c\dot{y} - ky + f(t) \quad (12)$$

The differential analyzer circuit is built up by assuming that at some place in the circuit there is a voltage $m\ddot{y}$. This can be converted to the voltage $-\dot{y}$ by passing it through an integrator with an RC time constant equal to m , as shown in Figure 5. This voltage is passed through a unit time-constant integrator, which then has the output y . Next the voltages representing y , $-\dot{y}$, and $f(t)$ are summed to give a voltage proportional to the right-hand side

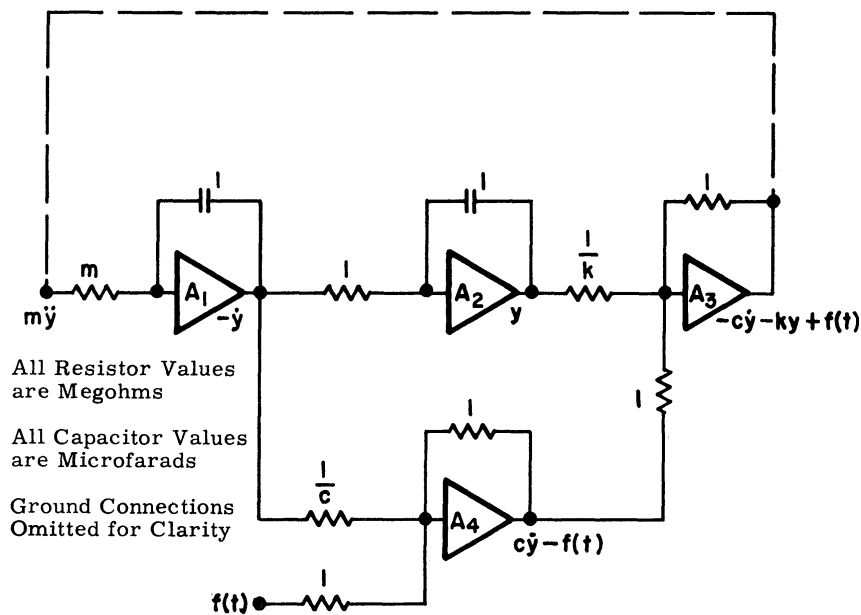


Figure 5. Synthesis of the Differential Equation
by Means of Operational Amplifiers

of Equation (12), namely $-c\dot{y} - ky + f(t)$, as shown in Figure 5. But the equation states that this voltage must equal $m\ddot{y}$; the connection of the output of amplifier A_3 in Figure 5 to the input $m\ddot{y}$ of amplifier A_1 then causes the equation to be satisfied.

In order to obtain a solution to our mass-spring-damper problem it is necessary to specify the initial displacement $y(0)$ and velocity $\dot{y}(0)$. These initial conditions on Equation (12) for time $t = 0$ are imposed by charging the integrating capacitors to the appropriate voltages, as shown in Figure 6. When the initial-condition switches are released simultaneously, the computer proceeds to generate the solution y as a time-varying

voltage. A somewhat different technique than that shown in Figure 6 is used to impose initial conditions in the actual analyzer circuit (See Section 3.2).

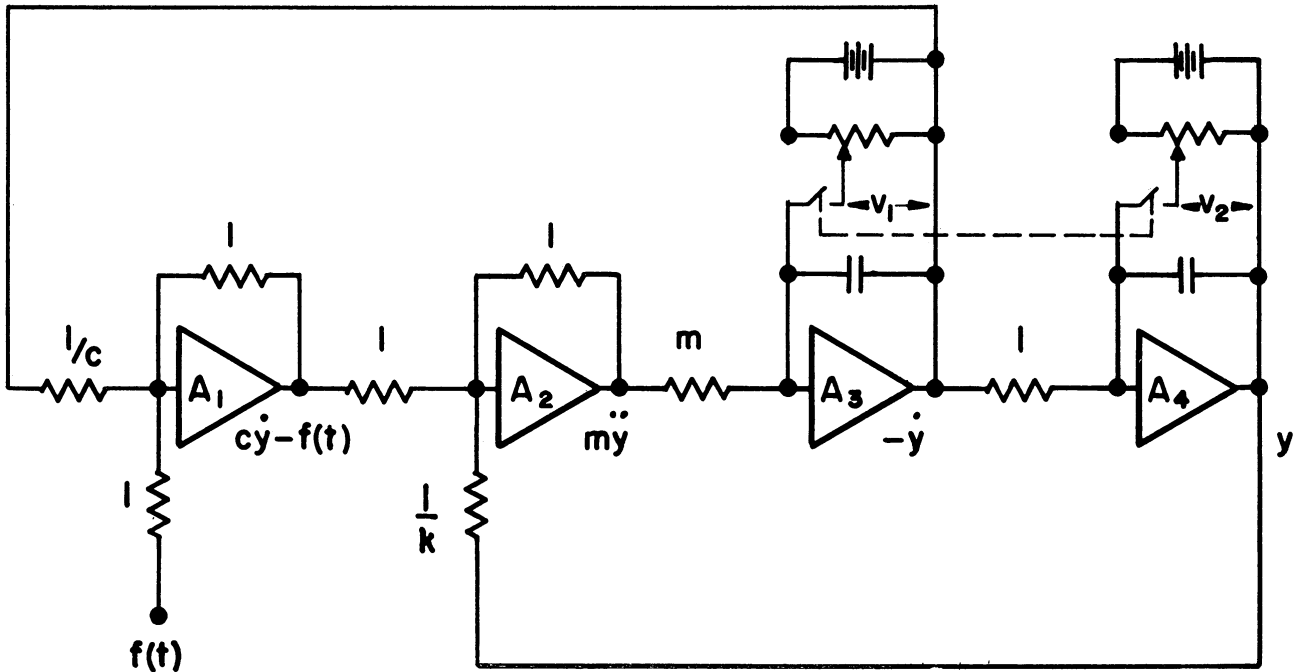
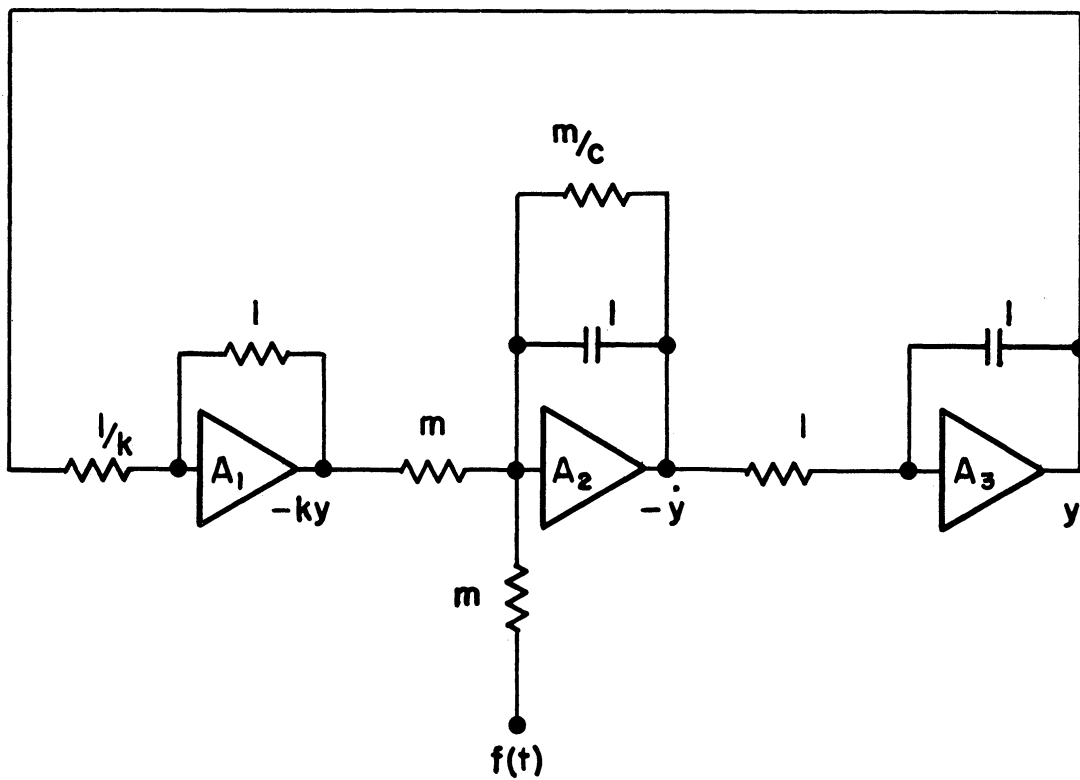


Figure 6. Analyzer Circuit for Mass-Spring-Damper System

The circuit shown in Figure 7 is equivalent to that in Figure 6 except that it uses only three amplifiers. The reader can confirm the equivalence of the circuits. Voltage-recordings made with a Sandborn Model 60 1300 Galvanometer are shown in Figure 8 for a step input $f(t)$. The latter is applied by switching a voltage onto the $f(t)$ terminal. Note that for zero damping the response is a pure sinusoidal oscillation of frequency $\sqrt{k/m}$ radians per second.

In Section 4 the conversion from problem units to computer units is discussed, along with additional illustrative problems.



INITIAL CONDITION CIRCUITS OMITTED FOR CLARITY

Figure 7. Three-Amplifier Circuit Equivalent to Figure 6

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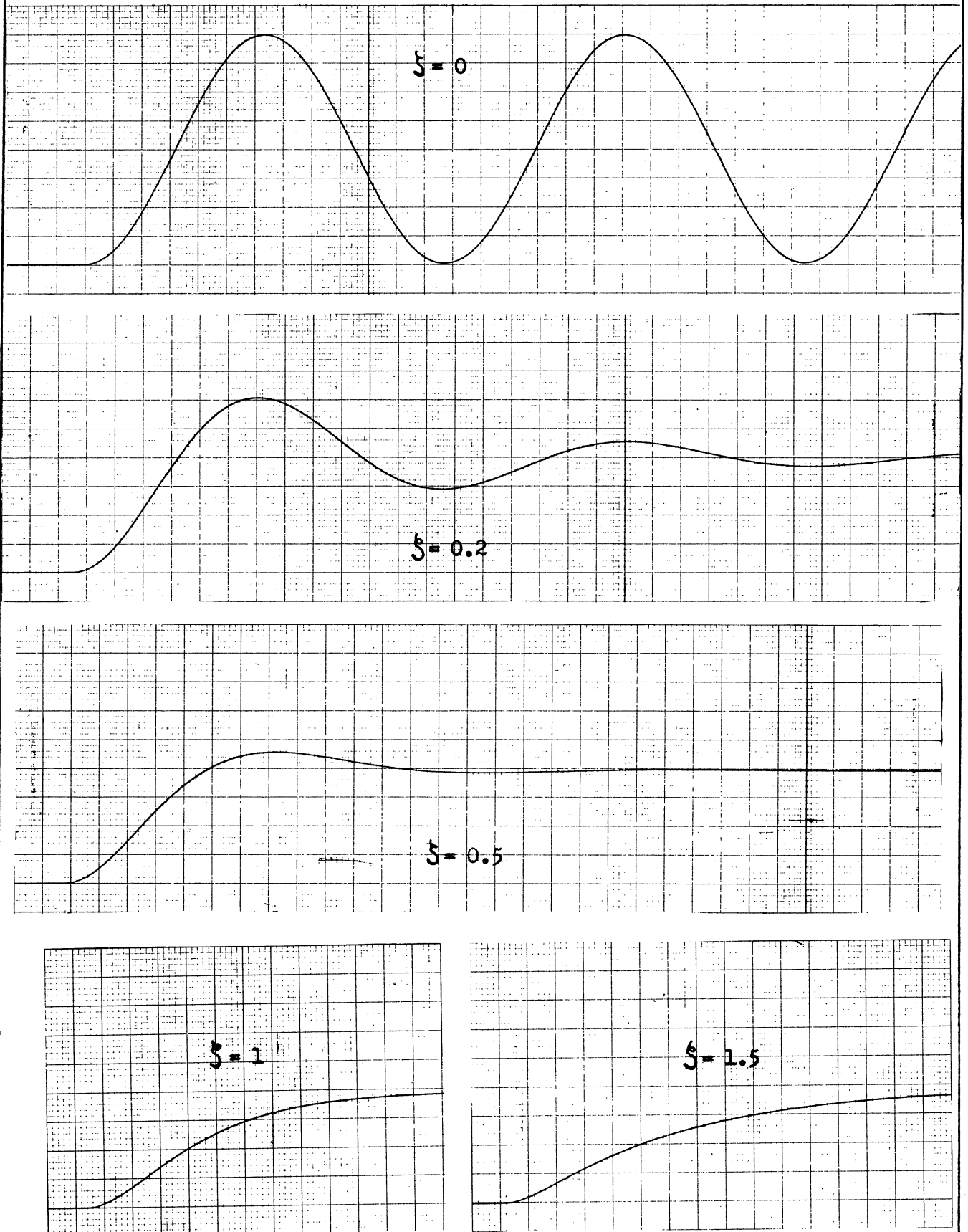


Figure 8. Step Response of Mass-Spring-Damper System
for Various Damping Constants

2. Operational Amplifiers

2.1 DC Amplifier Circuit

In the previous section we saw how operational amplifiers, consisting of high-gain dc amplifiers with feedback and input impedances, could be used to solve ordinary linear differential equations. The dc-amplifier circuit used in the 6-amplifier computer discussed in this report is shown in Figure 9. It consists of three stages of direct-coupled vacuum-tube amplification and a cathode-follower output stage. The first two stages of amplification utilize a single twin-triode envelope, an RCA 5691 "red tube" (long-life equivalent of a 6SL7). The third-stage and cathode-follower output employ a second twin triode, a 6SL7. The overall dc gain of the amplifier is approximately 10,000.

The first stage of amplification is designed so that the grid bias will be about -2 volts; this bias value will give the minimum grid current. A 5 K potentiometer in the self biasing circuit allows the bias to be varied so that output balance of the amplifier can be adjusted. The first two stages of amplification, including all capacitors and resistors except the 5 K potentiometer, are contained in a Vector Turret Can, No. C 12- OKas shown in Figures 11 and 12. The circuit layout for the can is shown in Figure 10.

A plot of maximum amplifier output voltage (positive or negative) versus external load resistance is shown in Figure 13. Note that for a load resistance of about 250 K the amplifier saturates at the same voltage level for positive or negative outputs. This load (250 K) should be employed when the amplifier is to be used to simulate a symmetrical saturation function on a "bang-bang" sign-sensitive type of output (see Section 4.5).

