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PROPAGATION OF UNDERWATER SOUND IN A BILINEAR VELOCITY GRADIENT

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PREFACE

This report summarizes the results of some theoretical investigations and differential analyzer solutions for the problem of wave propagation in a medium with varying indices of refraction. In particular, the problem of wave propagation underwater is considered when the index of refraction changes as a function of depth and the effect of the bottom is neglected. The electronic differential analyzer solutions were limited to the determination of the eigenvalues and eigenfunctions associated with the depth-dependent wave potential.

The above work was sponsored by the Office of Naval Research. Under the same contract, an electronic differential analyzer was designed and constructed to allow further study of the underwater-sound problem. The description of this equipment is given in a separate report.[16]

The author would like to acknowledge the assistance given by Dr. C. E. Howe in obtaining the differential analyzer solutions presented in this report. Dr. J. R. Sellars was mainly responsible for the approximate eigenvalue formulas developed in Chapter 2 from the asymptotic forms of the Hankel functions. The theoretical investigation was carried out by Dr. C. L. Dolph, who has written the portion of Chapter 1 which summarizes this effort. The author is also indebted to Dr. H. R. Alexander of the Acoustics Branch, Office of Naval Research, for his help in clarifying the derivation of the bilinear-gradient equations, and to Dr. H. W. Marsh, Jr., of the U. S. Navy Underwater Sound Laboratory, New London, without whose interest and support this research program would not have been possible.

R. M. Howe

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CHAPTER 1
OUTLINE OF THE PROBLEM

1.1 Introduction

The propagation of waves in a semi-infinite medium having varying indices of refraction has been the subject of a very considerable number of researches. This report is concerned with a small part of that problem, namely, the determination of the normal modes making up the depth-dependent wave-potential function describing propagation of sound waves underwater. The problem is complicated by the fact that the velocity of sound varies with the depth of the water.[1] In particular, we shall consider the special case where a positive velocity gradient exists from the surface to some finite depth, at which point the velocity gradient reverses and becomes a negative constant for all lower depths. For this reason the problem treated here is known as the bilinear gradient. The medium of propagation is considered to be semi-infinite, i.e., the effect of the ocean bottom is not included.

The electronic differential analyzer, along with tabulated solutions to Stokes' equation, is used to solve the depth-dependent equation. No attempt is made to normalize or interpret the eigenfunctions obtained; this task is left to other workers in the field.

1.2 Equations to be Solved

Cylindrical coordinates will be utilized to describe the wave potential function $\overline{\Psi}(r,z,t)$ where r is the radial distance from the origin, z is depth below the surface, and t is time. The wave equation is

$$\nabla^2_{r,z}\overline{\Psi} = \frac{1}{c^2}\frac{\partial^2\overline{\Psi}}{\partial t^2} \qquad (1\text{-}1)$$

where c is the velocity of propagation, and is a function only of the depth z. Note that the wave potential $\overline{\Psi}$ is assumed independent of the polar angle. Assuming that the time variation of $\overline{\Psi}$ is sinusoidal with frequency ω, so that $\bar{\psi} = \psi e^{j\omega t}$, we have

$$\nabla_{r,z}^{2}\psi + k^{2}\psi = 0 \tag{1-2}$$

where

$$k = \frac{\omega}{c} \quad , \tag{1-3}$$

and where ψ is a function only of r and z. By separating variables in the usual way the following types of solutions are obtained:

$$\psi(r,z) = H_0^{(1)}(\gamma r)\, U(z) \tag{1-4}$$

where U(z) satisfies the equation

$$\frac{d^2U}{dz^2} + (k^2 - \gamma^2)\, U = 0 \tag{1-5}$$

Here γ is an eigenvalue to be determined by the boundary conditions on U(z). These are

$$U(0) = 0 \tag{1-6}$$

and

$$\lim_{z \to \infty} U(z) \rightarrow \text{outgoing wave} \tag{1-7}$$

That is, the wave potential vanishes at the surface and corresponds to an outgoing wave at infinity.

Next we assume a linear variation of velocity with depth. Thus

$$c = c_0\,(1+bz) \tag{1-8}$$

where c_0 is the velocity at the surface and where b is a constant. Assuming that $bz \ll 1$

$$k^2 = \frac{\omega^2}{c^2} \cong \frac{\omega^2}{c_0^2}(1-2bz) = k_0^2\,(1-2bz) \tag{1-9}$$

We are interested in the case where $b = b_1$ for $z < z_0$, and $b = b_2$ for $z > z_0$. If we define a dimensionless depth variable ξ by

$$\xi = \frac{z}{z_0} \tag{1-10}$$

equation (1-5) becomes

$$\frac{d^2U(\xi)}{d\xi^2} + z_0^2 \left[k_0^2 \left(1 - 2b_1 z_0 \xi\right) - \gamma^2 \right] U(\xi) = 0 \qquad 0 \leq \xi < 1 \tag{1-11}$$

and

$$\frac{d^2U(\xi)}{d\xi^2} + z_0^2 \left[k_0^2 \left(1 - 2b_2 z_0 \xi + 2b_2 z_0 - 2b_1 z_0\right) - \gamma^2 \right] U(\xi) = 0 \qquad \xi > 1 \tag{1-12}$$

Note that equations (1-11) and (1-12) are written so that there is no discontinuity in the velocity at $\xi = 1$.

Equations (1-11) and (1-12) can be rewritten as

$$\frac{d^2U}{d\xi^2} + 2z_0^3 k_0^2 b_2 \left[\frac{1}{2b_2 z_0} - \frac{b_1}{b_2}\xi - \frac{\gamma^2}{2z_0 k_0^2 b_2} \right] U(\xi) = 0 \qquad 0 \leq \xi < 1 \tag{1-13}$$

and

$$\frac{d^2U}{d\xi^2} + 2z_0^3 k_0^2 b_2 \left[1 + \frac{1}{2b_2 z_0} - \frac{b_1}{b_2} - \xi - \frac{\gamma^2}{2z_0 k_0^2 b_2} \right] U(\xi) = 0 \qquad \xi > 1 \tag{1-14}$$

Let us now define the following parameters.

$$a = 1 - \frac{b_1}{b_2} \tag{1-15}$$

$$s^3 = -2z_0^3\, k_0^2\, b_2$$

$$Y = -\frac{k_0^2 - \gamma^2}{2z_0 k_0^2 b_2} \qquad (1\text{-}17)$$

Equations (1-13) and (1-14) finally become

$$\frac{d^2U}{d\xi^2} + s^3\left[f(\xi) + Y\right] U(\xi) = 0 \qquad \xi \geq 0 \qquad (1\text{-}18)$$

where

$$f(\xi) = (1-a)\xi \qquad 0 \leq \xi < 1 \qquad (1\text{-}19)$$

$$= \xi - a \qquad \xi > 1$$

These equations are subject to the end conditions $U(0) = 0$ and $U(\infty) \rightarrow$ outgoing wave. Since normally $b_2 < 0$, we see that s is a positive real parameter. For the reverse gradient case, $b_1 > 0$ and hence a is a positive real parameter greater than 1. The independent variable ξ is real, but in general the wave potential $U(\xi)$ and the eigenvalue Y will be complex.

The boundary conditions can be met only for certain discrete values of Y. We denote these eigenvalues by Y_m ($m = 1,2,\ldots$), where Y_1 is the smallest allowable value of Y, Y_2 is the next smallest Y, etc. The wave potentials associated with each Y_m are called normal modes or eigenfunctions $U_m(\xi)$. The problem is to find Y_m and $U_m(\xi)$ for the lowest modes (we will consider the first three modes in this report).

1.3 Method of Solution

For $\xi > 1$ equation (1-18) becomes

$$\frac{d^2U_m}{d\xi^2} + s^3\left[\xi - a + Y_m\right] U_m = 0 \qquad (1\text{-}20)$$

If we let

$$p = s\,(\xi - a + Y_m) \quad , \qquad (1\text{-}21)$$

equation (1-20) becomes

$$\frac{d^2U_m}{dp^2} + pU_m = 0 \qquad (1\text{-}22)$$

which is known as Stokes' equation. The solutions to this equation are modified Hankel functions of order 1/3. These functions (there are two types, $h_1(p)$ and $h_2(p)$) are tabulated.[2] It turns out that the function $h_2(p)$ satisfies the boundary condition at infinity, namely that $U_m(p)$ correspond to an outgoing wave. Thus in terms of our original variable ξ we have as a solution

$$U_m(\xi) = h_2\left\{s\,(\xi - a + Y_m)\right\} \qquad \xi > 1 \qquad (1\text{-}23)$$

where the solution is valid only for $\xi > 1$ and is subject to a prior knowledge of Y_m.

Thus the problem becomes one of patching onto the solution (1-23) a solution valid in the region $0 \leq \xi < 1$ and at the same time selecting the proper eigenvalue Y_m so that this second solution vanishes at $\xi = 0$. The differential analyzer will, therefore, be utilized to solve the equation

$$\frac{d^2U_m}{d\xi^2} + s^3\left[(1-a) + Y_m\right]U_m = 0 \qquad (1\text{-}24)$$

subject to the end conditions

$$U_m(0) = 0 \qquad (1\text{-}25)$$

$$U_m(1) = h_2\left\{s(1-a + Y_m)\right\} \qquad (1\text{-}26)$$

$$\frac{1}{s}\,\frac{dU_m}{d\xi}(1) = h_2'\left\{s(1-a + Y_m)\right\} \qquad (1\text{-}27)$$

The method of attack is to assume a trial Y_m, find U_m (1) and $\frac{1}{s} \frac{dU_m}{d\zeta}$ (1) from the Harvard tables, and with these starting conditions at $\zeta = 1$, integrate toward $\zeta = 0$. In general the resulting U (0) will not be zero. It is then necessary to assume new trial values of Y_m and redetermine U (0) in each case. In this way we can interpolate to the Y_m which yields a solution for which U (0) = 0, as required.

1.4 Review of the Status of the Mathematical Theory

by C. L. Dolph

Although considerable time and effort has been expended on various aspects of the complex eigenvalue problem encountered in propagation theory, nothing like a satisfactory mathematical theory has yet been devised. Since there are a number of sources such as Sommerfeld (3), Kerr (4), Marsh (1), and Friedman (5) which develop this problem from the physical situation, this report will be limited to a few observations concerning it.

In the course of examining the theory of anomalous propagation, an attempt was made to understand somewhat more clearly why the variational process introduced by MacFarlane (6) was capable of giving correct results. An examination of the curves of Ament and Pekeris (7) who also used this same formal process led to the observation that the imaginary part of the eigenvalues, if different from zero, was of constant sign. Subsequent investigation showed that the paper of Hartree (8) contained an argument, which, by a slight reinterpretation, made it apparent that this must necessarily be the case. Although the usual Rayleigh quotient leads to a saddle point, the definiteness of the imaginary part of the eigenvalues has the possibility of leading to a one sided estimate for the imaginary part of the eigenvalues and a min-max principle as in the usual positive-definite real case where a minimum is involved. A corresponding estimate for the real part does not appear possible. Moreover, it was shown that the usual theorems concerning the reduction of a real symmetric or hermitian quadratic form to the sum of squares by means of real orthogonal transformations were capable of generalization to this case. Here the coefficients of the form are symmetric but complex-valued, and the complex sum of squares remains invariant provided that the matrix of the coefficients has a minimum function which possesses simple roots. It should be noted that this condition on the roots can be shown to be automatically satisfied in the real symmetric or hermitian cases. It is interesting to note in the finite dimensional complex case that the proof of the above spectral theorem makes use of the fact that an orthogonal basis can be constructed. The vectors in it are orthogonal to each other in that the sum of their pairwise complex products vanishes so that a vector may be orthogonal to itself. The fact that such an orthogonal basis can be chosen in an infinite dimensional space has been established by J. McLaughlin at the University of Michigan. This result has apparently been obtained previously and independently by Kaplansky (9).

The existence of this spectral theorem naturally led to consideration of possible infinite dimensional generalizations in a sequence space having an inner product of the above form. Such a generalization as well as a theory of operators in such a space appears necessary before the calculus of variations method can be considered rigorously established. A basic difficulty occurs at once in such a generalization in that the space of vectors with the property that the sum of the squares of their complex components is finite does not form a linear space. It is easy to see that the sum of two vectors of this space may not lie in this space. Although the subset of this space consisting of all vectors whose components are zero except for a finite number at the beginning does form an infinite dimensional linear sub-space, it appears to be a difficult matter to complete this space in the proper way. Work in this direction is still continuing but little progress is expected until the right completion has been found. The difficulty can perhaps be summarized by remarking that these considerations apparently lead to a conditionally convergent situation rather than the more usually treated one of absolute convergence. As has been suggested by J. Dieudonne, one possible way out of the difficulty might be to employ the spaces of Kothe (10). Here the basic approach would be to start with the denumerable set of eigenfunctions and build up the largest complete space. That there are possibilities in this direction for the conditionally convergent situation has already been indicated by Kothe.

Many researchers (5,11) believe that the occurence of complex orthogonality in these problems can be best understood from the viewpoint of a hermitian inner product and the introduction of an adjoint differential problem. If this is properly done, the complex orthogonality can be shown to result from the bi-orthogonal relation that exists between the given problem and its adjoint. Friedman and his student, M. Kotik, have obtained some interesting unpublished results in this direction for a special class of problems; in fact they have been able to deduce a point-wise convergence theorem for some twice differentiable functions. Their results are also interesting in that they have investigated a case of isotropic eigenfunctions in some detail and have made a start toward an elementary divisor type of theory. R. S. Phillips (11), working in the usual Hilbert space framework, has given a discussion of second-order differential equations subject to complex-homogeneous linear boundary conditions. In this he has shown by use of Weyl's notion of the limit point and limit circle that in some cases the operator may be essential real while in others its spectrum may not be contained in any strip of one half of the complex eigenvalue plane. R. Phillips has a student continuing this work but he has reported that no progress has yet been made toward an expansion theorem. The basic tool in all of these investigations appears to be that of the resolvent. This can be constructed, formally at least, if all the eigenvalues can be confined to one-half of the complex plane. The use of the calculus of residues and Cauchy's integral formula are then available provided that the necessary estimates can be made. These have been successfully accomplished by Titchmarch (12) in the real case and work is continuing on the problem of finding appropriate estimates for this problem.

From the above it is apparent that considerable doubt exists as to the proper framework for the complex eigenvalue problem even though it dates back to the work performed by Watson in 1910. Thus, no one knows for sure whether it would be better to view problems of this sort as non-hermitian, non-normal operators in the usual Hilbert space framework or whether it would be better to treat them as symmetric operators in a space with a symmetric complex valued inner product. It is also not known whether the difficulties encountered in the evaluation of the norm of the eigenfunctions of ref. 7 in the plane-earth approximation are basic or whether they are another manifestation of the anamalous behavior of the wave equation in two-dimensions. In any event there are many unanswered questions concerning the whole problem. In view of the quite wide spread interest and importance of problems of this type it is to be hoped that some answers will soon be forthcoming. The present writer has had the keen interest and help of his colleagues, Dr. I. Marx and Dr. J. McLaughlin and present plans call for our continued collaboration on these problems.

CHAPTER 2
APPROXIMATE NORMAL-MODE SOLUTION USING ASYMPTOTIC FORMS

2.1 Solution for the Linear Gradient

By use of the asymptotic forms of the modified Hankel-function solutions, we can get approximation formulas for the eigenvalues Y_m for the bilinear gradient problem. These approximate eigenvalues give an excellent point of departure for the differential analyzer solutions.

Let us consider first the simplified case of a linear velocity gradient (a=0). The depth-dependent equation becomes from (1-18) and (1-19)

$$\frac{d^2U(\xi)}{d\xi^2} + s^3\left[\xi + Y\right]U(\xi) = 0 \tag{2-1}$$

with end conditions U(0) = 0 and $U(\infty) \to$ outgoing wave. The solutions of equation (2-1) can be rewritten as

$$U(\xi) = h_2\left\{s(\xi + Y)\right\} \tag{2-2}$$

where h_2 is the modified Hankel function of the second kind and satisfies the outgoing-wave boundary condition at $\xi = \infty$. To determine the eigenvalue Y it is necessary to impose the boundary condition U(0) = 0.

The $h_2\left\{s(\xi + Y_m)\right\}$ function given in equation (2-2) can be represented approximately by the first term of the asymptotic expansions for h_2.[2] Since the imaginary part of the eigenvalue Y_m is known to be positive,[7,8] the argument s $(\xi + Y_m)$ always lies in quadrants 1 and 2. The asymptotic expression valid in this region is

$$h_2(p) \cong \alpha\, p^{-\frac{1}{4}}\left[e^{-\frac{2}{3}ip^{3/2} + \frac{5\pi i}{12}} + e^{\frac{2}{3}ip^{3/2} + \frac{11\pi i}{12}}\right] \tag{2-3}$$

or

$$U(\xi) \cong \alpha\left\{s(\xi + Y)\right\} e^{\frac{2}{3}i\left\{s(\xi+Y)\right\}^{3/2} - \frac{\pi i}{12}}\left[e^{-\frac{4}{3}i\left\{s(\xi+Y)\right\}^{3/2} + \frac{\pi i}{2}} - 1\right] \tag{2-4}$$

The boundary condition U(0) = 0 is met when

$$- \frac{4}{3} (sY)^{3/2} + \frac{\pi}{2} - 2m\pi = 0, \; m=0,\pm1,\pm2,\ldots \tag{2-5}$$

The equation for the eigenvalues Y_m is then

$$Y_m \cong \frac{1}{s} \left[\frac{3\pi}{2}(m-\frac{1}{4})\right]^{\frac{2}{3}} e^{\frac{2}{3}\pi i}, \; m=1,2,3,\ldots \tag{2-6}$$

where we have discarded m=0, and negative m values, since the imaginary part of Y_m is always positive. The accuracy of formula (2-6) can be seen in the following table, which compares the eigenvalues Y_m gotten from equation (2-6) with the exact values obtained by interpolation from the Harvard Tables.

Comparison of Eigenvalues Obtained from Approximation Formula Equation (2-6) with Exact Values for Linear Gradient (a=0).

Mode	sY_m from Equ.(2-6)	Exact Value of sY_m
1	-1.16+i2.01	-1.170+i2.025
2	-2.04+i3.53	-2.044+i3.540
3	-2.76+i4.78	-2.761+i4.781

Evidentally the eigenvalues obtained from the first terms of the asymptotic expressions are quite accurate for the case of a linear velocity gradient.

2.2 Solution for the Bilinear Gradient

Consider next the case of the bilinear velocity gradient. From equations (1-18) and (1-19)

$$\frac{d^2U}{d\xi^2} + s^3\left[(1-a)\xi + Y\right] U=0 \qquad 0 \leq \xi \leq 1 \tag{2-7}$$

and

$$\frac{d^2U}{d\xi^2} + s^3\left[\xi - a + Y\right] U=0 \qquad 1 \leq \xi \tag{2-8}$$

with the end conditions $U(0) = 0$ and $U(\infty) \longrightarrow$ outgoing wave. The solutions can be written as

$$U(\xi) = Ah_1\left\{s(1-a)^{-2/3}\left[(1-a)\xi + Y\right]\right\} + Bh_2\left\{s(1-a)^{-2/3}\left[(1-a)\xi + Y\right]\right\} \qquad 0 \leq \xi \leq 1 \qquad (2\text{-}9)$$

and

$$U(\xi) = Ch_2\left\{s(\xi - a + Y)\right\} \qquad 1 \leq \xi \qquad (2\text{-}10)$$

Note that for $\xi > 1$ only the h_2 function is included, since it satisfies the outgoing-wave boundary condition at $\xi = \infty$.

The modified Hankel functions given in equations (2-9) and (2-10) can be represented approximately by the first terms of their asymptotic expansions.[2] Again since the imaginary part of Y is always positive, it is necessary to use the expansions for h_1 and h_2 valid in the first and second quadrant. Taking only the first terms of those expansions, we have

$$h_1(p) \cong p^{\frac{1}{4}} e^{-\frac{2}{3} ip^{3/2} - \frac{5\pi}{12} i} \qquad (2\text{-}11)$$

and equation (2-3) for $h_2(p)$. Thus

$$U(\xi) \cong A'\left[\xi(1-a) + Y\right]^{-1/4} e^{-\frac{2}{3}i \frac{s^{3/2}}{1-a}\left[(1-a)\xi + Y\right]^{3/2}} + B' e^{\frac{2}{3}i\frac{s^{3/2}}{1-a}\left[(1-a)\xi + Y\right]^{3/2}} \qquad 0 \leq \xi \leq 1 \qquad (2\text{-}12)$$

where the phase constants in the exponents have been included in the constants A' and B'. Differentiating equation (2-12) and neglecting the contribution of $\frac{d}{d\xi}\left[\xi(1-a) + Y\right]^{-1/4}$, we have

$$\frac{dU(\xi)}{d\xi} \cong is^{3/2} A'\left[\xi(1-a) + Y\right]^{1/4}\left\{-e^{-\frac{2}{3}i\frac{s^{3/2}}{1-a}\left[(1-a)\xi + Y\right]^{3/2}} + B' e^{\frac{2}{3}i\frac{s^{3/2}}{1-a}\left[(1-a)\xi + Y\right]^{3/2}}\right\} \qquad 0 \leq \xi \leq 1 \qquad (2\text{-}13)$$

In the region $\xi \geq 1$, the approximate solution is

$$U(\xi) \cong C'(\xi-a+Y)^{-1/4}\left[e^{-\frac{2}{3}is^{3/2}(\xi-a+Y)^{3/2}} + ie^{\frac{2}{3}is^{3/2}(\xi-a+Y)^{3/2}}\right] \qquad \xi \geq 1 \tag{2-14}$$

If we take $\frac{dU(\xi)}{d\xi}$ and neglect $\frac{d}{d\xi}(\xi-a+Y)^{-1/4}$ as before, it follows that

$$\frac{dU(\xi)}{d\xi} \cong C'is^{3/2}(\xi-a+Y)^{-1/4}\left[-e^{-\frac{2}{3}is^{3/2}(\xi-a+Y)^{3/2}} + ie^{\frac{2}{3}is^{3/2}(\xi-a+Y)^{3/2}}\right] \qquad \xi \geq 1 \tag{2-15}$$

We must now match at $\xi = 1$ the solutions for $U(\xi)$ and $U'(\xi)$ for $0 \leq \xi \leq 1$ with the solutions for $U(\xi)$ and $U'(\xi)$ for $\xi \geq 1$. Since our task here is to find the eigenvalues Y_m for which $U(0)$ vanishes, we are not concerned with evaluating the constant A' which sets the magnitude of the solution in the region $0 \leq \xi \leq 1$, but only with evaluating the constant B' in equation (2-12). The constant B' determines the ratio of $U'(1)/U(1)$ from equations (2-12) and (2-13). This ratio must equal the $U'(1)/U(1)$ given by equations (2-14) and (2-15), which is

$$\frac{U'(1)}{U(1)} = \frac{is^{3/2}\left[ie^{\frac{4}{3}is^{3/2}(1-a+Y)^{3/2}} - 1\right]}{\left[ie^{\frac{4}{3}is^{3/2}(1-a+Y)^{3/2}} + 1\right]} \tag{2-16}$$

On the other hand, from equations (2-12) and 2-13)

$$\frac{U'(1)}{U(1)} = \frac{is^{3/2}\left[B'e^{\frac{4}{3}i\frac{as^{3/2}}{1-a}(1-a+Y)^{3/2}}\, e^{\frac{4}{3}is^{3/2}(1-a+Y)^{3/2}} - 1\right]}{\left[B'e^{\frac{4}{3}i\frac{as^{3/2}}{1-a}(1-a+Y)^{3/2}}\, e^{\frac{4}{3}is^{3/2}(1-a+Y)^{3/2}} + 1\right]} \tag{2-17}$$

After comparison of equations (2-16) and (2-17) it is obvious that

$$B' = ie^{-\frac{4}{3}i\frac{as^{3/2}}{1-a}(1-a+Y)^{3/2}} \tag{2-18}$$

which, substituted in equation (2-12), gives

$$U(0) \cong A'Y^{-\frac{1}{4}}\left\{ e^{-\frac{2}{3}i\,\frac{s^{3/2}\,Y^{3/2}}{1-a}} + i e^{-\frac{4}{3}i\,\frac{as^{3/2}}{1-a}(1-a+Y)^{3/2}}\, e^{\frac{2}{3}i\,\frac{s^{3/2}\,Y^{3/2}}{1-a}} \right\}$$

$$= A'Y^{-\frac{1}{4}}\, e^{-\frac{2}{3}i\,\frac{s^{3/2}\,Y^{3/2}}{1-a}} \left\{ 1-e^{-\frac{4}{3}i\,\frac{as^{3/2}}{1-a}(1-a+Y)^{3/2} + \frac{4}{3}i\,\frac{s^{3/2}\,Y^{3/2}}{1-a} - \frac{\pi i}{2}} \right\} \tag{2-19}$$

From the boundary condition U(0)=0 it follows that

$$-\frac{4i}{3}\,\frac{as^{3/2}}{1-a}(1-a+Y)^{3/2} + \frac{4i}{3}\,\frac{s^{3/2}\,Y^{3/2}}{1-a} - \frac{\pi i}{2} + 2m\pi i = 0 \quad m=0,\pm1,\pm2,\ldots,$$

and

$$Y^{3/2} - a(1-a+Y)^{3/2} + \frac{3\pi}{2}\,\frac{(a-1)}{s^{3/2}}\left(\frac{1}{4} - m\right) = 0 \tag{2-20}$$

For the linear velocity gradient(a=0), equation (2-20) reduces to

$$Y = \frac{1}{s}\left[\frac{3\pi}{2}\left(m-\frac{1}{4}\right)\right]^{2/3} e^{\frac{2\pi}{3}i} \tag{2-6}$$

which agrees with our previous result for a = 0, from which we conclude that m=1,2,3,... In equation, (2-20), then, m=1 yields the eigenvalue Y_1 corresponding to the first mode, m=2 yields Y_2 for the second mode, etc.

