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VISCOELASTIC FLOW PAST A WEDGE WITH A SOLUBLE COATING

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I. INTRODUCTION

In recent years the study of viscoelastic fluids has become important and active because of the broad applications of these fluids in industry and science. These applications vary greatly in scope, including such diverse topics as the transporting and processing of substances, reduction of drag forces on moving bodies or in pipe lines, use as lubricants with unusual properties, and for substitution of body fluids. Experiments have shown that several phenomena observable in these fluids are not predicted by the classical theory of viscous flow. This has led to the formulation of rheological equations of state of greater complexity than those for the Newtonian fluid.

Many rheological models have been proposed to describe the mechanical behavior of viscoelastic materials. In 1962 Williams and Bird[1] discussed these proposed models and concluded that of the relatively simple ones, Oldroyd's model[2] is the most reasonable one to represent viscoelastic materials at the present time. They also used the model to study steady viscoelastic flow in tubes. By a proper choice of material constants they obtained results showing good agreement with experimental data up to moderate rates of shear.[1,3] The model used exhibits the main non-Newtonian flow properties observed in flowing viscoelastic liquids such as polymers and colloidal solutions. Those properties are: a variable apparent viscosity which decreases with increasing rate of shear in simple shear, a Weissenberg climbing effect, and a Robert-Weissenberg normal stress pattern.

Leslie[4] used Oldroyd's model to study the creeping flow past a sphere by using perturbation techniques. His calculated drag force agrees with experimental results at low flow rates. Due to the complexity of the model there have been no other solutions for complicated flow reported in the literature. However, some problems using special cases of Oldroyd's model (assuming some of the material properties to be zero in order to make the problem mathematically tractable) have been reported. In 1961 Jones[5] considered inelastic liquids in the study of flow past a plate. In late 1962, Kulshrestha[6] considered a very special case of Oldroyd's model for helical flow. Tanner[7] also reported a solution of the Rayleigh-Benard problem for Oldroyd's fluid B.

In this study Oldroyd's model (1958) has been used to describe the mechanical behavior of viscoelastic materials for a steady, two dimensional, incompressible flow past a semi-infinite flat plate coated with viscoelastic materials. For the purpose of analysis, it is assumed that the coating is soluble in the main flow and that the mixture of the coating in the main flow is of small enough concentration to have constant diffusivity as well as constant density.

The equations of motion and diffusion are obtained by a boundary layer analysis. It is found that the set of partial differential equations has a similarity solution only when the external stream velocity is proportional to the cube of the distance along the plate. This represents a flow of Falkner-Skan type past a wedge of 90 degrees.

The main purpose of this study is to investigate how the frictional force is affected by the properties of the viscoelastic material and by diffusion. A detailed investigation is preformed for a flow past the wedge of 90 degrees with and without diffusion. The method of steepest descent, which has been shown to be a highly efficient and accurate method in solving boundary layer equations,[8] is employed here to solve the set of complicated ordinary non-linear differential equations.

II. CONSTITUTIVE EQUATIONS

For the idealized viscoelastic liquids considered here, the stress s_{ij} at any point in the flow may be considered as the superposition of two independent stress systems, that is

$$s_{ij} = - p\, g_{ij} + p_{ij}$$

in which g_{ij} are the components of the metric tensor, p is a scalar (not necessarily the pressure), and p_{ij} contains the non-isotropic part of the stress tensor. In 1958 Oldroyd[2] proposed a mathematical model for p_{ij} which qualitatively describes many effects observed in real viscoelastic fluids. The proposed rheological equations of state relating p_{ij} and the rate of deformation tensor

$$d_{ij} \equiv \frac{1}{2} \left(\frac{\partial u_i}{\partial x^i} + \frac{\partial u_j}{\partial x^j} \right)$$

are

$$p_{ik} + \lambda_1' \frac{\mathscr{D} p_{ik}}{\mathscr{D} t} - \mu_1' \left(p_i^{\ j} d_{jk} + p^j_{\ k} d_{ij} \right) + \nu_1' \, p_{j\ell} \, d^{j\ell} \, g_{ik} + \mu_o' \, p_{jj} d_{ik}$$

$$= 2 \mu' \left[d_{ik} + \lambda_2' \frac{\mathscr{D} d_{ik}}{\mathscr{D} t} - 2 \mu_2' \, d_{ij} \, d^j_{\ k} + \nu_2' \, d_{j\ell} d^{j\ell} g_{ik} \right] \qquad (1)$$

Here μ', λ_1', and λ_2' are the viscosity, relaxation time, and retardation time of the material, respectively, at very small rates of strain, and μ_1', μ_2', ν_1', ν_2' and μ_0' are five material constants with the dimension of time. $\frac{\mathscr{D}}{\mathscr{D}t}$ is the Jaumann derivative, and u_i are the components of the velocity vector.

The Jaumann derivative is a time derivative of the components of a tensor as measured with respect to a rigid coordinate system which translates and rotates with a fluid particle. This derivative (as well as Oldroyd's convected derivative) satisifies the requirement of invariance of response in rheological equations of state. When the Jaumann derivative operates on a second order tensor with components b_{ij} and is transformed to the fixed coordinates x^i, one has in cartesian coordinates

$$\frac{\mathcal{D} b_{ij}}{\mathcal{D} t} = \frac{\partial b_{ij}}{\partial t} + u^k \frac{\partial b_{ij}}{\partial x^k} - \omega_{ik} \, b^k{}_j + \omega_{kj} b_i{}^k$$

in which ω_{ij} are the components of the vorticity tensor; i.e.

$$\omega_{ij} \equiv \frac{1}{2} \left(\frac{\partial u_i}{\partial x^j} - \frac{\partial u_j}{\partial x^i} \right)$$

The Jaumann derivative operating on either contravariant or covariant components of a tensor will result in the same form since

$$\frac{\mathcal{D} g_{ij}}{\mathcal{D} t} = 0 \quad \text{and} \quad \frac{\mathcal{D} g^{ij}}{\mathcal{D} t} = 0$$

where g_{ij} are the components of the metric tensor. Thus, the use of the Jaumann derivative removes an objection against Oldroyd's convected derivative, which in general gives different forms for contravariant and covariant components of a tensor. In any case either the Jaumann or the Oldroyd derivative may be used in Equation (1); the only difference is in interpretation of the parameters μ_0' and μ_2'.

