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ELECTRONIC DIFFERENTIAL ANALYZER SOLUTION OF BEAMS WITH NONLINEAR DAMPING

By

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PREFACE

In a previous report, the solution of linear beam-vibration problems by difference techniques using the electronic differential analyzer was described.[1] This report extends the application of the same techniques to lateral-beam vibrations where nonlinear damping terms are present. Examples considered include cantilever beams with velocity-squared damping and Coulomb damping. Analyzer solutions give recorded output voltages representing lateral displacement, velocity, and bending moment as a function of time and at various stations along the beam. Approximate theoretical checks of the computer accuracy are given in several cases.

The computer solutions were obtained with the electronic differential analyzer facility of the Department of Aeronautical Engineering, University of Michigan.

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INTRODUCTION

Nonlinear partial differential equations are almost impossible to solve exactly in all but a few special cases. Unlike linear partial differential equations they cannot be treated by separation of variables, since for the nonlinear equations superposition of normal mode solutions does not result in another solution. Hence at the onset we are led to computing techniques in order to solve nonlinear partial differential equations. Solutions can be obtained by replacing all derivitives with finite differences and by using digital machines, or they can be accomplished by replacing derivitives with respect to all variables but one by finite differences and by using electronic differential analyzers. In this latter method the original nonlinear partial differential equation is converted to a system of simultaneous ordinary nonlinear differential equations.

1.1 Equation for Lateral Vibration of Beams

In this report the nonlinear partial differential equation which we shall consider is the description of lateral (transverse) vibration of beams having nonlinear damping terms. Consider the cantilever beam shown in Figure 1-1. Let $\bar{y}$ denote the transverse motion of the neutral axis of the beam, $\bar{x}$ equal distance along the beam, and $\bar{t}$ be the time variable. The equation describing the dynamic behavior of the beam is[2]

$$\frac{\partial^2}{\partial \bar{x}^2} EI(\bar{x}) \frac{\partial^2 \bar{y}}{\partial \bar{x}^2} + \bar{c}(\bar{x})\, f_c \left(\frac{\partial \bar{y}}{\partial \bar{t}}\right) + \rho(\bar{x}) \frac{\partial^2 \bar{y}}{\partial \bar{t}^2} = \bar{f}(\bar{x},\bar{t}) \tag{1-1}$$

Here $EI(\bar{x})$ is the flexural rigidity, $\rho(\bar{x})$ is the mass per unit length, and $\bar{f}(\bar{x},\bar{t})$ is the external force per unit length. The damping force per unit length is $\bar{c} f_c\ (\partial \bar{y}/\partial \bar{t})$, where f_d is a nonlinear function of the transverse velocity $(\partial \bar{y}/\partial \bar{t})$. We recall that the bending moment M is given by

$$M(\bar{x},\bar{t}) = EI(\bar{x}) \frac{\partial^2 \bar{y}}{\partial \bar{x}^2} \tag{1-2}$$

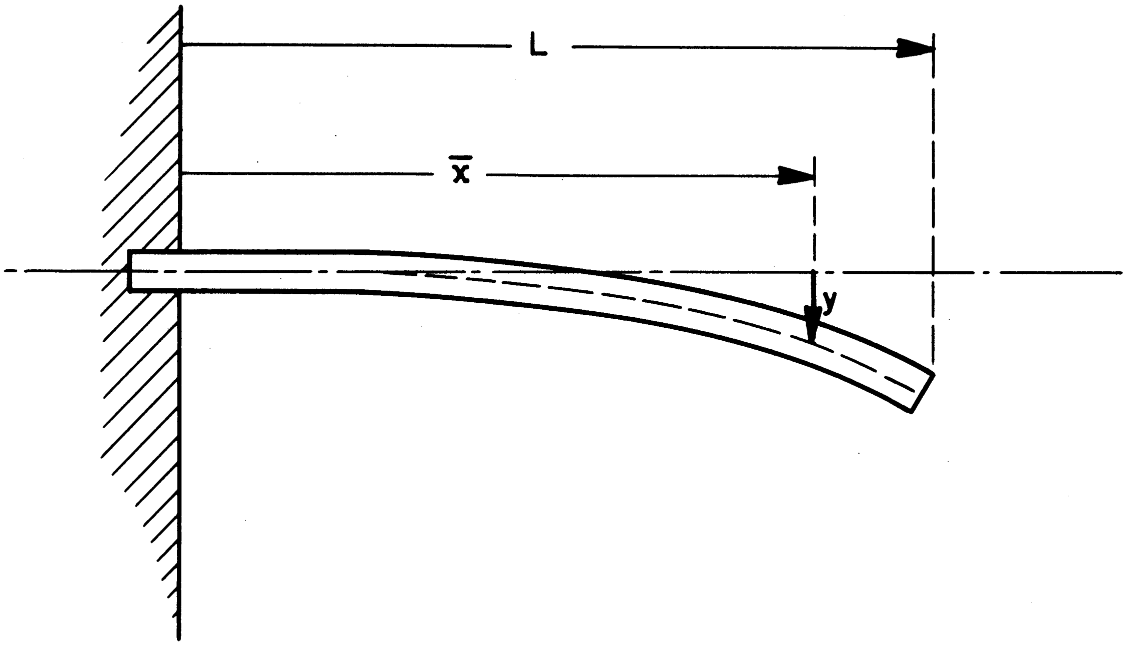


Figure 1-1. Cantilever Beam.

while the shear force V is

$$V(\bar{x},\bar{t}) = \frac{\partial M(\bar{x},\bar{t})}{\partial \bar{x}} \qquad (1\text{-}3)$$

For the cantilever beam of length L shown in Figure 1-1 the boundary conditions are

$$\bar{y}(o,\bar{t}) = \frac{\partial \bar{y}(o,\bar{t})}{\partial \bar{x}} = 0 \quad \text{(clamped end)} \qquad (1\text{-}4)$$

and

$$M(L,t) = V(L,t) = 0 \quad \text{(free end)} \qquad (1\text{-}5)$$

In writing Equation (1-1) we have neglected deflection due to transverse shear or rotary inertia, which means that the transverse dimensions of the beam must be small compared with the beam length L. The effect of transverse shear can be included if necessary[1,3].

Let us lump the variable characteristics of the flexural rigidity $EI(\bar{x})$ into a dimensionless variable $\phi_f(\bar{x})$, so that

$$EI(\bar{x}) = EI_o \, \phi_f(\bar{x}) \tag{1-6}$$

Here EI_o could represent the maximum value of $EI(\bar{x})$. In the same way we let

$$\rho(\bar{x}) = \rho_o \, \phi_d(\bar{x}) \tag{1-7}$$

and

$$\bar{c}(\bar{x}) = \bar{c}_o \, \phi_c(\bar{x}) \tag{1-8}$$

It is also convenient to define a dimensionless distance variable x such that the beam length in x is unity. Thus

$$x = \frac{\bar{x}}{L} \tag{1-9}$$

from which

$$\frac{\partial}{\partial \bar{x}} = \frac{\partial}{\partial x}\frac{dx}{d\bar{x}} = \frac{1}{L}\frac{\partial}{\partial x} \; ; \; \frac{\partial^2}{\partial \bar{x}^2} = \frac{1}{L^2}\frac{\partial^2}{\partial x^2} \; ; \text{ etc.} \tag{1-10}$$

From Equations (1-6), (1-7), (1-8), and (1-10) the beam Equation (1-1) becomes

$$\frac{\partial^2}{\partial x^2} \phi_f(x) \frac{\partial^2 \bar{y}}{\partial x^2} + \frac{L^4 c_o}{EI_o} \phi_c(x) \, f_c\left(\frac{\partial \bar{y}}{\partial \bar{t}}\right) + \frac{\rho_o L^4}{EI_o} \phi_d(x) \frac{\partial^2 \bar{y}}{\partial \bar{t}^2} = \frac{L^4}{EI_o} \bar{f}(x,\bar{t}) \tag{1-11}$$

Next we introduce a dimensionless time variable t given by

$$t = \frac{1}{L^2}\sqrt{\frac{EI_o}{\rho_o}} \tag{1-12}$$

Since Equation (1-11) is nonlinear, the behavior of the solution will in general depend on the magnitude of the displacement $\bar{y}$. For this reason we consider a dimensionless displacement y defined as

$$y(x,t) = \frac{\bar{y}(x,t)}{y_o}, \tag{1-13}$$

where y_o is a reference value of $\bar{y}$; y_o might be defined as the maximum expected value of $\bar{y}$.