Equation (2-20) is only an approximation, and its usefulness should not be overemphasized, since it can be solved for Y_m only by trial and error, in any case. However, it leads to an important simplified formula for Y_m when $Y_m \gg 1-a$. In this case we can write,

$$(Y_m+1-a)^{3/2} \cong Y_m^{3/2} + \frac{3}{2} Y_m^{1/2}(1-a) \tag{2-21}$$

Substituting this equation into (2-20), we obtain

$$Y_m^{3/2} - \frac{3}{2} a Y_m^{1/2} \cong \frac{3\pi}{2s^{3/2}} \left[m - \frac{1}{4}\right] e^{\pi i} \tag{2-22}$$

Taking the 2/3 power of both sides of equation (2-22), we find that

$$\left(Y_m^{3/2} - \frac{3}{2} a Y_m^{1/2}\right)^{2/3} \cong Y_m - a \cong \frac{1}{s} \left[\frac{3\pi}{2}\left(m - \frac{1}{4}\right)\right]^{2/3} e^{\frac{2}{3}\pi i} \tag{2-23}$$

or $$Y_m \cong \frac{1}{s} \left[\frac{3\pi}{2}\left(m - \frac{1}{4}\right)\right]^{2/3} e^{\frac{2}{3}\pi i} + a, \quad Y \gg a, \; Y \gg 1. \tag{2-24}$$

Thus for large eigenvalues (as is the case when $s \ll 1$ or when the mode number m is large), equation (2-24) gives a very simple approximate formula for the eigenvalue. To illustrate this, the following table compares the eigenvalues Y_m obtained from equation (2-24) with the exact values (subject, of course, to computer errors) obtained with the differential analyzer.

<u>Comparison of Eigenvalues Obtained from Approximation Formula (Equ. 2-24) with Computer Values.</u>

	Mode	Equ.(2-24)	Differential Analyzer		Mode	Equ.(2-24)	Differential Analyzer
s=0.5	1	-1.22+i4.02	-1.23+i4.05	s=0.5	1	-0.32+i4.02	-0.32+i4.05
a=1.1	2	-2.98+i7.06	-2.98+i7.08	a=2.0	2	-2.08+i7.06	-2.06+i7.08
	3	-4.42+i9.56	-4.41+i9.56		3	-3.52+i9.56	-3.50+i9.56
s=1.0	1	-0.06+i2.01	0.03+i1.98	s=1.0	1	+0.84+i2.01	1.01+i1.91
a=1.1	2	-0.94+i3.53	-0.86+i3.45	a=2.0	2	-0.04+i3.53	0.13+i3.40
	3	-1.66+i4.78	-1.57+i4.68		3	-0.76+i4.78	-0.65+i4.57

Evidentally for $s \leq 0.5$, equation (2-24) is an excellent approximation to Y_m.

CHAPTER 3
PRINCIPLES OF OPERATION
OF THE ELECTRONIC DIFFERENTIAL ANALYZER

3.1 Introduction to Operational Amplifiers

The basic component of the electronic differential analyzer is the operational amplifier, which is shown schematically in Figure 3-1. It consists of a dc voltage amplifier of high gain, an input impedance Z_i, and a feedback impedance Z_f.

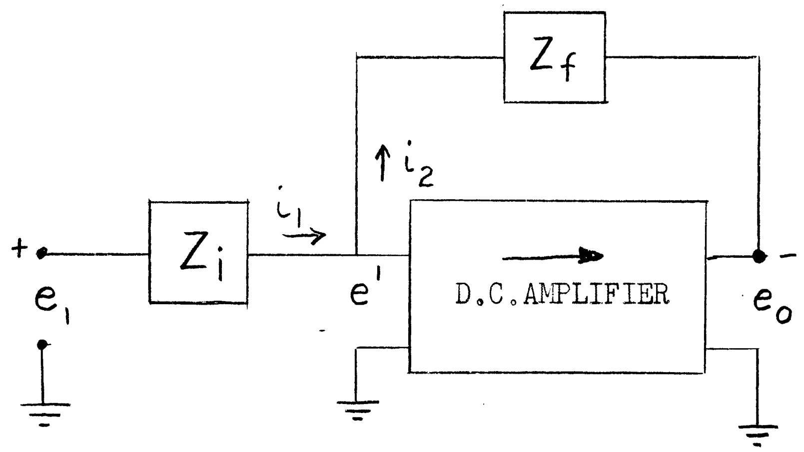


Figure 3-1. Operational Amplifier.

If we neglect the current into the dc amplifier itself (i.e., neglect the current to the grid of the input tube), it follows that $i_1 = i_2$. Let us also neglect the voltage input e' to the dc amplifier in comparison with the output voltage e_2 or the input voltage e_1 to the operational amplifier. We then have

$$i_1 = i_2$$

or

$$\frac{e_1}{Z_i} = -\frac{e_2}{Z_f}$$

from which

$$e_2 = -\frac{Z_f}{Z_i} e_1 , \qquad (3\text{-}1)$$

which is the fundamental equation governing the behavior of the operational amplifier. In general Z_f/Z_i is made the order of magnitude of unity. We shall now consider the scheme by which the operational amplifier can be used to perform three different functions:

(a) Multiplication by a constant.

If we wish to multiply a certain voltage e_1 by a constant factor k, we need only make $Z_f/Z_i = k$. From equation (3-1), then, the output voltage e_2 of the operational amplifier will be given by

$$e_2 = - k\, e_1. \tag{3-2}$$

Thus the required multiplication by a constant has been achieved, except for a reversal of sign. For example, if we wish k to be 10, we may let Z_i = 1 megohm resistance, Z_f = 10 megohms resistance. If we also desire the sign of e_2 to be the same as e_1, we must feed e_2 through an additional operational amplifier with $Z_i = Z_f$ = 1 megohm. This second operational amplifier merely acts as a sign changer by multiplying any voltage by -1.

(b) Addition.

In order to add a number of voltages, say e_a, e_b, and e_c, the arrangement shown in Figure 3-2 is used. Hence $i_a + i_b + i_c = i_2$, and if we

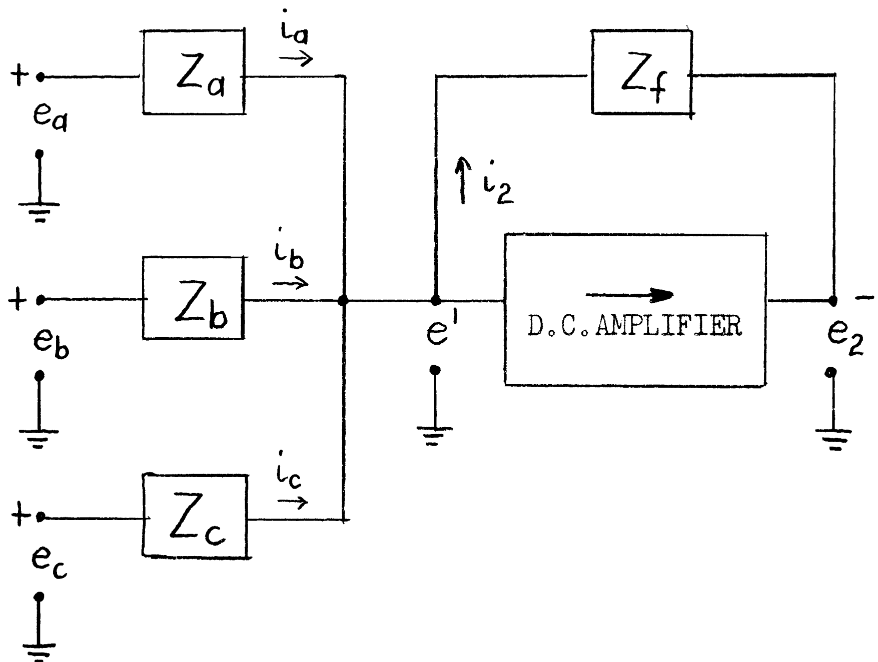


Figure 3-2. Operational Amplifier Used for Summation.

neglect e' as small compared with e_2, we have

$$\frac{e_a}{Z_a} + \frac{e_b}{Z_b} + \frac{e_c}{Z_c} = -\frac{e_2}{Z_f}$$

or

$$e_2 = -\left\{\frac{Z_f}{Z_a} e_a + \frac{Z_f}{Z_b} e_b + \frac{Z_f}{Z_c} e_c\right\} . \qquad (3\text{-}3)$$

Thus the output voltage e_2 is the sum of the three input voltages, each multiplied respectively by a constant $-\frac{Z_f}{Z_n}$ (n = a,b, or c). The operational amplifier can, of course, be used in general to sum any number of input voltages.

(c) Integration.

If we make the input impedance Z_1 a resistor and the feedback impedance Z_f a capacitor, then the operational amplifier serves as an integrator. Referring to Figure 3-3, we see that if we neglect e' and let $i_1 = i_2$ as before, we have

$$e_2 = -\int \frac{i_1 dt}{C} \text{ and } i_1 = \frac{e_1}{R} \quad ,$$

from which

$$e_2 = -\frac{1}{RC}\int e_1 dt \quad . \qquad (3\text{-}4)$$

The output voltage e_2 is then the integral with respect to time of the input voltage e_1 (multiplied by a constant factor $-\frac{1}{RC}$).

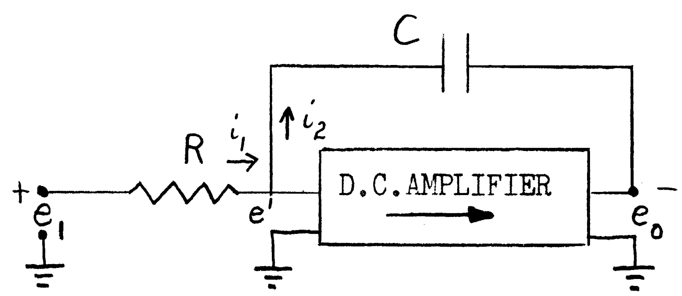


Figure 3-3. Operational Amplifier as an Integrater

3.2 Solution of an Ordinary Differential Equation with Constant Coefficients

In order to demonstrate how operational amplifiers performing the above three functions can be combined to solve ordinary linear differential equations, we will now set up the amplifier circuits required to solve the following differential equation:

$$a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_o x = 0 \qquad (3\text{-}5)$$

subject to the initial conditions

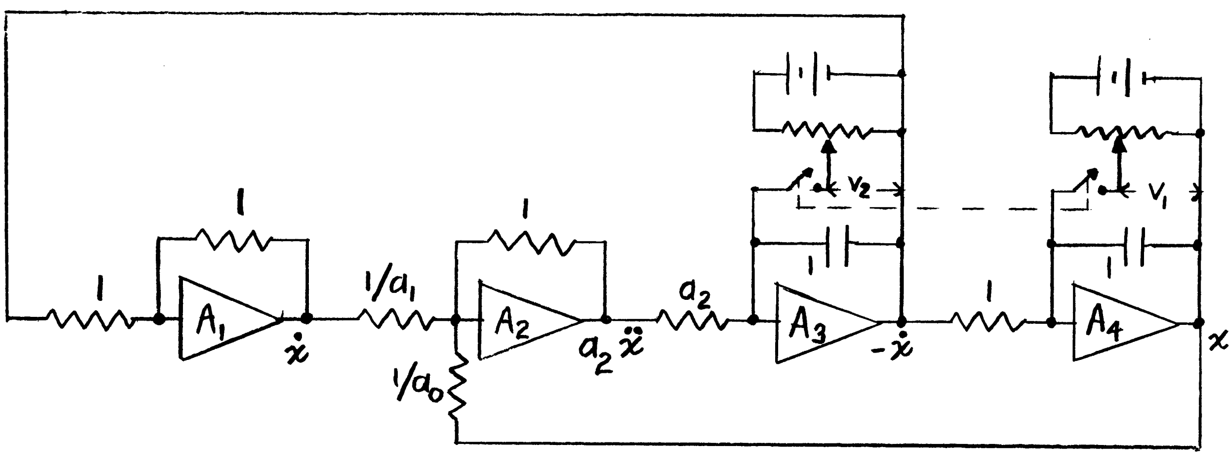
$$x(0) = V_1$$

and

$$\frac{dx}{dt}(0) = V_2 \quad . \qquad (3\text{-}6)$$

The constants a_2, a_1, and a_o are assumed positive. Since the electronic differential analyzer integrates with respect to time, the independent variable t in equation (3-5) above will be time. The dependent variable x is represented by voltage.

The computer circuit for solving equation (3-5) subject to initial conditions (3-6) is shown schematically in Figure 3-4.



All Resistor Units are Megohms
All Capacitors are 1Mfd.
Ground Connections are Omitted for Clarity

Figure 3-4. Computer Circuit for Solving $a_2\ddot{x} + a_1\dot{x} + a_o x = 0$.

If we assume that the output of amplifier A_2 is $a_2\ddot{x}$, then this voltage gets multiplied by $-\frac{1}{a_2}$ and integrated once in passing through amplifier A_3, the output of which is therefore $-\dot{x}$. This voltage is multiplied by -1 and integrated once to give x as the output of A_4. In order to obtain $+\dot{x}$ instead of $-\dot{x}$ it is necessary to pass $-\dot{x}$ through sign-reversing amplifier A_1. $\dot{x}$ and x are then multiplied by $-a_1$ and $-a_o$ respectively and summed in amplifier A_2, the output of which is now $-a_1\dot{x} - a_o x$. But we originally assumed the output of A_2 to be $a_2\ddot{x}$. Hence, $a_2\ddot{x} = - a_1\dot{x} - a_o x$, which is just the equation which we wish to solve.

The initial conditions (3-6) are imposed as voltages impressed across the integrating condensors of A_3 and A_4 in Figure (3-4). When the two switches holding the initial voltages across the condensors are simultaneously opened, the solution of the problem as a function of time begins, i.e., the voltage output of A_4 represents x(t).

3.3 Solution of Differential Equations with Variable Coefficients

Suppose the coefficients a_2, a_1, and a_o in equation (3-5), instead of being constant, are functions of the independent variable t. Then it is apparent that the resistors marked a_2, $1/a_1$, and $1/a_o$ in Figure 3-4 must vary as a function of time. If we can accomplish this, then the differential analyzer can solve the more general problem of ordinary differential equations with coefficients which are functions of the independent variable [for example, the bilinear gradient equation (1-18)].

It is often more convenient to vary a resistance with time in discrete steps instead of continuously.[14,15] As an example, a resistance varying linearly with time can be approximated by the staircase function shown in Figure 3-5. The staircase function is arranged so that at the end of each time interval Δt the area under the stepped curve is equal to that under the linear curve. The accuracy attainable by using such a crude approximation is surprisingly good, even when the time increments Δt are made relatively large.[14] It is this step-method of approximating variable coefficients which is used in obtaining the differential analyzer solutions to the bilinear gradient problem. For a complete description of the circuitry involved, the reader is referred to other reports.[14,15,16]

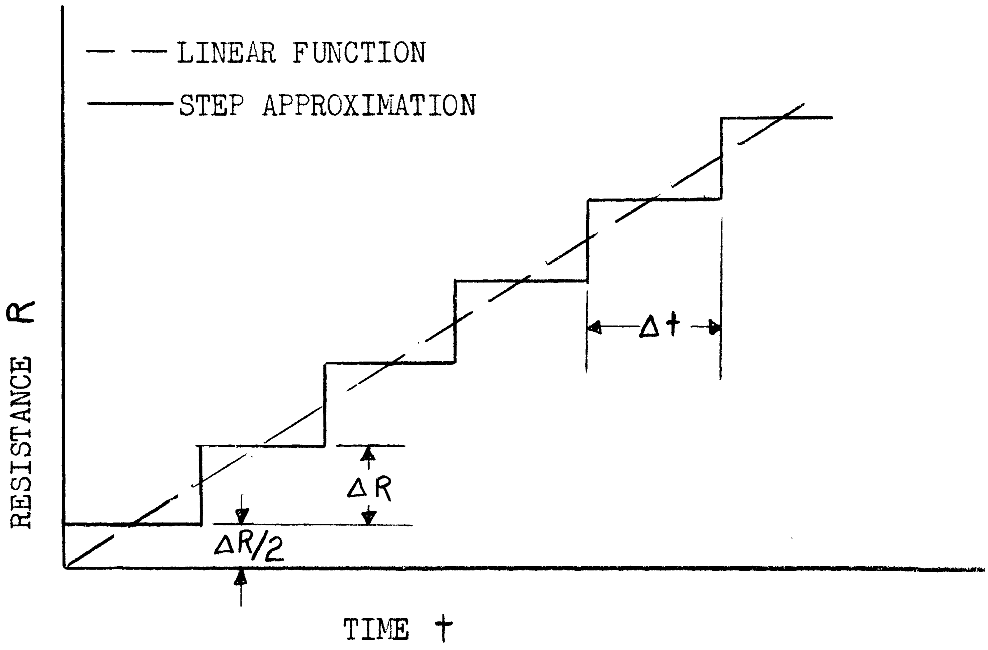


Figure 3-5. Step-Method of Approximating a Linear Function.

CHAPTER 4
SOLUTION OF THE BILINEAR GRADIENT PROBLEM BY THE ELECTRONIC DIFFERENTIAL ANALYZER

4.1 Transformation of the Equation into Computer Units

In the bilinear gradient differential equation (1-18) the independent variable ξ starts with the value 1 and runs to the value 0. Actually, the computer independent variable, which is time t, will start at t = 0 and run to t = 1, where one computer time unit is now the length (or duration) of the solution. Thus we make the following change of independent variable

$$\xi = 1 - t \qquad (4\text{-}1)$$

and

$$\frac{d}{d\xi} = \frac{-d}{dt} , \quad \frac{d^2}{d\xi^2} = \frac{d^2}{dt^2} . \qquad (4\text{-}2)$$

Equation (1-1) remains

$$\frac{d^2U(t)}{dt^2} + s^3 \left[f(t) + Y \right] U(t) = 0 \qquad (4\text{-}2a)$$

but now

$$f(t) = (1-a)\ (1-t), \quad 0 \leq t \leq 1. \qquad (4\text{-}3)$$

4.2 Separation into Real and Imaginary Parts

We have already pointed out that in general the wave potential U is complex, as is the eigenvalue Y. To solve the problem with the differential analyzer it is necessary to break the complex functions into real and imaginary parts. Thus we let

$$U = U_r + iU_i \tag{4-4}$$

$$Y = Y_r + iY_i \tag{4-5}$$

After substituting (4-4) and (4-5) into equation (4-2a) and equating real and imaginary parts to zero, we find that

$$\frac{1}{s^3}\frac{d^2 U_r}{dt^2} + f(t)U_r + Y_r U_r - Y_i U_i = 0 \tag{4-6}$$

and

$$\frac{1}{s^3}\frac{d^2 U_i}{dt^2} + f(t)U_i + Y_r U_i + Y_i U_r = 0 . \tag{4-7}$$

Initial conditions for the above equations become from (1-26), (1-27), and (4-1)

$$\begin{aligned} U_r(0) &= R\left[h_2\left\{s(1-a+Y)\right\}\right] \\ U_i(0) &= I\left[h_2\left\{s(1-a+Y)\right\}\right] \\ -\frac{1}{s}U'_r(0) &= R\left[h'_2\left\{s(1-a+Y)\right\}\right] \\ -\frac{1}{s}U'_i(0) &= I\left[h'_2\left\{s(1-a+Y)\right\}\right] . \end{aligned} \tag{4-8}$$

4.3 Computer Circuit for Solving the Bilinear-Gradient Problem

The electronic differential analyzer circuit used to solve equations (4-6) and (4-7) for the case when $a = 0$ (linear gradient) is shown in Figure 4-1. Note in this case that $Y_r < 0$ and $Y_i > 0$. The circuit required for $a > 1$ (bilinear gradient), $Y_r > 0$, $Y_i > 0$ is shown in Figrre 4-2. If $Y_r < 0$, it is only necessary to restore the $R/s^2 Y_r$ input to the positions shown in Figure 4-1.

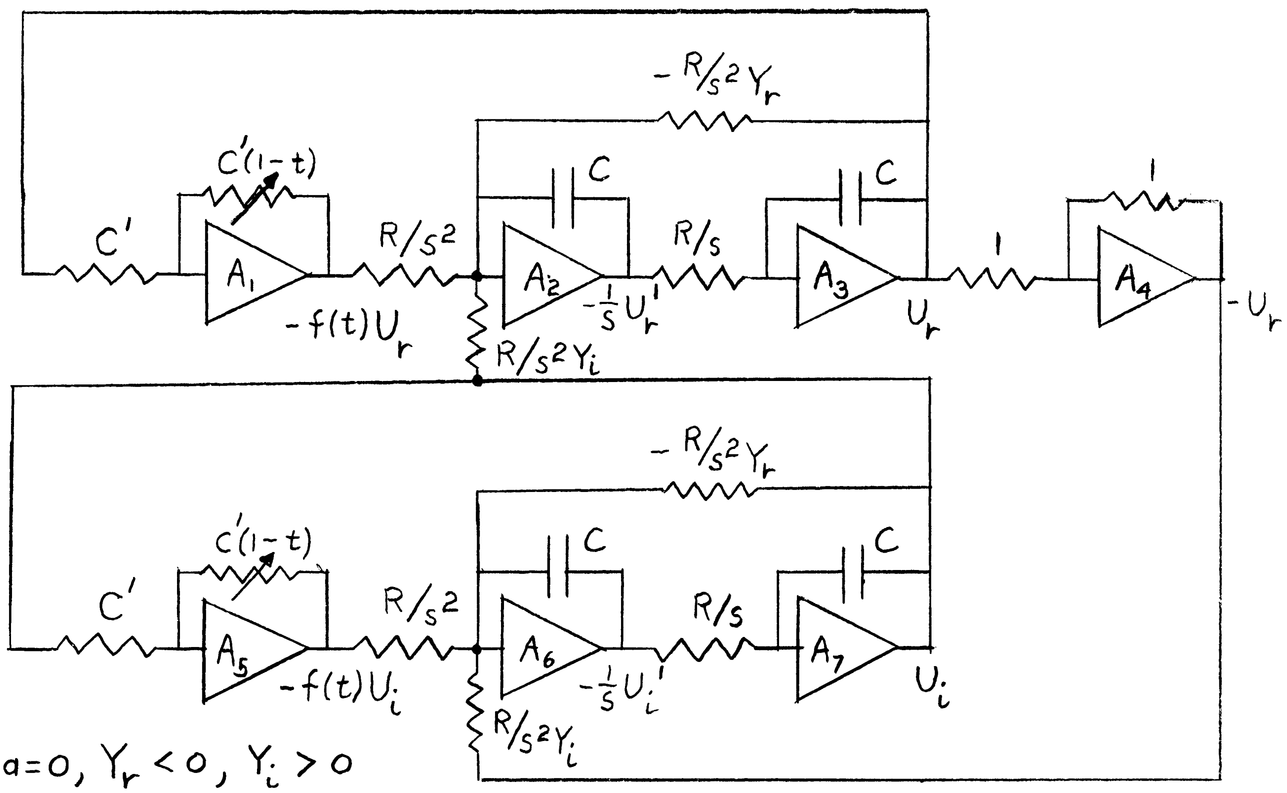


Figure 4-1.