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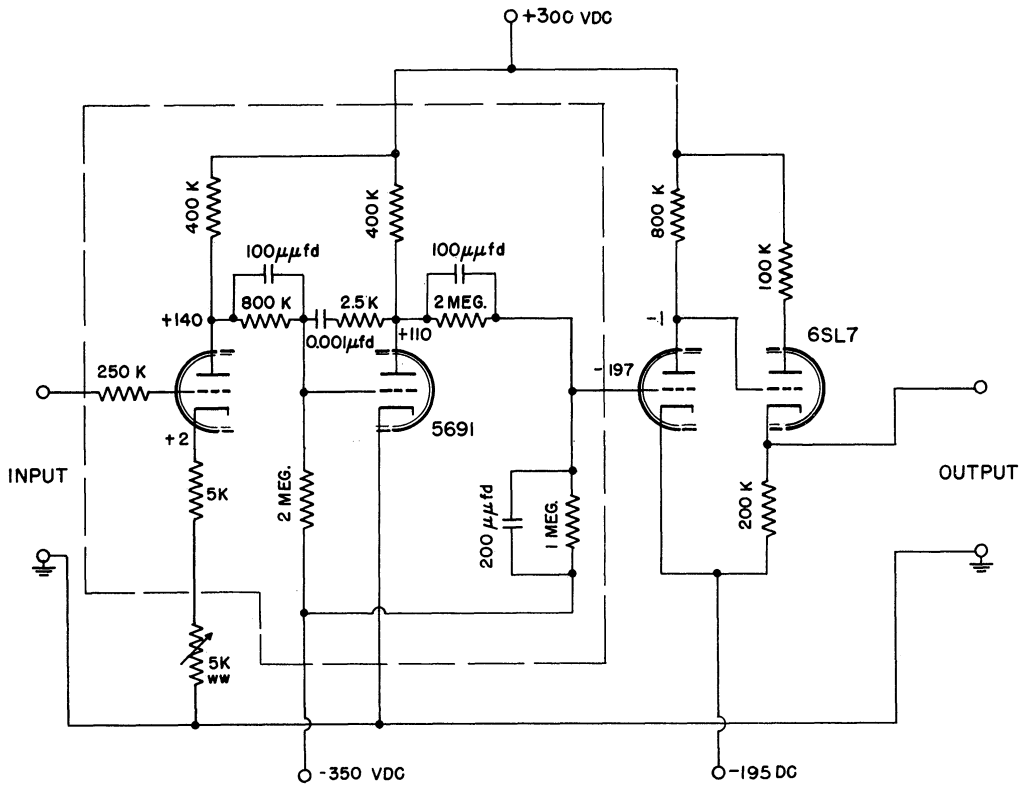


Figure 9. DC Amplifier Circuit

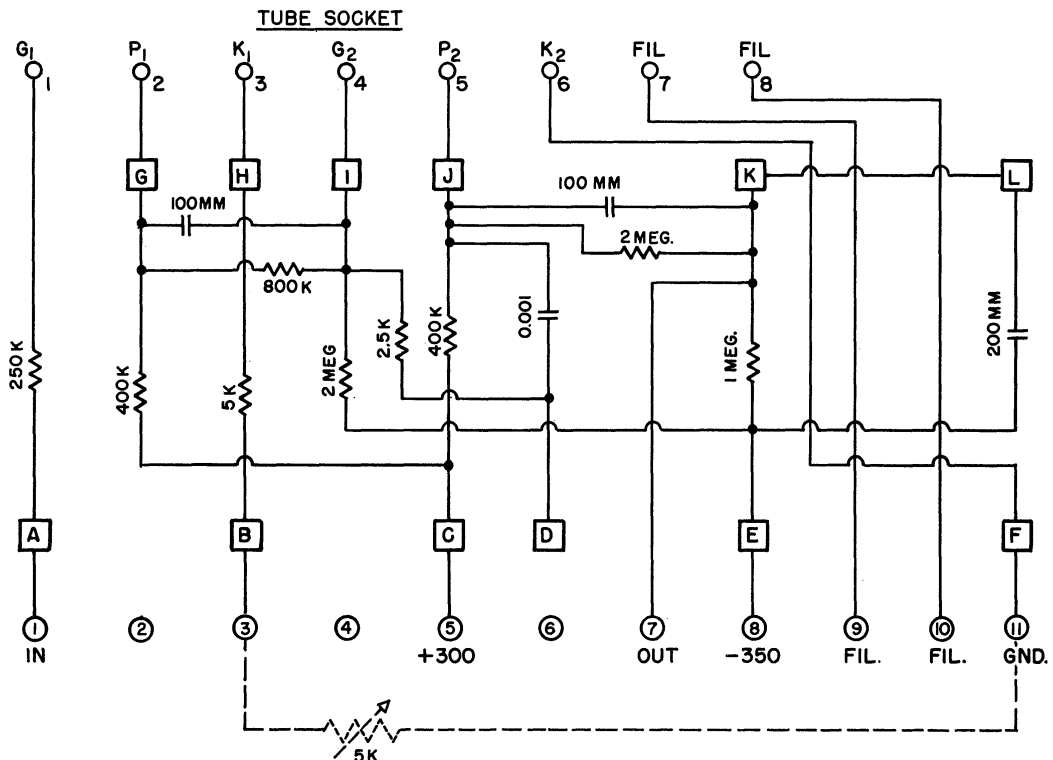


Figure 10. Circuit Layout for Amplifier Can

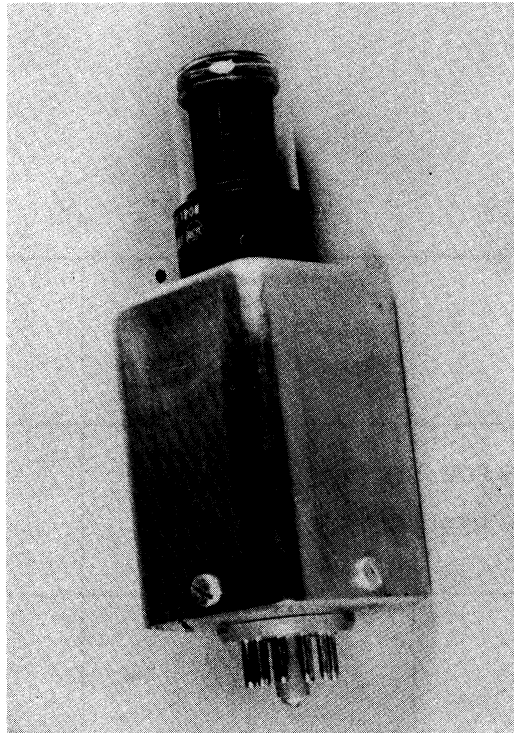


Figure 11. External View of Amplifier Can

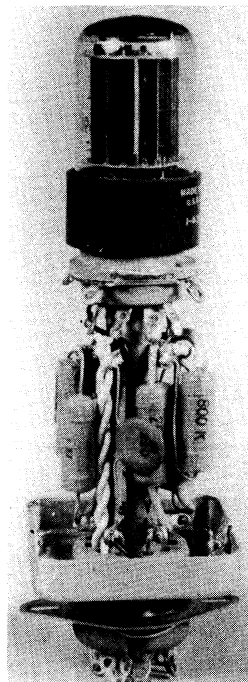


Figure 12. Internal View of Amplifier Can

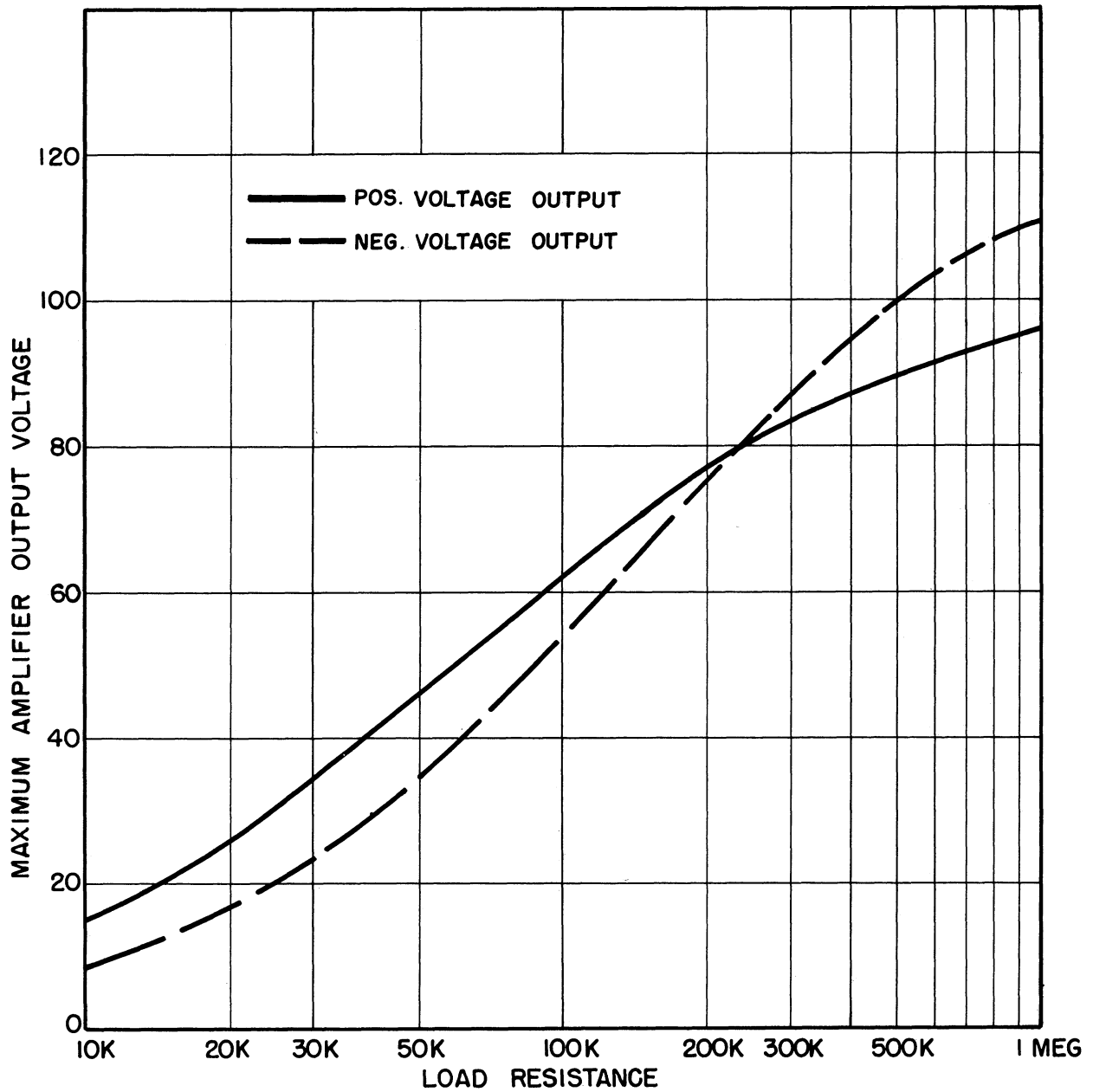


Figure 13. Maximum Output Voltage as a Function of Load Resistance

2.2 Stability Consideration

Since the operational amplifier is essentially a feedback device, it is important to design the attenuation-frequency characteristic of the dc amplifier so that it will be stable when the feedback and input impedances are added. From Equation (4) the output voltage e_2 is given by

$$e_2 = -\frac{Z_f}{Z_i} \frac{\mu(p)}{\mu(p) + (1 + \frac{Z_f}{Z_i})} e_1 \quad (13)$$

where the dc amplifier gain $\mu(p)$ actually includes time-derivative terms, as indicated by the fact that it is a function of the differential operator p . If the operational amplifier is to be stable, the denominator of Equation 13 must not vanish for any values of p with positive real part. This is apparent when we realize that the roots of the denominator are the characteristic roots of the equation of motion of the operational amplifier.

When resistors R_f and R_i are used for feedback and input impedances respectively, the roots of the denominator of Equation 13 are the values of p for which $\mu(p) = -(1 + R_f/R_i)$. If none of these p values is to have positive real parts, the phase shift of the amplifier gain $\mu(j\omega)$ for sinusoidal inputs ($p = j\omega$) must not be as negative as 180 degrees at the frequency where the magnitude of $\mu(j\omega) = 1 + R_f/R_i$. This in turn implies that a db gain versus log frequency plot of $\mu(j\omega)$ for sinusoidal inputs must not have a slope as negative as -12 db per frequency octave in the region where $\mu(j\omega) = 1 + R_f/R_i$. For a conservative design, a slope of -6 db/octave is preferred. (This corresponds to about -90 degrees phase shift). If the operational amplifier is to be stable for all possible ratios R_f/R_i , then the db gain versus log frequency characteristic of the dc amplifier proper should have a slope of -6 db/octave down to below unity (zero db) gain. The capacitors shown in Figure 9 have been chosen so that this is true. The frequency response of the dc amplifier is shown in Figure 14.

When the dc amplifier is used as an integrator (feedback capacitor C , input resistor R), the ratio $Z_f/Z_i = \frac{1}{j\omega RC}$ for sinusoidal inputs. At the frequency ω where $|\mu(j\omega)| = 1 + Z_f/Z_i$ the term Z_f/Z_i is negli-

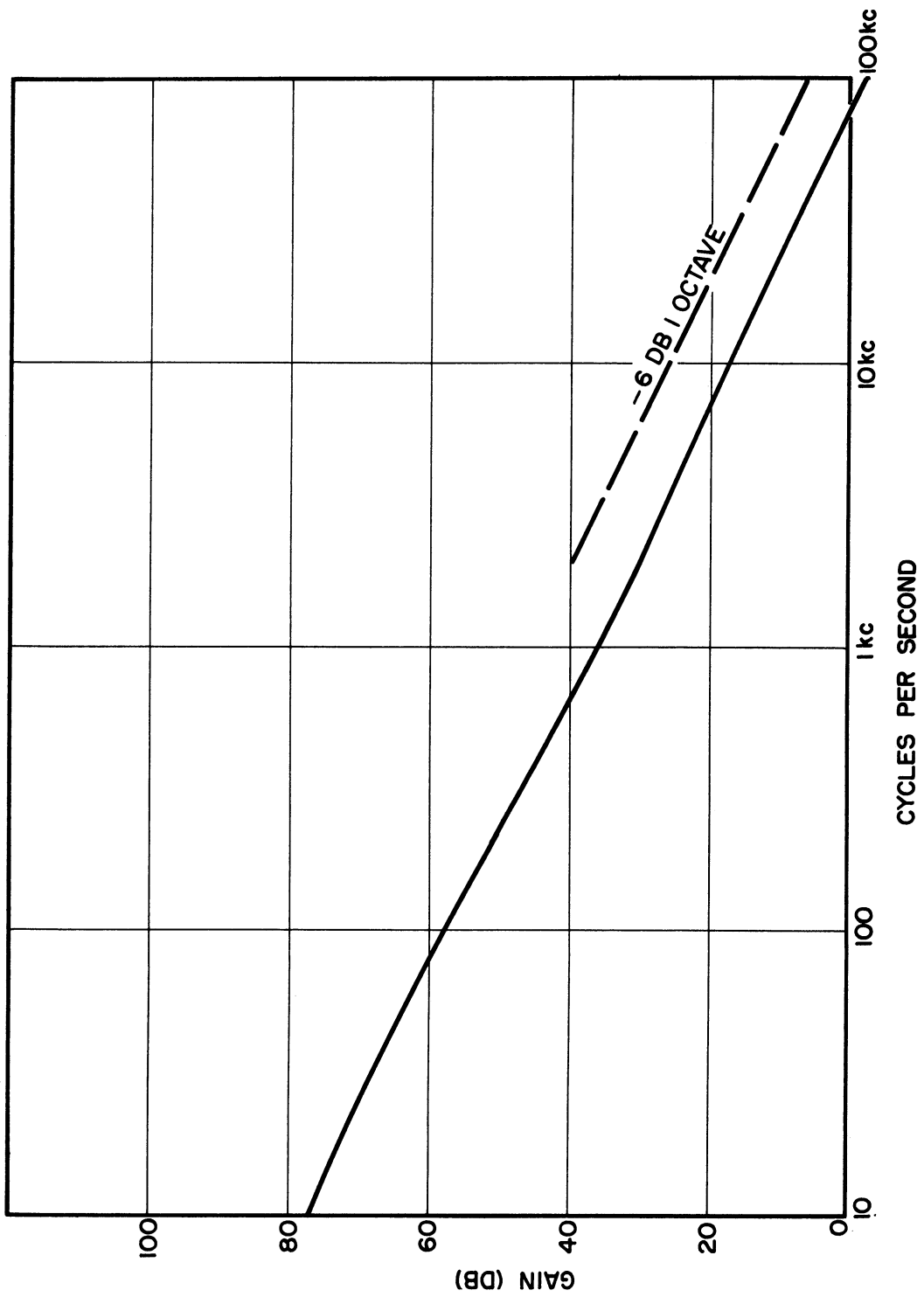


Figure 14. Frequency Response of the DC Amplifier

gible compared with 1, and the stability consideration is virtually the same as for zero feedback resistance in the previous case. Thus the amplifier attenuation characteristic shown in Figure 14 insures that the operational amplifier will be stable for integration as well as for summation and multiplication by a constant.