In terms of Equations (1-12) and (1-13) Equation (1-11) becomes

$$\frac{\partial^2}{\partial x^2} \phi_f(x) \frac{\partial^2 y}{\partial x^2} + \frac{L^4 c_o}{EI_o\, y_o} \phi_c(x)\, f_c\left(\frac{\partial y}{\partial t}\right) + \phi_d(x) \frac{\partial^2 y}{\partial t^2} = f(x,t) \qquad (1\text{-}14)$$

where

$$f(x,t) = \frac{L^4}{EI_o\, y_o} \bar{f}(x,t) \qquad (1\text{-}15)$$

Equation (1-14) is the equation describing beam vibration with nonlinear damping which we will solve in this report. For a cantilever beam it is subject to the boundary conditions

$$y(o,t) = \frac{\partial y(o,t)}{\partial x} = 0 \qquad (1\text{-}16)$$

and

$$\phi_f(1) \frac{\partial^2 y(1,t)}{\partial x^2} = \frac{\partial}{\partial x} \phi_f(1) \frac{\partial^2 y(1,t)}{\partial x^2} = 0 \qquad (1\text{-}17)$$

We will denote the initial conditions as

$$y(x,o) = Y(x) \qquad (1\text{-}18)$$

and

$$\frac{\partial y(x,o)}{\partial t} = \dot{Y}(x) \qquad (1\text{-}19)$$

Two examples will be considered. For the first example the damping force is proportional to the square of the transverse velocity. Thus

$$f_c\left(\frac{\partial \bar{y}}{\partial \bar{t}}\right) = \frac{\partial \bar{y}}{\partial \bar{t}} \left| \frac{\partial \bar{y}}{\partial \bar{t}} \right| \text{ (example 1)} \qquad (1\text{-}20)$$

Here the sign of the damping term is always the same as that of the transverse velocity, and as a result energy is always extracted from the system. For the second example the damping force is of the Coulomb type. Thus

$$f_c\left(\frac{\partial \bar{y}}{\partial t}\right) = 1, \quad \frac{\partial \bar{y}}{\partial t} > 0$$

$$= -1, \quad \frac{\partial \bar{y}}{\partial t} < 0 \qquad (1\text{-}21)$$

Here the damping force is always constant but depends in sign on the sign of the transverse velocity.

1.2 Finite Difference Method for Approximating Derivatives

Instead of considering the transverse displacement y of our beam at all points along the beam, let us consider y only at certain stations along x, as shown in Figure 1-2. Further, let the distance between x stations be Δx. Thus we define y_1 as the transverse displacement at $x = \Delta x$, y_2 as the displacement at $x = 2\Delta x$, y_n as the displacement at $x = n\Delta x$. Clearly a good approximation to $\partial y/\partial x|_{n+1/2}$ (i.e., the partial derivitive of y with respect to x evaluated at the n+1/2 station) is simply

$$\left.\frac{\partial y}{\partial x}\right|_{n+1/2} = \frac{y_{n+1} - y_n}{\Delta x} \qquad (1\text{-}22)$$

Indeed, in the limit as $\Delta x \to 0$ Equation (1-22) defines $\partial y/\partial x$ at $x = (n+1/2)\Delta x$. In the same way

$$\left.\frac{\partial^2 y}{\partial x^2}\right|_n = \frac{1}{\Delta x}\left|\left.\frac{\partial y}{\partial x}\right|_{n+1/2} - \left.\frac{\partial y}{\partial x}\right|_{n-1/2}\right| \qquad (1\text{-}23)$$

or

$$\left.\frac{\partial^2 y}{\partial x^2}\right|_n = \frac{y_{n+1} - \partial y_n + y_{n-1}}{(\Delta x)^2} \qquad (1\text{-}24)$$

Higher order derivitives are computed in the same manner.

The displacement y_n at the nth station is a function only of time. Hence if we replace x derivitives in Equation (1-14) with finite differences, a system of ordinary nonlinear differential equations will result, equations which can be solved directly by the electronic differential analyzer.

1.3 Principles of Operation of the Electronic Differential Analyzer

The reader unfamiliar with the theory of operation of the electronic differential analyzer is directed to other references[4,5]. To review briefly, we recall that the basic unit of this type of computer is the operational amplifier, which consists of a high-gain dc amplifier along with feedback impedance Z_f and one or more input impedances, as shown in Figure 1-3.

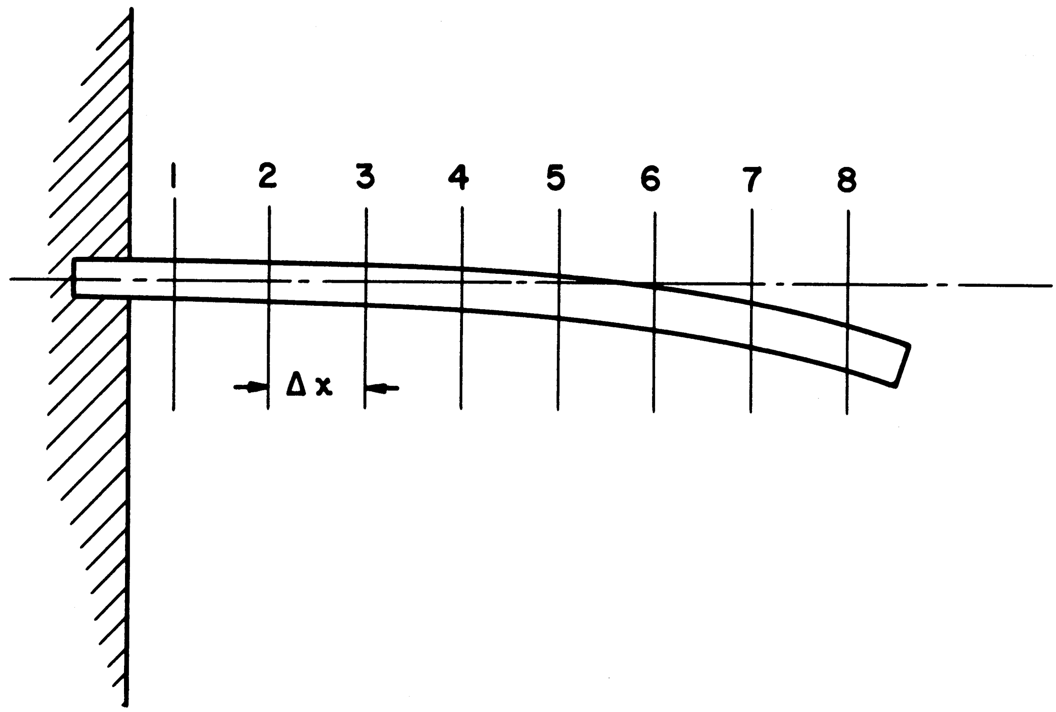
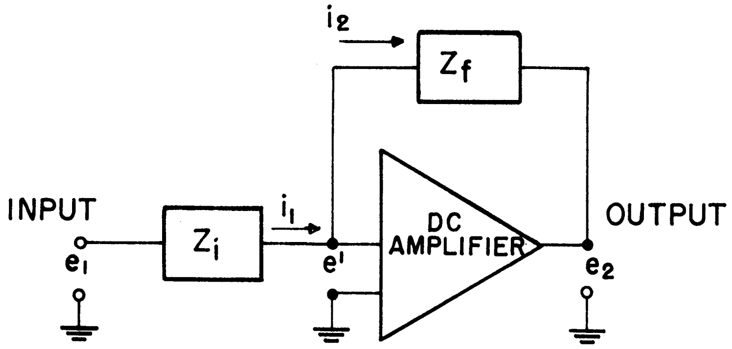


Figure 1-2. Cantilever Beam Divided into Stations.

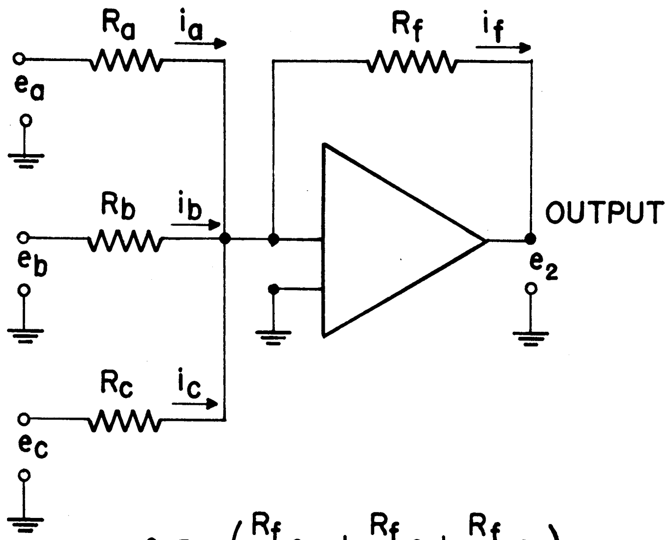
To a high degree of approximation the output voltage e_o of an operational amplifier is equal to the input voltage divided by the ratio of feedback to input impedance, with a reversal of sign (Figure 1-3a). If several input resistors are used, the output voltage is proportional to the sum of the input voltages (Figure 1-3b). If an input resistor and feedback capacitor are used, the output voltage is proportional to the time integral of the input voltage (Figure 1-3c).