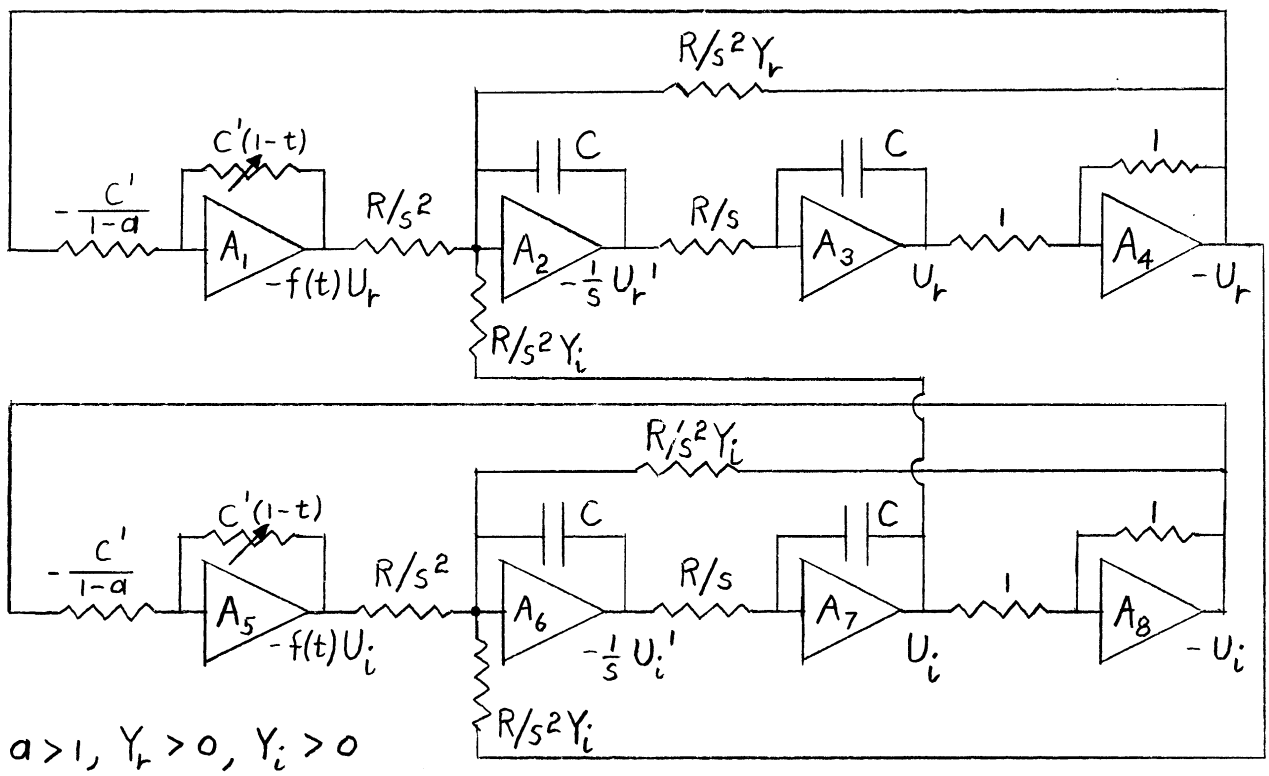


Figure 4-2.

The (1-t) factor in the f(t) function of equation (4-3) is incorporated into the feedback resistor of amplifiers A_1 and A_5 as a staircase type of function (see Section 3.3). The interval $0 \leq t \leq 1$ is broken into 20 equal time intervals. The first resistance step is 0.975 megohms, etc., finally down to 0.025 megohms on the 20th step. Thus the constant C' in Figures 4-1 and 4-2 is actually unity.

The initial-condition voltage circuit is omitted in Figures 4-1 and 4-2. Actually, the initial conditions are applied in a somewhat different manner than that shown in Figure 3-4.[16]

The unit of computer time is RC seconds, so that one unit of t corresponds to RC seconds in real time. For the work done on the bilinear gradient, a feedback capacity C of 5 microfarads was employed, along with an R value of 2 megohms. Thus one unit of t corresponds to 10 seconds, and the length of a computer solution is 10 seconds. The interval of time between steps on the staircase resistance simulation of f(t) is $10 \div 20$ or 0.5 seconds.

4.4 Measurement Techniques

Initial conditions were set in as voltages across the integrating capacitors by reading the output voltages U_r, U_i, $-\frac{1}{s}U'_r$, $-\frac{1}{s}U'_i$ with a type K-2 Leeds and Northrup potentiometer. This allowed the voltages to be set with a precision of 0.01%. Since the K-2 potentiometer can only measure voltages up to 1.6 volts whereas we might wish to set in initial voltages as high as 100 volts, a potential divider arrangement was connected across each output whenever that particular voltage was read. Then only about 1/50 of the actual output voltage was read by the potentiometer.

The general technique for obtaining the normal modes was discussed at the end of Section 1.3. It involves measuring $U_r(\xi)$ and $U_i(\xi)$ when $\xi = 0$, or for the computer variable t, measuring $U_r(t)$ and $U_i(t)$ when $t = 1$. The computer voltages representing $U_r(1)$ and $U_i(1)$ are measured by actually stopping the solution at the end of one unit of computer time (10 seconds of real time) and by reading the U_r and U_i voltages held on integrators A_3 and A_7 respectively. These readings can again be made with the K-2 potentiometer to a high precision. The integrators are made to "hold" their output voltages by means of a relay which disconnects the input resistors to the respective dc amplifiers. This "hold" relay is energized 10 seconds after the solution is begun by control circuits originating in the synchronous equipment running the resistor steps (see Ref 16 for complete circuit description).

If we have chosen the eigenvalue $Y_r + iY_i$ properly, then $U_r(1)$ and $U_i(1)$ should both be zero (corresponding to zero wave potential at the surface). Actually, a finite value for $U_r(1)$ and $U_i(1)$ will in general exist, and the interpolation method described in the next section must be employed to find the correct eigenvalue.

The resistors in the computer circuit were calibrated to the order of 0.01% accuracy. Capacitors were calibrated by connecting three amplifiers (two integrators and one summer) to solve the equation $(RC)^2 \ddot{x} + x = 0$, and by measuring very accurately the resulting period of sinesoidal oscillation.[15] A 100 kilocycle frequency standard stepped down to 1,2,2.5,5, or 10 cycles per second was utilized as a time reference and as a means for driving the resistor-stepping equipment.[16] Most of the solutions were obtained with the amplifiers balanced manually; in the unit being delivered to the Navy the amplifiers are drift stabilized, and frequent rebalancing should be unnecessary.

CHAPTER 5
COMPARISON OF COMPUTER RESULTS WITH THEORETICAL RESULTS FOR A LINEAR GRADIENT

5.1 Theoretical Solutions for the Linear Gradient

In order to check the solutions of the electronic differential analyzer, we considered first the problem of a linear gradient (no discontinuity). This is equivalent to letting the parameter a equal zero in equation (1-19), so that now $f(\xi) = \xi$ over the entire range in depth variable . The Hankel function solution (1-23) which in general is valid only for $\xi > 1$ is now valid for the whole range $0 < \xi < \infty$ for $a = 0$. We have seen in Section 2.1 that in this case the eigenvalues Y_m are given approximately by

$$Y_m \cong \frac{1}{s} \left[\frac{3\pi}{2} \left(m - \frac{1}{4}\right) \right]^{2/3} e^{2\pi i/3} \qquad (2\text{-}6)$$

Thus for a given value of the parameter s we can calculate theoretically the approximate eigenvalues Y_m and compare these eigenvalues with those obtained by the computer. Furthermore, for the a=o case we can directly cross-check the U and U' values obtained from the computer with the entries in the Harvard tables.

5.2 Method of Interpolation to the Exact Eigenvalues

The computer was set up to solve the a = 0 case in the range $0 \leq \xi \leq 1$. The circuit of Figure 4-1 was used, and initial voltages were set in according to equations (1-26) and (1-27). In order to compute these initial voltages it was necessary to go to the Harvard tables of the $h_2(z)$ function[2], where $z = x + iy$. The increments in x and y for these tables are 0.1, and since it is inconvenient to interpolate the functions, the smallest increments in assumed eigenvalues Y which we used were 0.1/s. Thus for s = 1, Y_1 (for the first mode) should be $-1.16 + i2.02$ according to equation (2-6). Instead of trying to get the computer solution for $Y = -1.16 + i2.02$ (this necessitates the determination from the tables of

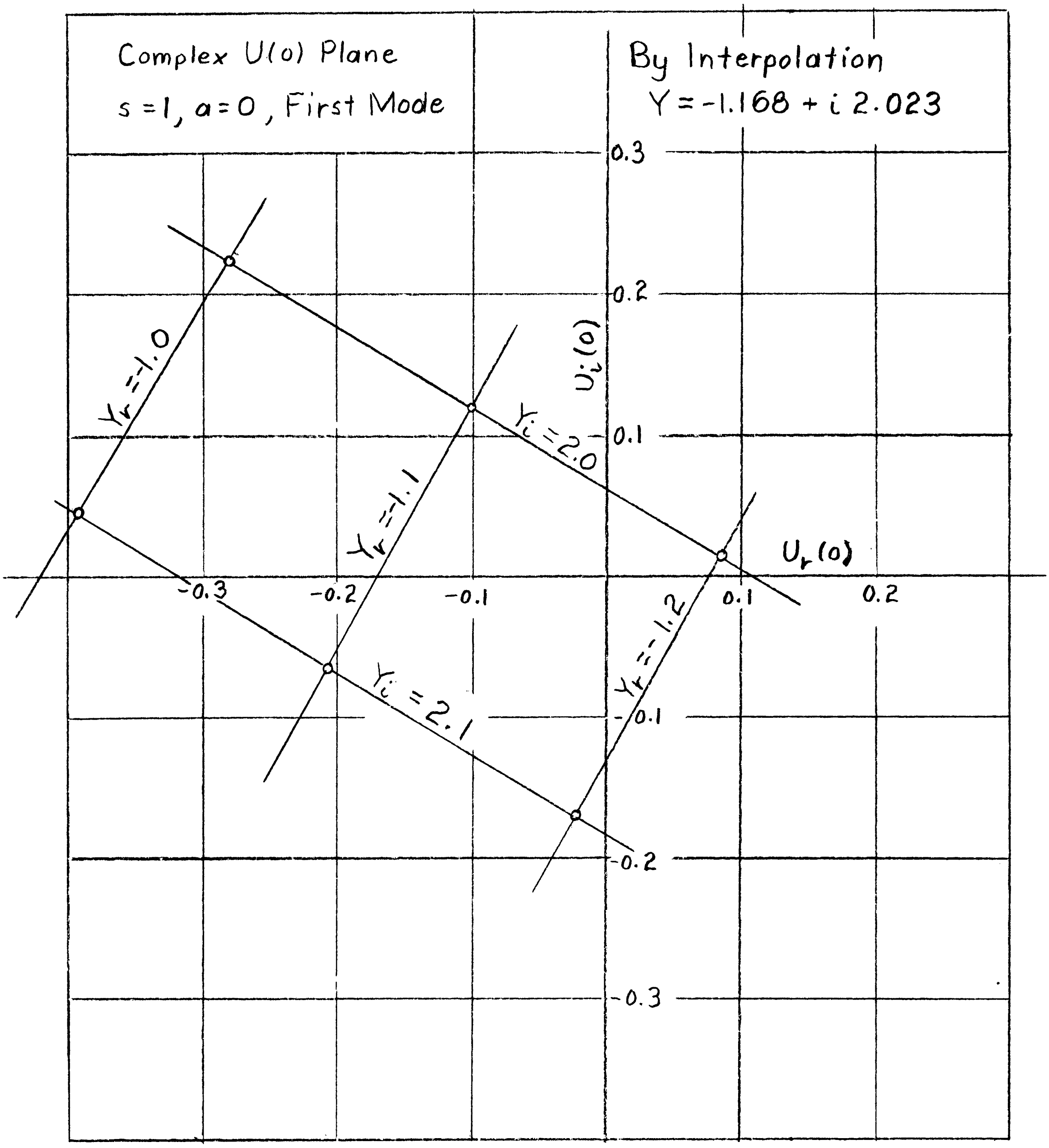


Figure 5-1. Interpolation Diagram.

h_2 $[-0.16 + i\,2.02]$ and h_2 $= [0.16 + i2.02]$), we obtained computer solutions for $Y = -1.1 + i\,2.0$, $-1.2 + i\,2.0$, $-1.1 + i\,2.1$, and $-1.2 + i\,2.1$. This entailed setting in initial conditions involving $h_2(z)$ and $h'_2(z)$ where $z = -0.1 + i\,2.0$, $-0.2 + i\,2.0$, $-0.1 + i\,2.1$, and $-0.2 + i\,2.1$ respectively, which involve no interpolation in the tables. By bracketing the true eigenvalue Y_1 in this manner we ought to be able to interpolate to Y. This has been done in Figure 5-1, where U(0) for various assumed eigenvalues Y has been plotted in the complex U plane. In each case the U(0) value was taken from U_r and U_i computer solutions using the technique described in Section 4.4.

The eigenvalue obtained from the differential analyzer results shown in Figure 5-1 is $Y_1 = -1.168 + i\,2.023$ compared with a theoretical value of $-1.16 + i\,2.01$ from equation (2-6). Since equation (2-6) is only approximately true, it seemed desirable to check the computer Y_1 against a more exact theoretical value for Y_1. With this thought in mind a plot similar to Figure (5-1) was made using the values of U(0) obtained from the Harvard tables. By interpolation from this graph a value $Y_1 = 1.170 + i\,2.025$ was obtained, which shows better agreement with the value obtained from the electronic differential analyzer.

5.3 Comparison of Differential Analyzer Solution and Harvard Tables.

We have seen in the previous section how, by starting at $\xi = 1$ and integrating to $\xi = 0$, we can obtain $U_r(0)$ and $U_i(0)$ from the differential analyzer for a given $Y_r + iY_i$. By chosing a number of different values for $Y_r + iY_i$, we can make the plot shown in Figure 5-1 and interpolate to the correct eigenvalue for which $U_r(0) = U_i(0) = 0$. A check of the analyzer solution for a = o, s = 1 with the theoretical eigenvalue showed agreement to the order of 0.1%. A more direct check of the computer accuracy is obtained by comparing computer values for U(0) and U'(0) for a given $Y_r + iY_i$ with the equivalent entries in the Harvard tables. This has been done in the following table:

Comparison of U(0) and U'(0) as Obtained with the Differential Analyzer with Values from the Harvard Tables.

s = 1, a = o, First Mode

	Y	U(0)	U'(0)
Differential Analyzer	-1.1+i2.0	-0.098+i0.110	-1.819+i1.075
Harvard Tables	-1.1+i2.0	-0.100+i0.119	-1.829+i1.070

Another method of checking the accuracy of the differential analyzer involves solving the linear gradient problem exactly. With s = 1, a = o, for example, interpolation from the Harvard tables indicates that Y_1 = -1.170 + i2.025. If we wish to run the solution from ζ = 1 to ζ = 0 with this eigenvalue, we must find h_2 (-0.0170 + i2.025) and h_2' (-0.0170 + i2.025) from the tables. To do this, the following interpolation formulas are used:[2]

$$h_2(z_0+t) = h_2(z_0) + h_2'(z_0)t \tag{5-1}$$

and

$$h_2'(z_0+t) = - h_2(z_0)t + h_2'(z_0) \tag{5-2}$$

where z_0 is the table entry and z_0 + t is the desired entry. Equations (5-1) and (5-2) are accurate to the order of 0.1% or better. The initial conditions obtained from these equations, along with the correct eigenvalues, are then set into the computer. The computer solution for $U_r(0)$ and $U_i(0)$ should now vanish. The table below shows the results obtained with the differential analyzer to corroborate this.

Summary of Differential Analyzer Results, Linear Gradient(a=o,s=1), for Correct Eigenvalues.
Solution Started at ζ = 1.

Mode	Y (Interpolation from Harv. Tables)	Analyzer Solution $U_r(0)$	$U_i(0)$
1	-1.170+i2.025	-0.0055	-0.0005
2	-2.044+i3.540	-0.0027	+0.0100

Evidentally $U(\zeta)$ practically vanishes at ζ = 0 as required, when the correct eigenvalue is utilized. Having thoroughly checked the differential analyzer solutions against the theoretical solutions for the linear velocity gradient(a = o), we next proceeded to solve the bilinear gradient problem with the analyzer.

CHAPTER 6
DIFFERENTIAL ANALYZER SOLUTIONS TO THE BILINEAR GRADIENT PROBLEM

6.1 Determination of the Eigenvalues

In Section 5 we described the method for obtaining differential analyzer solutions when $a = 0$ (linear gradient). The method of solution for $a > 0$ (i.e., a bilinear gradient) is exactly the same. The approximation formula (2-24) for the eigenvalues is utilized as a starting point. Note that this formula can be written as

$$Y_m = Y_{mo} + a \, , \qquad (6\text{-}1)$$

where Y_{mo} is the eigenvalue for the mth mode when $a = 0$. For a given a and s the eigenvalue from (6-1) is computed, and then bracketed with integral eigenvalues so that the corresponding initial conditions given in equations (1-26) and (1-27) can be looked up directly in the Harvard tables. Note that the computer variable $t = 1-\xi$, so that $-\frac{1}{s}\frac{dU}{dt} = \frac{1}{s}\frac{dU}{d\xi}$. Hence the initial condition applied to amplifiers A_2 and A_6 in Figures (4-1) and 4-2 is merely given by $h_2{}' \left[s(1-a+Y_m) \right]$, as shown in equation (1-27). Analyzer solutions for each of the trial eigenvalues are run off, and U(0) is recorded by "holding" the solution 10 seconds after it has begun and by reading U_r and U_i with the K-2 potentiometer. A plot similar to that in Figure 5-1 is made, and the correct eigenvalue for which U(0) vanishes is obtained by interpolation. This eigenvalue, along with the appropriate initial conditions interpolated from the Harvard tables with equations (5-1) and (5-2), is set into the differential analyzer and a solution run off. This solution should then represent an exact solution to the problem, that is, $U_r(0)$ and $U_i(0)$ should vanish. Again this is checked with the potentiometer while the solution is being "held" at $\xi = 0$ (t=1). At the same time, $\frac{1}{s} U'_r(0)$ and $\frac{1}{s} U'_i(0)$ are carefully recorded with the K-2 potentiometer. By knowing the derivitives of the wave potential at the surface, we can later turn the problem around backwards and integrate from the surface on down.

The reader is referred to Appendix I for a complete sample calculation for a given a and s.

Eigenvalues were obtained for a = 1.1 and a = 2 and s values of 0.5, 1, and 2. The results are tabulated in Appendix II. For $s < 0.5$, the eigenvalues are given quite accurately by equation (6-1), and the normal-mode solutions are practically identical with those for s = 0.5, except for the necessary scale change in independent variable. For s = 2 the solutions are somewhat critical, particularly for the higher modes. There is some question as to how practical it would be to attempt solutions with the differential analyzer for s values much above s = 2. The asymptotic solutions ought to get better for these high values of s, however.

6.2 Rerun of Analyzer Solution from the Surface on Down

Originally it seemed desirable to have plots of the $U(\xi)$ and $U'(\xi)$ functions over the range $0 \leq \xi \leq 2$. The method for finding the eigenvalues involves differential analyzer solutions from $\xi = 1$ to $\xi = 0$. However, by recording the derivitive of wave potential at the surface $\left(\text{i.e., } \frac{1}{s}\frac{dU(0)}{d}\right)$ we are able to turn the problem around backwards on the analyzer and rerun it from $\xi = 0$ to $\xi = 2$, since now we know the eigenvalue and all initial conditions at the surface $[U(0) = 0]$.

Note that the computer time variable t is now actually ξ, and not $1 = \xi$. The only change needed in the computer circuit involves readjustment of the f(t) function which appears as a variable feedback resistor in amplifiers A_1 and A_5 of Figures (4-1) and (4-2). The $f(\xi)$ given in equation (1-19) is shown in Figure (6-1). For a = 2 the function $f(\xi)$ is always negative, and hence $-f(\xi)$ can be represented by the staircase approximation described in Section 3.3. For a = 1.1, the function $f(\xi)$ changes sign, but by setting up $0.1 + f(\xi)$ with the stair case approximation, we are able to simulate a function which is always positive. The net result is an output $[-0.1 - f(\xi)]U_r$ from amplifier A_1, to which $0.1U_r$ must be added to obtain $(\xi)U_r$. A similar method is utilized for obtaining $f(\xi)U_i$ when a = 1.1.

Again a computer time constant of 10 seconds was employed, so that now the length of computer solution is 20 seconds, corresponding to

ξ going from 0 to 2. A total of 20 steps were utilized to simulate $f(\xi)$ as before, but in this case the resistor values were changed once per second instead of twice per second.

The $U_r(\xi)$, $U_i(\xi)$, $-\frac{1}{s}U_r'(\xi)$ and $-\frac{1}{s}U_i'(\xi)$ curves obtained from the electronic differential analyzer for $0 \leq \xi \leq 2$ are shown in Appendix III. The first three modes for a = 1.1 and 2, and s = 0.5,1, and 2 are included. The computer output voltages were recorded with a Sandborn, Model 60, Two-Channel Recorder. Accuracy of the recordings should be the order of 2% of full scale.

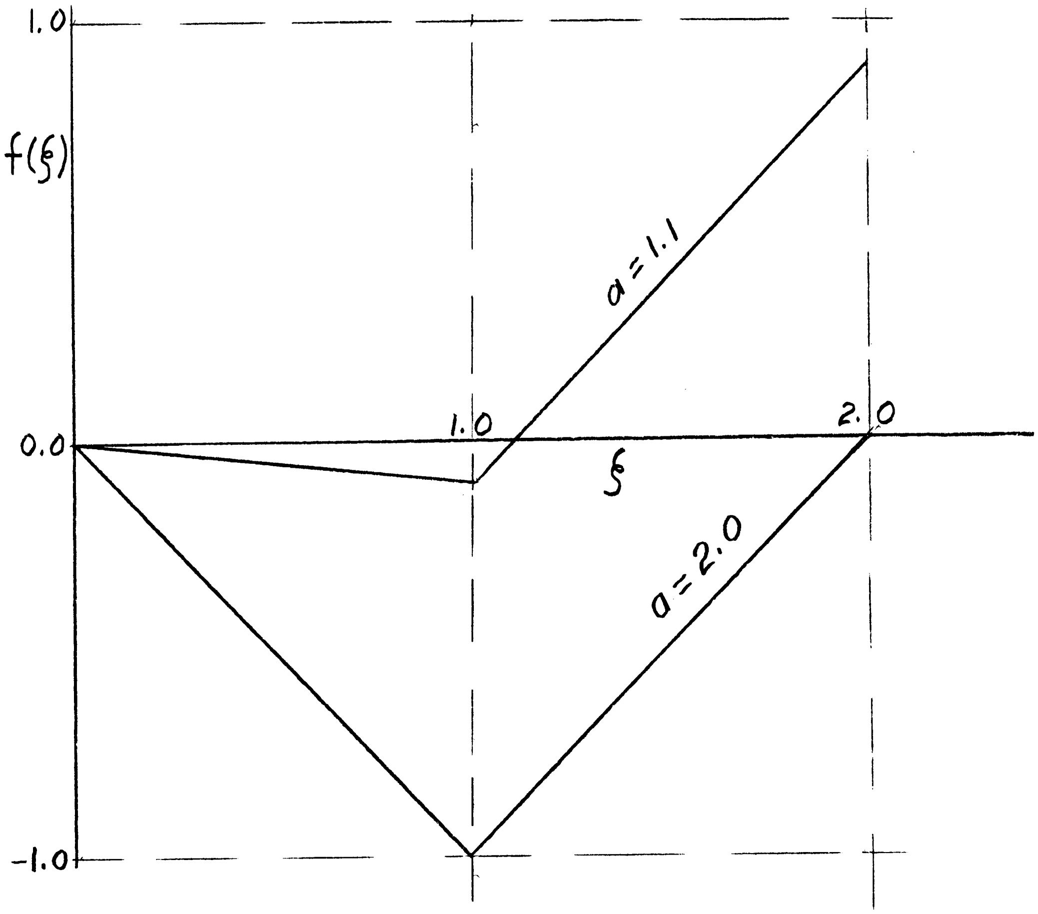


Figure 6-1. Bilinear Velocity Gradient.