To understand some of the implications of the constitutive equation (1), a few simple solutions will be described:

(a) Simple shearing flow[2]: If the velocity components are taken to be of the form

$$v_1 = v_2 = 0 , \quad v_3 = \gamma x_2$$

where γ = constant, then the constitutive equations give

$$\begin{aligned} p_{11} &= [\nu_2'\mu' - \nu_1' F(\gamma)]\, \gamma^2 \\ p_{22} &= p_{11} + [(\lambda_2' - \mu_2')\, \mu' - (\lambda_1' - \mu_1')\, F(\gamma)]\, \gamma^2 \\ p_{33} &= p_{11} - [(\lambda_2' + \mu_2')\, \mu' - (\lambda_1' + \mu_1')\, F(\gamma)]\, \gamma^2 \\ p_{23} &= \gamma\, F(\gamma) \quad , \quad p_{13} = p_{12} = 0 \end{aligned} \tag{2}$$

in which

$$\begin{aligned} F(\gamma) &\equiv \mu'(1 + \sigma_2\gamma^2)/(1 + \sigma_1\gamma^2) \\ \sigma_1 &\equiv \lambda_1'^2 + \mu_0'(\mu_1' - \tfrac{3}{2}\nu_1') - \mu_1'(\mu_1' - \nu_1') \\ \sigma_2 &\equiv \lambda_1'\lambda_2' + \mu_0'\,(\mu_2' - \tfrac{3}{2}\nu_2') - \mu_1'(\mu_2' - \nu_2') \end{aligned} \tag{3}$$

The results obtained in Equation (2) show that the normal stresses are in general unequal and, therefore, in order to maintain a simple shearing flow, not only a shear stress but also normal stresses should be applied. If the material constants are related in the manner $\lambda_1' = \lambda_2'$, $\mu_1' = \mu_2'$, and either $\nu_1' = \nu_2'$ or $3\mu_0' = 2\mu_1'$, none of the non-Newtonian effects will be observed in this flow.

(b) Steady flow in a circular pipe[1]: In this flow cylindrical coordinates (r,θ,z) are convenient and the velocity components are taken to be

$$v_r = v_\theta = 0 , \quad v_z = v_z(r)$$

The equations of motion and constitutive equations (1) then give

$$- \frac{\partial p}{\partial z} = P \quad , \quad p_{rz} = \frac{1}{2} P r \qquad (4)$$

with shear stresses and normal stresses given by Equation (2), the suffixes 1, 2, 3 being replaced by Θ , r , z , γ now denoting $- dv_z/dr$. For a given pressure gradient P , Equation (4) shows that the shear stress profile in a pipe flow is exactly the same as that of Newtonian liquids. However, Equation (2) shows that the normal stresses and the velocity profile are different from that of the Newtonian case. Because of this the volumetric rate of flow of a pseudoplastic liquid ($\sigma_2/\sigma_1 < 1$) is larger than that of a Newtonian liquid at a given pressure gradient; in the case of a dilatant fluid ($\sigma_2/\sigma_1 > 1$), it is smaller. Representative values for the material parameters are[1]:

Material	$\sigma_1(sec^2)$	σ_2/σ_1	μ'(poise)
4.0% Aqueous Carboxymethy-cellulose at 85° F (low flow rate)	2.53×10^{-4}	0.67	1.52
Cholesterylbutyrate at 100° C (high flow rate)	0.0297	0.45	0.862

The tensile normal stress p_{zz} is one of the causes of the Merrington effect[3] -- a jet swelling as it exits from a tube.

(c) Flow between rotating vertical cylinders[2] : In a cylindrical coordinate system the only non-vanishing velocity component for this problem is taken to be $v_\Theta = r\Omega(r)$. The stresses obtained from the constitutive equations (1) are given by Equation (2) with the suffixes 1, 2, 3 replaced by z , r, Θ respectively and where γ now denotes $r d\Omega/dr$. From the equations of motion one has

$$- s_{zz} = p - p_{zz} = - \rho g z + \int \rho r \Omega^2 dr + p'(r) + \text{const.} \qquad (5)$$

in which

$$p'(r) = p_{rr} - p_{zz} + \int \frac{1}{r} (p_{rr} - p_{\Theta\Theta})\, dr \qquad (6)$$

If the liquid has the properties $\sigma_1 = \sigma_2 = 0$, one obtains

$$p'(r) = 2\mu'(2\mu_1' - \lambda_1' - 2\mu_2' + \lambda_2')\, M\, r^{-4} \qquad (7)$$

where M is a constant relating to the couples applied on the cylinders. (This is a result including Rivlin's liquid C (1948) and Oldroyd's liquids A and B (1950).) Equation (7) indicates the deviation of s_{zz} from that found in a Newtonian fluid; this may cause the climbing or sinking of the free surface near the inner cylinder (Weissenberg effect).

(d) Flow between a flat plate and a rotating wide-angled cone[2,3]: Assuming that the only non-vanishing velocity component in spherical coordinates (r,Θ,ϕ) is $v_\phi = r\Omega(\Theta)\sin\Theta$, the constitutive equations (1) then give the stresses shown in Equation (2) with suffixes 1, 2, 3 replaced by r , Θ , ϕ and where γ now denotes $\sin\Theta \; d\Omega/d\Theta$. Many experimental measurements have been made on the normal stresses and some of these show that

$$p_{rr} = p_{\Theta\Theta} \neq p_{\phi\phi} \qquad (8)$$

This relation is known as the Roberts-Weissenberg normal stress relation. Equation (2) can also give the relation (8) if one sets $\lambda_1' = \mu_1'$ and $\lambda_2' = \mu_2'$.

From the simple examples above it is seen that Oldroyd's model gives at least a qualitative representation of some of the effects observed in viscoelastic liquids; it is also probably the simplest model available at present time which will describe all of these effects.