The operational amplifiers shown in Figure 1-3 can be used to multiply a voltage by a constant factor, invert signs, sum voltages, and integrate a voltage with respect to time. To multiply several voltages a servomechanism which drives potentiometers is the most commonly used device. In Figure 1-4 the block diagram of a servo multiplier is shown. It consists of a number of linear potentiometers ganged together and driven by a servo motor. The reference voltage $\pm V_R$ is connected across one of the pots, and the variable tap voltage αV_R is subtracted from the voltage Z. The resulting error signal $\epsilon = Z - \alpha V_R$ is sent through a high-gain servo amplifier and applied to the servo motor. The motor drives the variable tap in the proper direction to reduce the error to zero, i.e., to make $\alpha V_R = Z$. In this way the tap position on all of the ganged pots is proportional to the voltage Z. If $\pm$ X and $\pm$ Y are applied across each of the remaining two pots shown in Figure 1-4, it is apparent that the variable tap voltages will be XZ/V_R and YZ/V_R respectively. Thus the servo multiplier can generate output voltages proportional to the product of input voltages.



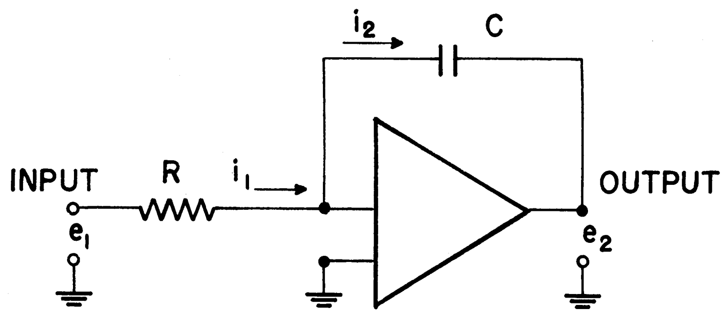
$$e_2 = -\frac{Z_f}{Z_i} e_1$$

a.) OPERATIONAL AMPLIFIER



$$e_2 = -\left(\frac{R_f}{R_a} e_a + \frac{R_f}{R_b} e_b + \frac{R_f}{R_c} e_c\right)$$

b.) OPERATIONAL AMPLIFIER AS A SUMMER



$$e_2 = -\frac{1}{RC} \int e_1 \, dt$$

C.) OPERATIONAL AMPLIFIER AS AN INTEGRATOR

Figure 1-3. Operational Amplifier.

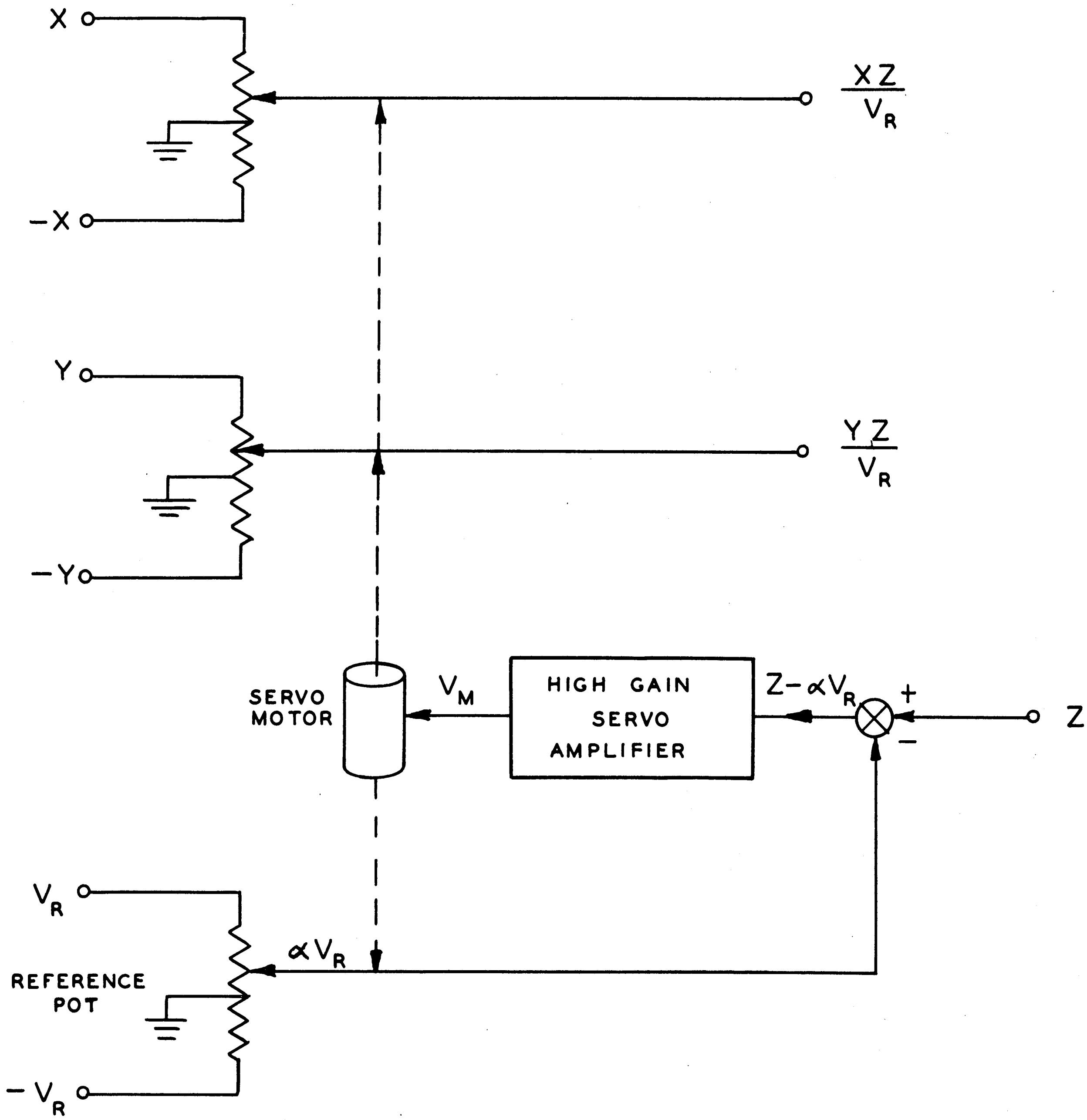


Figure 1-4. Servo Multiplier.

For the electronic differential analyzer solutions obtained in this report REAC* Servo Unit S-101 Mod 4 servos were used. Accuracy of multiplication is about 0.1% of full scale (± 100 volts). Drift-stabilized dc amplifiers of our own design[6] were used along with computing resistors calibrated to 0.02%. Amplifier gain is about 12 x 10^6 and average offset referred to input is approximately 10^{-4} volts.

By employing operational amplifiers for summation and integration, and servos for multiplication, we are able to solve the cantilever beam with velocity-squared damping.

* Reeves Electronic Analog Computer, Reeves Instrument Corp., New York 28, New York

CHAPTER 2

CANTILEVER BEAM WITH VELOCITY-SQUARED DAMPING

2.1 Beam Equation Including Velocity-Squared Damping

For flexural vibration of a beam with damping proportional to the square of the transverse velocity, we have the following equation from Equations (1-14) and (1-20):

$$\frac{\partial^2}{\partial x^2} \phi_f(x) \frac{\partial^2 y}{\partial x^2} + \frac{L^4 c_o}{EI_o y_o} \phi_c(x) \frac{\partial \bar{y}}{\partial \bar{t}} \left| \frac{\partial \bar{y}}{\partial \bar{t}} \right| + \phi_d(x) \frac{\partial^2 y}{\partial t^2} = f(x, t) \qquad (2\text{-}1)$$

Since $\bar{y} = y_o\, y$ and $\bar{t} = L^2 \sqrt{\rho_o/EI_o}\, t$, Equation (2-1) becomes

$$\frac{\partial^2}{\partial x^2} \phi_f(x) \frac{\partial^2 y}{\partial x^2} + c\phi_c(x) \frac{\partial y}{\partial t} \left| \frac{\partial y}{\partial t} \right| + \phi_d(x) \frac{\partial^2 y}{\partial t^2} = f(x, t) \qquad (2\text{-}2)$$

where

$$c = \frac{c_o\, y_o}{\rho_o} \qquad (2\text{-}3)$$

For a cantilever beam the boundary conditions are given in Equation (1-16) and (1-17). We recall that the transverse displacement y, distance along the beam x, and time t in Equation (2-2) are all dimensionless.

2.2 Equivalence of Damping-Coefficient Size and Amplitude of Vibration

In any nonlinear equation the behavior of the solution is not independent of the magnitude of the dependent variables, as with a linear system. At first thought one might therefore assume that our nonlinear beam Equation (2-2) must be solved not only for different damping constants c but also for different amplitudes of transverse vibrations. Actually, this is not true. Consider first the case where $f(x, t) = 0$ and where we know the solution $y(x, t)$ for a given damping constant c and for given initial conditions. If we now double the size of the initial conditions, the solution will be simply $2y(x, t)$ providing the damping constant is $c/2$. This is evident from the damping term in Equation (2-2).