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APPENDIX I

SAMPLE CALCULATION OF EIGENVALUES AND EIGENFUNCTIONS

Third Mode $s = 1.0$, $a = 2.0$

From equation (6-1), $Y_3 \cong -0.76 + i4.78$

Trial Eigenvalues	Initial Conditions from Harvard Tables		From Analyzer Solution
Y_3	$U(1)$	$\frac{1}{s}U'(1)$	$U(0)$
$-0.6 + i4.5$	$1.196 + i3.650$	$6.509 + i4.765$	$-0.110 + i0.163$
$-0.7 + i4.5$	$0.632 + i3.180$	$4.825 + i4.607$	$+0.120 + i0.152$
$-0.7 + i4.6$	$0.091 + i3.645$	$4.415 + i6.226$	$+0.116 - i0.087$
$-0.6 + i4.6$	$0.624 + i4.294$	$6.294 + i6.722$	$-0.120 - i0.074$

From interpolation plot similar to Figure 5-1, $Y_3 = -0.650 + i4.566$.
For an exact solution, $U(1) = h_2(-1.650+i4.566)$, $\frac{1}{s}U'(1) = h_2'(-1.650+i4.566)$.
In equations (5-1) and (5-2), let $z_o = -1.7 + i4.6$, $t = 0.050 - i0.034$.
Then

$$U(1) = h_2(z_o+t) = (0.091+i3.645)+(4.415+i6.226)(0.050-i0.034)$$
$$= 0.524+i3.806$$

$$\frac{1}{s}U'(1) = h_2'(z_o+t) = (4.415+i6.226)+(0.091+i3.645)(1.7-i4.6)(0.050-i0.034)$$
$$= 5.459+i5.941$$

Using these values for intitial conditions when $Y_3 = -0.650 + i4.566$, we find from the analyzer solution

$$U(0) = -0.012 + i0.007 \quad \text{(ideally, } U(0) \text{ should vanish)}$$

and

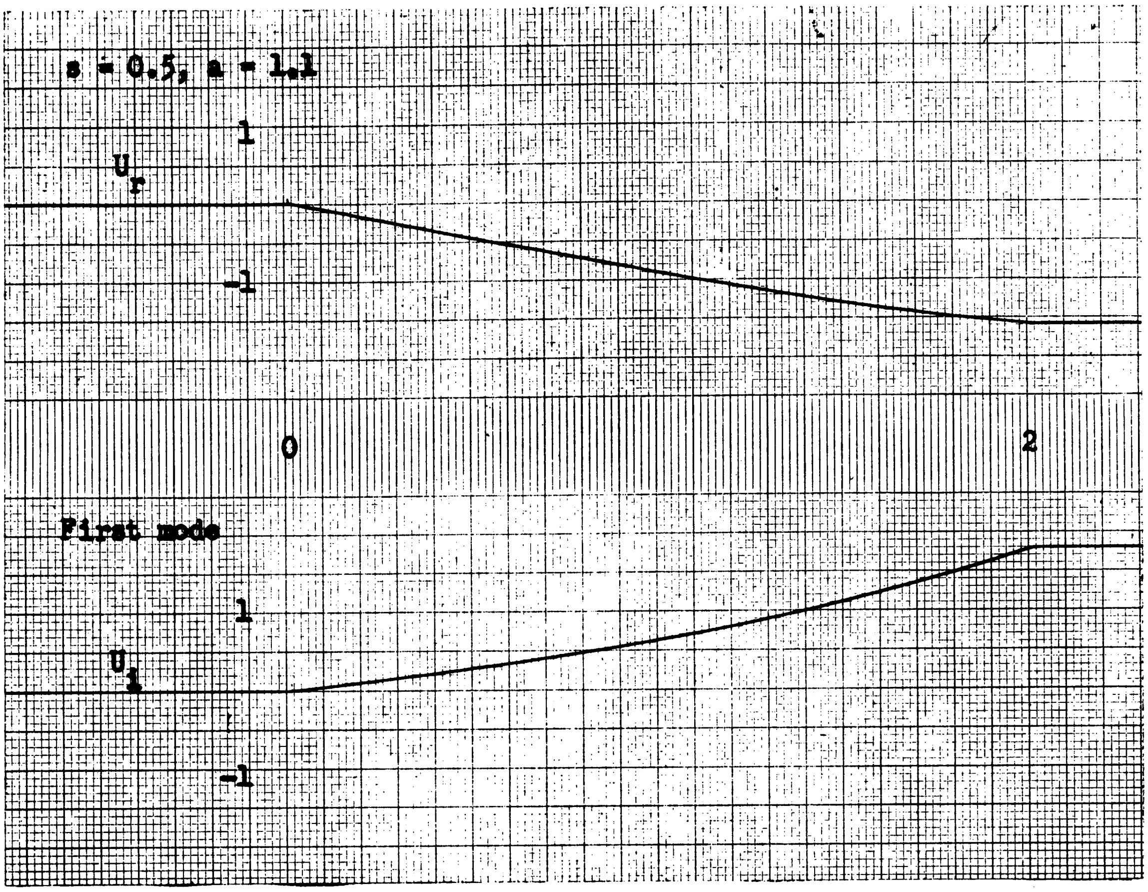
$$\frac{1}{s}U'(0) = -1.515 + i2.450$$

To solve the problem in reverse, we use $U(0) = 0$ and $-\frac{1}{s}U'(0) = 1.515 - i2.450$ as initial conditions for the analyzer. We then proceed to integrate from $t = 0$ to $t = 2$ (i.e., from $\delta = 0$ to $\delta = 2$).

APPENDIX II

SUMMARY OF COMPUTER RESULTS

s	Mode	a	Y_r	Y_i	$\frac{1}{s} U_r'(0)$	$-\frac{1}{s} U_i'(0)$
0.2	1	0.0	- 5.850	10.125	- 1.835	1.060
	1	1.1	- 4.750	10.125	- 1.835	1.061
	1	2.0	- 3.850	10.125	- 1.836	1.062
	2	0.0	- 10.22	17.70	2.121	- 1.214
	2	1.1	- 9.12	17.70	2.121	- 1.214
	2	2.0	- 8.22	17.70	2.121	- 1.214
	3	0.0	- 13.805	23.905	- 2.279	1.300
	3	1.1	- 12.705	23.905	- 2.279	1.300
	3	2.0	- 11.805	23.905	- 2.279	1.300
0.5	1	0.0	- 2.340	4.050	- 1.835	1.060
	1	1.1	- 1.225	4.049	- 1.868	1.088
	1	2.0	- 0.315	4.048	- 1.923	1.123
	2	0.0	- 4.088	7.080	2.121	- 1.214
	2	1.1	- 2.978	7.078	2.165	- 1.257
	2	2.0	- 2.062	7.077	2.197	- 1.295
	3	0.0	- 5.422	9.562	- 2.279	1.300
	3	1.1	- 4.408	9.558	- 2.322	1.341
	3	2.0	- 3.500	9.556	- 2.375	1.367
1.0	1	0.0	- 1.170	2.025	- 1.835	1.060
	1	1.1	0.032	1.980	- 2.132	1.581
	1	2.0	1.011	1.910	- 2.155	2.044
	2	0.0	- 2.044	3.540	2.121	- 1.214
	2	1.1	- 0.860	3.450	2.123	- 1.884
	2	2.0	0.130	3.40	1.933	- 2.594
	3	0.0	- 2.761	4.781	- 2.279	1.300
	3	1.1	- 1.573	4.677	- 2.056	2.126
	3	2.0	- 0.650	4.566	- 1.515	2.450
2.0	1	0.0	- 0.585	1.012	- 1.835	1.060
	1	1.1	0.44	0.46	- 0.293	1.513
	1	2.0	1.09	0.225	- 0.401	1.668
	2	0.0	- 1.022	1.770	2.121	- 1.214
	2	1.1	- 0.161	1.468	0.580	- 0.683
	2	2.0	0.679	1.400	0.679	- 0.657
	3	0.0	- 1.380	2.391	- 2.279	1.300
	3	1.1	- 0.583	2.171	- 0.553	0.332
	3	2.0	0.263	2.173	- 0.656	0.298



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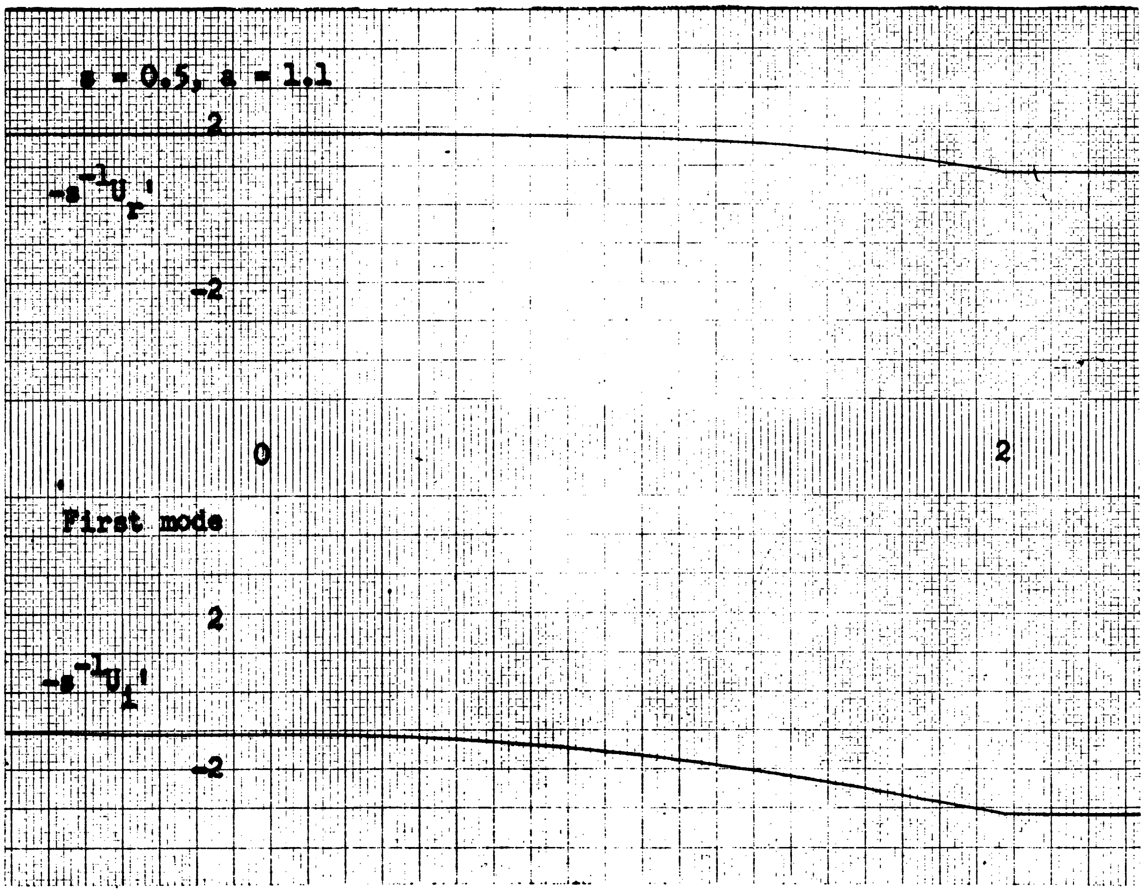
1
U_r
-1
0
2
First mode
1
U_i
-1



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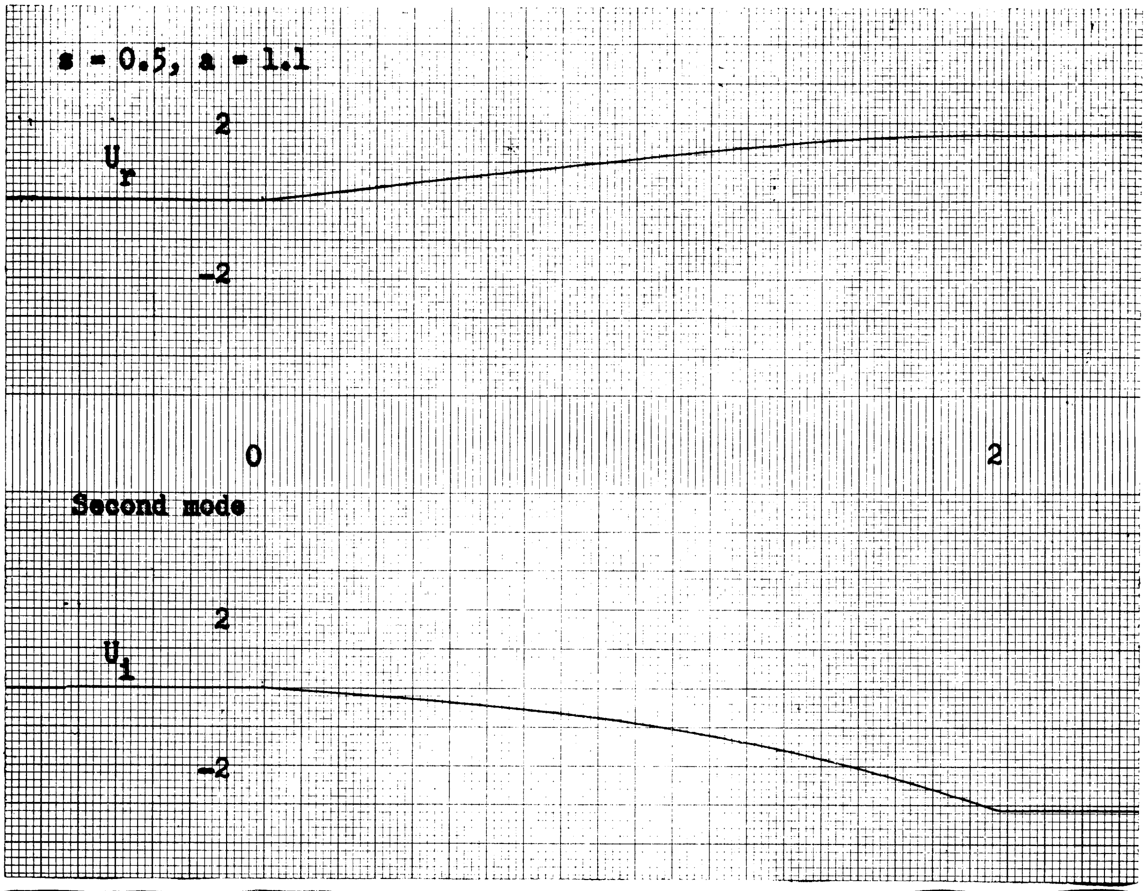
2
-s^-1 U_r'
-2
0
2
First mode
2
-s^-1 U_i'
-2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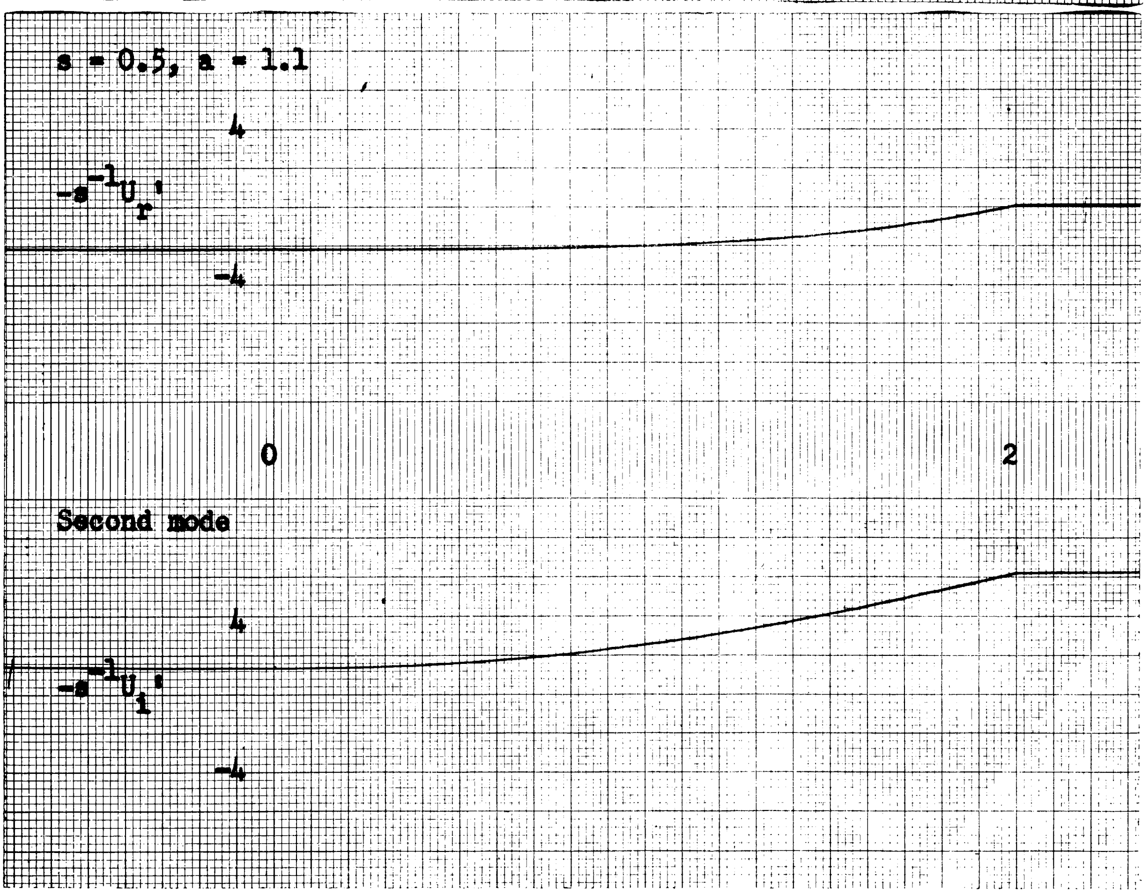
2
U_r
-2
0
2
Second mode
2
U_i
-2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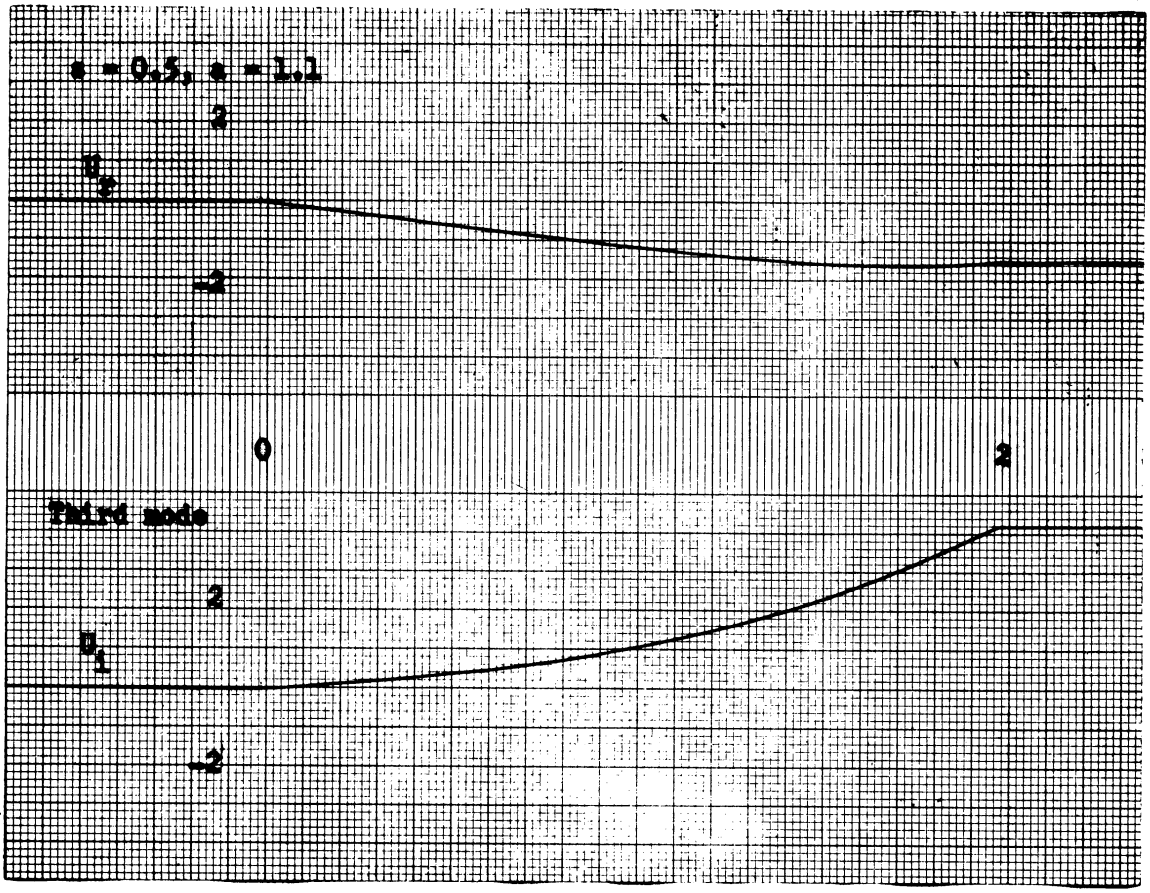
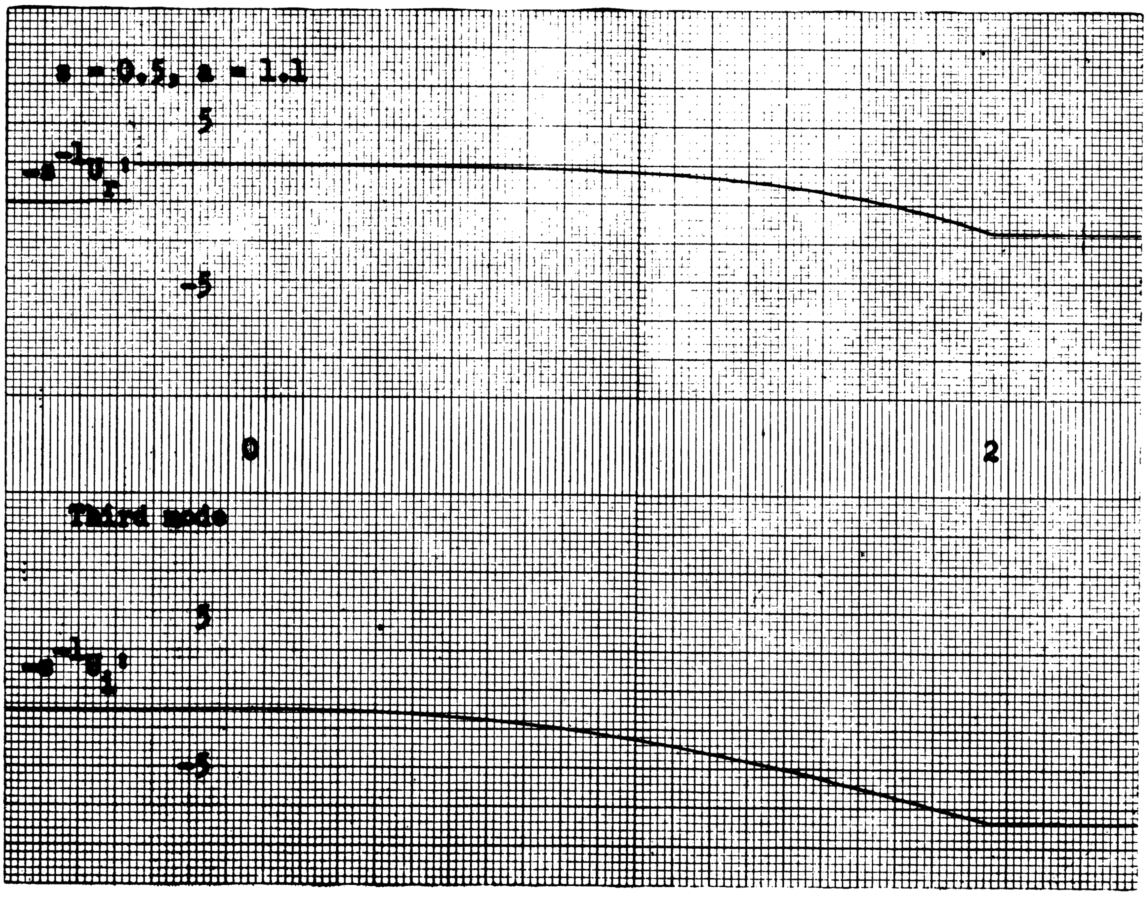
$-s^{-1}U_r'$
-4
0
2
Second mode
4
$-s^{-1}U_i'$
-4