In the present study a steady, two dimensional, incompressible fluid flow problem is considered. Using Cartesian coordinates (x', y') directed along and perpendicular to the body, the constitutive equations (1) along with the continuity equation yield

$$p_{xx} + \lambda_1' \left[\frac{Dp_{xx}}{Dt} + \left(\frac{\partial v'}{\partial x'} - \frac{\partial u'}{\partial y'}\right) p_{xy} \right] - \mu_1' \left[2\frac{\partial u'}{\partial x'} p_{xx} + \left(\frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'}\right) p_{xy} \right]$$

$$+ \nu_1' \left[\frac{\partial u'}{\partial x'} p_{xx} + \left(\frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'}\right) p_{xy} + \frac{\partial v'}{\partial y'} p_{yy} \right] + \mu_0' \frac{\partial u'}{\partial x'} \left(p_{xx} + p_{yy} + p_{zz}\right)$$

$$= 2\mu' \left\{ \frac{\partial u'}{\partial x'} + \lambda_2' \left[\frac{D}{Dt}\left(\frac{\partial u'}{\partial x'}\right) + \frac{1}{2}\left(\frac{\partial v'}{\partial x'} - \frac{\partial u'}{\partial y'}\right)\left(\frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'}\right) \right] - 2\mu_2' \left[\left(\frac{\partial v'}{\partial x'}\right)^2 \right. \right.$$

$$\left. \left. + \frac{1}{4}\left(\frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'}\right)^2 \right] + \nu_2' \left[\left(\frac{\partial u'}{\partial x'}\right)^2 + \frac{1}{2}\left(\frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'}\right)^2 + \left(\frac{\partial v'}{\partial y'}\right)^2 \right] \right\}$$

$$p_{xy} + \lambda_1' \left[\frac{Dp_{xy}}{Dt} + \frac{1}{2}\left(\frac{\partial v'}{\partial x'} - \frac{\partial u'}{\partial y'}\right)\left(p_{yy} - p_{xx}\right) \right] - \frac{\mu_1'}{2}\left(\frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'}\right)\left(p_{xx} + p_{yy}\right)$$

$$+ \frac{\mu_0}{2}\left(\frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'}\right)\left(p_{xx} + p_{yy} + p_{zz}\right)$$

$$= 2\mu' \left\{ \frac{1}{2}\left(\frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'}\right) + \lambda_2' \left[\frac{D}{Dt}\left(\frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'}\right) + \frac{1}{2}\left(\frac{\partial v'}{\partial x'} - \frac{\partial u'}{\partial y'}\right) \times \left(\frac{\partial v'}{\partial y'} - \frac{\partial u'}{\partial x'}\right)\right]\right\}$$

$$p_{yy} + \lambda_1' \left[\frac{Dp_{yy}}{Dt} + \left(\frac{\partial u'}{\partial y'} - \frac{\partial v'}{\partial x'}\right)p_{xy}\right] - \mu_1' \left[\left(\frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'}\right)p_{xy} + 2\frac{\partial v'}{\partial y'}p_{yy}\right]$$

$$+ \nu_1' \left[\frac{\partial u'}{\partial x'}p_{xx} + \left(\frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'}\right)p_{xy} + \frac{\partial v'}{\partial y'}p_{yy}\right] + \mu_0' \left[\frac{\partial v'}{\partial y'}\left(p_{xx} + p_{yy} + p_{zz}\right)\right]$$

$$= 2\mu' \left\{\frac{\partial v'}{\partial y'} + \lambda_2' \left[\frac{D}{Dt}\left(\frac{\partial v'}{\partial y'}\right) + \frac{1}{2}\left(\frac{\partial u'}{\partial y'} - \frac{\partial v'}{\partial x'}\right)\left(\frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'}\right)\right]\right.$$

$$\left. - 2\mu_2' \left[\left(\frac{\partial v'}{\partial y'}\right)^2 + \frac{1}{4}\left(\frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'}\right)^2\right] + \nu_2' \left[\left(\frac{\partial u'}{\partial x'}\right)^2 + \frac{1}{2}\left(\frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'}\right) + \left(\frac{\partial v'}{\partial y'}\right)^2\right]\right\}$$

$$p_{zz} + \lambda_1' \frac{Dp_{zz}}{Dt} + \nu_1' \left[\frac{\partial u'}{\partial x'}p_{xx} + \left(\frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'}\right)p_{xy} + \frac{\partial v'}{\partial y'}p_{yy}\right]$$

$$= 2\mu\nu_2' \left[\left(\frac{\partial u'}{\partial x'}\right)^2 + \frac{1}{2}\left(\frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'}\right)^2 + \left(\frac{\partial v'}{\partial y'}\right)^2\right]$$

$$p_{xz} + \lambda_1' \left[\frac{Dp_{xz}}{Dt} + \frac{1}{2}\left(\frac{\partial v'}{\partial x'} - \frac{\partial u'}{\partial y'}\right)p_{yz}\right] - \mu_1' \left[\frac{\partial u'}{\partial x'}p_{xz} + \frac{1}{2}\left(\frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'}\right)p_{yz}\right] = 0$$

$$p_{yz} + \lambda_1' \left[\frac{Dp_{yz}}{Dt} + \frac{1}{2}\left(\frac{\partial u'}{\partial y'} - \frac{\partial v'}{\partial x'}\right)p_{xz}\right] - \mu_1' \left[\frac{\partial v'}{\partial y'}p_{yz} + \frac{1}{2}\left(\frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'}\right)p_{xz}\right] = 0 \qquad (9)$$

where $\frac{D}{Dt}$ is the substantial derivative, u' and v' are the velocity components in the x' - and y' - directions, respectively.

To make equations (9) dimensionless, the density ρ , the viscosity η_0 , a characteristic free stream velocity U_∞ , and a characteristic length L are chosen as reference quantities. A Reynolds number Re can then be defined as

$$Re = \frac{\rho U_\infty L}{\eta_0} \equiv \frac{1}{\epsilon^2} \text{ , say,}$$

where $\epsilon << 1$ for the case under consideration. In order to perform a boundary layer analysis one lets

$$x' = Lx \text{ , } y' = \epsilon Ly, \; u' = U_\infty u, \; v' = \epsilon U_\infty v, \; \mu' = \eta_0 \mu \qquad (10)$$

$$(\lambda_1', \mu_1', \mu_0', \lambda_2', \mu_2', \nu_1', \nu_2') = \epsilon \frac{L}{U_\infty} (\lambda_1, \mu_1, \mu_0, \lambda_2, \mu_2, \nu_1, \nu_2) \qquad (11)$$

$$(p_{xx}, p_{xy}, p_{yy}, p_{zz}) = \epsilon \rho U_\infty^2 (\tau_{xx}, \tau_{xy}, \tau_{yy}, \tau_{zz}) \qquad (12)$$

where unprimed quantities in Equations (10) and (11) are assumed to be of order one or less. For the purposes of the present analysis, in writing Equation (11) it was assumed that the order of the seven dimensionless material constants are all equal to or less than that of ϵ . This requirement ensures that the flow of a Newtonian fluid will be a limiting case. The order of magnitude of stresses shown in Equation (12) is a consequence of Equation (11) for large Reynolds number. This can be verified by the substitution of Equations (10) and (11) into Equation (9).