2.3 Drift Analysis

Next let us consider the problem of zero drift in the dc operational amplifiers. A dc amplifier must be balanced so that with zero input voltage the output voltage is zero. This balance can, in a properly designed dc amplifier, usually be achieved by slight changes in the operating conditions of the first stage of vacuum-tube amplification. In the circuit shown in Figure 9 the balance is adjusted by changing a variable resistor in the cathode circuit of the first triode stage. Once the amplifier has been balanced, subsequent changes in heater voltage, B voltages, ambient temperature, etc., may cause the amplifier to drift off of balance so that zero voltage input no longer gives zero voltage output.

In Figure 1 assume the operational amplifier has an input resistor R_i and a feedback resistor R_f . If we consider the effect of amplifier unbalance, the output voltage e_2 is given by

$$e_2 = -\mu(e' + e_B), \quad (14)$$

where $-\mu$ is the amplifier gain, e' is the dc amplifier input voltage, and e_B is the voltage unbalance of the dc amplifier referred to its input. Equating input and feedback currents we have

$$\frac{e_1 - e'}{R_i} = \frac{e' - e_2}{R_f}, \quad (15)$$

where e_1 is the input voltage to the operational amplifier. Eliminating e' in Equation (15) by means of Equation (14) we have

$$e_2 \left[1 + \frac{1}{\mu} \left(1 + \frac{R_f}{R_i} \right) \right] = -\frac{R_f}{R_i} e_1 - \left(1 + \frac{R_f}{R_i} \right) e_B \quad (16)$$

For $\mu \gg R_f/R_i$

$$e_2 = -\frac{R_f}{R_i} e_1 - \left(1 + \frac{R_f}{R_i}\right) e_B \quad (17)$$

and the unbalance at the output (i. e., the output voltage when $e_1 = 0$) is given by

$$e_2(0) = -\left(1 + \frac{R_f}{R_i}\right) e_B, \quad \mu \gg R_f/R_i \quad (18)$$

where we recall that e_B is the voltage unbalance referred to the input. Thus the larger the ratio R_f/R_i (i. e., the larger the gain of the operational amplifier), the larger the voltage drift or unbalance at the output.

The voltage unbalance e_B referred to input may be quite small compared with the voltage change which caused e_B . For example, if e_B resulted from a 1 volt shift in plate voltage of the first stage of triode amplification, and if the gain of the first stage were 30, then e_B would be 1/30 volt. i. e., a 1/30 volt change in input voltage e' would produce an equivalent unbalance

In the 6-amplifier computer described in this report the amplifier balance is tested by connecting a 100:1 feedback-input resistance ratio onto the amplifier with zero input voltage. A typical drift over one hour would be 0.5 volt, which means about 5 millivolts referred to input, according to Equation 18. If the amplifier had been connected with a 1:1 feedback-input resistance ratio (unity gain), the output-voltage drift over the same period would have been 2 x 5 or 10 millivolts, again from Equation 18.

2.4 Amplifier Maintenance

The amplifier described in Section 2.1 has been found to be fairly reliable. Most of the circuitry which is apt to give any trouble is located in the Vector can shown in Figures 11 and 12. The rest of the amplifier circuit has only three resistors and a 6SL7 associated with it.

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The failure of an amplifier usually falls in one of the following two categories: (1) the amplifier output cannot be balanced, or (2) the amplifier output is excessively noisy. In category (2) the trouble is almost certain to be in the first or second amplifier stage (i. e. , in the Vector can circuitry). In either case the first step is to replace the can with a unit which is known to be working. This will immediately determine whether it is the can or the final 6SL7 circuitry which is at fault. It proves to be the latter, a test of the 6SL7 tube and resistance checks on the three resistors in the circuit will locate the trouble.

If the failure is in the Vector can, a check of the 5691 tube is obviously the first step. If the tube operates satisfactorily in another amplifier, then the trouble is in the amplifier components. The cover should be removed from the can so that voltages can be checked while the amplifier is operating with no feedback. The normal operating voltages are shown in Figure 9. It should be possible to bracket these voltages by changing the 5 K balance potentiometer. A high-impedance vacuum-tube voltmeter should be used to measure the dc voltages, and when grid voltages are read, they should be measured with respect to the cathode (this minimizes the voltmeter loading effect on the high-impedance grid networks). Using this technique it is usually easy to locate the faulty resistor when the amplifier output will not balance.

When the amplifier output balances but is noisy, the guilty component is more difficult to find. However, it is almost certain to be in the first stage of amplification. The trouble can usually be located by measuring the cathode and plate voltages of the first triode section and the grid voltage of the second, noting where the fluctuations are first introduced. A battery in series with the plate voltage can be used to buck down that voltage so that it can be read on the sensitive scale of the voltmeter. This is often necessary to observe the fluctuations at the plate.

It should be emphasized that in all the above tests the amplifier should be "wide open", i. e. , without any feedback and with the input grounded.

3. Auxiliary Circuits

3.1 General Panel Description

A front view of the 6-amplifier electronic differential analyzer is shown in Figure 15. The underside of the front panel is shown in Figure 16, and additional views of the entire unit, including the power supply, are shown in Figures 17 and 18. The front panel has banana sockets for plugging in feedback and input resistors for the six amplifiers; as can be seen in Figure 15, there are provisions for a single feedback resistor and five input resistors on each amplifier. The amplifiers are numbered 1 to 6 going from left to right. Numbers 2, 3, 5, and 6 can be used as integrators by plugging integrating capacitors into the appropriate sockets at the top of the panel. A switch in the upper left-hand corner applies initial conditions to the integrators when in the "reset" position, and releases the initial conditions when in the "operate" position. The magnitude of each initial condition is controlled by a ten-turn helipot, while the sign (plus or minus) is controlled by a toggle switch. The voltmeter in the center of the panel can be connected to the output of any of the 6 amplifiers. Full meter scale can be selected as 5, 25, or 100 volts. Finally, a step-function voltage outlet is provided in the upper right-hand corner of the 6 amplifier panel shown in Figure 15.

Schematic wiring diagrams for the front panel are shown in Figures 19 and 20. More detailed descriptions of some of the circuits are contained in the following sections.

3.2 Initial-Condition Circuits

In Section 1.3 it was pointed out that the feedback capacitor across an integrating amplifier must be charged to a voltage representing the equivalent initial condition in the differential equation being solved. If the scheme indicated in Figure 6 were used to apply initial conditions, a separate floating power supply would be necessary for each initial-condition circuit. In order to avoid this, the technique shown in Figure 19 is used. When the initial-condition relay is in the reset position (relay energized), the external input circuits on the panel front are disconnected and 500 K input and feedback resistors are connected onto the amplifier. In this