s = 0.5, a = 1.1
2
U_r
-2
0
2
Third mode
2
U_i
-2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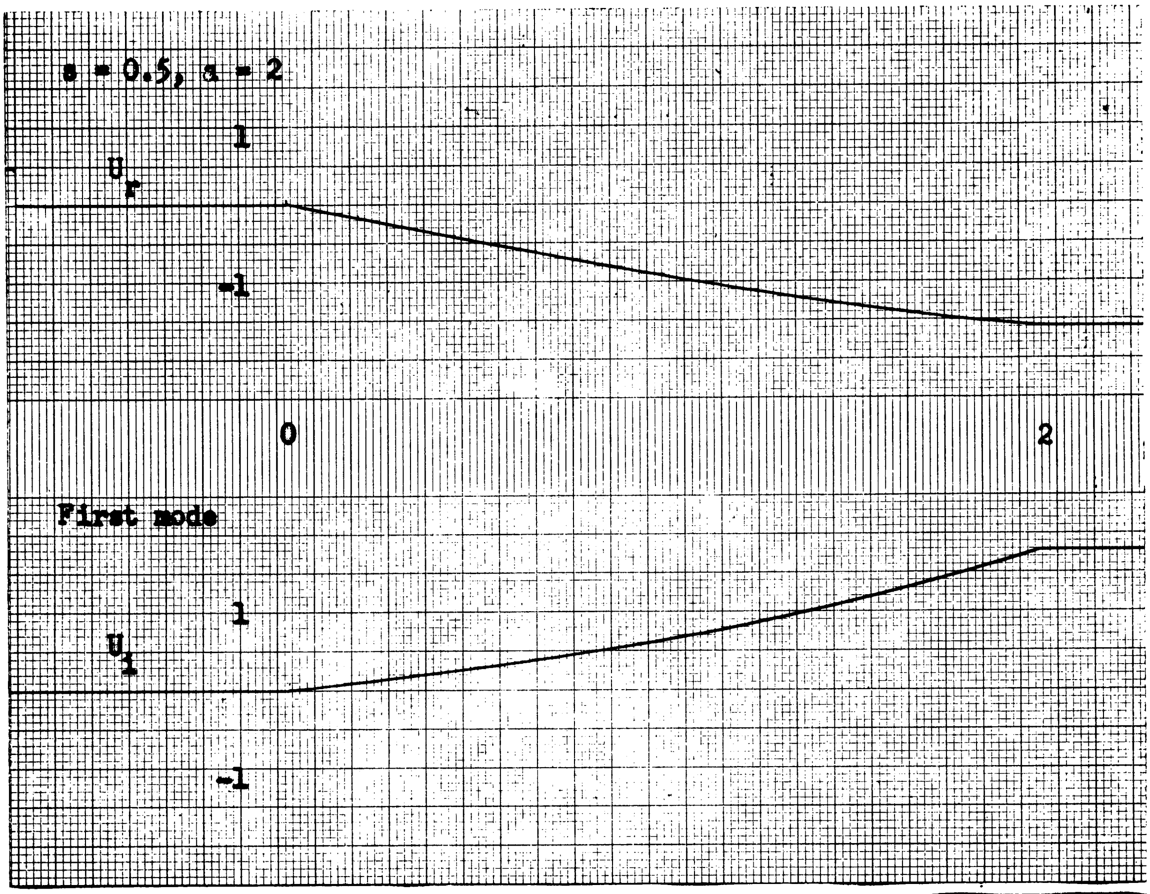
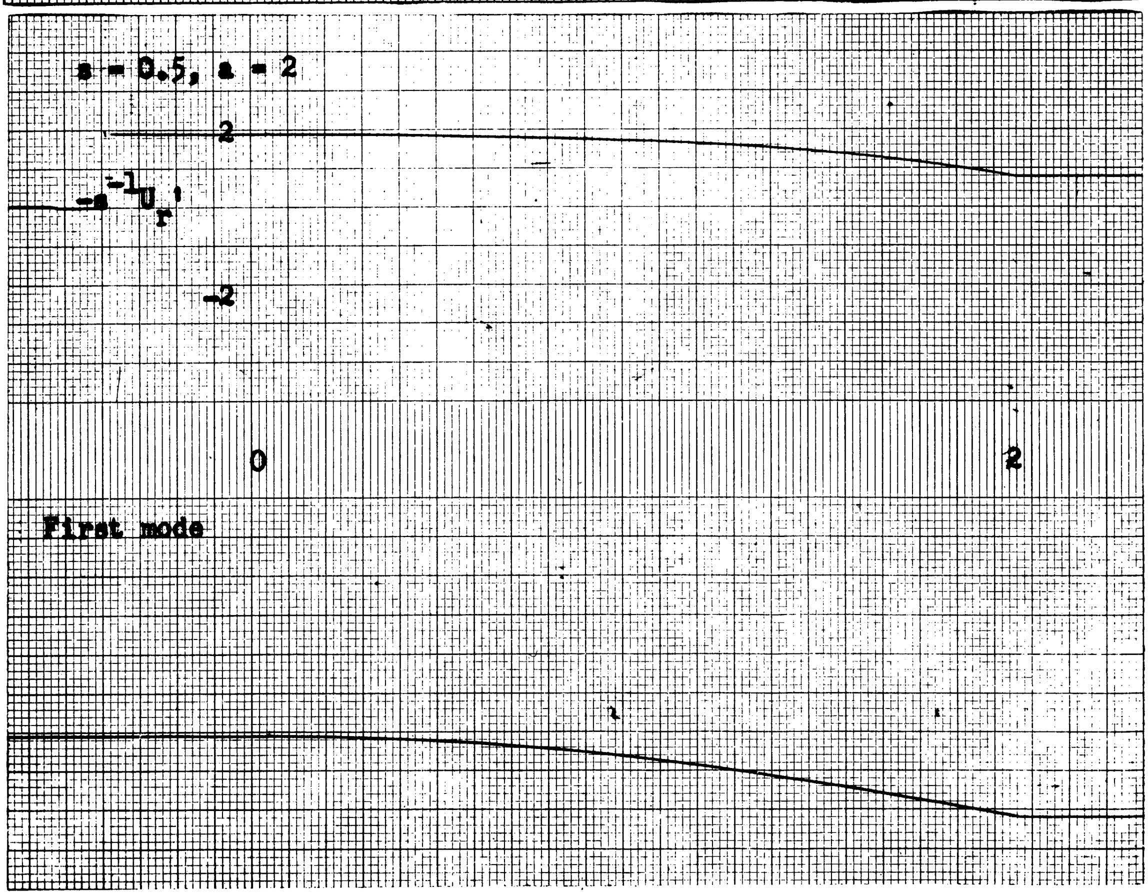
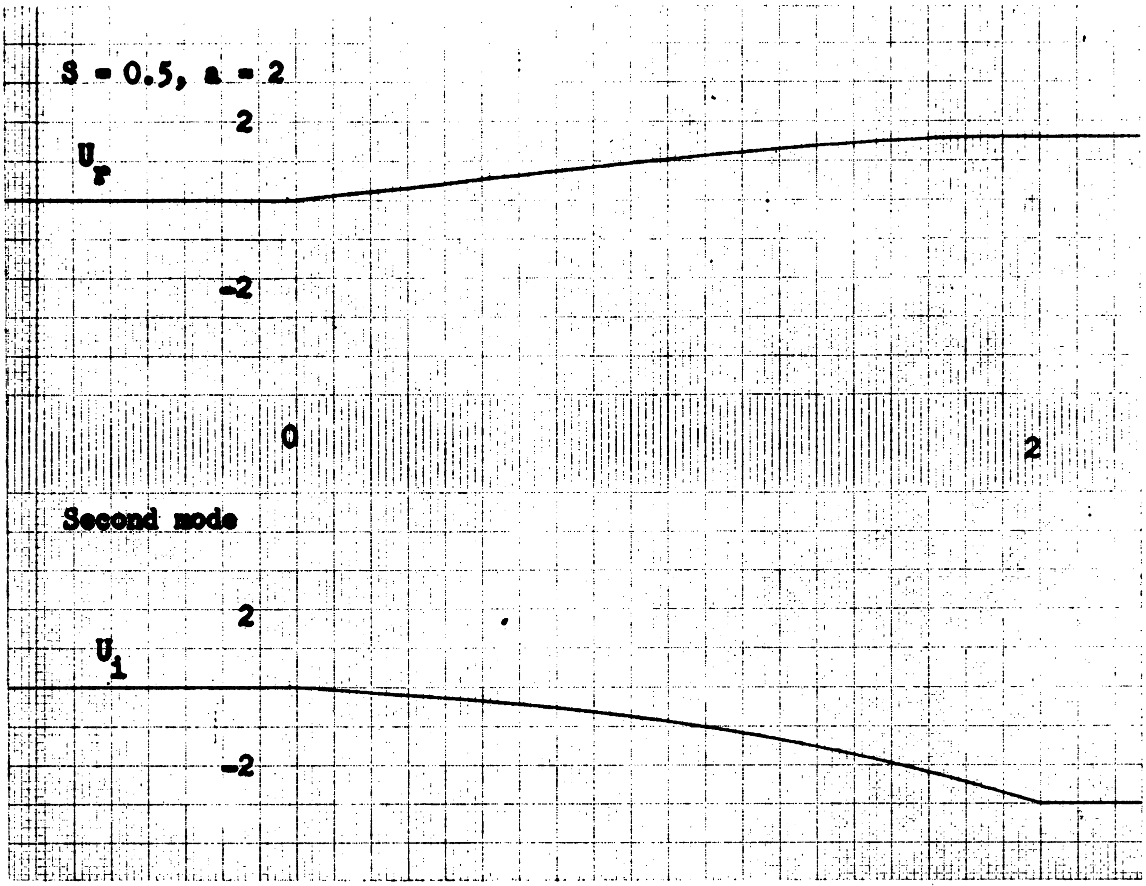
5
-s^-1 U_r'
-5
0
2
Third mode
5
-s^-1 U_i'
-5


s = 0.5, a = 2
1
U
r
-1
0
2
First mode
1
U
i
-1


s = 0.5, a = 2
2
-s
-1
U
r
'
-2
0
2
First mode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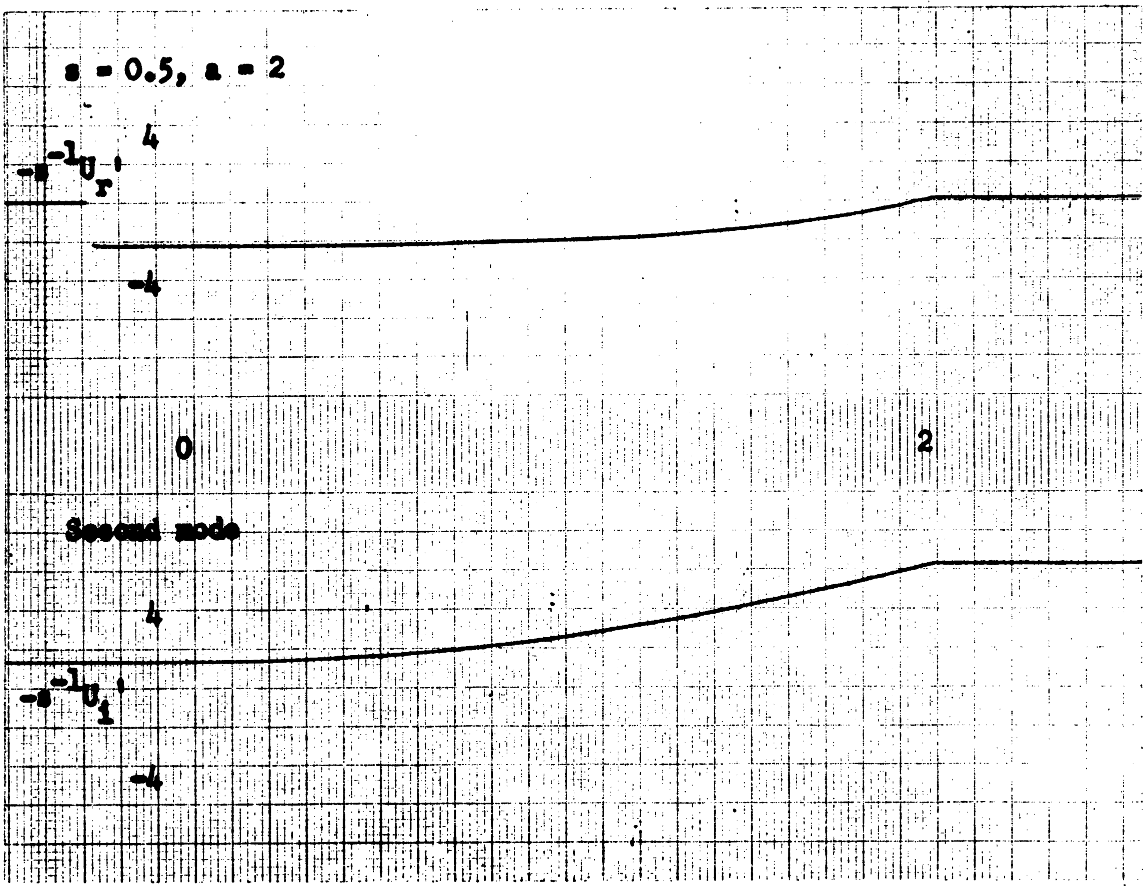
2
U_r
-2
0
2
Second mode
2
U_i
-2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4
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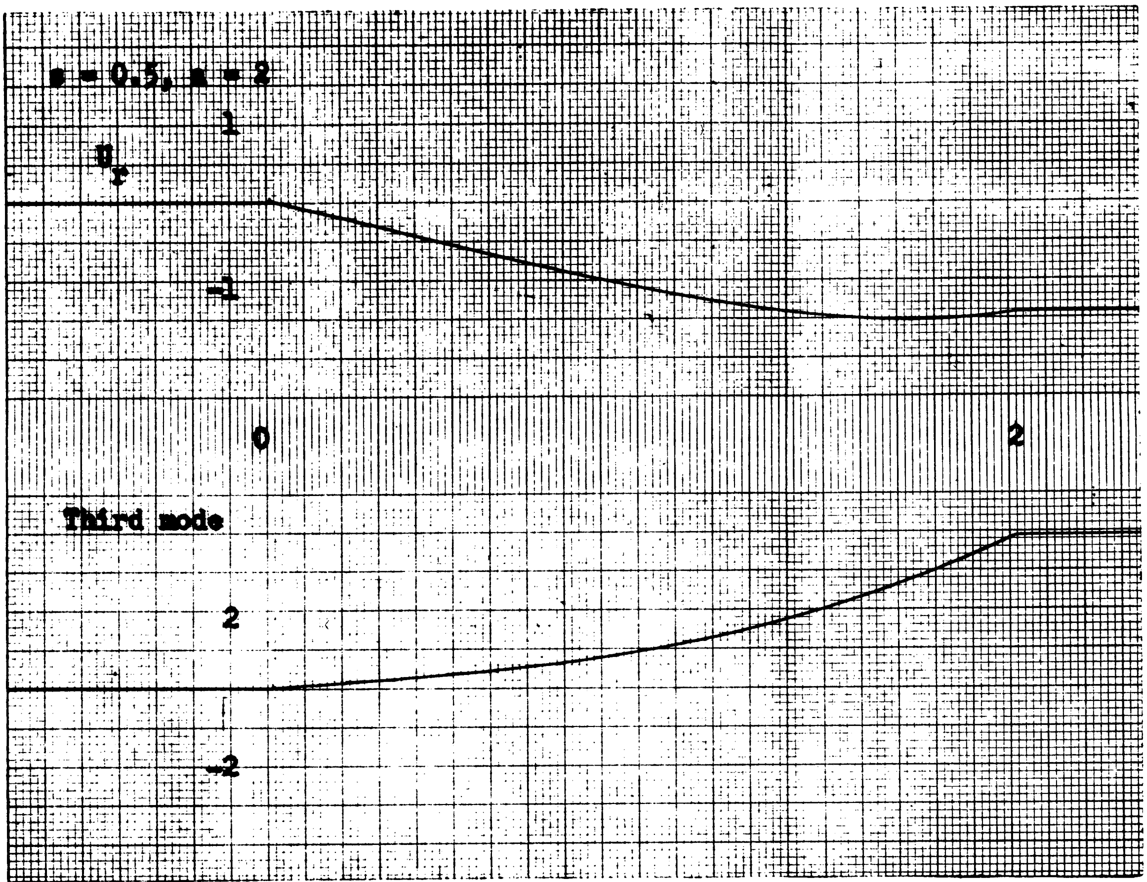
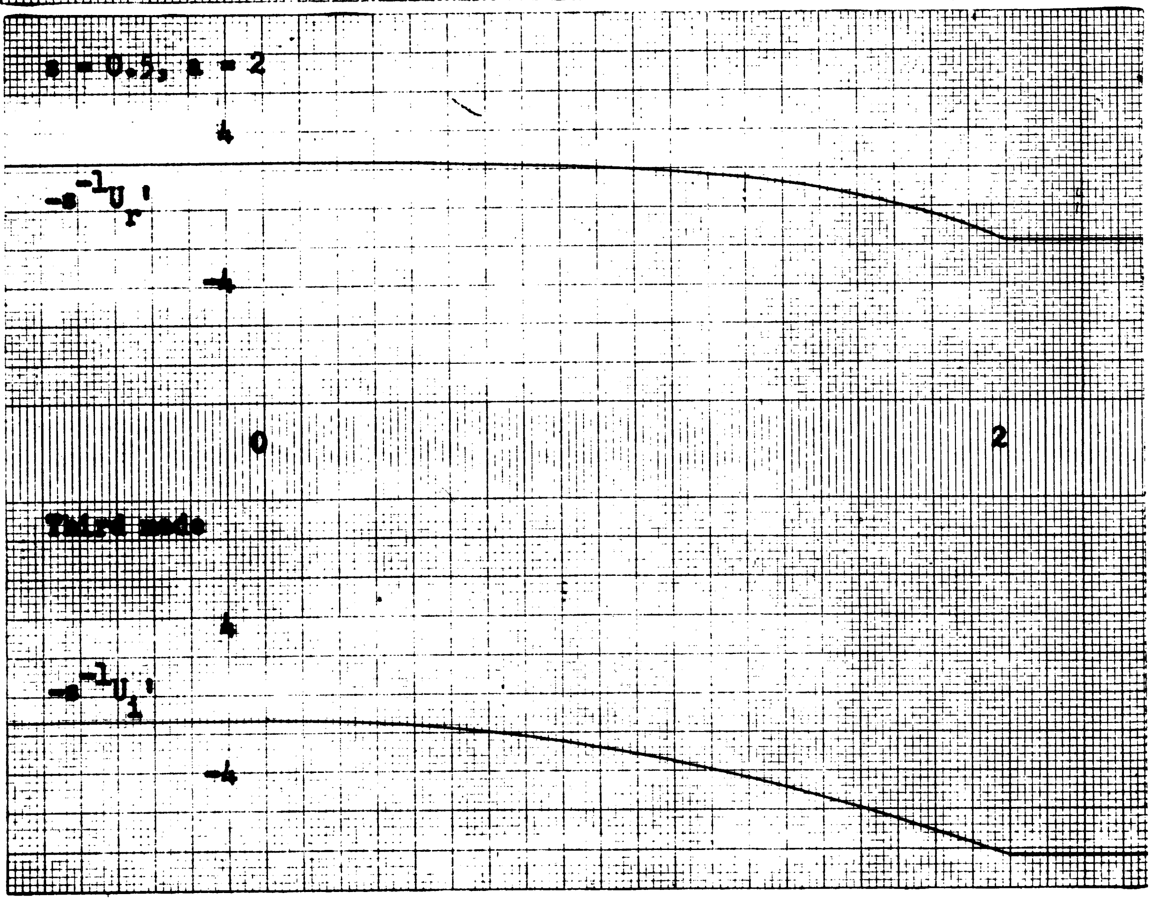
-4
0
2
Second mode
4
$-s^{-1}U_i'$
-4