Now the substitution of Equations (10) - (12) into Equation (9), for large Reynolds number, yields

$$\left.\begin{aligned}
&\tau_{xx} - (\lambda_1 + \mu_1 - \nu_1) \frac{\partial u}{\partial y} \tau_{xy} = -\mu(\lambda_2 + \mu_2 - \nu_2) \left(\frac{\partial u}{\partial y}\right)^2 + O(\epsilon) \\
&\tau_{xy} + \frac{1}{2}(\lambda_1 - \mu_1 + \mu_0) \frac{\partial u}{\partial y} \tau_{xx} - \frac{1}{2}(\lambda_1 + \mu_1 - \mu_0) \frac{\partial u}{\partial y} \tau_{yy} + \frac{\mu_0}{2} \frac{\partial u}{\partial y} \tau_{zz} = \\
&\mu \frac{\partial u}{\partial y} + O(\epsilon) \\
&\tau_{yy} + (\lambda_1 - \mu_1 + \nu_1) \frac{\partial u}{\partial y} \tau_{xy} = \mu(\lambda_2 - \mu_2 + \nu_2) \left(\frac{\partial u}{\partial y}\right)^2 + O(\epsilon) \\
&\tau_{zz} + \nu_1 \frac{\partial u}{\partial y} \tau_{xy} = \mu\nu_2 \left(\frac{\partial u}{\partial y}\right)^2 + O(\epsilon) \\
&p_{xz} - \frac{1}{2}(\lambda_1 + \mu_1) \frac{\partial u}{\partial y} p_{yz} = O(\epsilon\, p_{xz}) \\
&p_{yz} + \frac{1}{2}(\lambda_1 - \mu_1) \frac{\partial u}{\partial y} p_{xz} = O(\epsilon\, p_{xz})
\end{aligned}\right\} \quad (13)$$

From Equation (13) one can obtain explicit forms for the τ_{ij} in terms of the shear rate $\frac{\partial u}{\partial y}$. They are, upon neglecting higher order terms in ϵ,

$$\tau_{xx} = -\mu \left[(\lambda_2 + \mu_2 - \nu_2) - (\lambda_1 + \mu_1 - \nu_1) \frac{1 + \Lambda_2 \left(\frac{\partial u}{\partial y}\right)^2}{1 + \Lambda_1 \left(\frac{\partial u}{\partial y}\right)^2} \right] \left(\frac{\partial u}{\partial y}\right)^2 \quad (14)$$

$$\tau_{xy} = \mu \frac{\partial u}{\partial y} \frac{1 + \Lambda_2\left(\frac{\partial u}{\partial y}\right)^2}{1 + \Lambda_1\left(\frac{\partial u}{\partial y}\right)^2} \tag{15}$$

$$\tau_{yy} = \mu \left[(\lambda_2-\mu_2+\nu_2) - (\lambda_1-\mu_1+\nu_1) \frac{1 + \Lambda_2\left(\frac{\partial u}{\partial y}\right)^2}{1 + \Lambda_1\left(\frac{\partial u}{\partial y}\right)^2} \right] \left(\frac{\partial u}{\partial y}\right)^2 \tag{16}$$

$$\tau_{zz} = \mu \left[\nu_2-\nu_1 \frac{1 + \Lambda_2\left(\frac{\partial u}{\partial y}\right)^2}{1 + \Lambda_1\left(\frac{\partial u}{\partial y}\right)^2} \right] \left(\frac{\partial u}{\partial y}\right)^2 \tag{17}$$

in which

$$\Lambda_1 \equiv \lambda_1^2 + \mu_0(\mu_1 - \frac{3}{2} \nu_1) - \mu_1(\mu_1-\nu_1) \tag{18}$$

$$\Lambda_2 \equiv \lambda_1\lambda_2 + \mu_0(\mu_2 - \frac{3}{2} \nu_2) - \mu_1(\mu_2- \nu_2) \tag{19}$$

The last two equations of Equation (13) show that

$$p_{xz} = p_{yz} = 0 \tag{20}$$

Equation (15) indicates that in general the apparent viscosity, defined by

$$\mu_{\text{apparent}} \equiv \frac{\tau_{xy}}{\frac{\partial u}{\partial y}} = \mu \left[1 + \Lambda_2 \left(\frac{\partial u}{\partial y}\right)^2\right]/\left[1 + \Lambda_1 \left(\frac{\partial u}{\partial y}\right)^2\right] \tag{21}$$

depends on the shear rate and has the limiting values

$$\mu_{\text{app.}} \rightarrow \mu \quad \text{as} \quad \frac{\partial u}{\partial y} \rightarrow 0 \quad \text{and} \quad \mu_{\text{app.}} \rightarrow \mu \frac{\Lambda_2}{\Lambda_1} \quad \text{as} \quad \frac{\partial u}{\partial y} \rightarrow \infty$$

Equation (14), Equation (16), and Equation (17) also show that the normal stresses are in general unequal, and that to obtain a two dimensional flow a stress τ_{zz} normal to the flow has to be provided.

The seven material constants λ_1 , μ_1 , etc. and the viscosity μ are normally functions of the concentration of the viscoelastic component of the solution and hence can be represented by power series in the concentration c. Since the seven material constants will approach zero and the dimensionless viscoelastic μ will approach unity when the concentration c approaches zero, it is reasonable to assume for a dilute material solution (i.e. for small values of c) that

$$\mu = 1 + \gamma c \quad , \quad \lambda_1 = ac \quad , \quad \mu_1 = bc \quad , \; --- \; , \text{etc.} \qquad (22)$$

Consequently Equations (18) and (19) can be written as

$$\Lambda_1 = \beta c^2 \quad , \quad \Lambda_2 = \alpha c^2 \qquad (23)$$

In Equations (22) and (23), α , β , γ , etc. are constants for a given viscoelastic material. A more general representation of μ , λ_1 , etc. in terms of c is possible within the scope of the similarity solution (see Chapter III), since any function of c alone is allowed by similarity for certain boundary distribution of c. However the linearized form is felt sufficient to give an indication of the effects of the variation of the material parameters.