In the same way if we know the solution for a given force $f(x, t)$ and damping-constant c, the solution for a force $\alpha f(x, t)$ will be α times as big as the previous solution providing the damping constant is c/α.

2.3 Difference Equations for the Cantilever Beam with Velocity-Squared Damping

As explained in Section 1.2 we will consider the transverse displacement y only at stations along the beam. In this way derivitives with respect to x can be replaced by finite differences. Following the procedure of our previous report[1] we introduce a new distance variable χ such that the distance $\Delta\chi$ between stations is unity. If the beam is divided into N stations or cells, then

$$\chi = Nx \text{ and } \frac{\partial}{\partial x} = N \frac{\partial}{\partial \chi} . \qquad (2\text{-}4)$$

It is also convenient to introduce a new time variable τ given by

$$\tau = N^2 t \text{ and } \frac{\partial}{\partial t} = N^2 \frac{\partial}{\partial \tau} \qquad (2\text{-}5)$$

Equation (2-2) then becomes

$$\frac{\partial^2}{\partial \chi^2} \phi_f(\chi) \frac{\partial^2 y}{\partial \chi^2} + c\phi_c(\chi) \frac{\partial y}{\partial \tau} \left|\frac{\partial y}{\partial \tau}\right| + \phi_d(\chi) \frac{\partial^2 y}{\partial \tau^2} = \Phi(\chi, \tau) = \frac{1}{N^4} f(x,\tau) \qquad (2\text{-}6)$$

The difference equation at the nth cell is from Equation (1-24)

$$\phi_{d_n} \frac{d^2 y_n}{d\tau^2} + c\phi_{c_n} \frac{dy_n}{d\tau} \left|\frac{dy_n}{d\tau}\right| = - m_{n+1} + 2m_n - m_{n-1} + \Phi_n(\tau) \qquad (2\text{-}7)$$

where m_n is proportional to the bending moment and is given by

$$m_n = \phi_{f_n} (m_{n+1} - 2m_n + m_{n-1}) \qquad (2\text{-}8)$$

Boundary conditions for the difference technique are discussed more completely in a previous report[1]. For the clamped end the condition of zero displacement and slope is approximated by letting

$$y_0 = y_1 = 0 . \text{ (clamped end at } \chi = 1/2) \qquad (2\text{-}9)$$

For the free end the condition of zero moment and shear is approximated by letting

$$m_N = m_{N+1} = 0 \text{ (free end at } \chi = N + 1/2) \qquad (2\text{-}10)$$

Note that the N-cell cantilever beam has its built-in (clamped) end at $\chi = 1/2$ and its free end at $\chi = N + 1/2$. From Equations (2-7), (2-8), (2-9), and (2-10) the complete set of difference equations for the cantilever beam with velocity-squared damping becomes

$$\begin{aligned}
\phi_{d_2} \frac{d^2 y_2}{d\tau^2} + c\phi_{c_2} \frac{dy_2}{d\tau}\left|\frac{dy_2}{d\tau}\right| &= - m_3 + 2m_2 - m_1 + \Phi_1(\tau) \\
\phi_{d_3} \frac{d^2 y_3}{d\tau^2} + c\phi_{c_3} \frac{dy_3}{d\tau}\left|\frac{dy_3}{d\tau}\right| &= - m_4 + 2m_3 - m_2 + \Phi_2(\tau) \\
\vdots \qquad\qquad & \qquad\qquad \vdots \\
\phi_{d_{N-2}} \frac{d^2 y_{N-2}}{d\tau^2} + c\phi_{c_{N-2}} \frac{dy_{N-2}}{d\tau}\left|\frac{dy_{N-2}}{d\tau}\right| &= - m_{N-1} + 2m_{N-2} - m_{N-3} + \Phi_{N-2}(\tau) \\
\phi_{d_{N-1}} \frac{d^2 y_{N-1}}{d\tau^2} + c\phi_{c_{N-1}} \frac{dy_{N-1}}{d\tau}\left|\frac{dy_{N-1}}{d\tau}\right| &= 2m_{N-1} - m_{N-2} + \Phi_{N-1}(\tau) \\
\phi_{d_N} \frac{d^2 y_N}{d\tau^2} + c\phi_{c_N} \frac{dy_N}{d\tau}\left|\frac{dy_N}{d\tau}\right| &= - m_{N-1} + \Phi_N(\tau)
\end{aligned} \qquad (2\text{-}11)$$

where

$$\begin{aligned}
m_1 &= \phi_{f_1} y_2 \\
m_2 &= \phi_{f_2} (y_3 - 2y_2) \\
m_3 &= \phi_{f_3} (y_4 - 2y_3 + y_2) \\
&\vdots \\
m_{N-2} &= \phi_{f_{N-2}} (y_{N-1} - 2y_{N-2} + y_{N-3}) \\
m_{N-1} &= \phi_{f_{N-1}} (y_N - 2y_{N-1} + y_{N-2})
\end{aligned} \qquad (2\text{-}12)$$

Initial conditions $y_n(0)$ and $dy(0)/d\tau$ must of course be specified to define the complete problem.

2.4 Analyzer Circuit for the Cantilever Beam with Velocity-Squared Damping

The electronic differential analyzer circuit for solving the equation at the nth cell is shown in Figure 1-5. The velocity $dy_n/d\tau$ times its absolute

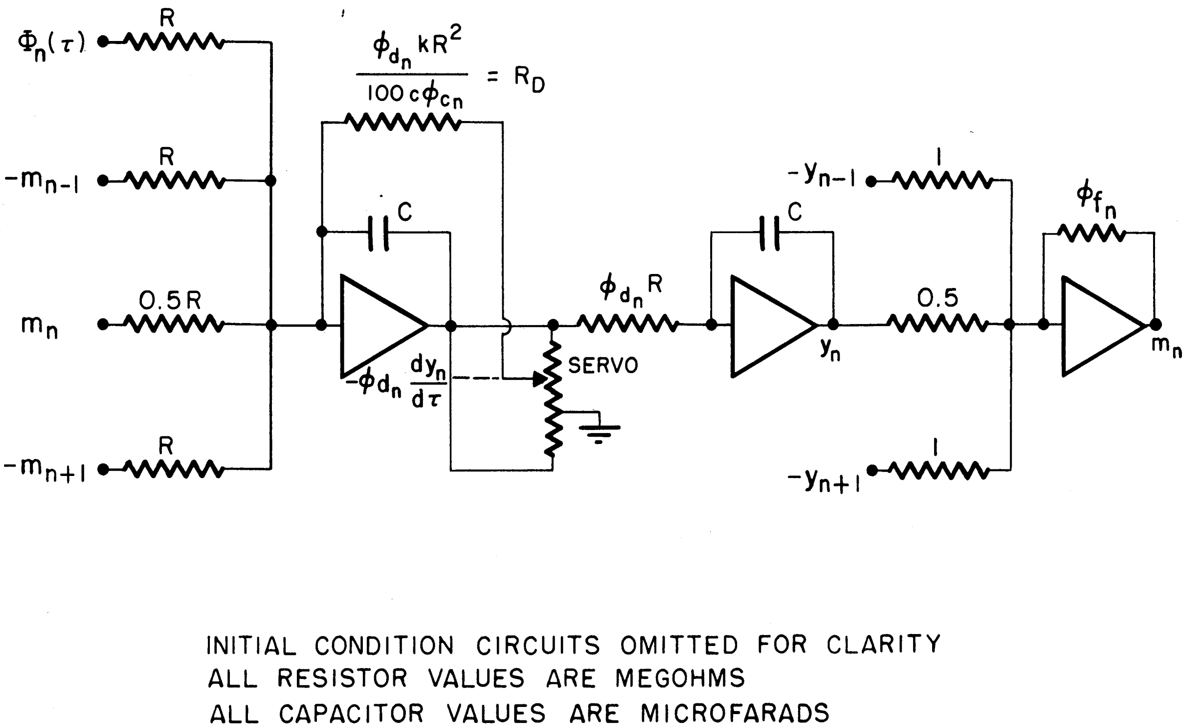


Figure 2-1. Analyzer Circuit at the nth Station for the Cantilever Beam with Velocity Squared Damping.

value is obtained by grounding the center tap of a servo-multiplier potentiometer and connecting $dy_n/d\tau$ to both ends of the pot. In the figure it is assumed that the servo reference voltage is 100 volts and that k volts on the computer equals unit y. The time scale of integration is RC seconds; thus one unit of τ equals RC seconds of actual time in the computer solution. The circuit of Figure 2-1 is iterated N-1 times to solve Equations (2-11) and (2-12). 3(N-1) operational amplifiers and N-1 servo multipliers are required to solve the N-1 simultaneous nonlinear ordinary differential equations. For a complete circuit diagram including the connections at the built-in and free ends the reader is referred to the previous report[1].