s = 0.5, a = 2
1
U_r
-1
0
2
Third mode
2
-2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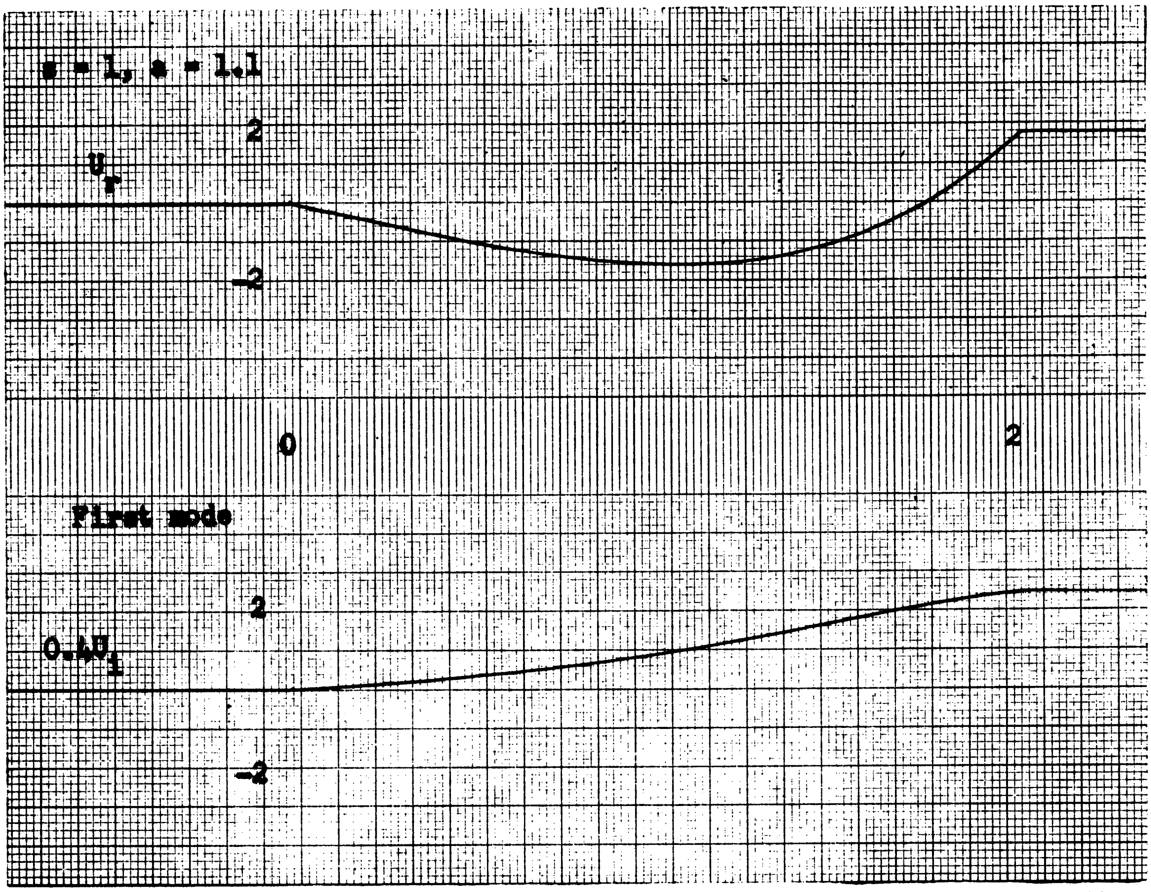
4
$-s^{-1}U_r'$
-4
0
2
Third mode
4
$-s^{-1}U_i'$
-4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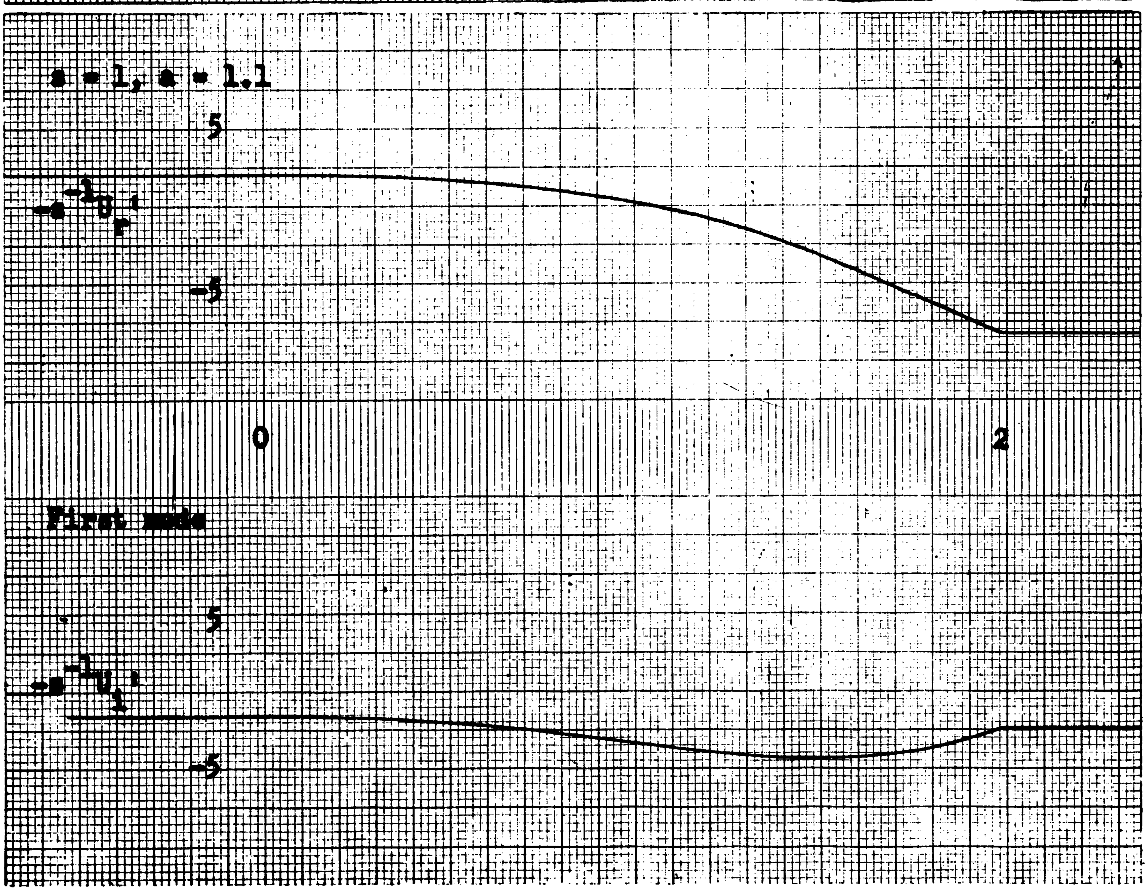
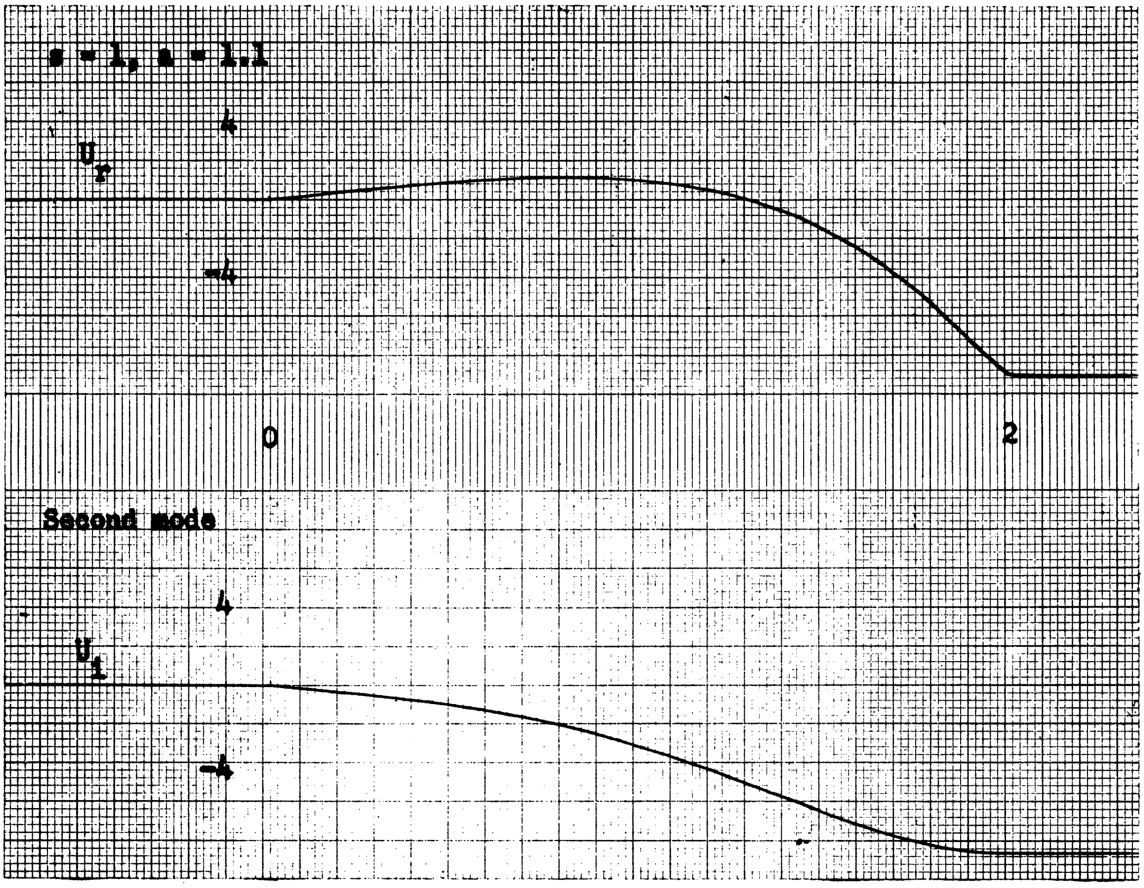
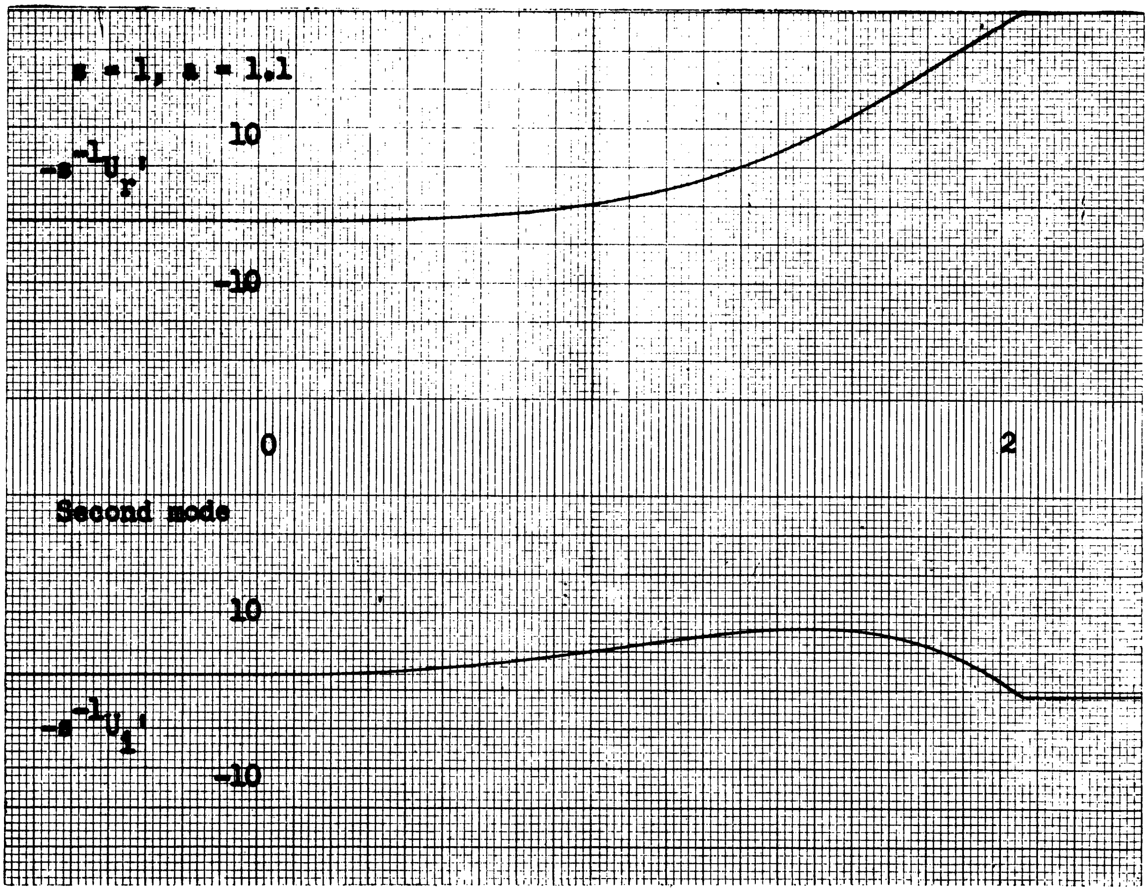
2
U_r
-2
0
2
First mode
2
0.4U_i
-2


s = 1, a = 1.1
5
-s^-1U_r'
-5
0
2
First mode
5
-s^-1U_i'
-5


s = 1, a = 1.1
4
U_r
-4
0
2
Second mode
4
U_i
-4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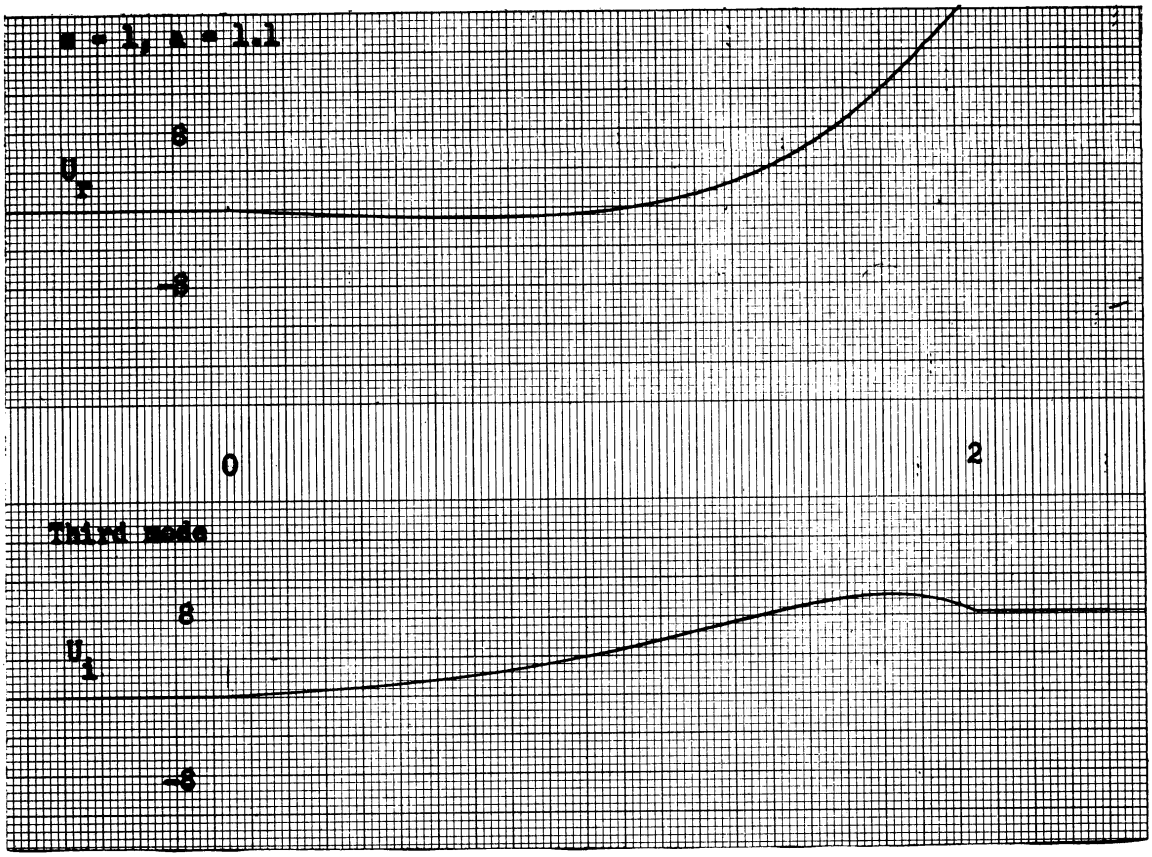
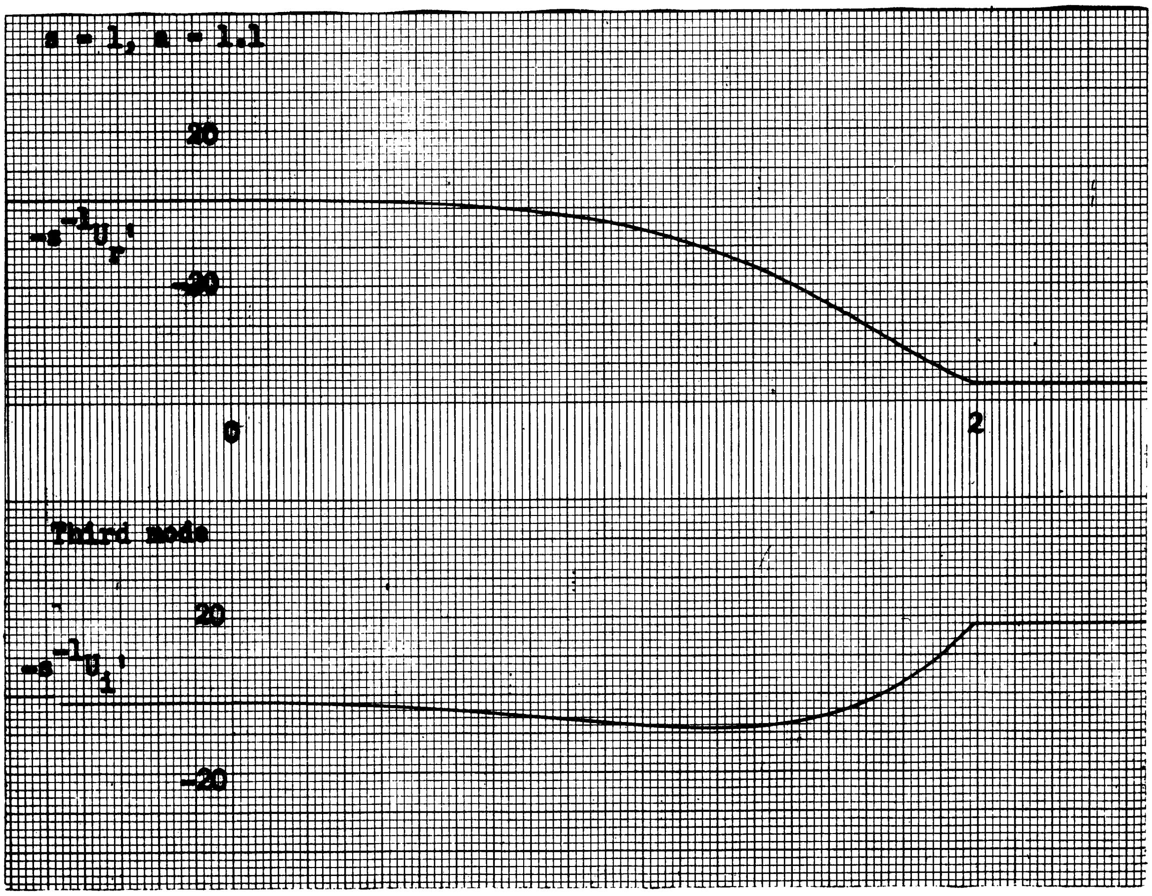
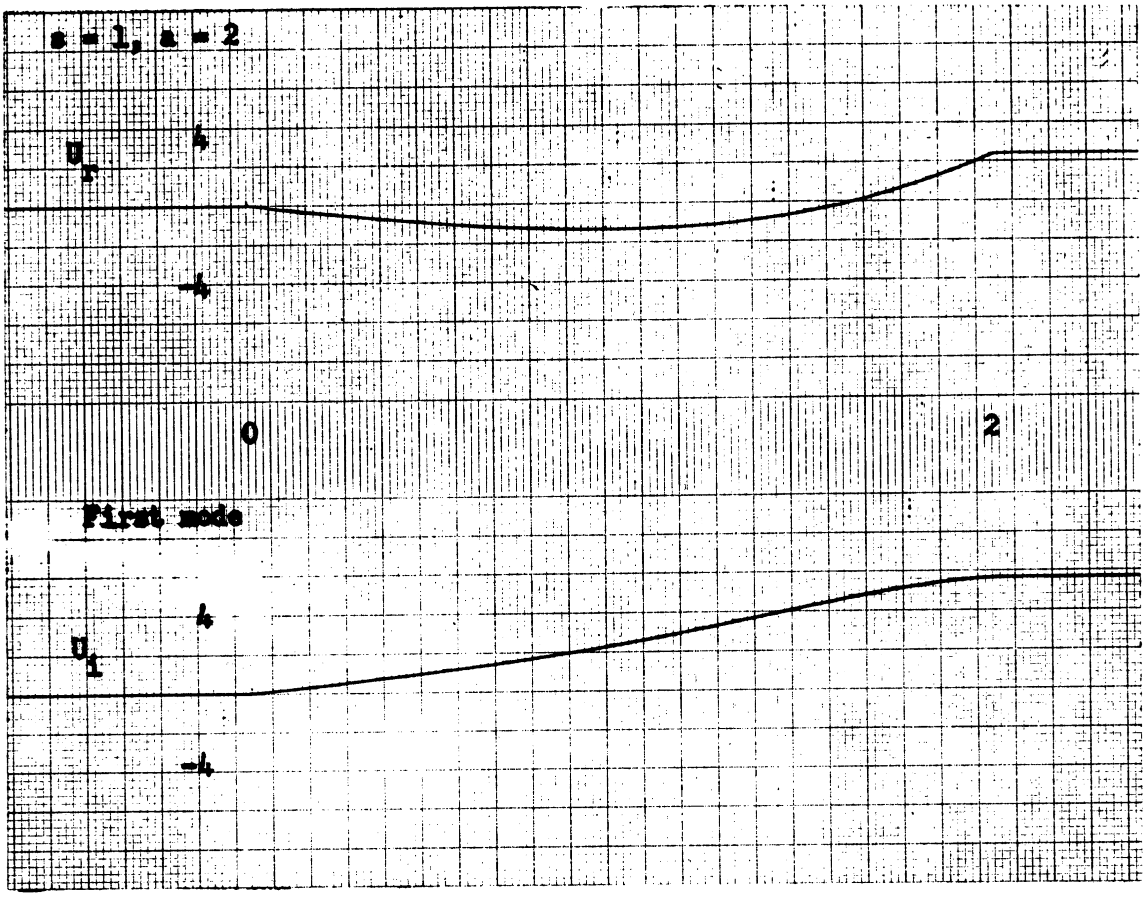
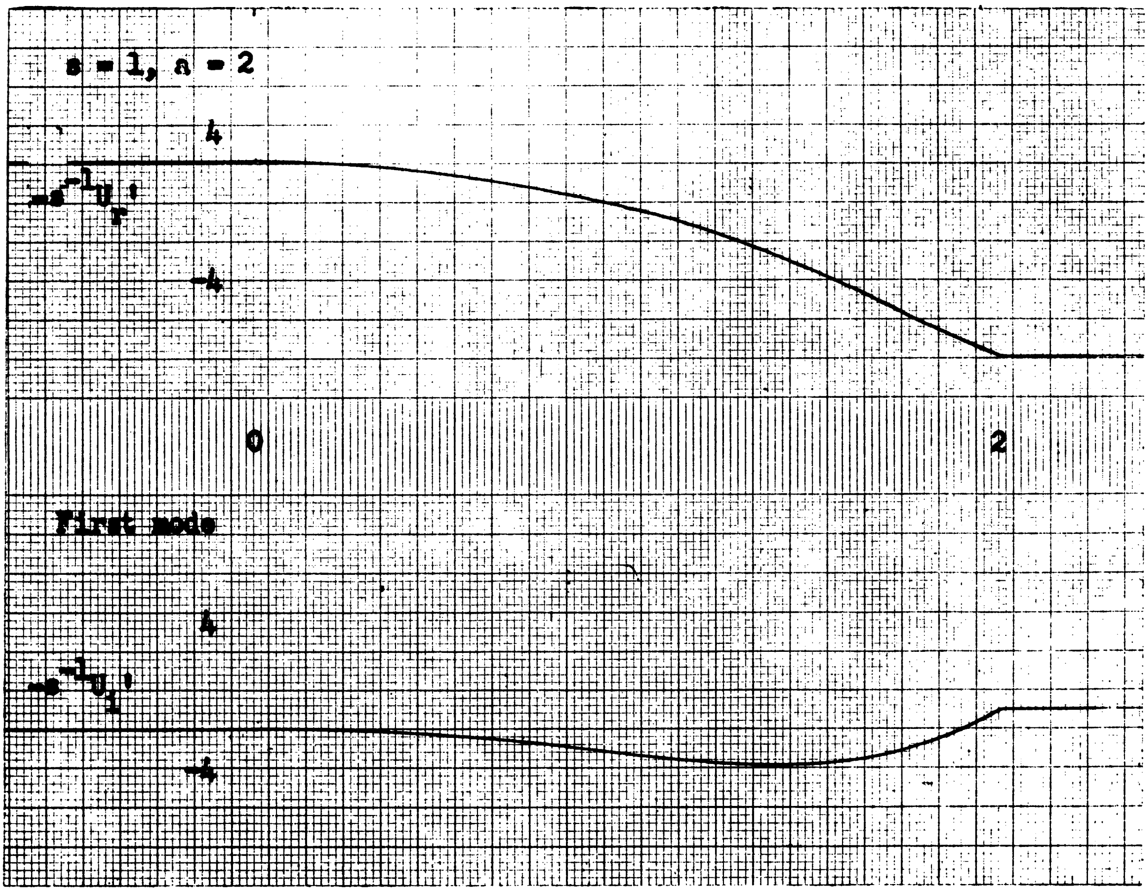
10
$-s^{-1}U_r'$
-10
0
2
Second mode
10
$-s^{-1}U_i'$
-10