III. THE GOVERNING DIFFERENTIAL SYSTEM

For the flow problem considered in this study the governing equations of continuity, motion, and diffusion are, respectively,

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0 \tag{24}$$

$$\rho \left(u' \frac{\partial u'}{\partial x'} + v' \frac{\partial u'}{\partial y'} \right) = - \frac{\partial p}{\partial x'} + \frac{\partial p_{xx}}{\partial x'} + \frac{\partial p_{xy}}{\partial y'} \tag{25}$$

$$\rho \left(u' \frac{\partial v'}{\partial x'} + v' \frac{\partial v'}{\partial y'} \right) = - \frac{\partial p}{\partial y'} + \frac{\partial p_{xy}}{\partial x'} + \frac{\partial p_{yy}}{\partial y'} \tag{26}$$

$$u' \frac{\partial \rho_c}{\partial x'} + v' \frac{\partial \rho_c}{\partial y'} = \kappa \left(\frac{\partial^2 \rho_c}{\partial x'^2} + \frac{\partial^2 \rho_c}{\partial y'^2} \right) \tag{27}$$

in which x' and y' are Cartesian coordinates along and perpendicular to the plate, u' and v' are the components of mass average velocity in the x' - and y' - directions, ρ is the mass average density of the solution, ρ_c is the mass density of the coating, and κ is the mass diffusivity of the binary system.

For flow rates at which Equations (10) - (12) holds, the substitution of Equations (10) - (12) into Equations (24) - (27) yields the following set of dimensionless boundary layer equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{28}$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U \frac{dU}{dx} + \frac{\partial \tau_{xy}}{\partial y} + O(\epsilon) \tag{29}$$

$$\frac{\partial (p - \tau_{yy})}{\partial y} = O(\epsilon) \tag{30}$$

$$u \frac{\partial c}{\partial x} + v \frac{\partial c}{\partial y} = \frac{1}{S} \left(\frac{\partial^2 c}{\partial y^2} + O(\epsilon^2 c) \right) \tag{31}$$

in which $U(x)$ is the dimensionless velocity of the inviscid flow, and

$$c \equiv \frac{\rho_c}{\rho} = \text{concentration of the coating}$$

$$S \equiv \frac{\eta_0}{\kappa\rho} = \text{Schmidt number}$$

The dimensionless shear stress τ_{xy} is given by

$$\tau_{xy} = \mu \frac{\partial u}{\partial y}\left[1 + \Lambda_2 \left(\frac{\partial u}{\partial y}\right)^2\right] \Big/ \left[1 + \Lambda_1 \left(\frac{\partial u}{\partial y}\right)^2\right] \tag{32}$$

where μ , Λ_1 , and Λ_2 are functions of the concentration c and are expressed by Equation (22) and Equation (23) for the approximation of a dilute solution.

The boundary conditions of the problem are

$$\text{at } y = 0 : \quad u = v = 0 \quad , \quad c = c_0 \tag{33}$$

$$\text{as } y \to \infty : \quad u \to U \quad , \quad c \to c_1 \tag{34}$$

It is assumed here that the dissolved viscoelastic material at the plate has a constant concentration c_0 , and that the external flow is a solution of the coating with concentration c_1. When the outer fluid is Newtonian, c_1 is zero.

A. Similarity Transformation

The general solution of the system of partial differential equations (28) - (31) with the complicated shear stress-shear rate relation (32) is extremely difficult. A similarity transformation is sought to simplify the mathematics of the problem and to illustrate the general behavior of the flow.

From the continuity equation (28), a stream function $\psi(x,y)$ can be introduced such that

$$u = \frac{\partial\psi}{\partial y} , \quad v = - \frac{\partial\psi}{\partial x} \tag{35}$$

Letting

$$\psi(x,y) = b(x)\ f(\eta) , \ \eta = \frac{yU(x)}{b(x)} , \quad c = a(x)\ g(\eta) \tag{36}$$

the boundary layer equations for the dilute viscoelastic solution take the forms

$$UU_x f_\eta^2 - U^2 \frac{b_x}{b} f f_{\eta\eta} - UU_x = \frac{U^3}{b^2} \frac{d}{d\eta} \left[(1+\gamma ag)\ f_{\eta\eta} \frac{1 + \alpha \left(\frac{aU^2}{b}\right)^2 \left(gf_{\eta\eta}\right)^2}{1 + \beta \left(\frac{aU^2}{b}\right)^2 \left(gf_{\eta\eta}\right)^2} \right] \tag{37}$$

$$a_x U f_\eta g - a\ U \frac{b_x}{b} f\ g_\eta = \frac{1}{S} \frac{aU^2}{b^2} g_{\eta\eta} \tag{38}$$

in which the subscripts x and η denote the differentiation with respect to x and η , respectively.

(i) If the velocity of the outer flow, U, varies with x , Equation (37) and Equation (38) can be written as

$$f_\eta^2 - mff_{\eta\eta} - 1 = \kappa_2 \frac{d}{d\eta}\left[(1+\gamma\kappa_1\kappa_3 U^{m-2} g) f_{\eta\eta} \quad \frac{1+\alpha\,\kappa_3^2(gf_{\eta\eta})^2}{1+\beta\,\kappa_3^2(gf_{\eta\eta})^2}\right] \tag{39}$$

$$(m-2)\, f_\eta g - mfg_\eta = \frac{\kappa_2}{S}\; g_{\eta\eta} \tag{40}$$

respectively, if one chooses

$$b(x) = \kappa_1 U^m(x)\ ,\ a(x) = \kappa_1\kappa_3 U^{m-2},\quad U = \left[\frac{(2m-1)\ x}{\kappa_1^2\ \kappa_2}\right]^{\frac{1}{2m-1}} \tag{41}$$

where κ_i and m are arbitrary constants. Equation (39) indicates that if the material constant γ is zero or if m is equal to 2, the set of partial differential equations (28) - (31) can be transformed into a set of ordinary non-linear differential equations.

(ii) When the external flow has no pressure gradient and U is a constant, Equation (37) and Equation (38) become

$$-\kappa_1 ff_{\eta\eta} = \frac{d}{d\eta}\left[(1+\gamma\kappa_2 bg)\ f_{\eta\eta} \quad \frac{1+\alpha U^4\kappa_2(gf_{\eta\eta})^2}{1+\beta U^4\kappa_2(gf_{\eta\eta})^2}\right] \tag{42}$$

$$f_\eta g - f\ g_\eta = \frac{1}{S\kappa_1}\, g_{\eta\eta} \tag{43}$$

with now

$$b(x) = (2\kappa_1 Ux)^{1/2}\ ,\ a(x) = \kappa_2 b(x) \tag{44}$$

where κ_1 and κ_2 are arbitrary constants. Again, if γ is zero, the boundary layer equation of motion and diffusion for flow past a semi-infinite flat plate with zero pressure gradient can be reduced to ordinary differential equations.