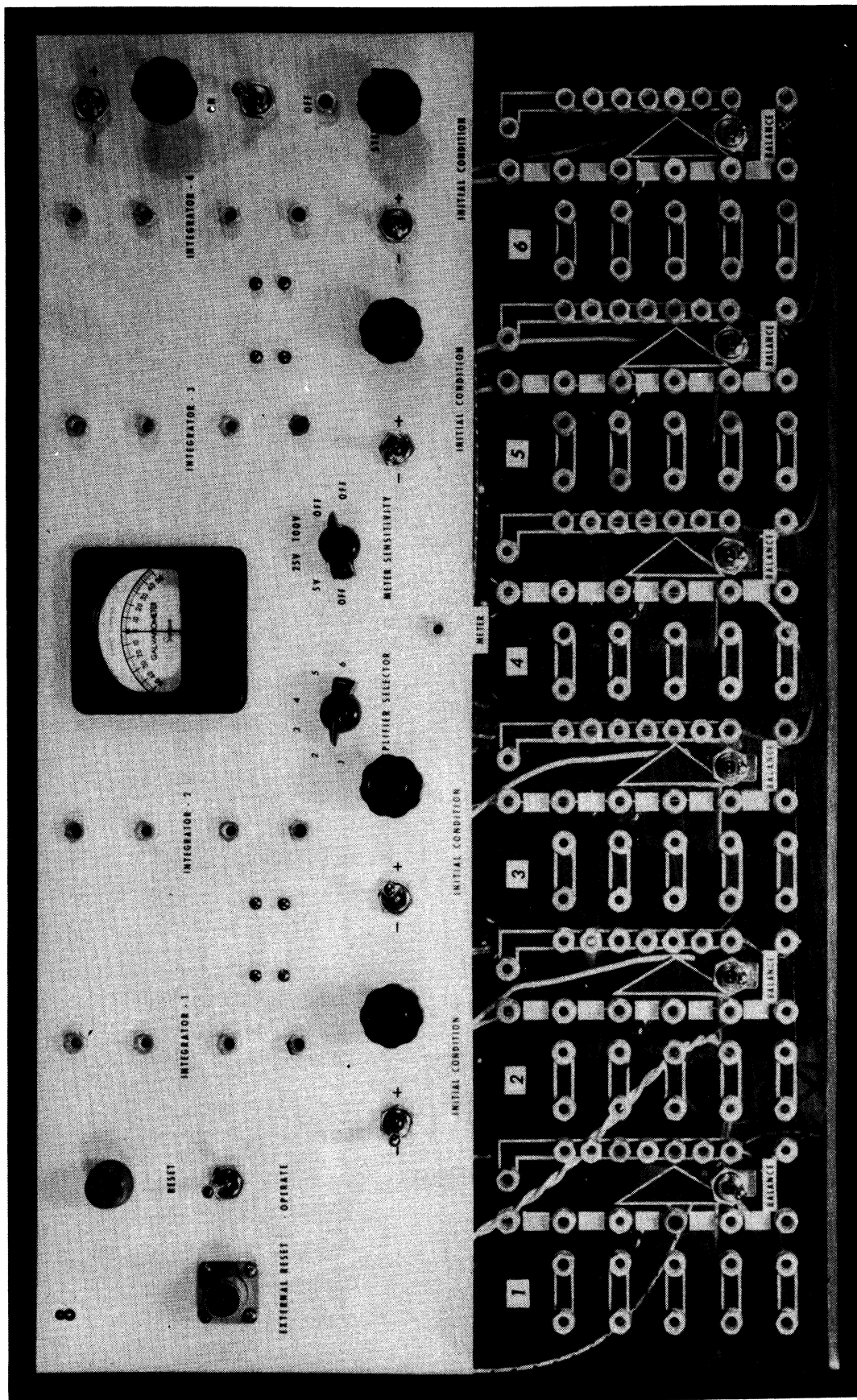


Figure 15. Front of the Six-Amplifier Computer

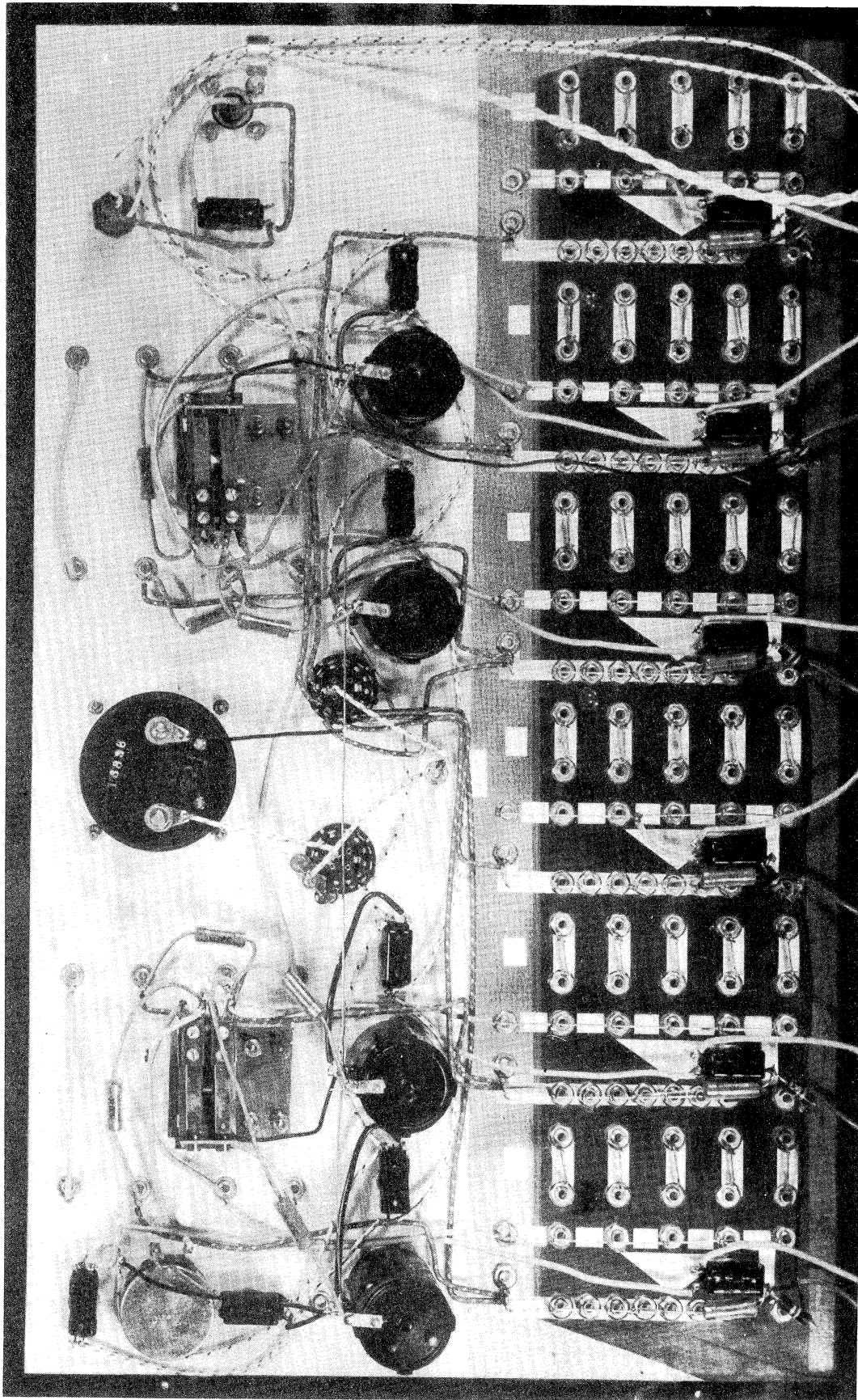


Figure 16. Underside of Front Panel

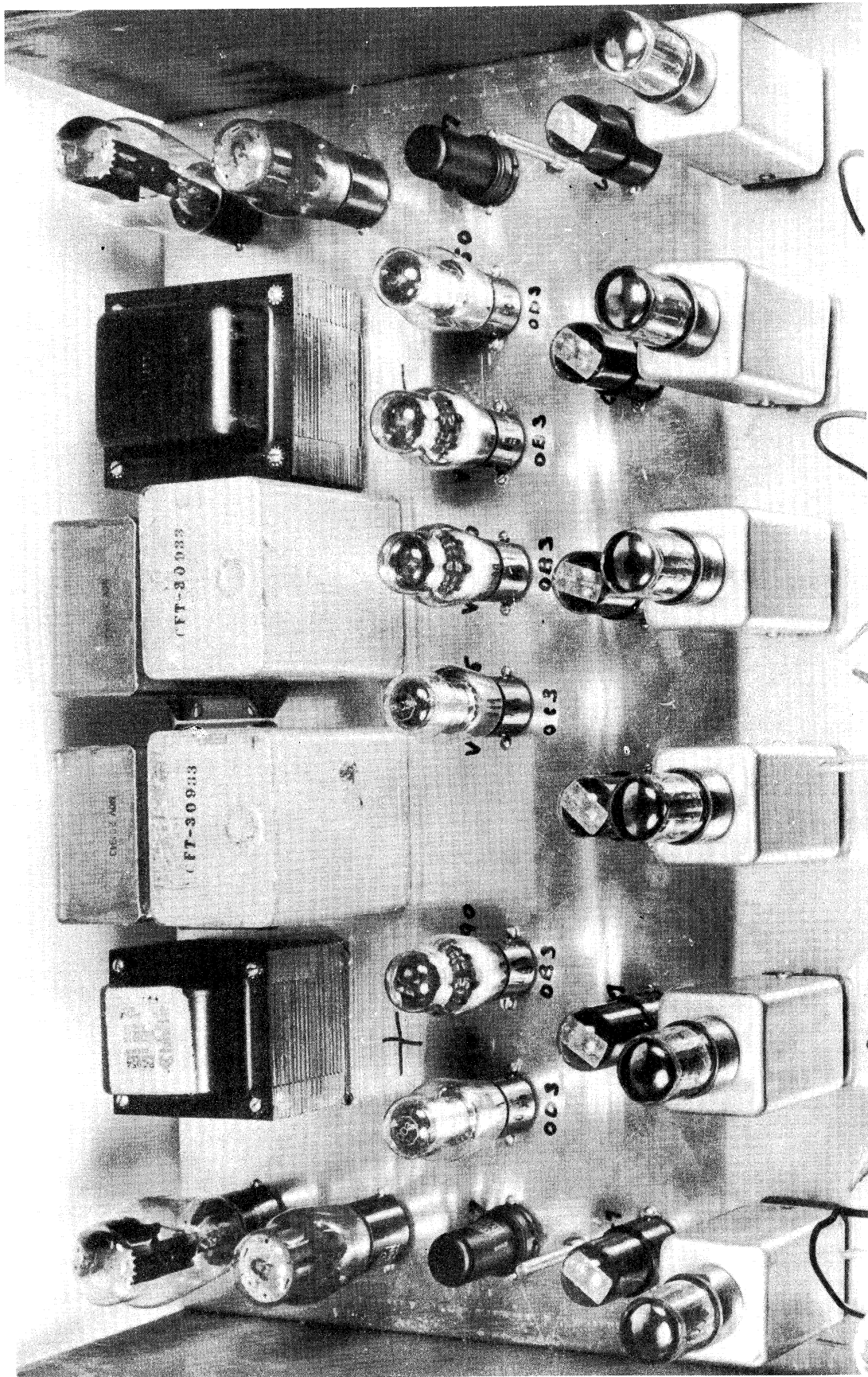


Figure 17. Top View of Chassis

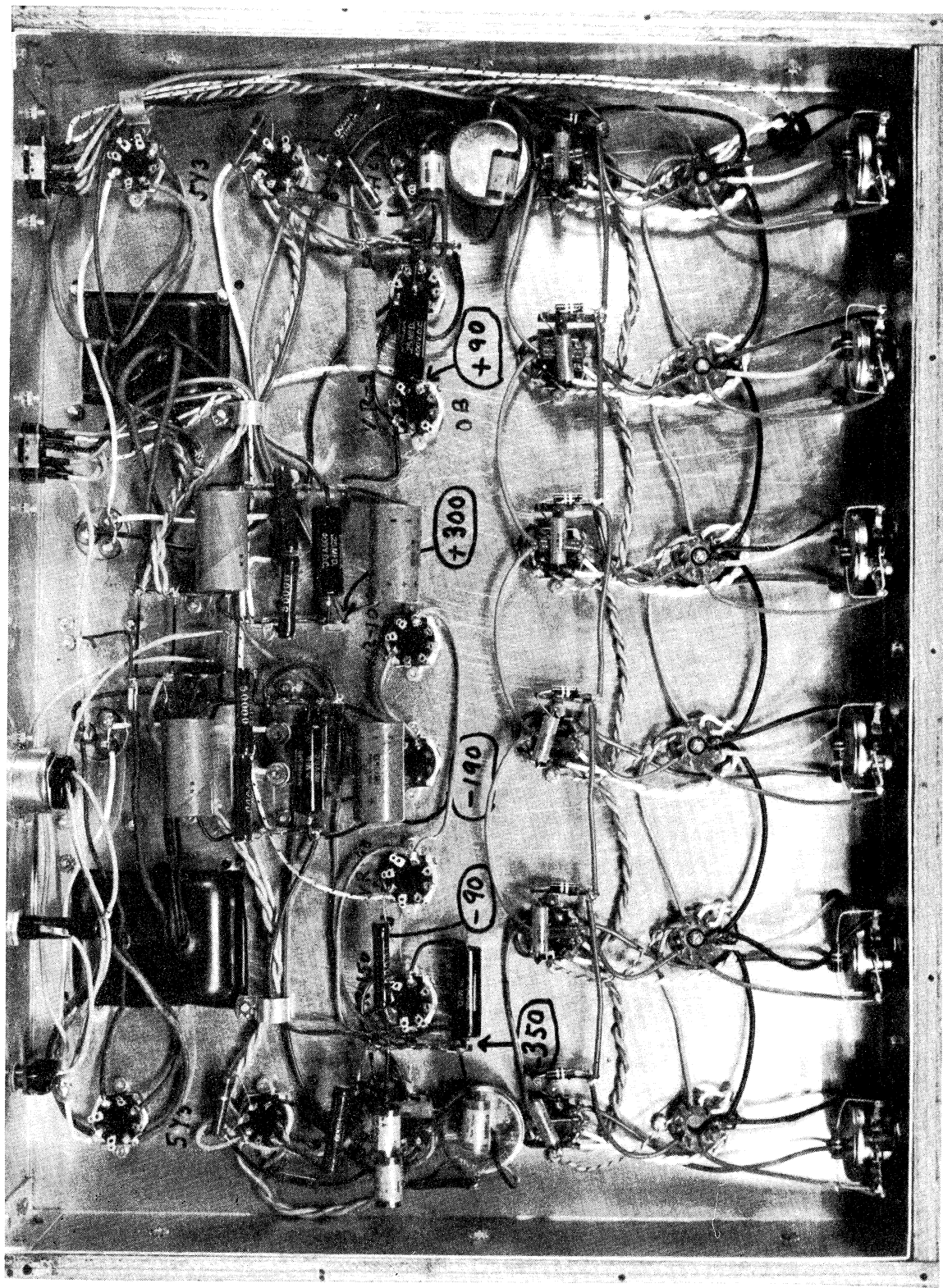


Figure 18. Bottom View of Chassis

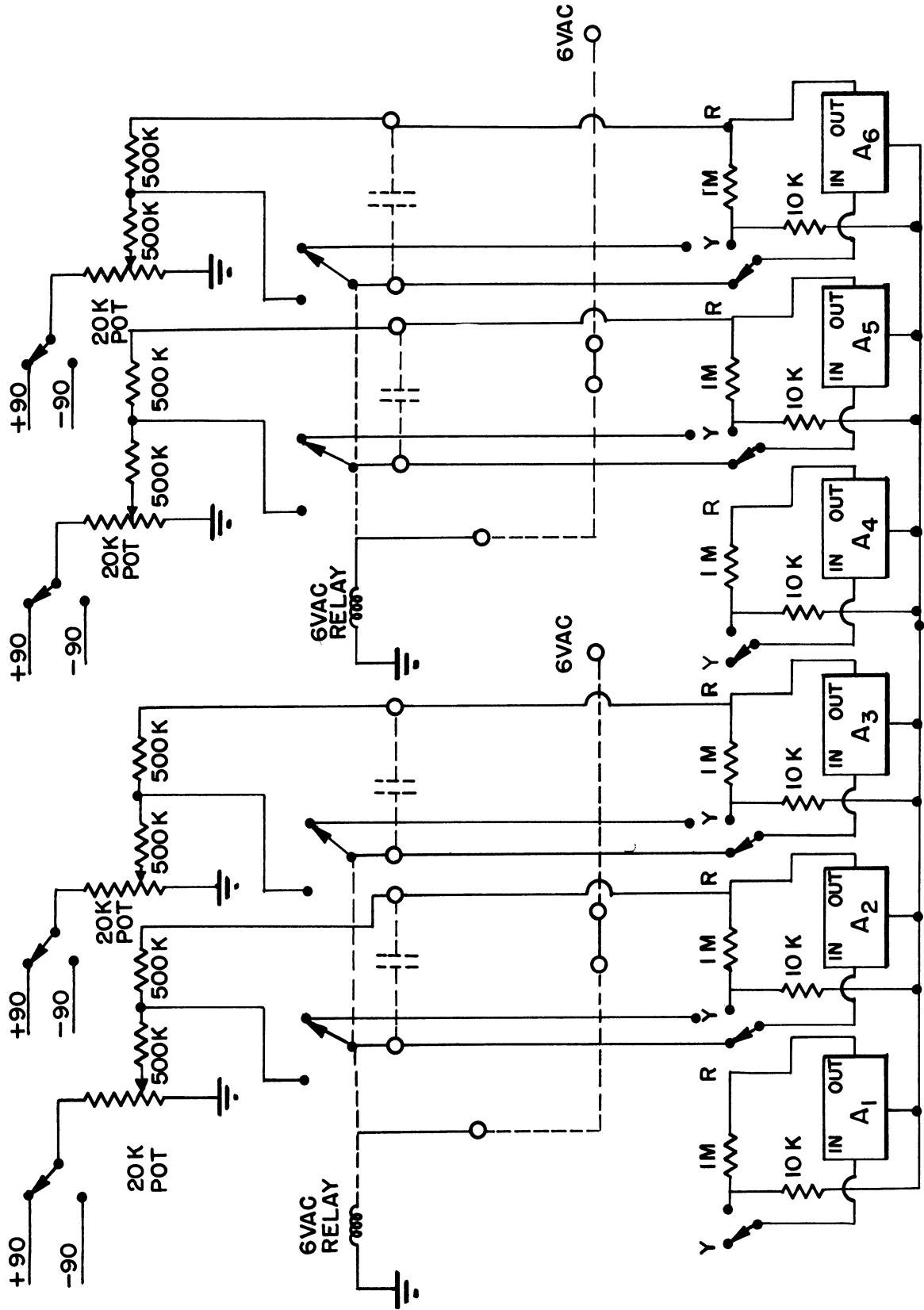
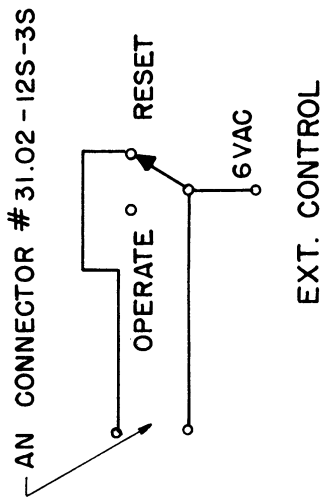
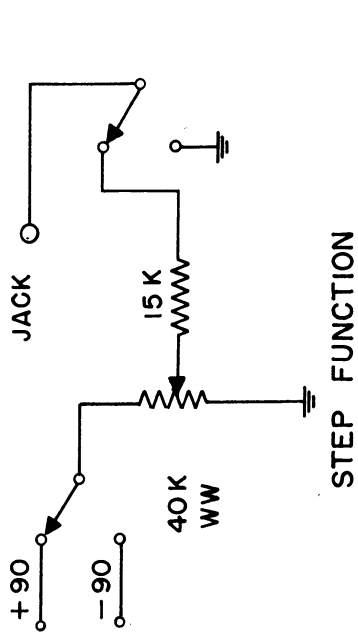
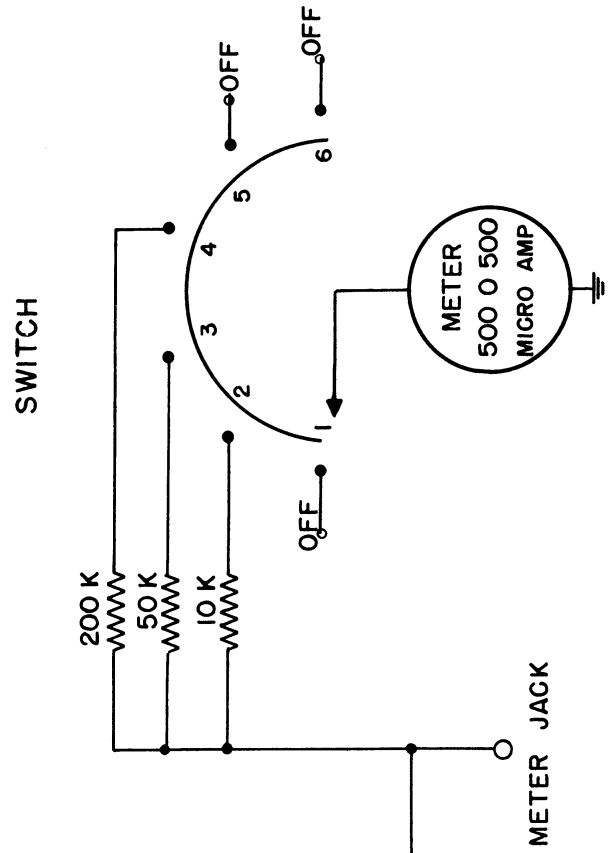