s = 1, a = 1.1
8
U_r
-8
0
2
Third mode
8
U_i
-8


s = 1, a = 1.1
20
$-s^{-1}U_r'$
-20
0
2
Third mode
20
$-s^{-1}U_i'$
-20


s = 1, a = 2
4
U_r
-4
0
2
First mode
4
U_i
-4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4
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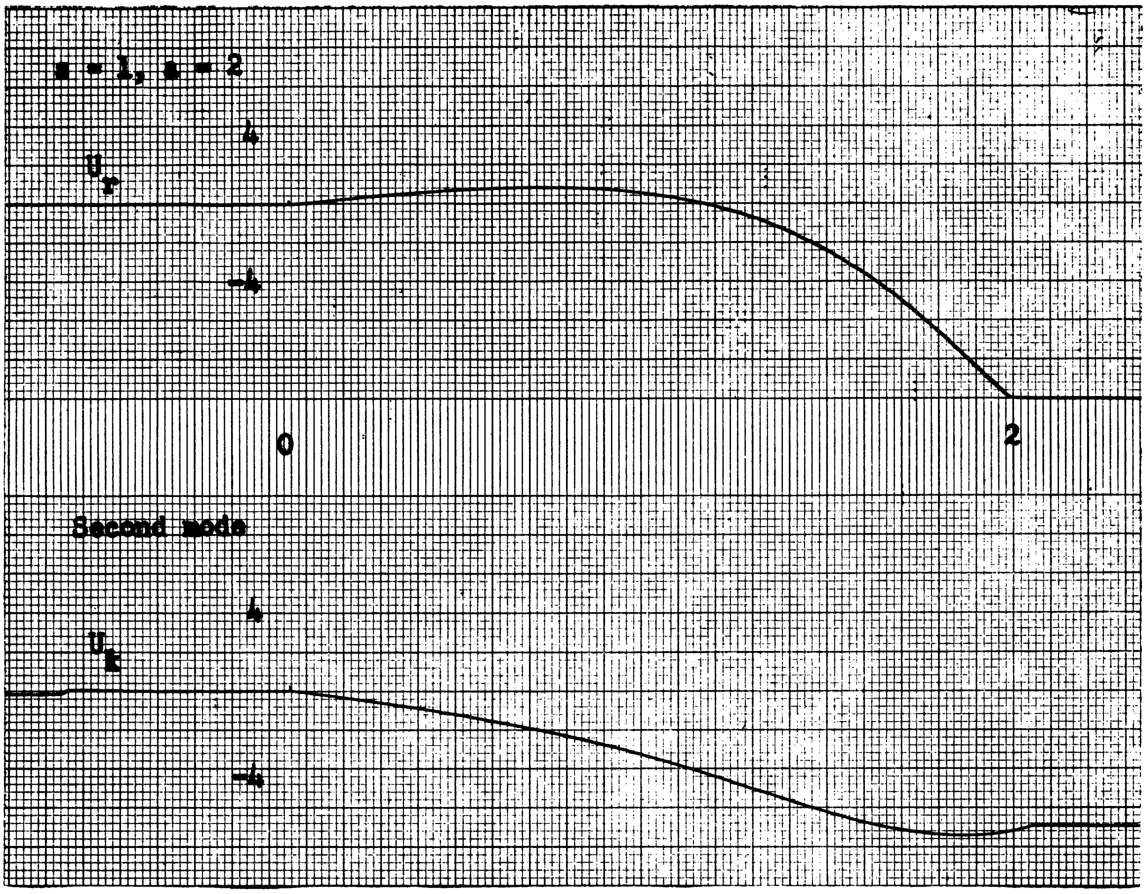
-4
0
2
First mode
4
-s^-1 U_i'
-4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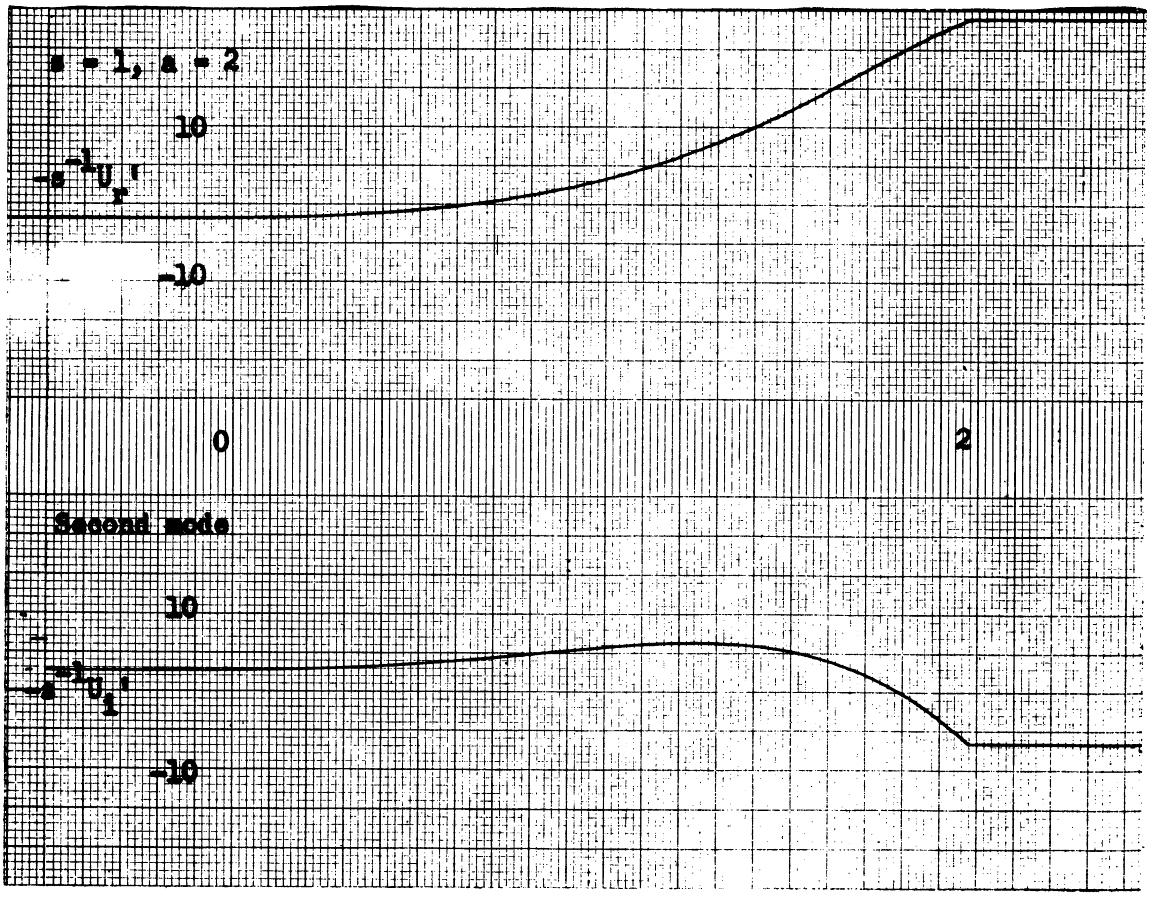
4
U_r
-4
0
2
Second mode
4
U_z
-4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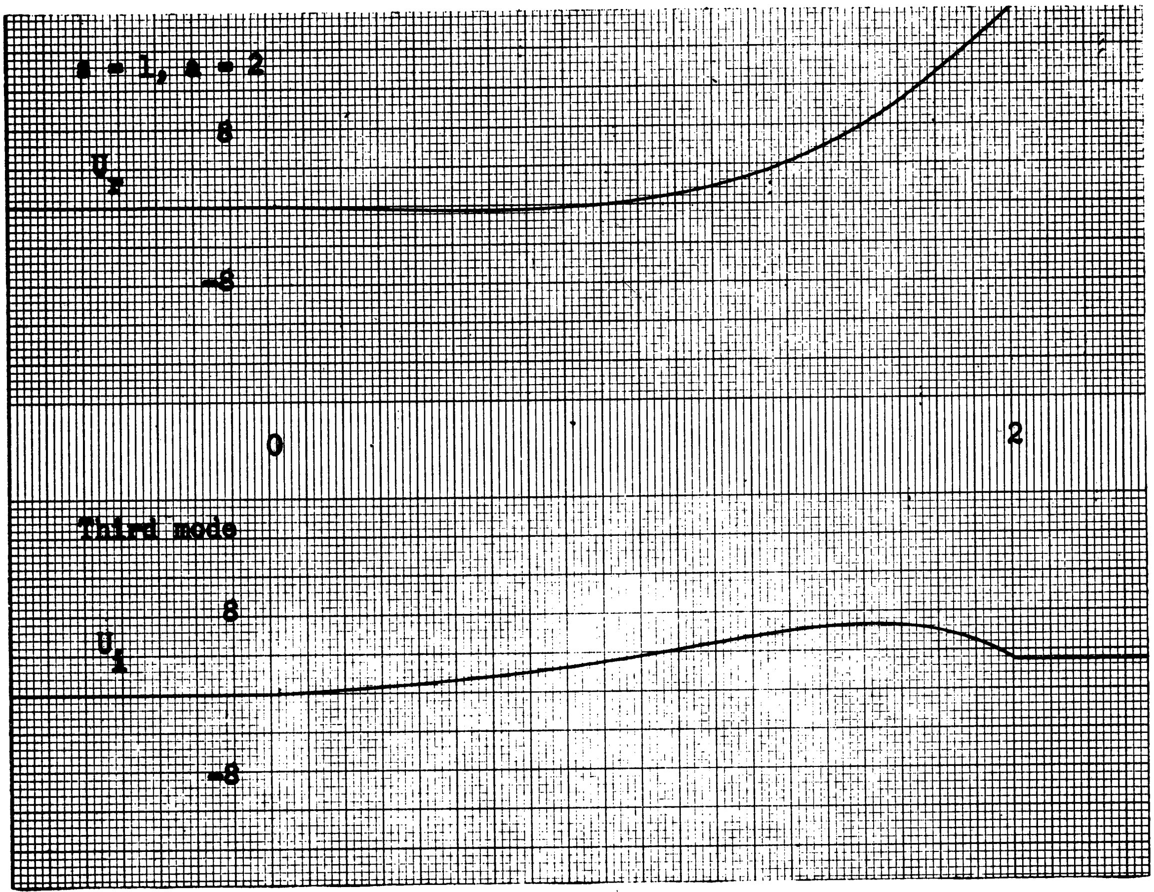
-10
0
2
Second mode
10
$-s^{-1}U_z'$
-10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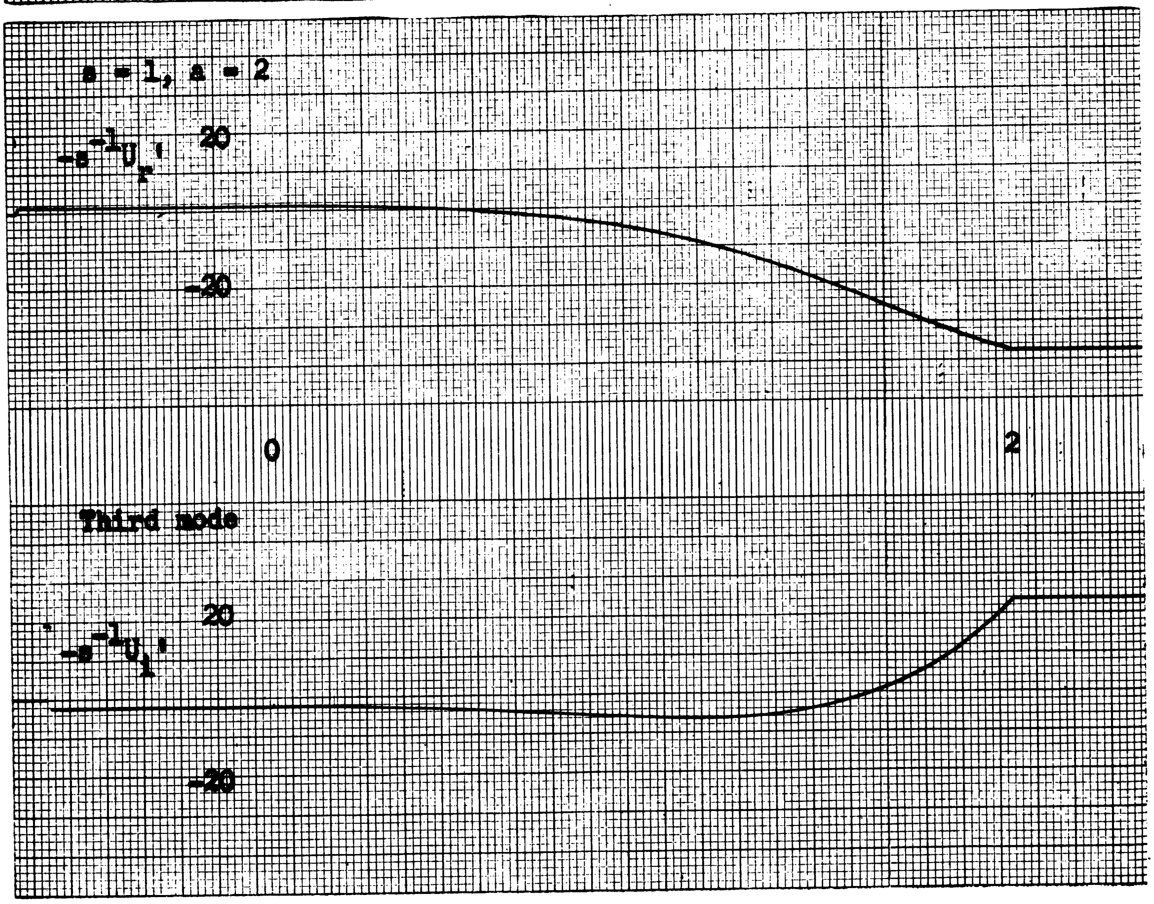
8
U_r
-8
0
2
Third mode
8
U_i
-8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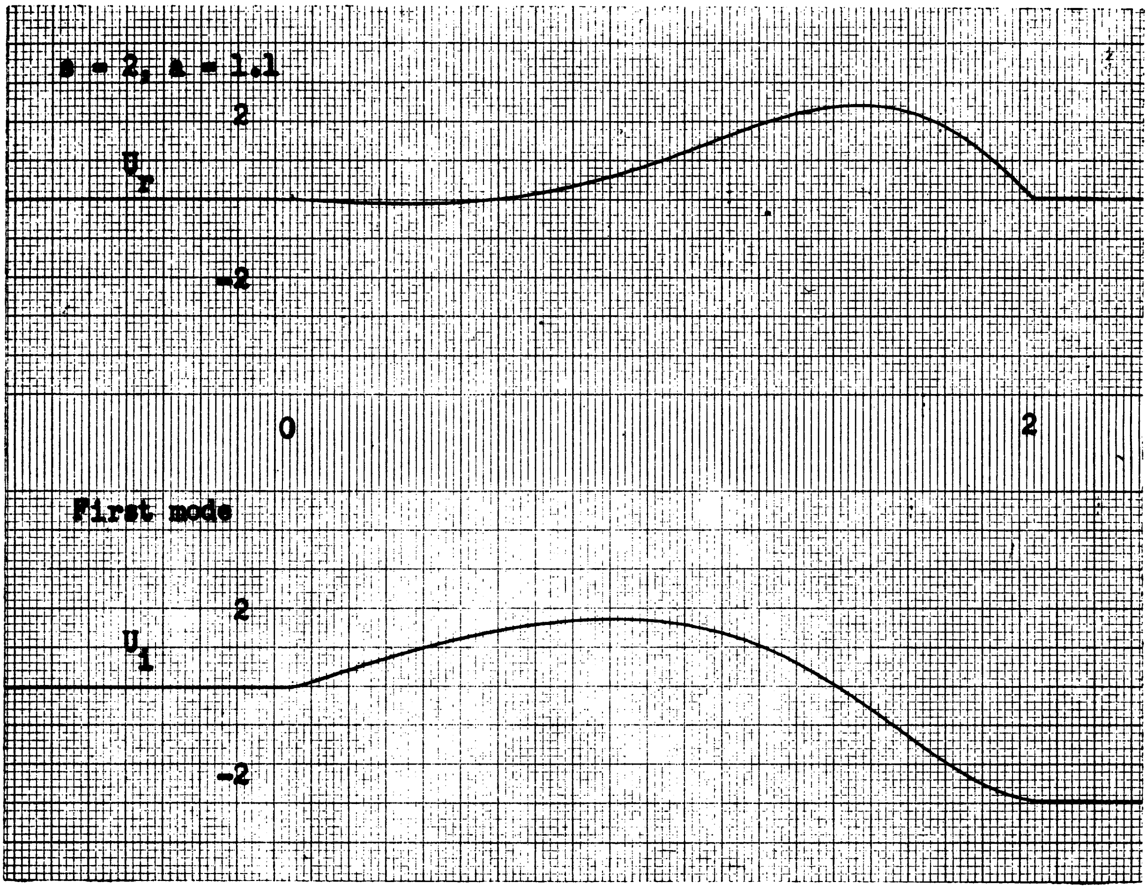
-20
0
2
Third mode
20
$-s^{-1}U_i'$
-20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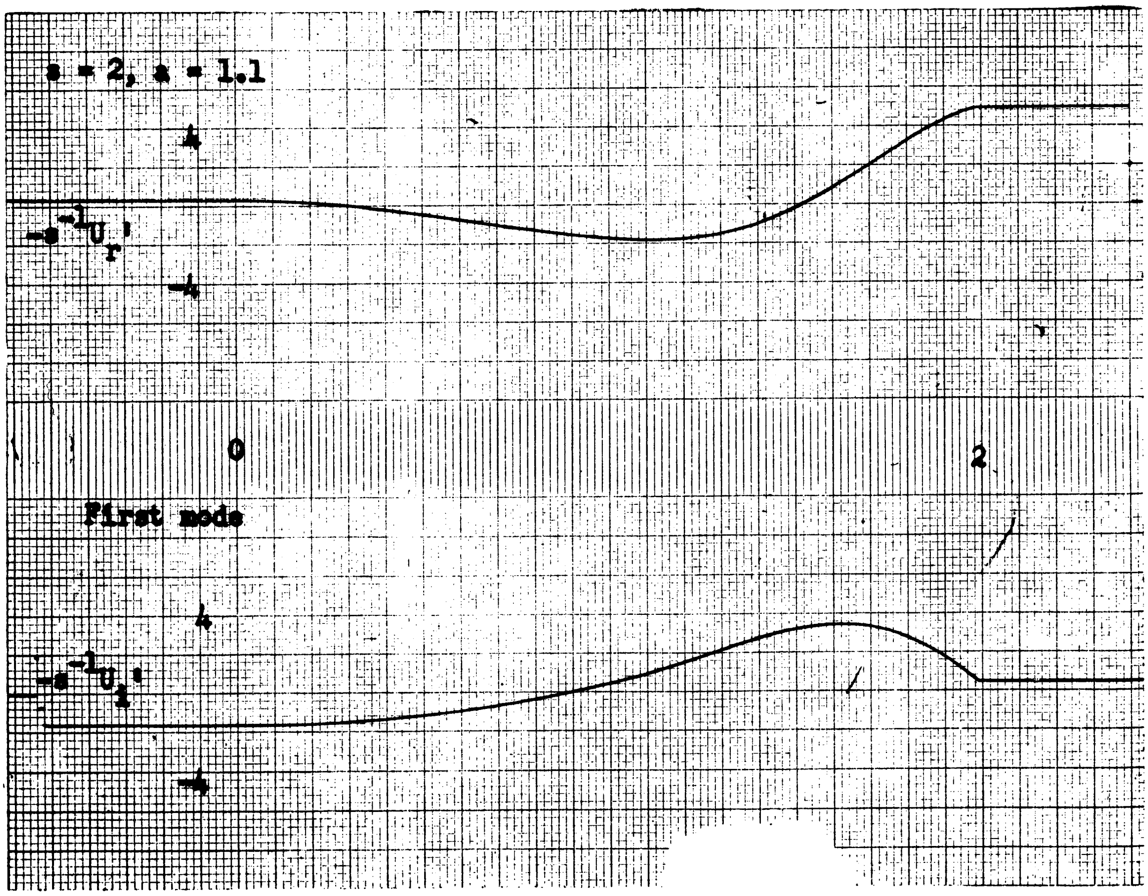
2
U_r
-2
0
2
First mode
2
U_1
-2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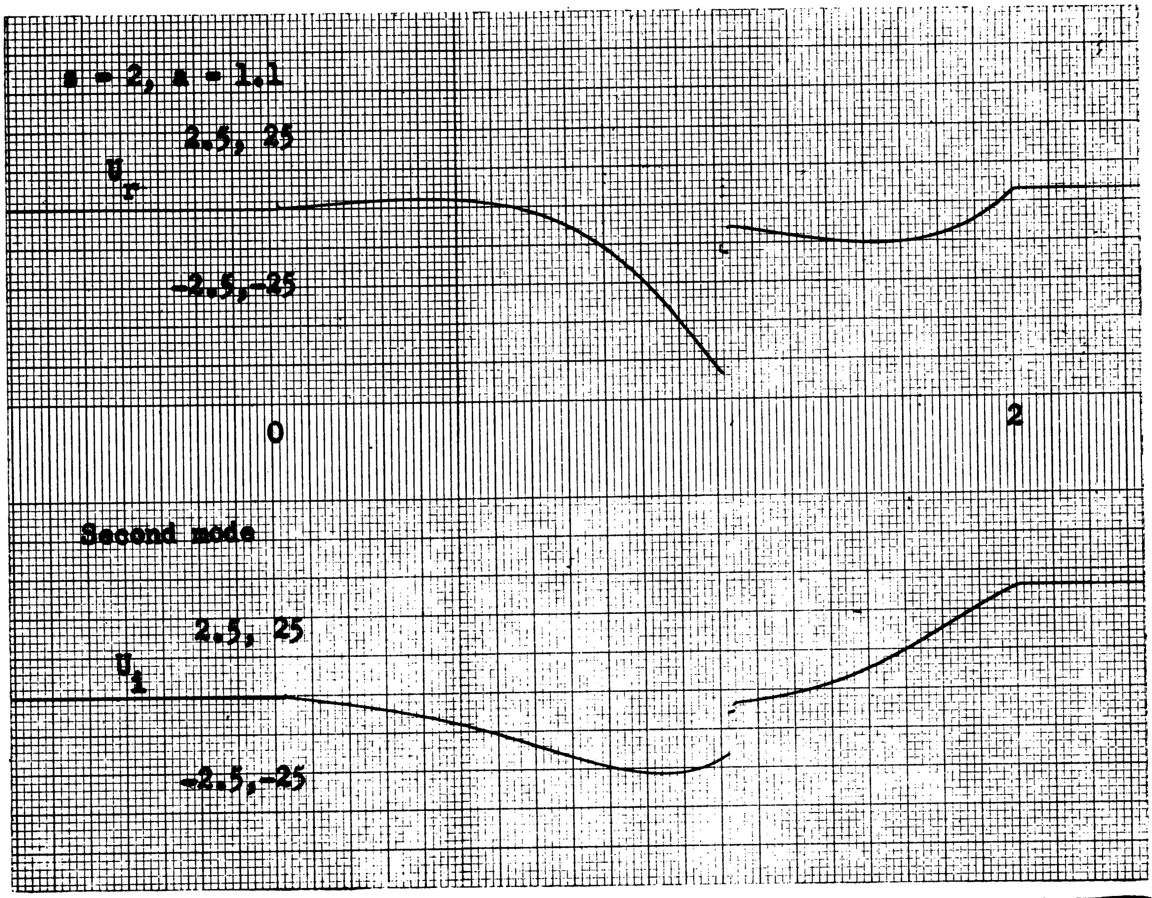
-4
0
2
First mode
4
-s^-1 U_1'
-4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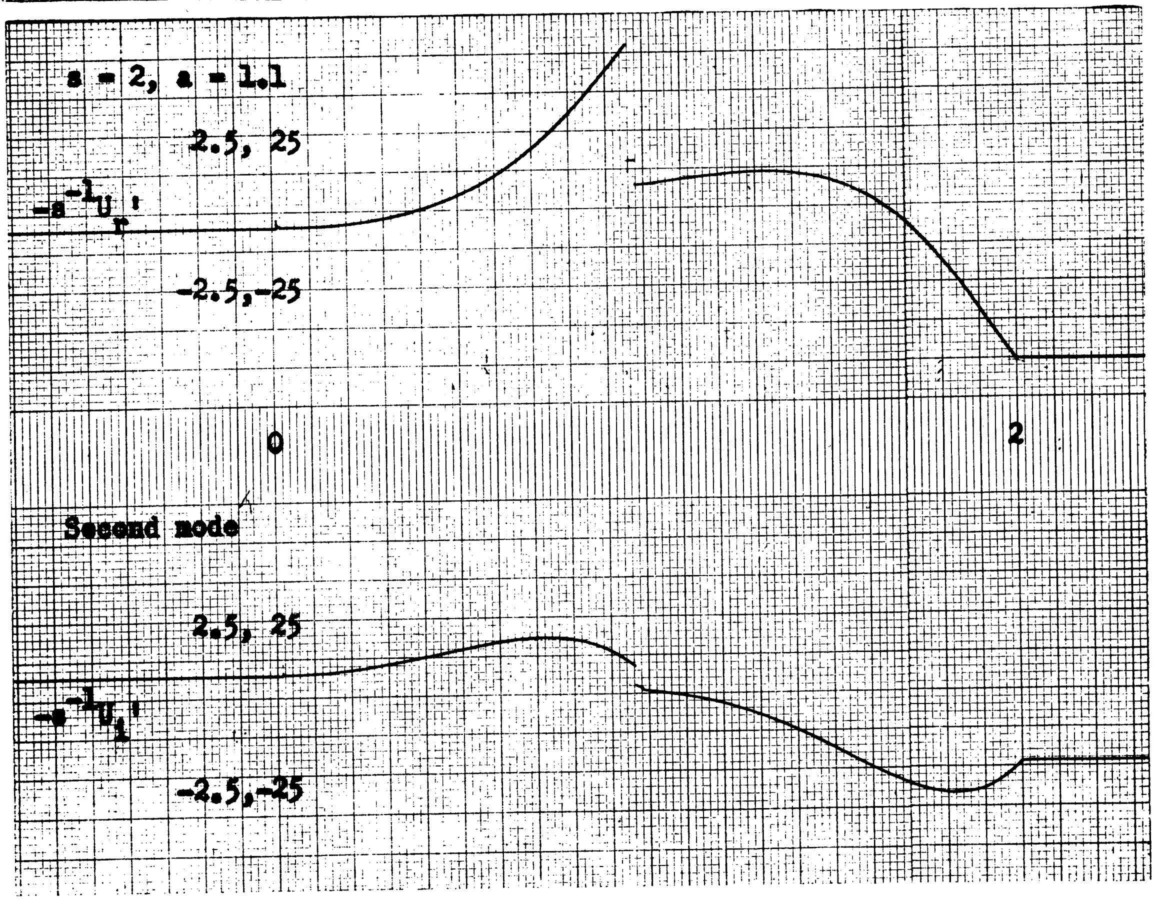
U_r
-2.5,-25
0
2
Second mode
2.5, 25
U_i
-2.5,-25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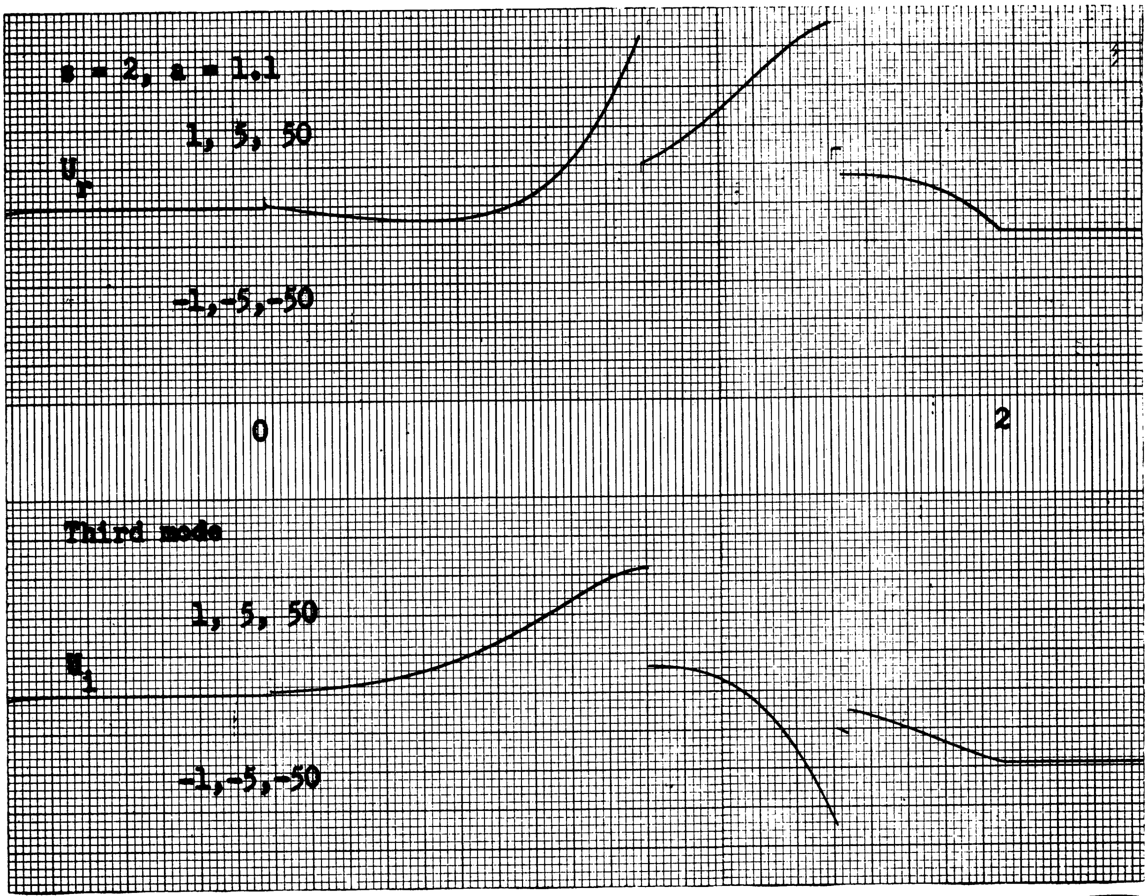
-2.5,-25
0
2
Second mode
2.5, 25
$-s^{-1}U_i'$
-2.5,-25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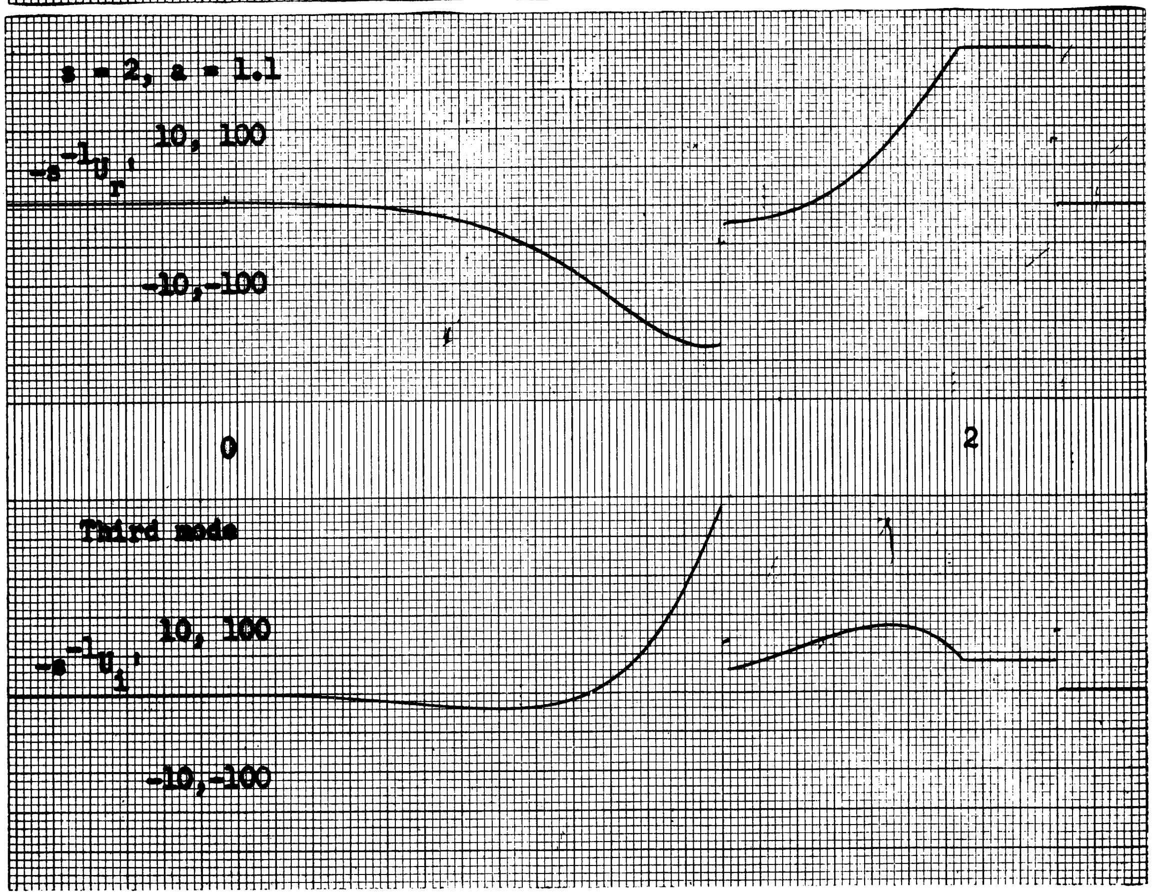
1, 5, 50
U_r
-1,-5,-50
0
2
Third mode
1, 5, 50
U_i
-1,-5,-50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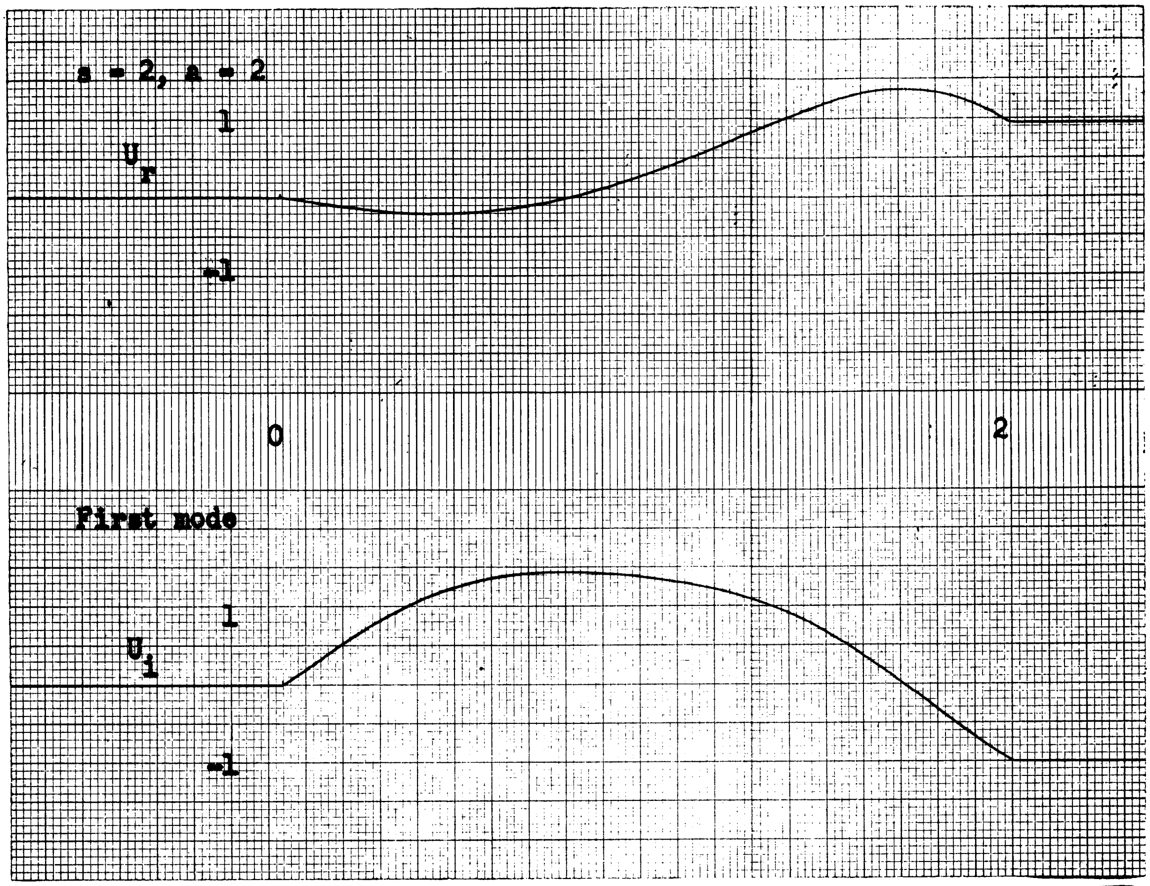
-s^-1 U_r'
-10,-100
0
2
Third mode
10, 100
-s^-1 U_i'
-10,-100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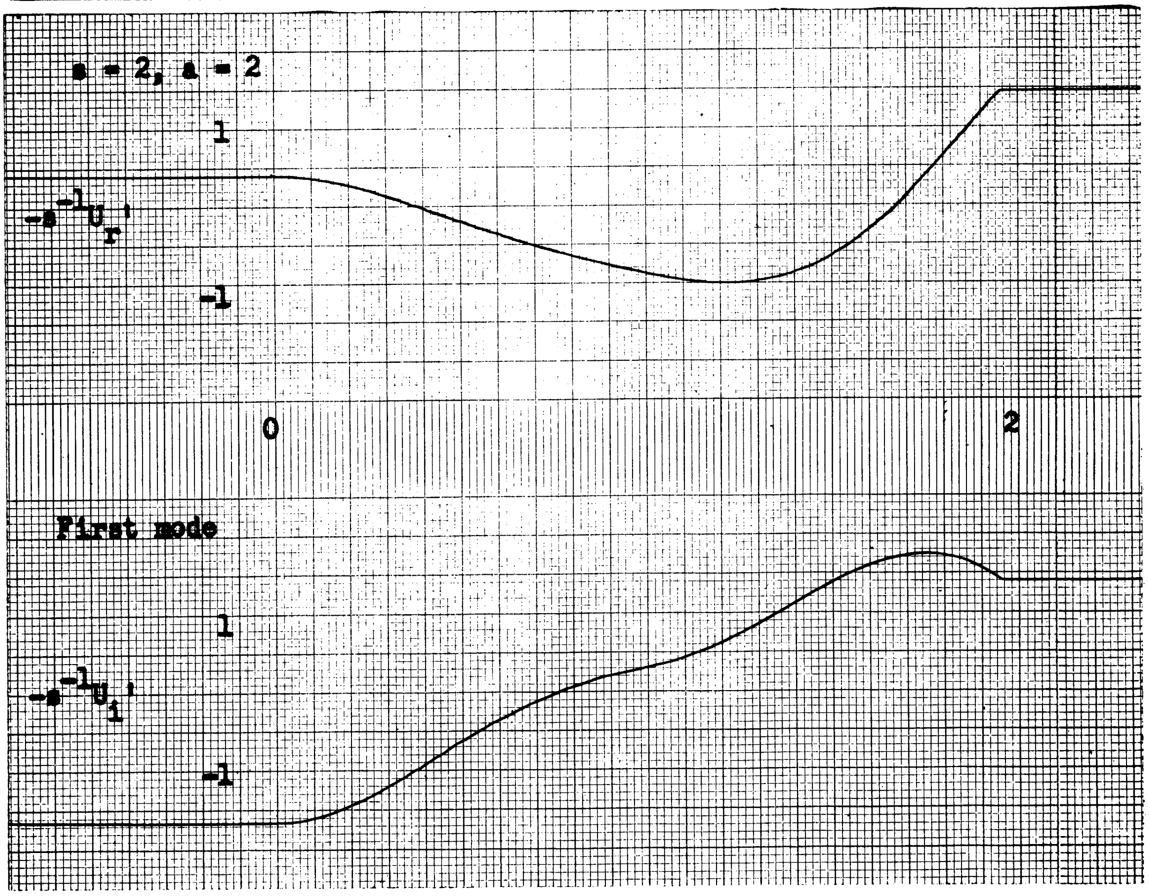
1
U_r
-1
0
2
First mode
1
U_i
-1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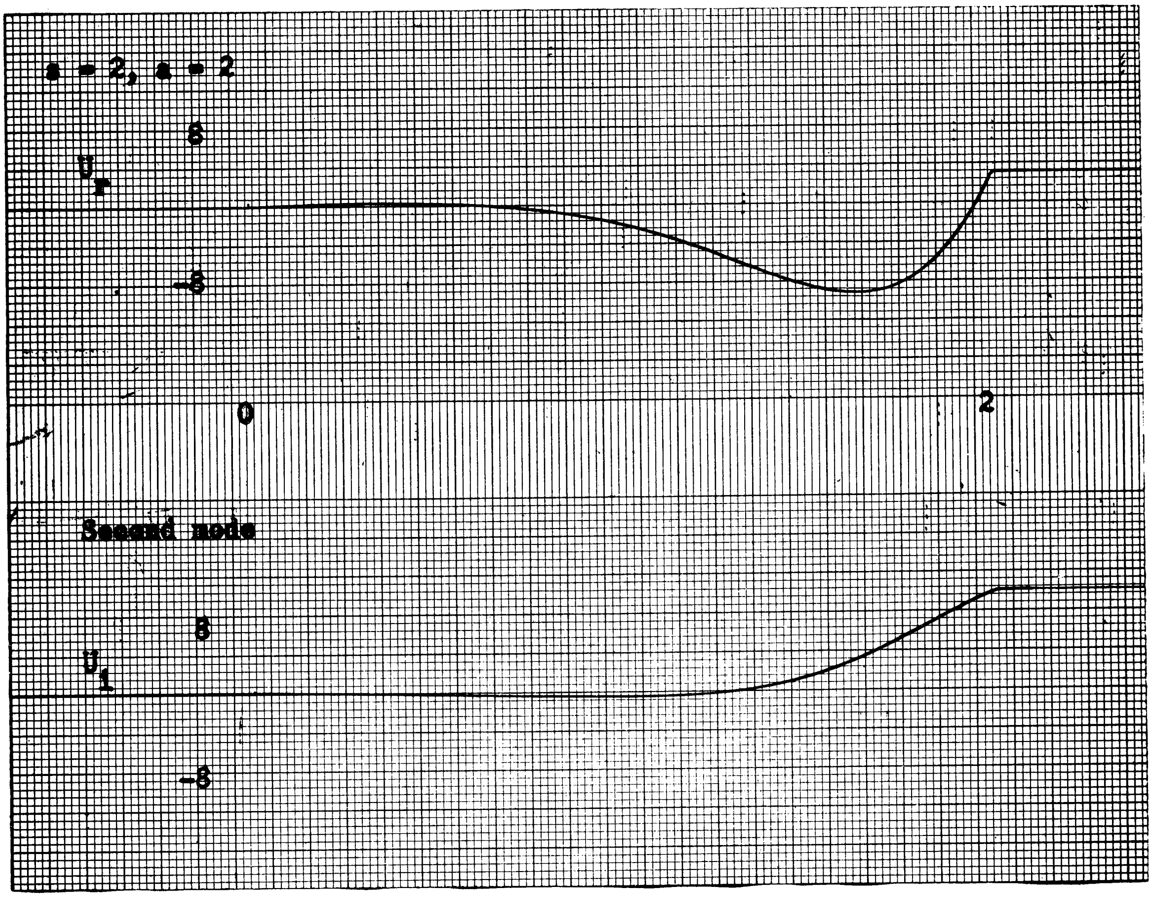
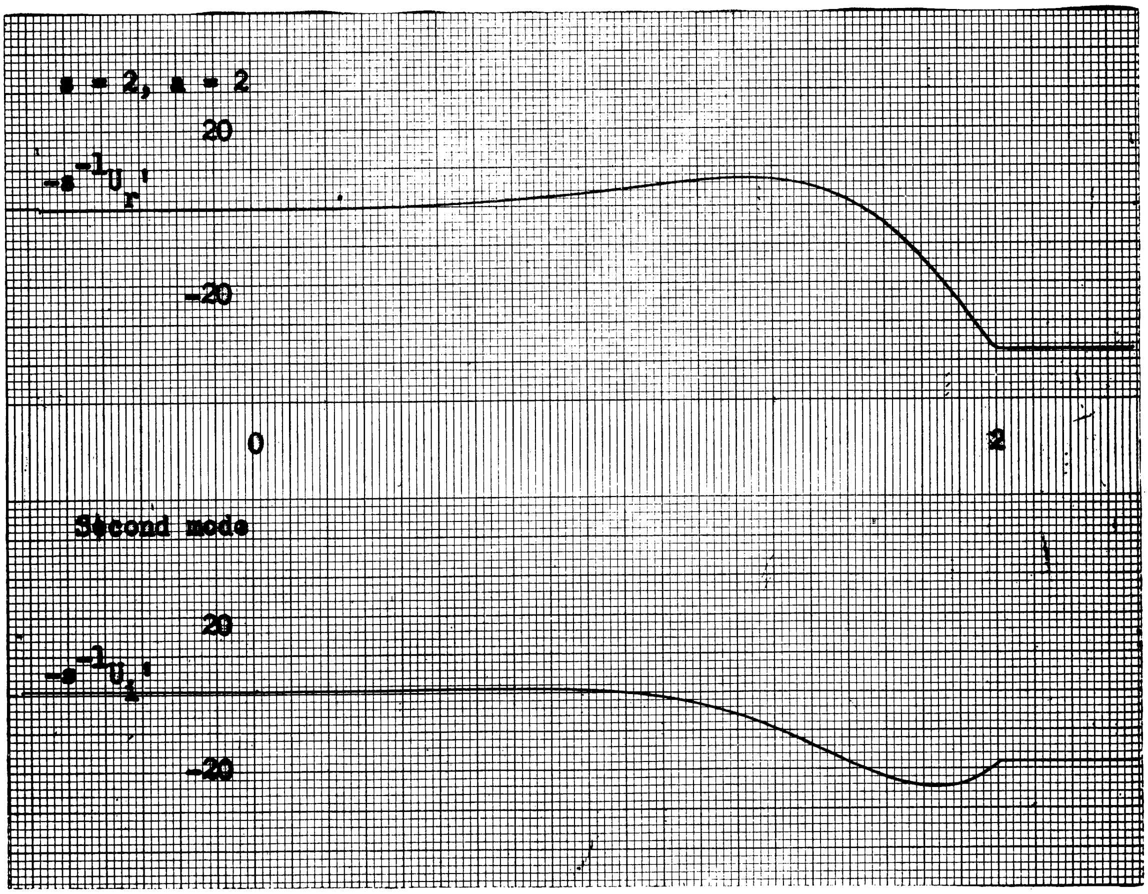
-1
0
2
First mode
1
-s^-1 U_i'
-1