B. Governing Differential System

In general the viscosity of a viscoelastic liquid depends on the concentration of the solution. Therefore, if this is to be included in the problem, from the above analysis the only similarity transformation one can have for the problem is that when m equals to 2. Now if the arbitrary constants κ_i are chosen such as

$$\kappa_1 = E^{-3/2} , \quad \kappa_2 = 3 , \quad \kappa_3 = {}^{c_0}/\kappa_1$$

then Equation (36) and Equation (41) give

$$U(x) = Ex^{1/3} , \quad \eta = y\left(\frac{U}{x}\right)^{1/2} , \psi(x,y) = (Ux)^{1/2} f(\eta) , \quad c = c_0 g(\eta) \qquad (45)$$

This transformation implies that the flow problem is a special type of flow, that is a flow of Falkner-Skan type past a 90 degree wedge.(9)

The transformed governing differential system for this special flow is

$$f_\eta^2 - 2\, ff_{\eta\eta} - 1 = 3 \frac{d}{d\eta}\left[(1 + Rg)\, f_{\eta\eta}\, (1 + Ag^2 f_{\eta\eta}^2)/(1+Bg^2 f_{\eta\eta}{}^2)\right] \qquad (46)$$

$$-\frac{2}{3}\, S\, f\, g_\eta = g_{\eta\eta} \qquad (47)$$

where $R \equiv \gamma c_0$, $A \equiv \alpha c_0^2 E^3$, and $B \equiv \beta c_0^2 E^3$. The boundary conditions (33) and (34) now become

$$\text{at} \quad \eta = 0 \quad : \quad f = f_\eta = 0 \ , \quad g = 1 \qquad (48)$$

$$\text{as} \quad \eta \to \infty \quad : \quad f_\eta \to 1 \quad . \quad g \to g(\infty) = \frac{c_1}{c_0} \qquad (49)$$

The coefficient of skin friction, C_d , will be given by

$$C_d \equiv \frac{\tau_{xy}|_{\eta=0}}{E^{3/2}} = (1+R)\ f_{\eta\eta}\ (0)\ [1+A\ f_{\eta\eta}^{\,2}\ (0)]/[1+B\ f_{\eta\eta}^{\,2}\ (0)] \qquad (50)$$

IV. METHOD OF SOLUTION

The governing differential equations (46) and (47) with the boundary consitions (48) and (49) are next solved. Since the general solution of Equations (46) - (49) can not be obtained in terms of known functions, it is necessary to use either purely numerical methods or series expansions.

The method of series expansions accompanied by the method of steepest descent is employed here to solve the problem. This method was first used by Meksyn[8] to solve the boundary layer equation for a Newtonian fluid. Several classical problems for Newtonian fluids have been reworked by this method, and the results obtained are very striking, in that only a few terms in the expansion are sufficient to obtain close agreement with accepted numerical results.

In applying the method to solve Equation (46) and Equation (47), one first expresses the dependent variables $f(\eta)$ and $g(\eta)$ in power series of η such that

$$f(\eta) = \sum_{n=0}^{\infty} \frac{A_n}{n!} \eta^n \tag{51}$$

$$g(\eta) = \sum_{n=0}^{\infty} \frac{B_n}{n!} \eta^n \tag{52}$$

in which A_n and B_n are constant coefficients to be determined. The expansions in Equations (51) and (52) are valid only for sufficiently small values of η. By using the boundary conditions (48) one finds, from Equations (51) and (52) that

$$A_0 = A_1 = 0, \quad B_0 = 1$$

Substituting the expansions (51) and (52) into Equation (47), the coefficients of the same power of η in both sides of the equation must be identical. This gives the relations between the B_n and A_n, which are found to be

$$
\left.\begin{aligned}
B_2 &= 0 \\
B_3 &= 0 \\
B_4 &= -2SA_2B_1/3 \\
B_5 &= -2SA_3B_1/3 \\
B_6 &= -2SA_4B_1/3 \\
B_7 &= -2SA_5B_1/3 + 40S^2A_2^2B_1/9 \\
&\text{etc.}
\end{aligned}\right\} \quad (53)
$$

Similarly, the substitution of Equations (51) - (53) into Equation (46) yields

$$
\left.\begin{aligned}
A_3 &= [1 + 3B_1A_2M_0 + BA_2^2 - 2B_1A_2^2I_0\,]/\text{Det} \\
A_4 &= 2[3B_1A_3M_0 - 3KBE_1^2 - 2A_2E_1I_1 - (E_1^2 + 2A_2B_1A_3)\,I_0]/\text{Det} \\
A_5 &= 6[1.5\,B_1A_4M_0 - 6KBE_1E_2 - 2A_2E_1I_2 - (2E_2A_2 + E_1^2)\,I_1 \\
&\qquad -(2E_1E_2 + A_2A_4B_1)\,I_0]/\text{Det} \\
A_6 &= 24[B_1M_1M_0 - G_4(1+BA_2^2) - 3KB(2E_1E_3 + E_2^2) - 2A_2E_1I_3 \\
&\qquad - (2A_2E_2 + E_1^2)\,I_2 - 2(A_2E_3 + E_1E_2)\,I_1 - (2M_1A_2B_1/3 \\
&\qquad + 2E_1E_3 + E_2^2)I_0]/\text{Det} \\
A_7 &= 120[B_1M_2M_0 - G_5(1+BA_2^2) - 6\,KB(E_1E_4 + E_2E_3) - 2A_2E_1I_4 \\
&\qquad - (2A_2E_2 + E_1^2)\,I_3 - 2(A_2E_3 + E_1E_2)\,I_2 - (2A_2E_4
\end{aligned}\right\} \quad (54)
$$

$$+ 2\, E_1E_3 + E_2^2)\, I_1 - 2(\tfrac{1}{3}\, B_1A_2M_2 + E_1E_4 + E_2E_3)I_0]/\text{Det} \quad (54)$$

etc.

