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SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS BY DIFFERENCE METHODS USING THE ELECTRONIC DIFFERENTIAL ANALYZER

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1. Introduction

During the third quarterly period some nonlinear partial differential equations were considered, as well as linear equations. Considerable effort was spent studying the application of difference techniques to the flexural vibrations of beams. A technical report which will cover all the differential analyzer and theoretical work on beams is being prepared.

2. Solution of Nonlinear Problems in Heat Transfer

In the previous report periods the dynamic equations of heat flow in one and two spacial dimensions were studied. For this report the problem of heat flow in a medium where the conductivity is a linear function of the temperature was considered. In this case the equation describing the temperature u within a slab of unit thickness is

$$\frac{\partial}{\partial x} K_o u \frac{\partial u}{\partial x} = c \delta \frac{\partial u}{\partial t} \tag{1}$$

with the boundary conditions

$$u(0,t) = 0, \quad \frac{\partial u(1,t)}{\partial x} = 0 \tag{2}$$

where x is distance through the slab, t is time, $K_o u$ is the conductivity, c is the specific heat, and δ is the density in the slab. The initial condition was assumed to be a constant temperature U_o.

As discussed in a previous report the problem is solved by breaking the slab into N cells and by considering the temperature only at the N stations in x. This transforms the partial differential equation (1) into a system of N simultaneous ordinary differential equations. At the n'th station the equation is

$$c\delta \frac{du_n}{dt} = \frac{K_o}{\Delta x}\left[\frac{u_{n+1} + u_n}{2\Delta x}(u_{n+1} - u_n) - \frac{u_n + u_{n-1}}{2\Delta x}(u_n - u_{n-1})\right]$$

or

$$c\delta \frac{du_n}{dt} = \frac{K_o}{2(\Delta x)^2}\left[u_{n+1}^2 - 2u_n^2 + u_{n-1}^2\right] \qquad (3)$$

The u^2 terms were computed by means of servo multipliers. The problem was solved for 4, 5, and 6 cells, and the resulting temperature distributions as a function of time were within several percent of being the same, indicating that higher numbers of cells would not increase the accuracy significantly. It is hoped that check solutions using a digital computer can be used to assess accurately the errors of the differential analyzer in solving this particular nonlinear problem.

3. Solution of Free-Free Beams

One of the most critical problems which occurs when using the difference method involves the transverse vibrations of a free-free beam. Here the beam is unrestrained by end fastenings (unlike a cantilever beam or a hinged-hinged beam). When the problem is set up on the differential analyzer, any unbalances in the many amplifiers used to solve the problem act as forces applied at the various stations along the beam. In general the net total force and total moment acting on the beam as a result of these separate forces will be finite. This causes the entire beam to drift and rotate slightly when the initial conditions are released and the problem solution begun.

A 7-cell approximation to a uniform free-free beam was set up on the electronic differential analyzer. It was found that when the amplifiers were balanced carefully, the lateral displacements along the beam would stay within the linear region of the computer (±100 volts) for a number of fundamental-mode periods. Thus it was possible to study the normal-mode frequencies and other dynamic effects with the free-free beam.

Normal-mode frequencies for the first three modes were measured from the 7-cell computer circuit and agreed within several tenths of a percent with theory for a seven-cell beam. Normal modes of the circuit were generated by establishing the mode shape as initial conditions or by driving the beam at the loop points with sinusoidal input forces having the normal-mode frequency.

The conclusion of these tests is that the free-free beam can definately be handled by the electronic differential analyzer using the difference technique. Thus a complete aircraft wing, for example, can be represented dynamically in lateral displacement on the computer.

Many of the operational amplifiers being used for these problems are currently being equipped with drift-stabilizing circuits. This should help even more the stability of the cellular representation of free-free beams.

4. Hinged-Hinged Beam with Concentrated Mass at Center

An 8-cell beam with hinged supports on both ends was set up using the difference method. Mode-shapes and frequencies were checked satisfactorily with theory. Then a concentrated mass was added to the center of the beam. For the differential analyzer circuit employing the difference method the mass was effectively distributed over 1/8 th of the beam instead of being concentrated precisely at the center. Results still showed good agreement (better than one percent with the theoretical first-mode frequency) for a large range of concentrated masses at the center.

5. Cantilever Beam with Concentrated Load at the Free End

Considerable time was spent studying the computer solutions to 8-cell uniform cantilever beams having a concentrated mass load at the free end. Here the difficulty is that for the 8-cell beam the mass representation must effectively be distributed over the last 1/8 of the beam instead of being at the very end. The equivalent concentrated mass would then seem to be about 1/16 th of the way from the free end. For mass loads up to one-half the total mass of the beam this error is not too serious(about 5% deviation in first-mode frequency from the value for a mass truly concentrated at the very end). However, for larger end masses the errors in normal-mode frequencies can get as high as 10%.

Unsuccessful attempts were made to rewrite the equations in such a way as to make the end-mass occur effectively at the free end itself. The basic difficulty is that the mass characteristics occur at integral stations, whereas the free end of the beam must occur at a half-integral station.

6. Cantilever Beam with Nonlinear Damping

At the suggestion of Dr. J. H. Giese, Ballistics Laboratories, Aberdeen Proving Ground, a uniform cantilever beam described by the following equation was set up.

$$\frac{\partial^4 y}{\partial x^4} + c\frac{\partial y}{\partial t}\left|\frac{\partial y}{\partial t}\right| + \frac{\partial^2 y}{\partial t^2} = f(x,t) \qquad (3)$$

with boundary condition

$$y(0,t) = \frac{\partial y}{\partial x}(0,t) = \frac{\partial^2 y}{\partial x^2}(1,t) = \frac{\partial^3 y}{\partial x^3}(1,t) = 0 \qquad (4)$$

This represents an aerodynamically damped beam of unit length and having lateral displacement y. The equation is nonlinear because of the velocity squared term, and hence the solution depends on the magnitude of the initial conditions as well as the damping-constant c. However, it turns out that change in the magnitude of initial conditions is equivalent to a change in c, so that when $f(x,t) = 0$ there is not the multiplicity of different solutions which one might at first expect.

This problem was set up for 5 cells on the electronic differential analyzer and a number of solutions were obtained for different initial conditions and various input forces $f(x,t)$. No difficulty was experienced and solutions for more cells will be run off in the future to allow an estimate of convergence to the exact solution.

These and other extensive beam studies will be summarized in a technical report currently being prepared.