8
U_r
-8
0
2
Second mode
8
U_1
-8


20
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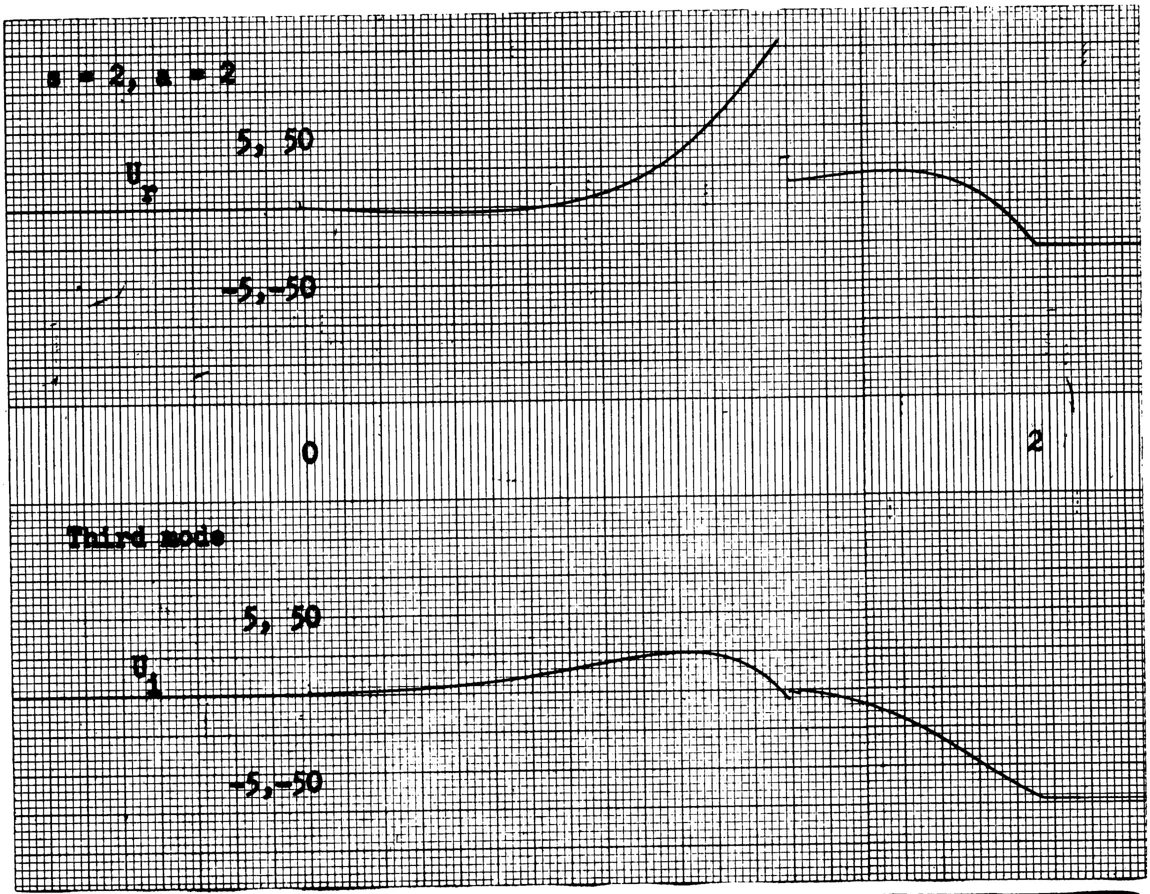
-20
0
2
Second mode
20
-s^-1 U_1'
-20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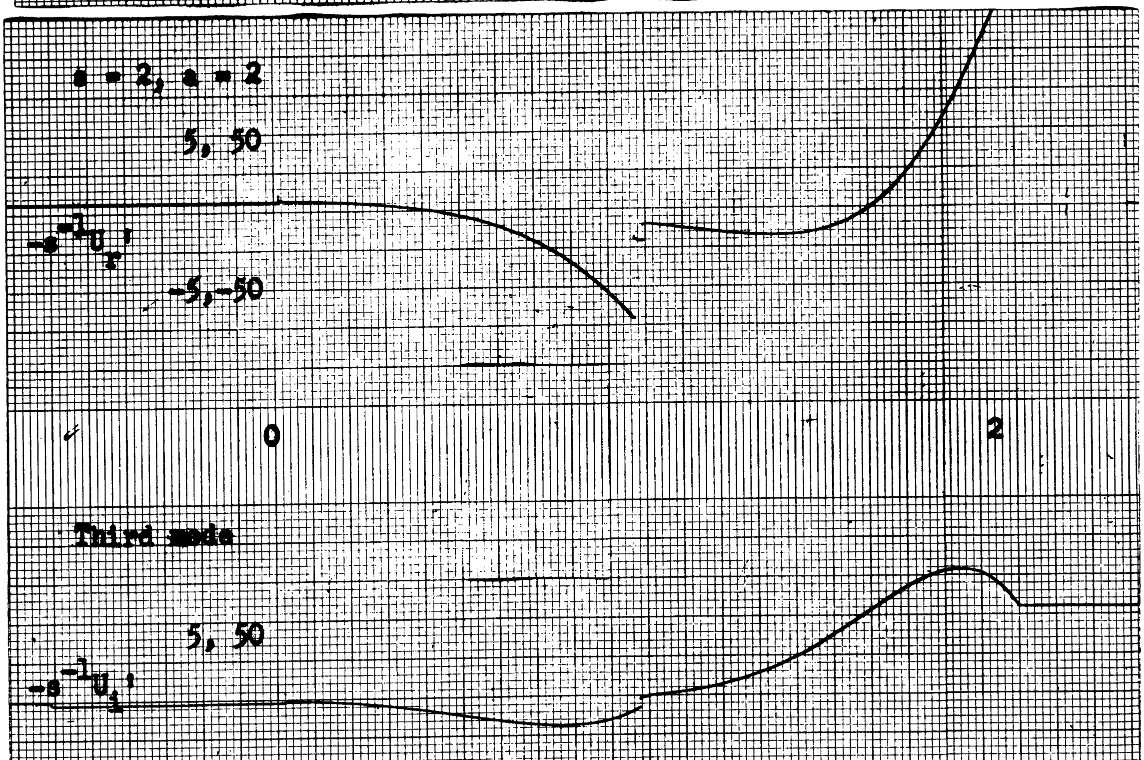
5, 50
U_r
-5,-50
0
2
Third mode
5, 50
U_i
-5,-50


s = 2, a = 2
5, 50
$-s^{-1}U_r'$
-5,-50
0
2
Third mode
5, 50
$-s^{-1}U_i'$
-5,-50

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