in which

$$\text{Det} \equiv 2A_2I_0 - 3AA_2^2(1+R) + 3(2KBA_2 - 1 - R)$$

$$I_0 \equiv -3AA_2(1+R), \qquad K \equiv \frac{A_2(1+R)(1+AA_2^2)}{1 + BA_2^2}$$

$$M_0 \equiv R(1+AA_2^2) - 2KBA_2 ,$$

$$I_1 \equiv -B - 3AA_3(1+R) - 3ARB_1A_2 , \qquad E_1 \equiv A_3 + A_2B_1$$

$$I_2 \equiv -3A(A_4/2 + RE_2) , \qquad E_2 \equiv A_4/2 + A_3B_1$$

$$M_1 \equiv A_5/2 - SA_2^2/12 , \qquad E_3 \equiv A_5/6 + B_1A_4/2$$

$$I_3 \equiv -3A(A_5/6 + RE_3) , \qquad G_4 \equiv -A_2A_3/12$$

$$M_2 \equiv A_6/8 - A_2A_3S/10 , \qquad E_4 \equiv A_6/24 + B_1M_1/3$$

$$I_4 \equiv BG_4 - 3A(A_6/24 + RE_4) , \qquad G_5 \equiv -A_2A_4/20 - A_3^2/60$$

Equation (53) and Equation (54) indicate that all of the expansion coefficients A_n and B_n can be related to A_2 and B_1 for given parameters A, B, R, and S. The remaining unknown coefficnents A_2 and B_1 are determined by the boundary conditions (49).

The method of steepest descent is next employed to determine A_2 and B_1. This is a method of evaluating a certain type of integral. The main idea of the method is that if an integral of the form

$$I = \int e^{kw(z)}\phi(z)dz$$

in which k is large and $\phi(z)$ is a slowly varying function of z has a col(saddle point) at $z = z_0$, then the main contribution to the integral when it is integrated along a Debye path (a path along which Im. $kw(z)$ = constant) comes from the region near z_0. The method furnishes an asymptotic expansion for the integral in a series of gamma functions.

In its application to the problem considered here, the variable z is the real variable η, and the Debye path is the one in the direction of η positive and real.

Integration of the diffusion equation (47) twice gives

$$g(\eta) = 1 + B_1 \int_0^{\eta} e^{-F(\eta)} \, d\eta \tag{55}$$

in which

$$F(\eta) = \frac{2S}{3} \int_0^{\eta} f(\eta) d\eta \ , \ B_1 = [g(\infty) - 1] / \int_0^{\infty} e^{-F(\eta)} \, d\eta \tag{56}$$

It is seen from the series expansion that the integral in Equation (55) has a col at $\eta = 0$ and that the function $F(\eta)$ is multiplied by a large parameter S. Thus the method of steepest descent can be used to provide an asymptotic expansion for $g(\eta)$.

In the course of evaluating $g(\eta)$ one first expands the function $F(\eta)$ around the col ($\eta = 0$ in this case), and obtains

$$F(\eta) = \frac{2S}{3} \eta^3 \sum_{n=o}^{\infty} \frac{A_{n+2}}{(n+3)!} \eta^n \tag{57}$$

(This shows that the integral has a col of order two at $\eta = 0$) Next, one lets

$$\tau = F(\eta) = \frac{2S}{3} \eta^3 \sum_{n=o}^{\infty} \frac{A_{n+2}}{(n+3)!} \eta^n \equiv \eta^3 \sum_{n=o}^{\infty} a_n \eta^n \tag{58}$$

It is well known from the theory of inverse function[11,12] that this equation can be solved for η in the form

$$\eta = \sum_{m=o}^{\infty} \frac{b_m}{m+1} \tau^{\frac{1}{3}(m+1)} \tag{59}$$

in which the coefficients b_m can be determined in terms of the a_n by Cauchy's theorem of residues. From the relation

$$\oint^{(0^+)} \frac{d\eta}{\tau^{\frac{1}{3}(m+1)}} = \frac{b_m}{3} \oint^{(0^+,0^+,0^+)} \frac{d\tau}{\tau} = 2\pi i\ b_m \tag{60}$$

where $\oint^{(0^+)}$ denotes an integration path around the 0^+ once, and $\oint^{(0^+,0^+,0^+)}$ denotes triple circuits around 0^+, it is seen that b_m is the coefficient of η^{-1} in the expression $\tau^{-\frac{1}{3}(m+1)}$. Hence, b_m is the coefficient of η^m in the expression

$$[a_0 + a_1\eta + a_2\eta^2 + a_3\eta^3 + \cdots]^{-\frac{1}{3}(m+1)} \tag{61}$$

It is found that

$$b_0 = \left(\frac{9}{SA_2}\right)^{1/3}$$

$$b_1 = -b_0^2 \frac{A_3}{6A_2}$$

$$
\left.
\begin{aligned}
b_2 &= b_0^3\left[-\frac{A_4}{20A_2}+\frac{1}{16}\left(\frac{A_3}{A_2}\right)^2\right]\\
b_3 &= b_0^4\left[-\frac{A_5}{90A_2}+\frac{7}{180}\frac{A_3A_4}{A_2^2}-\frac{35}{1296}\left(\frac{A_3}{A_2}\right)^3\right]\\
b_4 &= b_0^5\left[-\frac{A_6}{504A_2}+\frac{A_3A_5}{108A_2^2}+\frac{A_4^2}{180A_2^2}-\frac{5A_3^2A_4}{216A_2^3}+\frac{385}{31104}\left(\frac{A_3}{A_2}\right)^4\right]
\end{aligned}
\right\} \quad (62)
$$

etc.

The integral in Equation (55) can now be written in terms of τ and integrated asymptotically, giving

$$
\int_0^\eta e^{-F(\eta)}\, d\eta \sim \frac{1}{3}\int_0^\tau e^{-\tau}\sum_{m=o}^{\infty} b_m \tau^{\frac{1}{3}(m-2)}\, d\tau
$$

$$
= \frac{1}{3}\sum_{m=o}^{\infty} b_m \Gamma_\tau\left(\frac{1+m}{3}\right) \quad (63)
$$

in which the Γ_τ are incomplete gamma functions. Therefore $g(\eta)$ takes the form

$$
g(\eta) \sim 1 + \frac{1}{3} B_1 \sum_{m=o}^{\infty} b_m \Gamma_\tau\left(\frac{1+m}{3}\right) \quad (64)
$$

The corresponding value of η for a given τ can be obtained from Equation (59). Application of the boundary condition (49) gives

$$
B_1 \sim 3[g(\infty) - 1] / \sum_{m=o}^{\infty} b_m \Gamma\left(\frac{1+m}{3}\right) \quad (65)
$$

This is the first of the two relations which will be used to determine the unknown coefficients A_2 and B_1.