Figure 19. Circuit Diagram for Front Panel



METER SENSITIVITY SWITCH



AMPLIFIER SELECTOR SWITCH

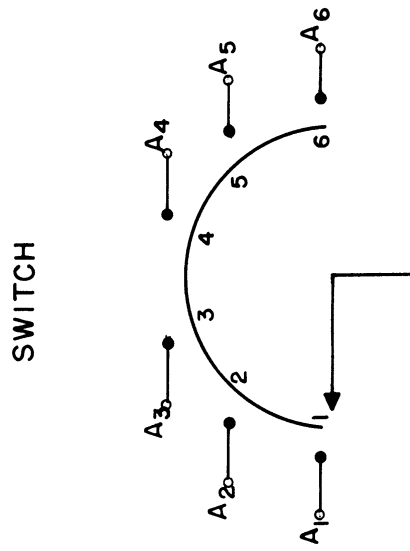


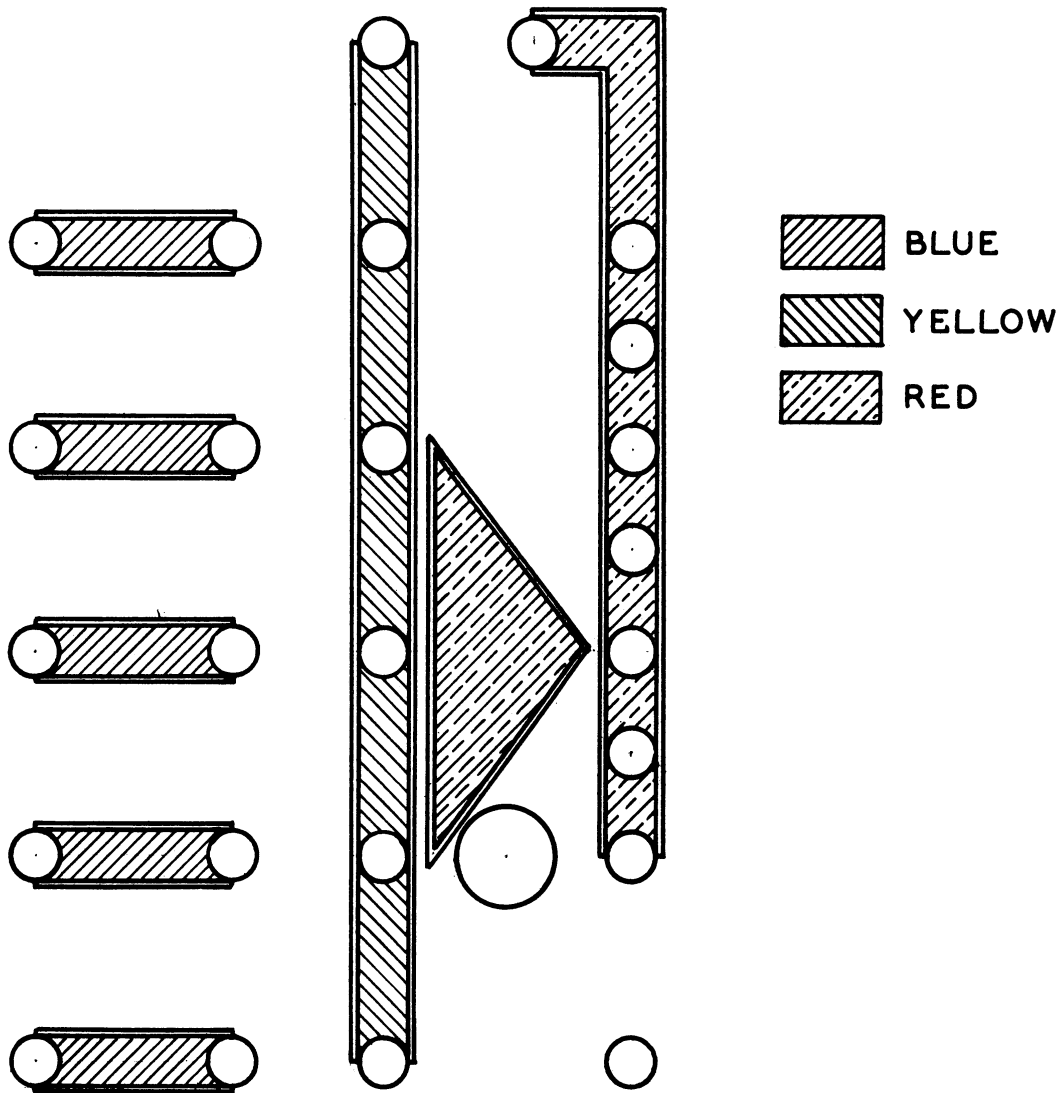
Figure 20. Auxiliary Circuits for Front Panel

condition the amplifier has a gain of unity with a time constant of 0.5 seconds, since the 1 microfarad integrating condenser is charged through a 0.5 megohm resistor. The negative of the desired initial-condition voltage is applied to the 500 K input resistor. Note that a common ground return is used for all the initial-condition input voltages. When the reset switch is opened, the initial condition relays release; this disconnects the initial-condition 500 K resistors and reconnects the external input resistors on the front panel.

Since double-pole relays are used, initial conditions for two integrators are applied with a single relay. When one of the amplifiers available for integration (2, 3, 5, or 6) is used as a summing amplifier, the initial-condition relay must be made inoperative. This is automatically achieved when the integrating condenser is unplugged, since the relay voltage is applied through the metal case of the condenser (see Figure 19). Actually, both condensers (2 and 3 or 5 and 6) must be plugged in before the relays operate. Thus the amplifiers available for integration must be used in pairs (2 and 3 together, 5 and 6 together) as either summers or integrators.

3.3 Amplifier Patch Panel

In Figure 15 the patch-connection arrangement for each operational amplifier can be seen. A detailed drawing of the banana-jack locations is shown in Figure 21. As explained earlier, there are jacks for one feedback resistor and five input resistors. The jacks are separated by $3/4$ inch in order to fit the standard, General Radio type plugs, on which the resistors themselves can be mounted. A color-coding scheme is used to identify the various connections, red for amplifier output, yellow for input to the dc amplifier proper, and blue for the connections to the input resistors. Below the triangular representation of the amplifier is a single-pole double-throw switch, which, in the down position disconnects the resistors on the panel and connects a 100:1 feedback-input resistor ratio with the input grounded (see Figure 19). This is used to adjust the amplifier balance.



FULL SCALE

Figure 21. Diagram of Amplifier Banana Jack Locations

3.4 Voltmeter Circuit

As is evident in Figure 20, the voltmeter used to monitor the amplifier outputs is a 500-0-500 microammeter with the appropriate series resistors. Actually, a 50-0-50 microammeter would be better, since the necessary series resistors would not load the amplifier outputs as seriously. An alternative circuit could employ a cathode follower to drive the meter. In any event, with the meter circuit shown in Figure 20 it is important not to leave the meter on the more sensitive scales if amplifier-saturation effects are to be avoided (it is also hard on the meter).

3.5 Power Supplies

Circuit diagrams for the B+ and B- power supplies are shown in Figure 22. The supplies are conventional electronically regulated circuits. The +300 volt regulated supply drives a VR-90 regulator tube which provides the +90 volt reference supply. Similarly, the -350 volt regulated supply drives a VR-90 regulator tube which provides the -90 volt reference supply, and VR-105 and VR-90 tubes in series which provide the -195 volt supply. The +300, +90, -90, -195, and -350 voltages, along with power ground and 6.3 volt ac filament voltage are distributed to the amplifiers and front-panel circuitry through an 8-conductor cable on the back of the main chassis. This feature allows an external, better-regulated power supply to be connected to the 6-amplifier unit if desired.

3.6 Computing Components

a) Resistors

For input and feedback elements either precision wire-wound or precision deposited-carbon resistors have been used by the authors. For extremely accurate computing the wire-wound resistors are essential, while for most demonstration and instruction-type problems one per cent accurate deposited-carbon resistors are adequate. A photograph of this type of resistor mounted on a General Radio Type 274-MB plug is shown in Figure 23.

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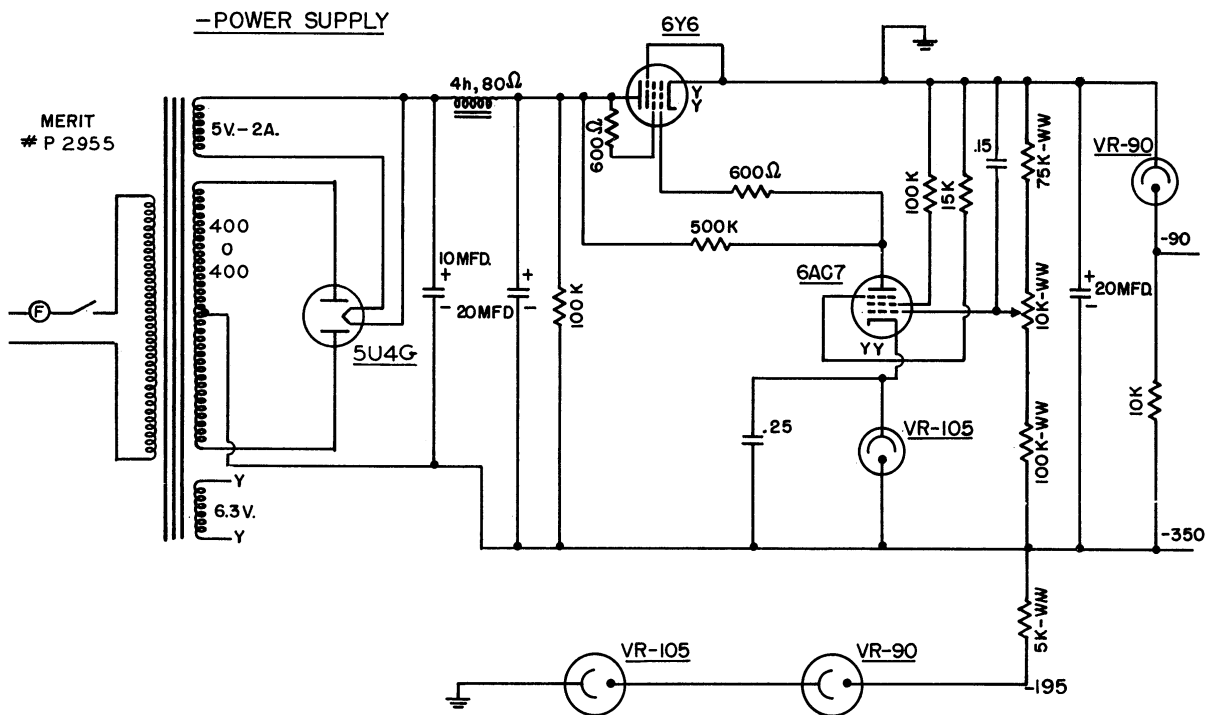
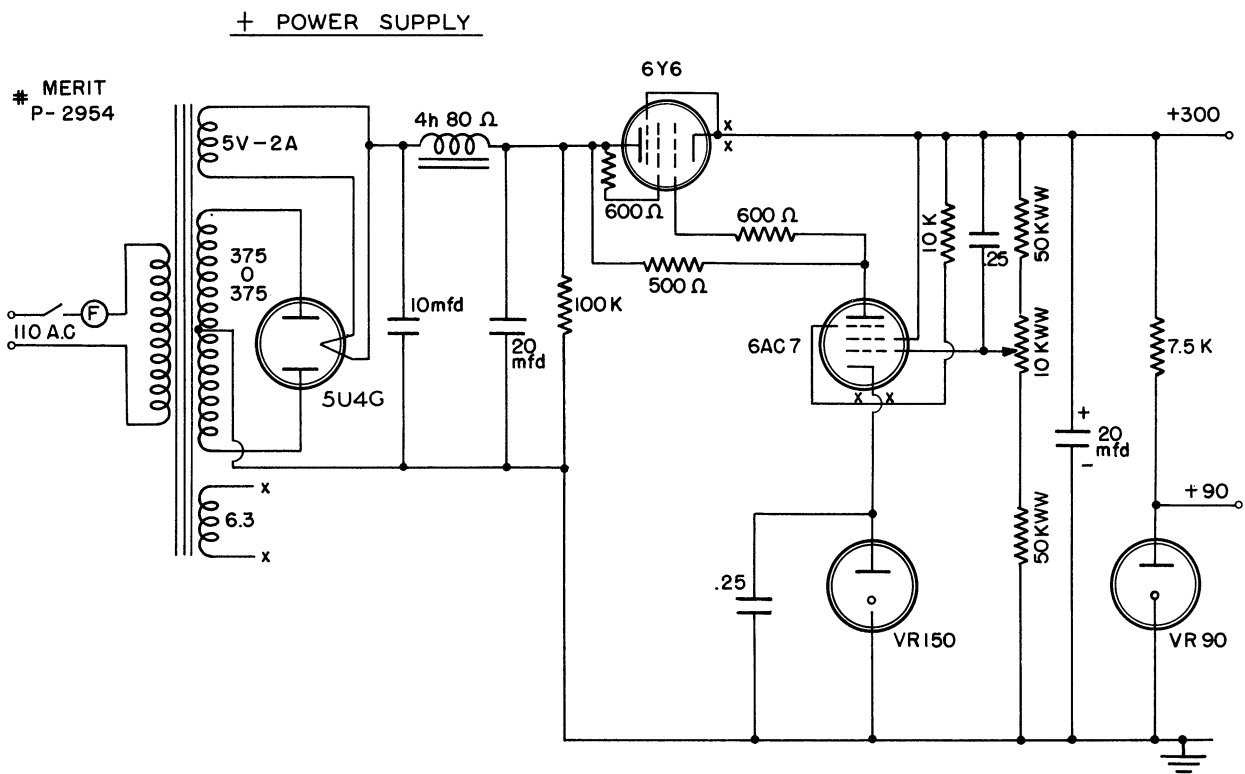