For the uniform cantilever beam which we will consider from now on $\phi_{cn} = \phi_{dn} = \phi_{fn} = 1$. An integrator time scale of 0.2 seconds (R = 0.2

megohms, C = 1 microfarad) was used in all cases. We let 50 volts equal a unit transverse displacement y, so that k = 50 in Figure 2-5.

2.5 Damped First-Mode Oscillation

An 8-cell uniform cantilever beam with velocity-squared damping was set up on the electronic differential analyzer. The correctness of the circuit is easily established by measuring the frequency of first-mode oscillation when no damping is present ($R_D = \infty$). This should agree closely with the theoretical value for an 8-cell beam, which is 0.70% higher than the frequency for a continuous beam[1]. First-mode oscillations were excited either by driving the beam circuit with a sinusoidal voltage $\Phi_8(\tau)$ having the first-mode frequency or by applying initial conditions representing the shape of the first-mode frequency[1]. For the latter case the displacement y_8 at the free end of the beam is recorded as a function of time in Figure 2-2 for several values of the damping-constant c. Note that the damping effect is large for big amplitudes of oscillation and decreases as the amplitude falls off. This is due to the velocity-squared damping.

2.6 Approximate Theoretical Solution

Let us consider the uniform cantilever beam with velocity-squared damping when the external force f(x, t) = 0. In this case

$$\frac{\partial^4 y}{\partial x^4} + c\frac{\partial y}{\partial t}\left|\frac{\partial y}{\partial t}\right| + \frac{\partial^2 y}{\partial t^2} = 0 . \qquad (2\text{-}13)$$

We have seen in Section 2.2 that increasing the damping constant c by a factor α is equivalent to keeping the same damping constant but increasing the initial amplitude y(x, 0) by the same factor α. The resulting solution is just α times the first solution. Thus if we solve Equation (2-13) for a given damping (say c = 1) but for a number of amplitudes of oscillation, we have also covered the solutions for different damping constants c.

Let us assume that the beam is vibrating periodically with frequency ω and is only lightly damped. A fairly accurate approximate solution to Equation (2-13) can be written by considering the energy E_d absorbed per cycle. This will be

$$E_d = \int_0^1 \int_{t_0}^{t_0 + 2\pi/\omega} c\frac{\partial y}{\partial t}\left|\frac{\partial y}{\partial t}\right|\frac{\partial y}{\partial t}\, dx\, dt , \qquad (2\text{-}14)$$

where ω is the frequency of oscillation and t_0 is the time at which the particular cycle in which we are interested starts. Let us assume next that the beam is

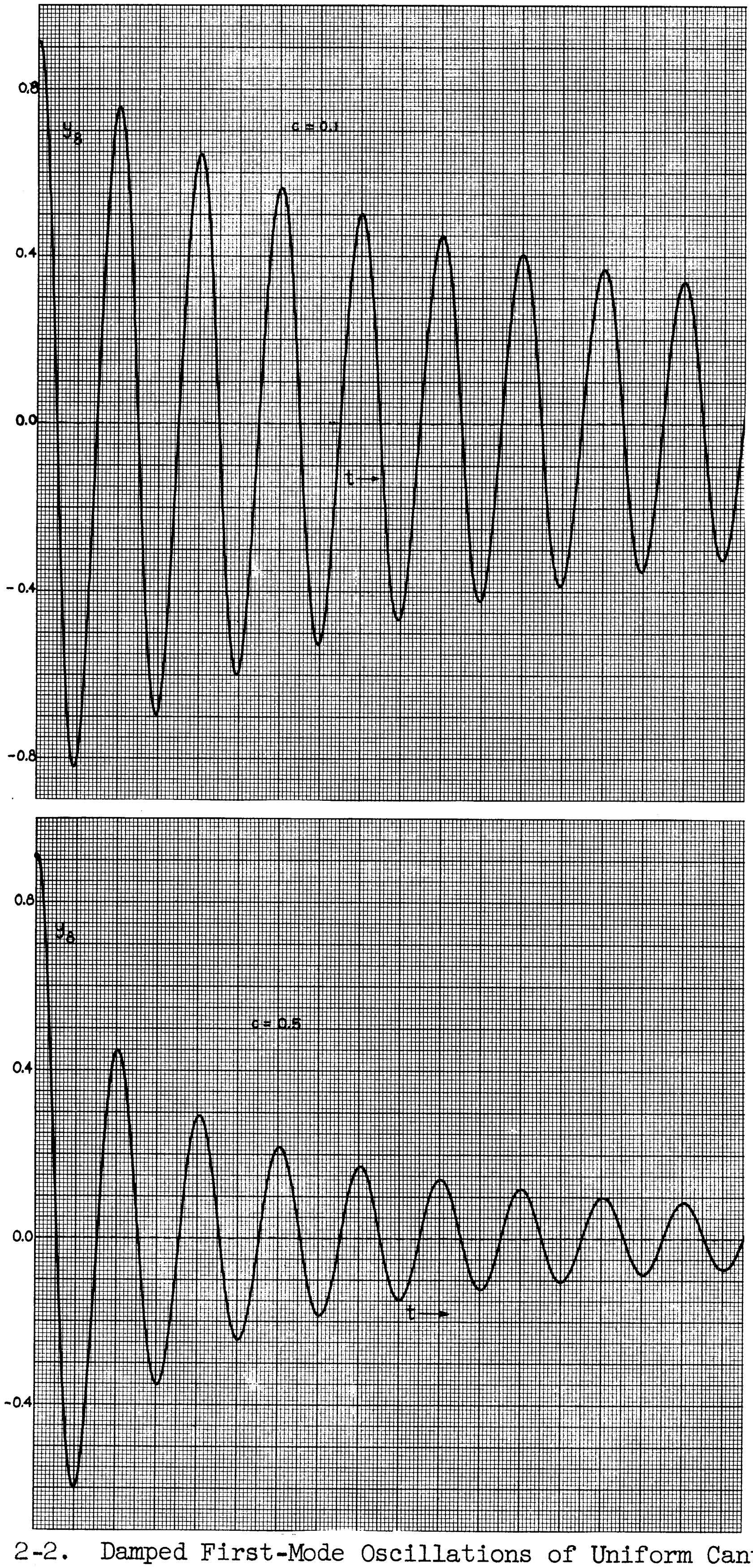


Figure 2-2. Damped First-Mode Oscillations of Uniform Cantilever Beam with Velocity-Squared Damping.

oscillating at one of its normal-mode frequencies, so that the motion is approximately sinusoidal. Also we will choose one time scale such that $t_o = 0$. Let $y_a(x)$ equal the amplitude at $t = 0$ and $y_b(x)$ equal the amplitude one cycle later. We will define an approximation $y_{ab}(x) \cos \omega t$ to $y(x, t)$ over the entire cycle as

$$y_{ab}(x) \cos \omega t = \frac{y_a(x) + y_b(x)}{2} \cos \omega t \tag{2-15}$$

Then E_{ka}, the kinetic energy at $t = 0$ is

$$E_{ka} = \frac{1}{2} \omega^2 \int_0^1 y_a^2 (x)\, dx \tag{2-16}$$

and E_{kb}, the kinetic energy at $t = 2\pi/\omega$ is

$$E_{kb} = \frac{1}{2} \omega^2 \int_0^1 y_b^2 (x)\, dx \tag{2-17}$$

We can calculate approximately the energy absorbed over one cycle from Equation (2-14), since we assumed $y(x, t) = y_{ab}(x) \cos \omega t$ for this period. Thus

$$E_d = c\omega^3 \int_0^1 y_{ab}^3 (x)\, dx \left[\int_0^{2\pi/\omega} \cos^2 \omega t \,|\cos \omega t|\, dt\right]$$

or

$$E_d = \frac{8c}{3} \omega^2 \int_0^1 y_{ab}^3 (x)\, dx \tag{2-18}$$

Let $y_m(x)$ equal the dimensionless mode shape having unit amplitude at the free end. Then

$$\int_0^1 y_a^2 (x)\, dx = y_a^2 (1) \int_0^1 y_m^2 (x)\, dx \tag{2-19}$$

$$\int_0^1 y_b^2 (x)\, dx = y_b^2 (1) \int_0^1 y_m^2 (x)\, dx \tag{2-20}$$

and

$$\int_0^1 y_{ab}^3 (x)\, dx = y_{ab}^3 (1) \int_0^1 y_m^3 (x)\, dx \tag{2-21}$$

Clearly the difference between kinetic energies before and after the cycle of oscillation is the energy absorbed over the cycle. Thus

$$E_{ka} - E_{kb} = E_d \tag{2-22}$$

or from Equations (2-16) through (2-21)

$$\frac{1}{2}\omega^2 y_a^2 (1) - \frac{1}{2}\omega^2 y_b^2 (1) = \frac{8c}{3}\omega^2 \left[\frac{y_a(1)+y_b(1)}{2}\right]^3 \frac{\int_0^1 y_m^3 (x)\, dx}{\int_0^1 y_m^2 (x)\, dx} \tag{2-23}$$