Figure 22. Power-Supply Circuit

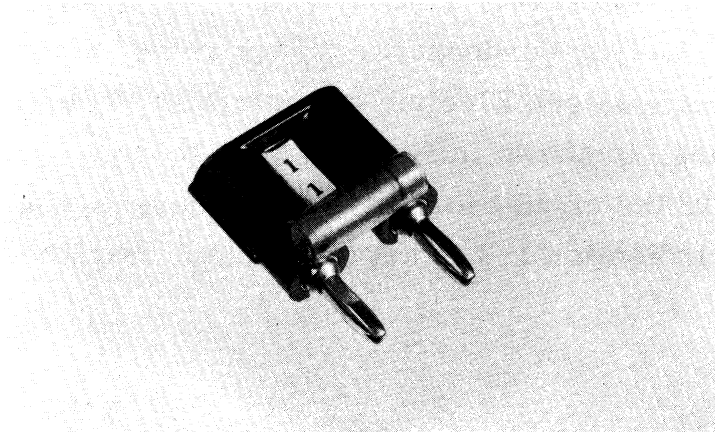


Figure 23. Plug-In Resistor Assembly

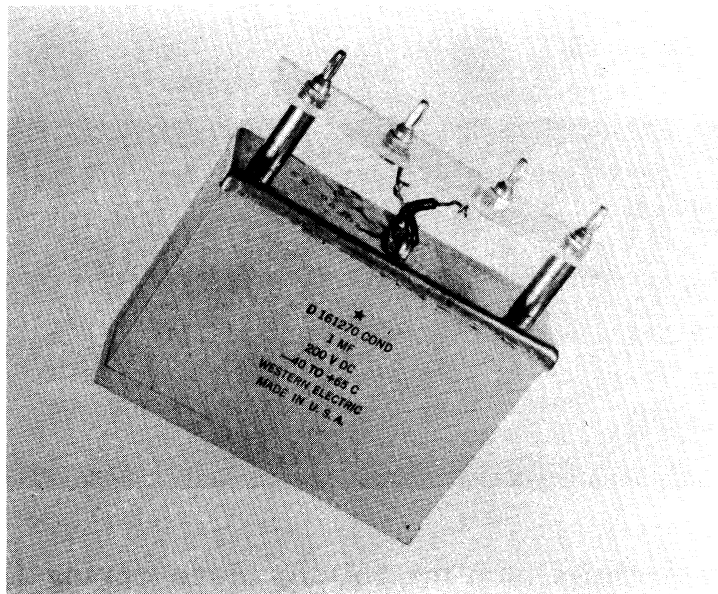


Figure 24. Plug-In Capacitor Assembly

b) Capacitors

For feedback impedances on integrating amplifiers it is important to use very high leakage capacitors having low dielectric absorption. Polystyrene capacitors formerly manufactured by Western Electric (Nos D 161270 or D 168233) and once widely available as war surplus items have proved very adequate. However, a number of commercially available polystyrene or polyethylene condensers, though rather expensive, have also been satisfactory. A Western Electric one microfarad capacitor, complete with plug-in mounting, is shown in Figure 24.

In Figure 25 the circuit for solving the mass-spring-damper problem discussed in Section 1.3 is set up on the 6-amplifier panel.

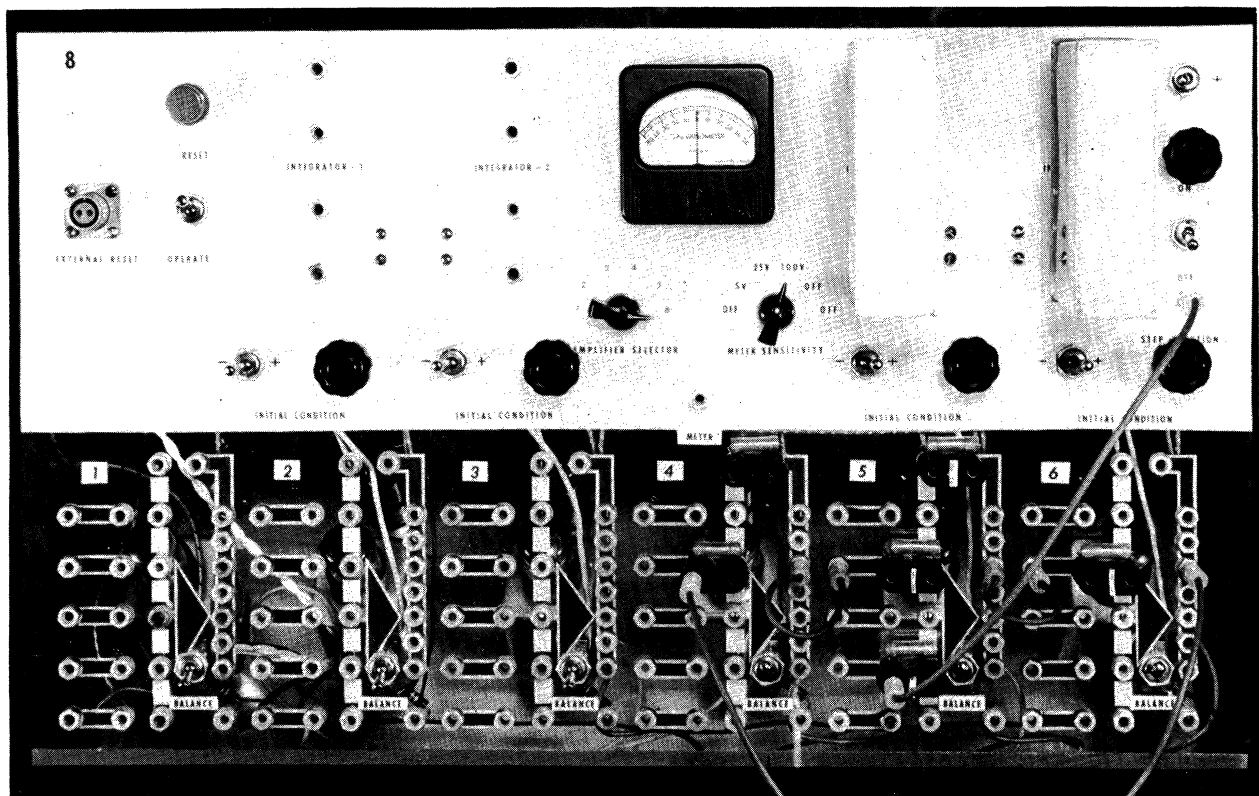


Figure 25. Computer Arrangement for Solving Mass-Spring-Damper Problem

4. Example Problems

4.1 Falling Body

As a first example of a physical problem which can be solved with the 6-amplifier analyzer, consider a body of mass m . Let y equal the height of the body above the earth and assume that the body is acted on by a constant force mg due to gravity, a force $c\dot{y}$ due to viscous drag and an inertial force $m\ddot{y}$. The equation of motion becomes

$$m\ddot{y} + c\dot{y} + mg = 0$$

or

$$\ddot{y} = -g - \frac{c}{m}\dot{y} \tag{19}$$

The computer circuit for solving Equation (19) is shown in Figure 26. Assume that initially the body starts at height y_0 and velocity \dot{y}_0

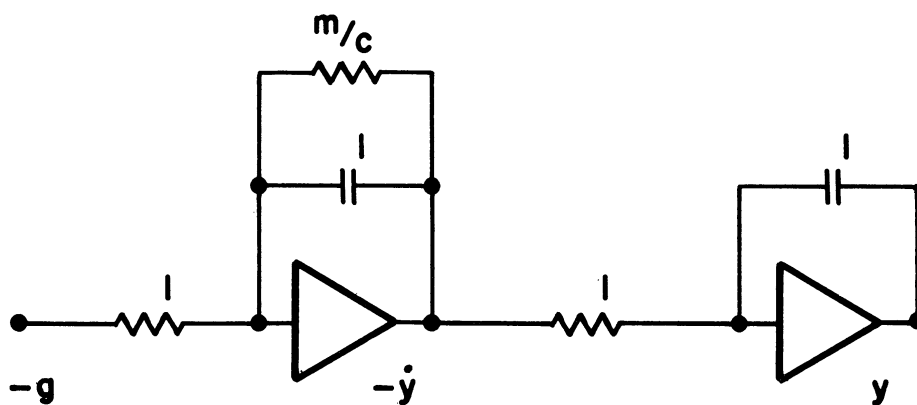


Figure 26. Falling-Body Circuit

One technique for converting from problem units to computer units simply requires the selection of a given number of volts (or recorder divisions, if more convenient) to represent a unit displacement, e. g., 0.1 volt =

1 foot (or perhaps, 0.1 recorder division = 1 foot). The problem is worked on a one to one time scale, i. e., one second of computer time equals one second of real time. Sample solutions showing vertical displacement y and velocity \dot{y} for $y_0 = 100$ feet, $\dot{y}_0 = 50$ feet/sec, and $c/m = 0$ and 2.5 sec^{-1} respectively are shown in Figures 27 and 28.

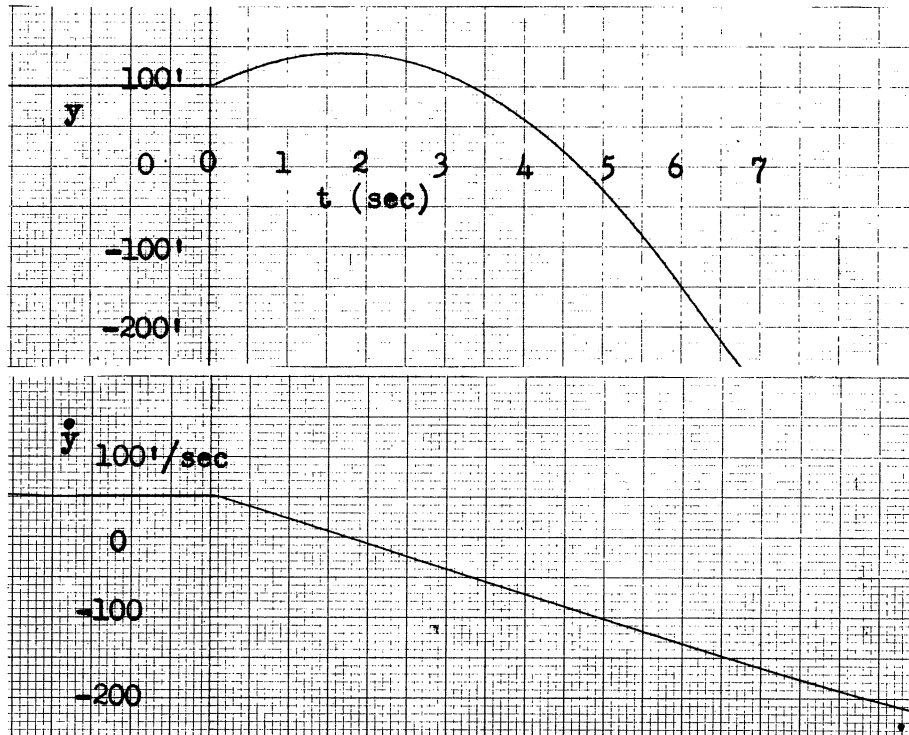


Figure 27. Solution for Falling Body, No Damping

4.2 Second-Order System; Use of Dimensionless Units

We have already discussed the analyzer circuit for solving the mass-spring-damper problem shown in Figure 4. The Equation of motion (11) can be rewritten as

$$\omega_n^2 y + \frac{2\xi}{\omega_n} \dot{y} + y = \frac{1}{k} f(t) \quad (20)$$

where

$$\omega_n = \sqrt{\frac{k}{m}} \quad \text{and} \quad \xi = \frac{c}{2\sqrt{mk}} \quad (21)$$

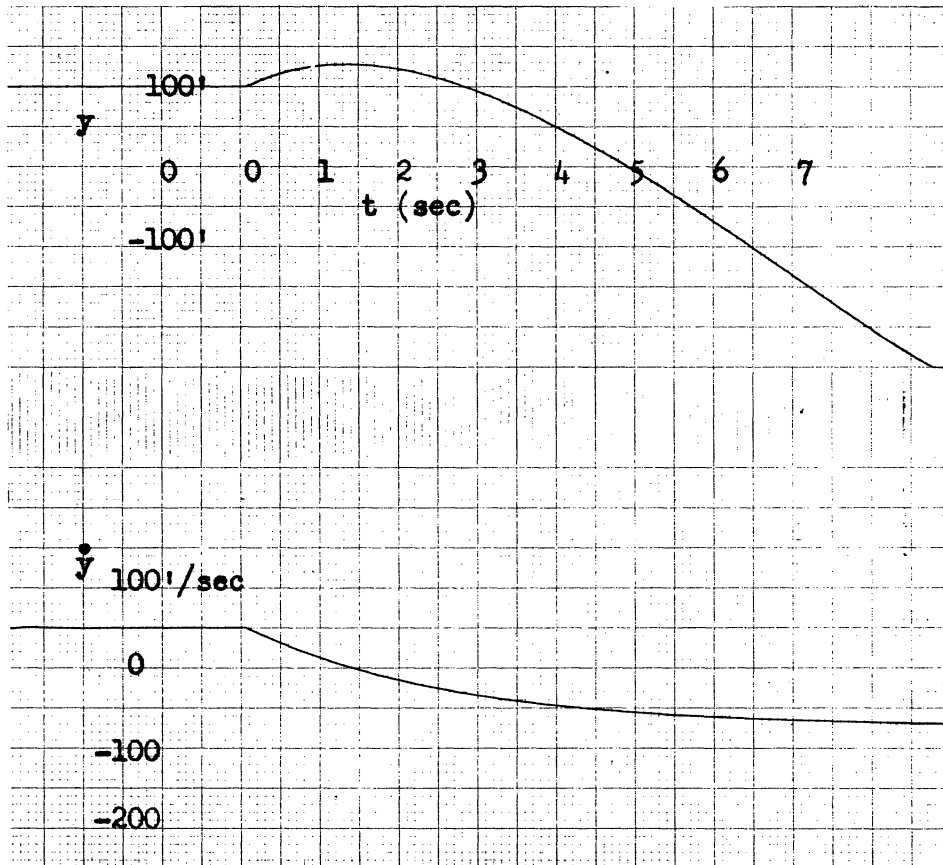


Figure 28. Solution for Body Falling Through a Viscous Medium

Here ω_n is known as the undamped natural frequency of the system, ξ is the dimensionless damping ratio, and $1/k$ is the static sensitivity. If we select a new dimensionless time variable τ given by

$$\tau = \omega_n t, \quad (22)$$

Equation (20) becomes

$$\frac{d^2 y}{d\tau^2} + 2\xi \frac{dy}{d\tau} + y = \frac{1}{k} f(\tau) \quad (23)$$

in terms of the new variable τ . The differential analyzer circuit for solving Equation (23) is shown in Figure 29.