For the first mode of a uniform cantilever beam

$$\frac{\int_0^1 y_m^3 (x)\, dx}{\int_0^1 y_m^2 (x)\, dx} \cong 0.736 \tag{2-24}$$

Thus for the first mode

$$y_b^2 (1) \cong y_a^2 (1) - 3.92\, c \left[\frac{y_a(1) + y_b(1)}{2}\right]^3 \tag{2-25}$$

Equation (2-25) can be used to solve for the amplitude y_b (1) at the free end following one cycle of oscillation starting with amplitude y_a (1). If the damping is very slight, $[y_a (1) + y_b (1)]/2 \cong y_b (1)$ and the logarithmic decrement δ is given by

$$\delta = \ln \left[\frac{y_a (1)}{y_b (1)}\right] \cong 1.96\, c y_a (1), \quad \delta << 1 \tag{2-26}$$

Equation (2-26) predicts that when the damping is slight, δ is directly proportional to the amplitude of oscillation. In Figure 2-3 δ is plotted as a function of amplitude of oscillation from Equations (2-25) and (2-26) and compared with computer results. Evidently Equation (2-25) is quite accurate and Equation (2-26) is accurate for small damping.

2.7 Impulse Response of the Cantilever Beam with Velocity Squared Damping

A number of solutions were recorded following unit impulses of one-fifth second duration applied simultaneously at each station along the beam. A five-cell uniform cantilever beam was used for these solutions. Shown in Figure 2-4 is the

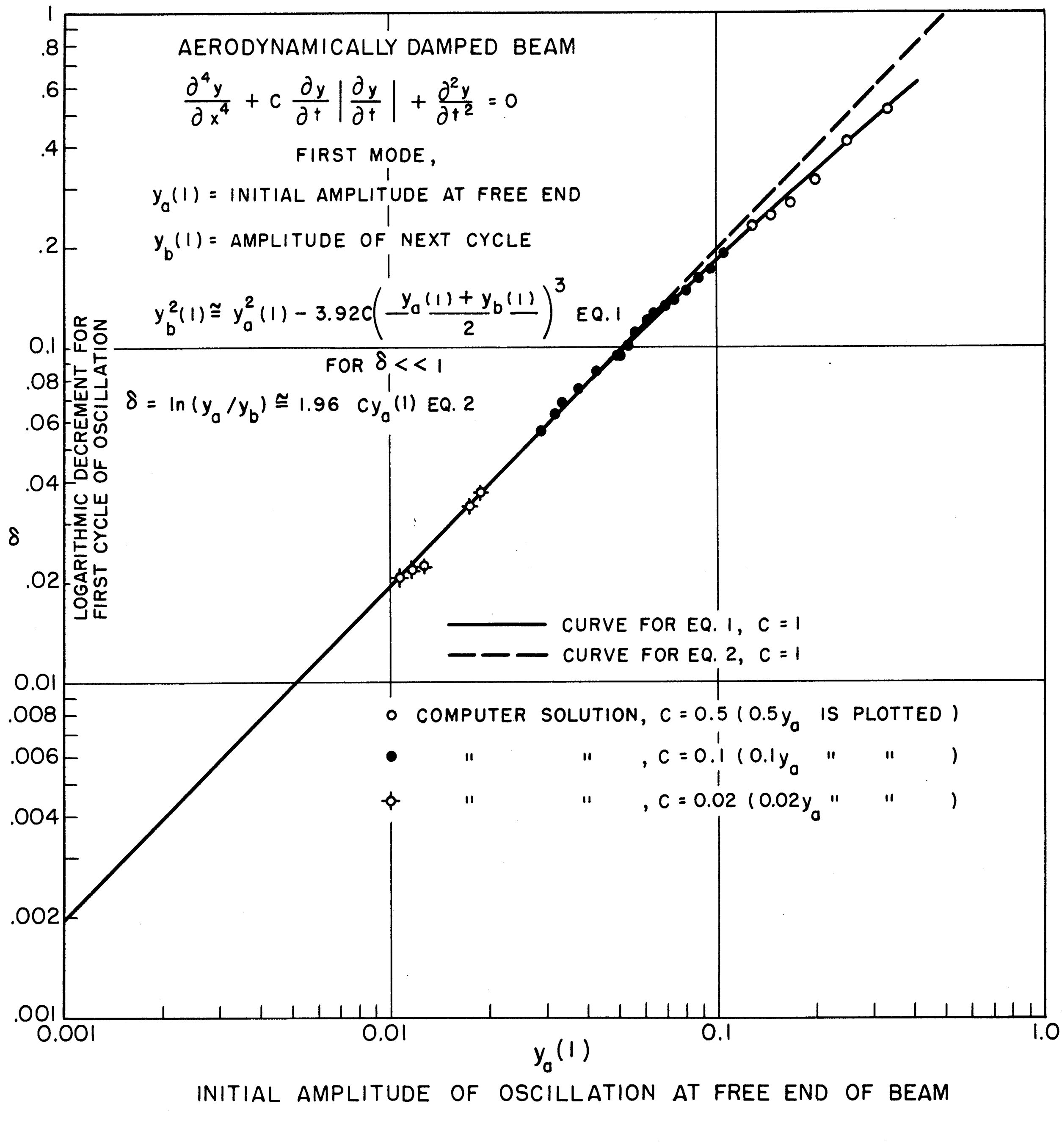


Figure 2-3 Variation of Logarithmic Decrement δ with Amplitude of Oscillation.

displacement at each of the stations following the unit impulse. In Figure 2-5 recordings of the bending moments are shown. Four different damping cases are shown.

The response of nonuniform cantilever beams or beams with other end fastenings could have been obtained with equal ease. Any arbitrary forces along the beam can be considered, as well as time dependent boundary conditions and transverse-shear effects. For a complete discussion of these and other cases for lateral vibration of linear beams the reader is directed to the previous report[1].

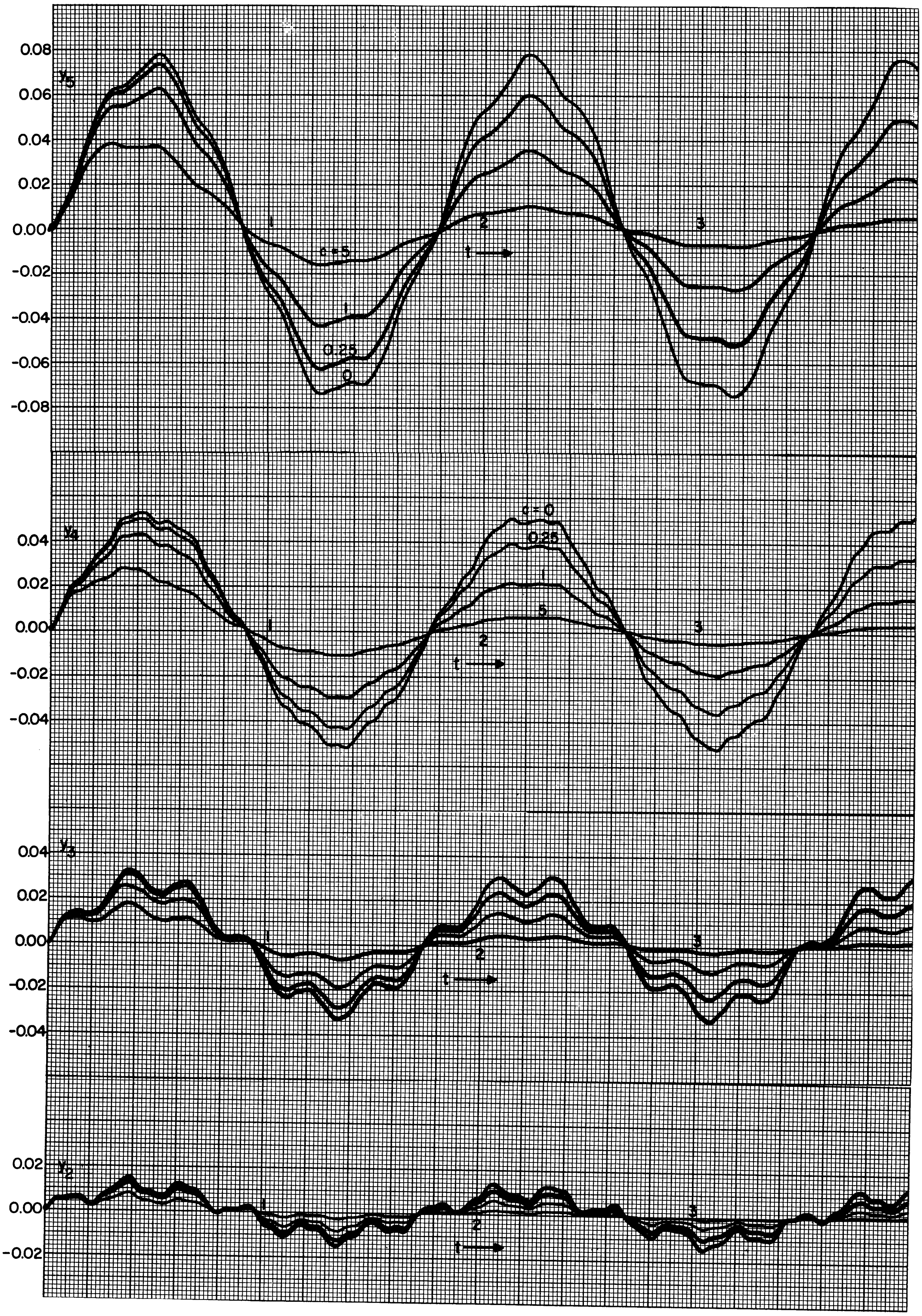


Figure 2-4. Unit Impulse Response of 5-Cell Uniform Cantilever Beam with Velocity-Squared Damping Displacements at Stations 2, 3, 4, and 5.