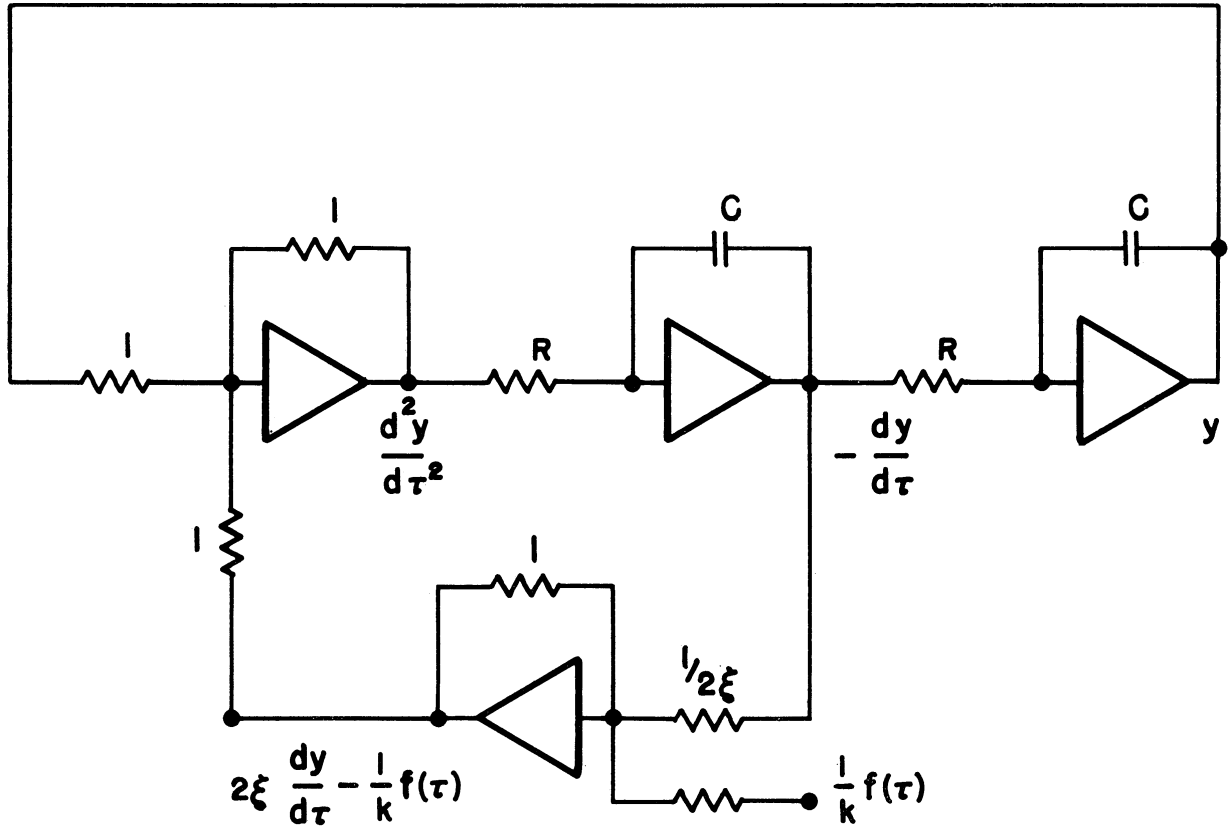


Figure 29. Circuit for Second-Order System

Note that the time constant for each of the integrators is RC seconds, so that if Equation (23) is to be solved by the computer, RC seconds of time on the computer equal one unit of the dimensionless time variable τ .

By varying the RC time constant of the integrators we can evidently select at will the relationship between computer time units and problem time units. Usually an integrator time constant RC of the order of one second is selected.

Note also in Figure 26 that $1/k f(t)$ and not $f(t)$ is fed into the circuit as the forcing function. This is convenient, since $1/k f(t)$ has units of displacement the same as y .

When no damping is present ($\xi = 0$) the analyzer makes an excellent low-frequency oscillator.⁸ The frequency of oscillation (radians per second) is equal to the reciprocal of the RC time constant of the integrators. By using the circuit of Figure 7 for the damped system (3 amplifiers required) and the circuit of Figure 8 with $\xi = 0$ and $f = 0$ for a sine wave generator (3 amplifiers required), the response of a second-order system to sinusoidal inputs can be studied with the 6-amplifier differential analyzer.

4.3 Second-Order System with Derivative Inputs

Often the equation of motion of a physical system may involve time rates of change of both input and response functions. This is particularly true in problems encountered in automatic control. An example of this is the following equation:

$$a_2 \ddot{y} + a_1 \dot{y} + a_0 y = b_2 \ddot{x} + b_1 \dot{x} + b_0 x \quad (24)$$

where x is the input and y is the response function (dependent variable). The easiest way to visualize the differential analyzer circuit needed to solve this equation is to integrate the equation twice, getting

$$a_2 y + a_1 \int y dt + a_0 \iint y dt = b_2 y + b_1 \int y dt + b_0 \iint y dt \quad (25)$$

The analyzer circuit is shown in Figure 30.

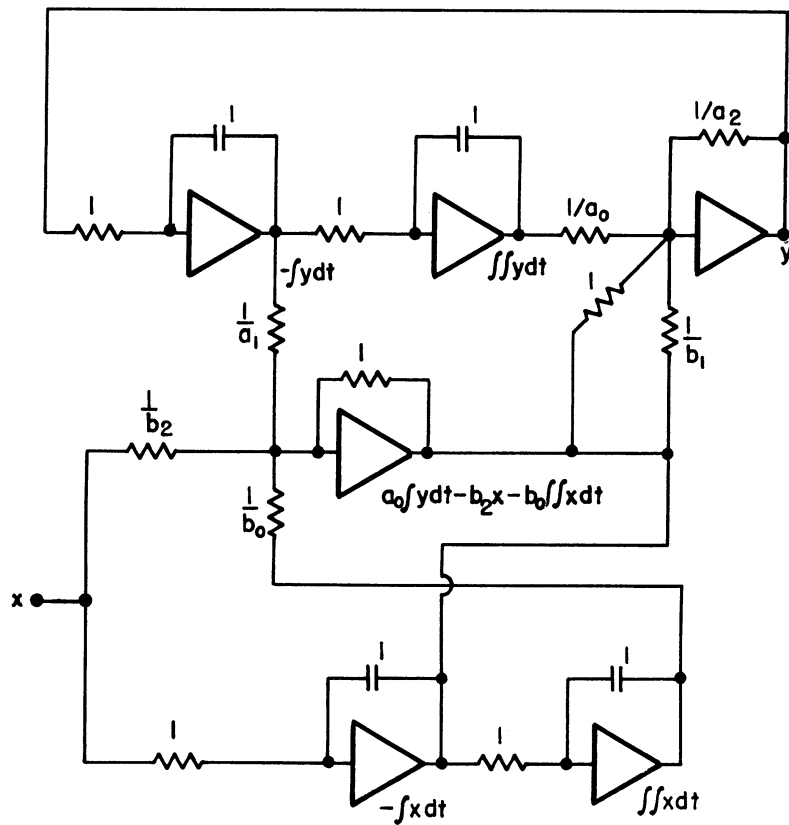


Figure 30. Analyzer Circuit for Solving the Equation

$$a_2 \ddot{y} + a_1 \dot{y} + a_0 y = b_2 \ddot{x} + b_1 \dot{x} + b_0 x$$

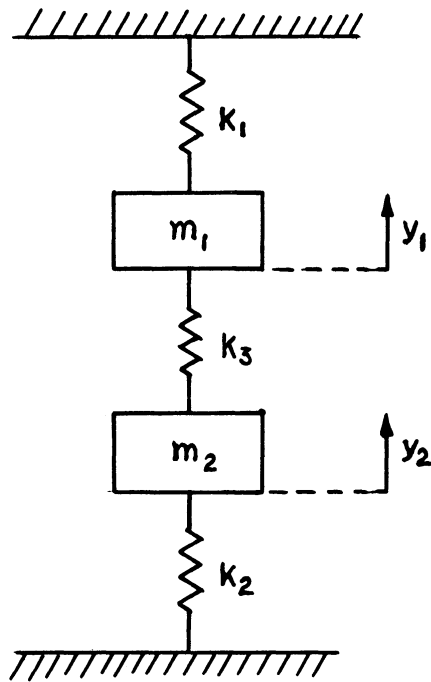
4.4 Two-Degree of Freedom System

The differential analyzer is not limited to the solution of problems where only one dependent variable is present. In Figure 31 the vertical displacements y_1 and y_2 of masses m_1 and m_2 respectively are considered. By summing forces on each of the masses the following two equations are obtained.

$$m_1 \ddot{y}_1 + (k_1 + k_3)y_1 - k_3 y_2 = 0 \quad (26)$$

$$m_2 \ddot{y}_2 + (k_2 + k_3)y_2 - k_3 y_1 = 0 \quad (27)$$

The computer circuit which solves these equations is also shown in Figure 31. The circuit is synthesized in exactly the same way as for the one-degree-of-freedom system discussed in Section 1.3, except that here two equations, including the cross-coupling term, are satisfied instead of one.



$$m_1 \ddot{y}_1 + (k_1 + k_3)y_1 - k_3 y_2 = 0$$

$$m_2 \ddot{y}_2 + (k_2 + k_3)y_2 - k_3 y_1 = 0$$

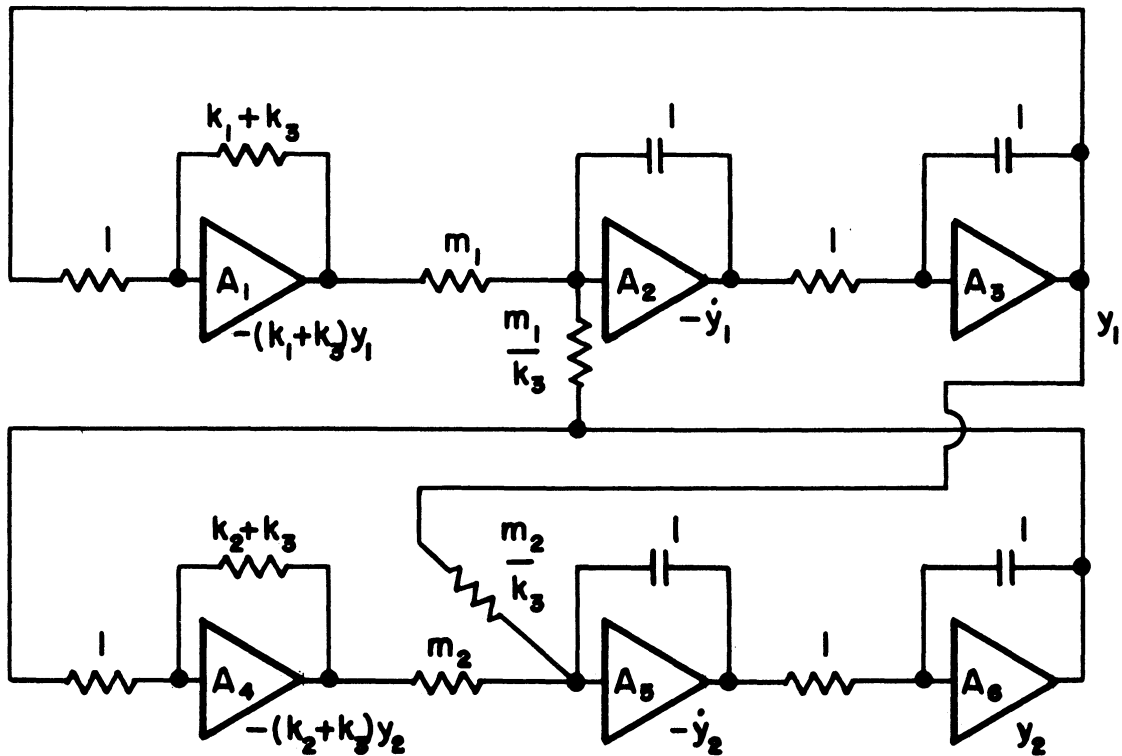


Figure 31. Two-Degree-of-Freedom System

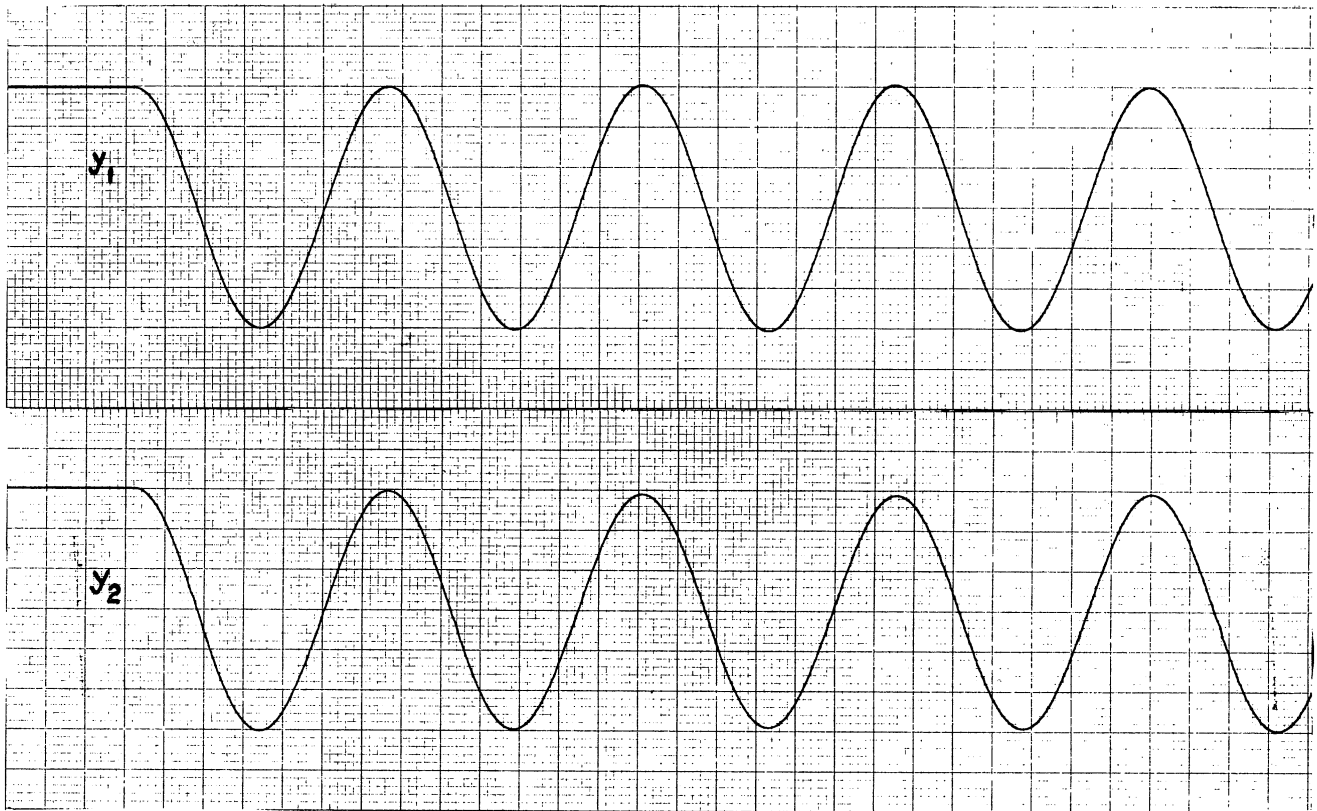


Figure 32. Two-Degree-of-Freedom System with Equal Initial Displacements

As a sample computer problem, let $m_1 = m_2 = 1$, $k_1 = k_2 = 1$, and $k_3 = 0.2$. In Figure 32 recordings of the displacements y_1 and y_2 with zero initial velocity conditions and equal initial displacements are shown. Note that the masses oscillate with pure sinusoidal motion at a frequency of 1 radian per second, as expected. This is one of the normal modes of the two-degree-of-freedom system. The other normal mode of oscillation is shown in Figure 33, where the masses have been started with equal but opposite initial displacement. Here the frequency is $\sqrt{1.4}$ or 1.18 radians per second.