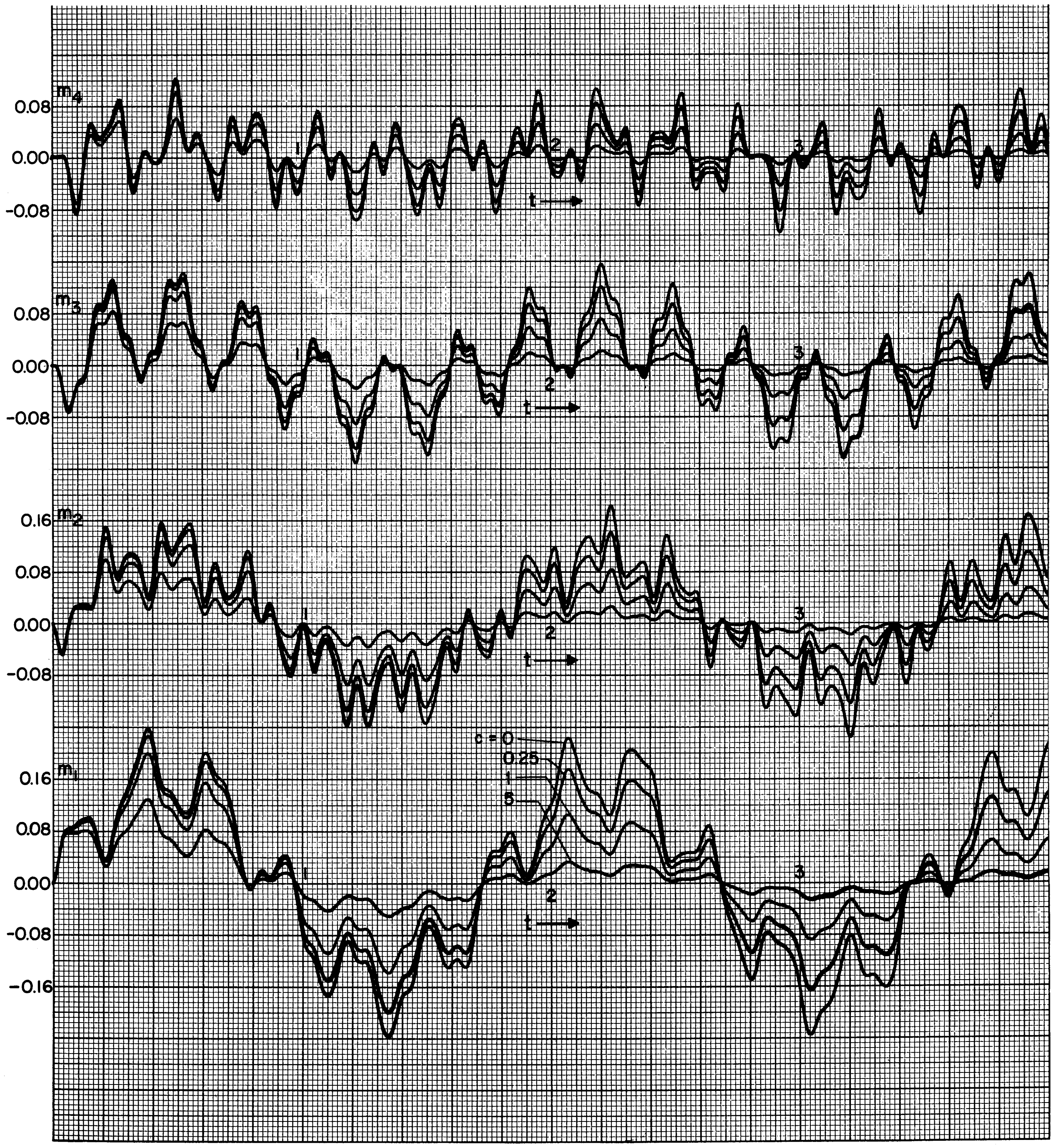


Figure 2-5. Unit Impulse Response of 5-Cell Uniform Cantilever Beam with Velocity-Squared Damping; Bending-Moment at Stations 1, 2, 3, and 4.

CHAPTER 3

CANTILEVER BEAM WITH COULOMB DAMPING

3.1 Beam Equation Including Coulomb Damping

From Equations (1-19) and (1-21) the equation for lateral vibrations of a beam with coulomb damping is given by

$$\frac{\partial^2}{\partial x^2} \phi_f(x) \frac{\partial^2 y}{\partial x^2} + c_d \, \phi_c(x) \, f_c\left(\frac{\partial y}{\partial t}\right) + \phi_d(x) \frac{\partial^2 y}{\partial t^2} = f(x,t) \qquad (3\text{-}1)$$

where

$$\begin{aligned} f_c\left(\frac{\partial y}{\partial t}\right) &= 1, \quad \frac{\partial y}{\partial t} > 0 \\ &= -1, \quad \frac{\partial y}{\partial t} < 0 \end{aligned} \qquad (3\text{-}2)$$

and where

$$c_d = \frac{L^4 c_o}{EI_o y_o} \qquad (3\text{-}3)$$

For a cantilever beam the boundary conditions are given in Equations (1-16) and (1-17). The lateral displacement y, distance along the beam x, and time t are all dimensionless.

If we know the solution $y(x,t)$ to Equation (3-1) for $f(x,t) = 0$, given initial conditions, and a given damping constant c_d, the solution for initial conditions α times as big will be simply $\alpha y(x,t)$ providing the damping constant is αc_d. Similarly, if we know the solution $y(x,t)$ for zero initial conditions, a given damping constant c_d, and a given external force $f(x,t)$, then the solution for a force $\alpha f(x,t)$ is simply $\alpha y(x,t)$, providing again that the damping constant is αc_d. Thus if we can find the beam response for a given $f(x,t)$ and all c_d values, we also know the solutions for any force $\alpha f(x,t)$, where α is a constant factor.

3.2 Difference Equations for the Cantilever Beam With Coulomb Damping

Following the procedure outlined in Section 2.3, we can rewrite Equation (3-1) as a set of simultaneous ordinary nonlinear differential equations by considering the lateral displacement only at discrete points along the beam. Thus at the nth station

$$\phi_{d_n} \frac{d\, y_n}{d\tau^2} + \frac{c_d}{N^4} \phi_{c_n}\, f_c(\frac{dy_n}{d\tau}) = -m_{n+1} + 2m_n - m_{n-1} + \Phi_n(\tau) \qquad (3\text{-}4)$$

where m_n is proportional to the bending moment and is given by

$$m_n = \phi_{f_n}\ (y_{n+1} - 2y_n + y_{n-1}) \qquad (2\text{-}8)$$

In Equation (3-4) we recall that N represents the number of cells into which the beam is divided. A new distance variable $\chi = Nx$ makes $\Delta\chi$, the distance between stations, equal to unity. The time variable τ in Equation (3-4) is equal to $N^2 t$ and

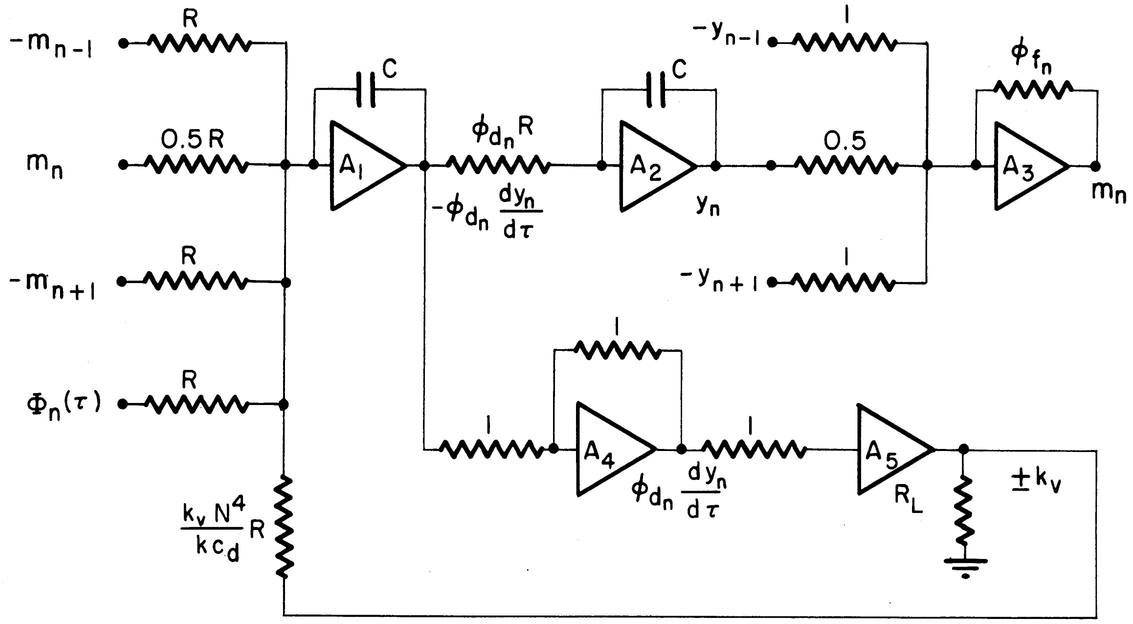
$$\Phi_n(\tau) = \frac{1}{N^4} f_n(\tau) \qquad (3\text{-}5)$$

The built-in boundary condition at $\chi = 1/2$ implies that $y_o = y\ \ = 0$. The free condition at $\chi = N + 1/2$ implies that $m_n = m_{n+1} = 0$. A set of N-1 equations similar to (2-11) and (2-12) is obtained for the complete cantilever beam with coulomb damping.

3.3 Analyzer Circuit for the Cantilever Beam with Coulomb Damping

The electronic differential analyzer circuit for solving the equation at the nth station is shown in Figure 3-1.

The $f_c(dy/d\tau)$ function is represented by amplifier A_5 in the figure. This amplifier has no feedback and is loaded with R_L, a resistor selected so that the amplifier output saturates at the same voltage k_v for either positive or negative outputs. A very small positive or negative input voltage (less than 4 millivolts) will produce a full output voltage k_v of negative or positive sign respectively. The result is an accurate simulation of the coulomb damping force represented by $f_c(dy/d\tau)$, which is summed into amplifier A_1 in Figure 3-1. The circuit is iterated N-1 times to solve the complete N-cell beam.



k VOLTS = UNITY

Figure 3-1. Analyzer Circuit at the nth Station for Cantilever Beam with Coulomb Damping.

3.4 Impulse Response of the Cantilever Beam with Coulomb Damping

A 5-cell uniform cantilever beam with various amounts of coulomb damping was set up on the differential analyzer with an integrator time scale of 0.5 seconds. Response y_5 at station 5 is shown in Figure 3-2 following a unit impulse of one-fifth second duration. Note the dead-space effect due to the coulomb damping; the displacement y_5 does not in general return to zero but ends up at some finite displacement for which the elastic forces are insufficient to overbalance the coulomb friction.

It seems hardly necessary to point out that arbitrary combinations of velocity-squared, coulomb, viscous and other types of damping can readily be handled by the electronic differential analyzer. For a more complete discussion of other types of beams, time-dependent boundary conditions, theoretical accuracy of the difference technique, etc., the reader is referred to a previous report.[1]

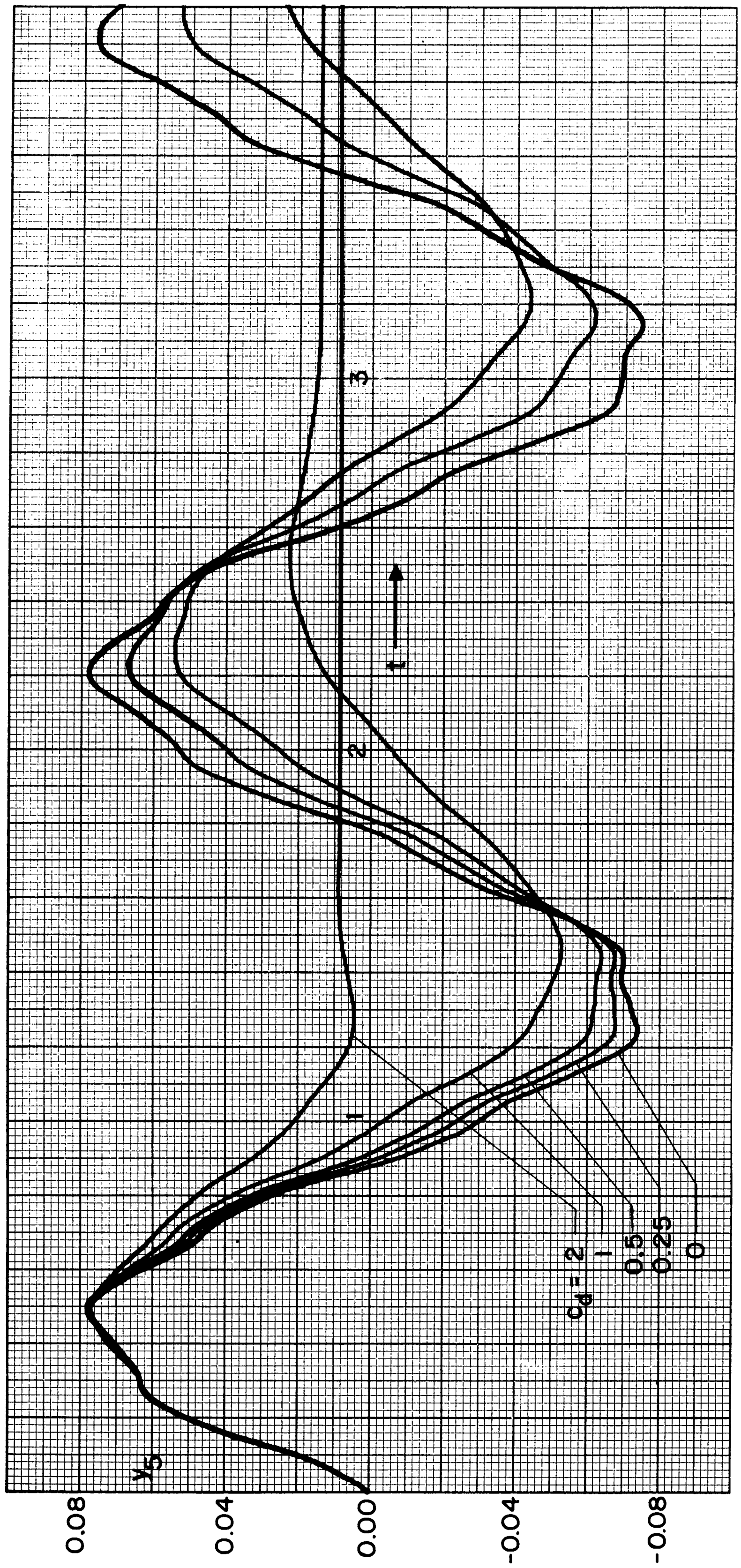


Figure 3-2. Impulse Response of a 5-Cell Uniform Cantilever Beam with Coulomb Damping.

BIBLIOGRAPHY

1. C. E. Howe and R. M. Howe, Application of Difference Techniques to the Lateral Vibration of Beams Using the Electronic Differential Analyzer, Report AIR-7, University of Michigan, Engineering Research Institute, OOR Contract No. DA-20-018-ORD-21811; April, 1954.

2. Timoshenko, Vibration Problems in Engineering, D. Van Nostrand (1937).

3. C. E. Howe, R. M. Howe, and L. L. Rauch, Application of the Electronic Differential Analyzer to the Oscillation of Beams, Including Shear and Rotary Inertia, External Memorandum UMM-67 (January 1951), University of Michigan Engineering Research Institute.

4. G. A. Korn and T. M. Korn, Electronic Analog Computers, McGraw Hill (1952).

5. D. W. Hagelbarger, C. E. Howe, and R. M. Howe, Investigation of the Utility of an Electronic Analog Computer in Engineering Problems, External Memorandum UMM-28 (April 1, 1949), University of Michigan Engineering Research Institute, AF Con. W33(038)ac-14222 (Project MX-794).

6. R. M. Howe, Theory and Operating Instructions for the Air Comp Mod 4 Electronic Differential Analyzer, Report AIR-4, University of Michigan, Engineering Research Institute, ONR Contract N6 onr 23223; March, 1953.