Finally, in Figure 34 one mass has been started with a finite, the other with zero, displacement. Energy is transferred back and forth between the masses as shown in the recording. The motion is actually a superposition of the two normal modes, and the beat frequency of 0.18 radians per second is evident in Figure 34.

Without using any additional amplifiers it is apparent that viscous damping effects or externally applied forces can be considered in solving the two-degree-of-freedom problem with the 6-amplifier analyzer.

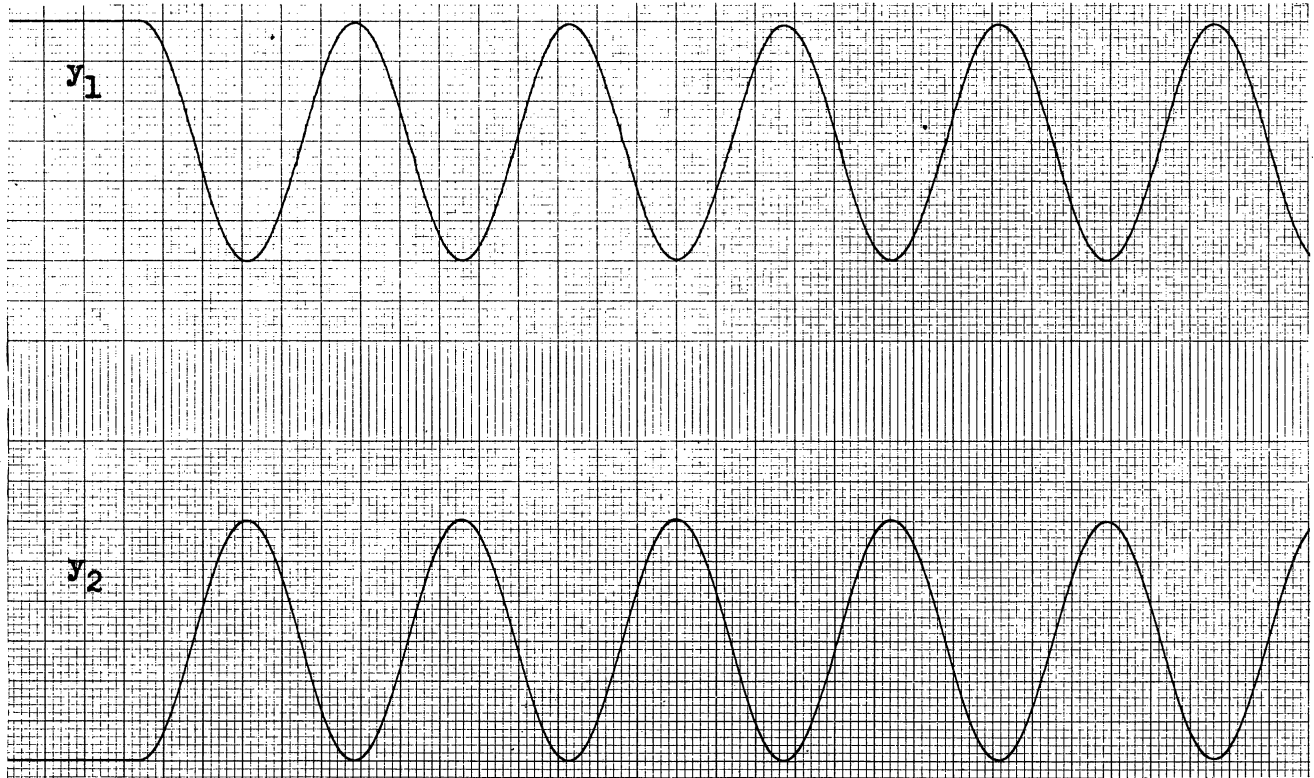


Figure 33

Two-Degree-of-Freedom System with Opposite Initial Displacements

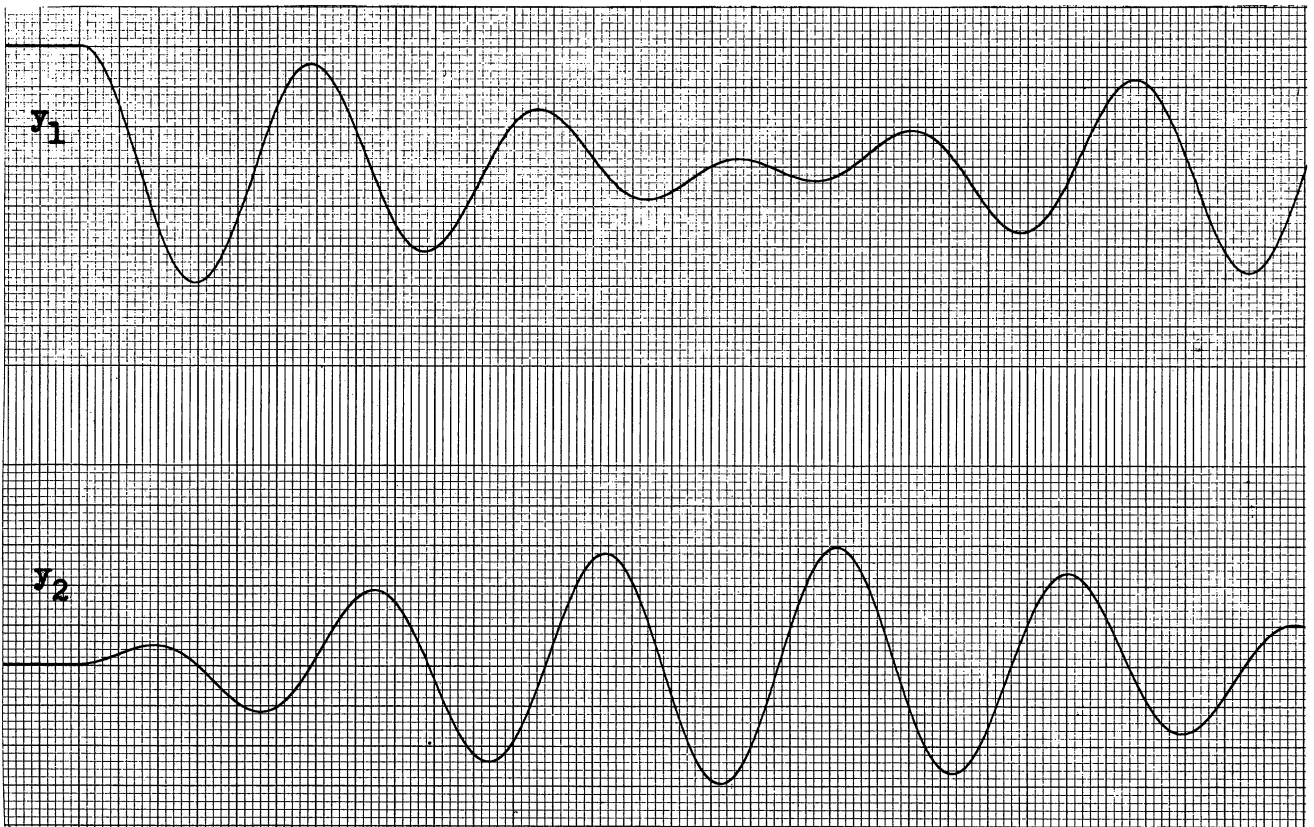


Figure 34. Two-Degree-of-Freedom System Showing Exchange of Energy

4.5 System with Coulomb Damping

One type of damping force which is often encountered in physical problems is coulomb damping or dry friction. Here the damping force, instead of being proportional to the velocity as in the viscous case, is constant in magnitude but changes sign whenever the velocity reverses sign. Thus for a second order system with coulomb damping the following equation is appropriate.

$$m\ddot{y} + f_c(\dot{y}) + ky = f(t) \quad (28)$$

where

$$f_c(\dot{y}) = + C, \dot{y} > 0 \quad (29)$$

$$= - C, \dot{y} < 0$$

The analyzer circuit is shown in Figure 35. Note that the f_c function representing the coulomb damping is simulated by feeding the \dot{y} voltage into amplifier A_4 , which has no feedback connection. The output of A_4 will therefore be saturated positive or negative depending on the sign of \dot{y} . From Figure 13 it is evident that a total load resistance of around 250 K will cause the amplifier output to saturate equally for both positive and negative inputs at about 80 volts. The output of A_4 is then summed into A_1 to represent the coulomb damping. Since the resistor labeled $80/C$ is normally much larger than 250 K, the proper loading on A_4 is obtained approximately by connecting a 250 K resistor from output to ground. The effect of the $80/C$ resistance in parallel is then usually negligible.

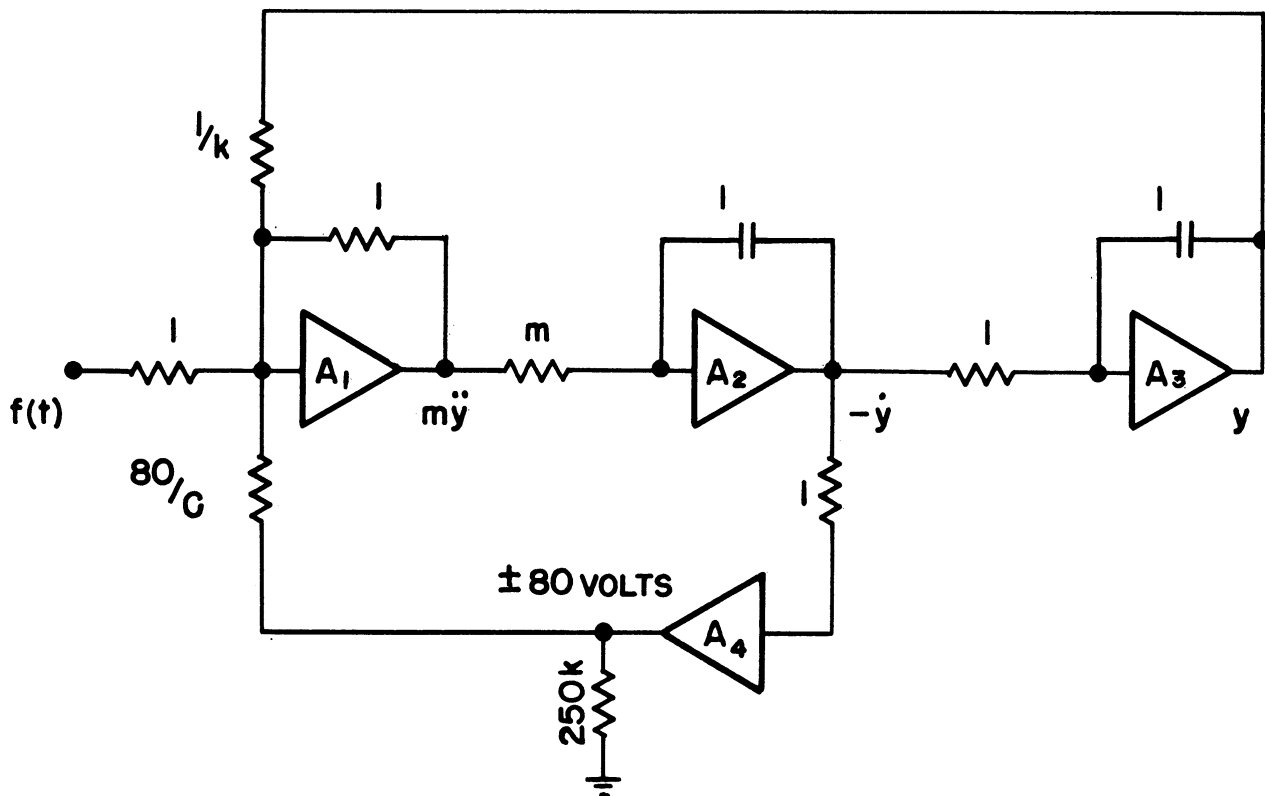


Figure 35. Second-Order System with Coulomb Damping

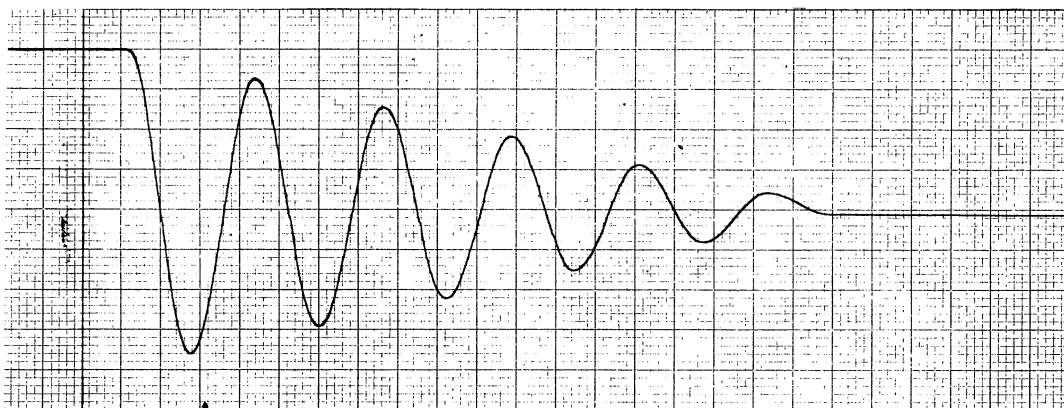


Figure 36. Response of Second-Order System with Coulomb Damping

Transient response of the second-order system with coulomb damping when displaced from equilibrium with zero force input is shown in

Figure 36. Note that the envelope of the decaying oscillations is linear rather than exponential, as it would be for a linear system with viscous damping.

4. 6 Additional Problems

Many additional linear and simple nonlinear problems can be solved with the 6-amplifier electronic differential analyzer. The unit has been used for problems in radioactive decay, consecutive chemical reactions, beam oscillation, heat flow, etc. For these and other examples the reader is referred to the bibliography.

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