To obtain the second relation the boundary-layer equation of motion, Equation (46), is used. This equation can be rewritten in the form

$$f_{\eta\eta\eta} + \frac{2f}{3(1+Rg)} f_{\eta\eta} = H(\eta) \tag{66}$$

in which

$$H(\eta) = \frac{1}{3(1+Rg)} \left\{ f_\eta^2 - 1 - 3Rg_\eta f_{\eta\eta} + \frac{d}{d\eta} \left[(1 + Rg)f_{\eta\eta} \times \frac{(B-A)g^2 f_{\eta\eta}{}^2}{1+B\ g^2 f_{\eta\eta}{}^2} \right] \right\}$$

Letting

$$F_1(\eta) = \frac{2}{3} \int_0^\eta \frac{f}{1+Rg} d\eta \tag{67}$$

$$\phi(\eta) = f_{\eta\eta}(0) + \int_0^\eta e^{F_1(\eta)} H(\eta)\, d\eta \tag{68}$$

integration of Equation (66) once yields

$$f_{\eta\eta}(\eta) = e^{-F_1(\eta)} \phi(\eta) \tag{69}$$

and a further integration gives

$$f_\eta(\eta) = \int_0^\eta e^{-F_1(\eta)} \phi(\eta)\, d\eta \tag{70}$$

a form which can be evaluated by the method of steepest descent.

Before applying the method to evaluate Equation (70), the properties of its integrand should be investigated. Since the form of $F(\eta)$ indicates that the integral in Equation (56) has a col of order two at $\eta = 0$, the function $F_1(\eta)$ will dictate that the integral (70) also has a col of order two at $\eta = 0$. $F_1(\eta)$ is also known to be a positive function. When η becomes very large, the equation of motion takes the form

$$f_{\eta\eta\eta} + \frac{2\eta}{3[1+Rg(\infty)]} f_{\eta\eta} \simeq 0 \tag{71}$$

because $f_\eta \to 1$, $f \to \eta$, $g \to g(\infty)$ and $f_{\eta\eta} \to 0$ as $\eta \to \infty$. Integration of Equation (71) yields

$$f_{\eta\eta} \sim \text{constant} \times e^{-\eta^2/3[1+Rg(\infty)]} \tag{72}$$

Thus, the comparison between Equation (69) and Equation (72) shows that $\phi(\eta)$ approaches a constant value as $\eta \to \infty$, and can be expected to be a slowly varying function of η throughout most of the region for small values of the Schmidt number. However, for a large Schmidt number Equation (68) and the function $H(\eta)$ show that $\phi(\eta)$ will be a rapidly varying function near the boundary because g_η changes very rapidly near the boundary. This implies that the integral in Equation (70) evaluated by the method of the steepest descent will be an extremely divergent series for a very large Schmidt number.

Now if one introduces a small parameter ξ and rewrite Equation (70) in the form

$$f_\eta(\eta,\xi) = \int_0^\eta e^{-\frac{1}{\xi^3}F_1(\eta)} \phi(\eta)d\eta \qquad (73)$$

then this integral can be evaluated asymptotically by the method of steepest descent for small Schmidt numbers.

To evaluate the integral in Equation (73) by the method of steepest descent, the functions $F_1(\eta)$ and $\phi(\eta)$ are first expanded in series forms about the col $\eta = 0$; that is

$$F_1(\eta) = \eta^3 \sum_{n=o}^{\infty} q_n\eta^n , \quad \phi(\eta) = \sum_{n=o}^{\infty} \ell_n\eta^n \qquad (74)$$

The coefficients q_n can be found by the substitution of Equations (51) and (52) in Equation (67). They are

$$\left.\begin{aligned}
q_0 &= \frac{1}{3!}\left[\frac{2A_2}{3(1+R)}\right] \\
q_1 &= \frac{1}{4!}\left[-\frac{2RB_1A_2}{(1+R)^2} + \frac{2A_3}{3(1+R)}\right] \\
q_2 &= \frac{1}{5!}\left[\frac{8A_2}{1+R}\left(\frac{RB_1}{1+R}\right)^2 - \frac{8A_3}{3(1+R)}\left(\frac{RB_1}{1+R}\right) + \frac{2A_4}{3(1+R)}\right] \\
q_3 &= \frac{1}{6!}\left[-\frac{40A_2}{1+R}\left(\frac{RB_1}{1+R}\right)^3 + \frac{40A_3}{3(1+R)}\left(\frac{RB_1}{1+R}\right)^2 - \frac{10A_4}{3(1+R)}\left(\frac{RB_1}{1+R}\right) + \frac{2A_5}{3(1+R)}\right] \\
&\text{etc.}
\end{aligned}\right\} \qquad (75)$$

The coefficients ℓ_n are obtained easire from Equation (69) rather than from Equation (68). Thus, the substitution of Equation (51) and Equation (74) into Equation (67) gives

$$\left.\begin{aligned} \ell_0 &= A_2 \\ \ell_1 &= A_3 \\ \ell_2 &= \frac{A_4}{2!} \\ \ell_3 &= \frac{1}{3!}\left[A_5 + \frac{2A_2^2}{3(1+R)} \right] \\ \ell_4 &= \frac{1}{4!}\left[A_6 + \frac{10A_2A_3}{3(1+R)} - \frac{2A_2^2}{1+R}\left(\frac{RB_1}{1+R} \right) \right] \\ &\text{etc.} \end{aligned}\right\} \quad (76)$$

Again one lets

$$\tau = F_1(\eta) = \eta^3 \sum_{n=o}^{\infty} q_n \eta^n \quad (77)$$

and obtains

$$\eta = \sum_{m=o}^{\infty} \frac{b'_m}{m+1} \tau^{\frac{1}{3}(m+1)} \quad (78)$$

As in obtaining the coefficients b_m, one finds that b'_m is the coefficient of η^m in the expression

$$[q_0 + q_1\eta + q_2\eta^2 + q_3\eta^3 + ---]^{-\frac{1}{3}(m+1)}$$

thus

$$
\left.\begin{aligned}
b'_0 &= q_0^{-\frac{1}{3}} \\
b'_1 &= -\frac{2}{3}\, b'^{2}_0\, \frac{q_1}{q_0} \\
b'_2 &= b'^{3}_0 \left[-\frac{q_2}{q_0} + \left(\frac{q_1}{q_0}\right)^2 \right] \\
b'_3 &= b'^{4}_0 \left[-\frac{q_3}{3q_0} + \frac{7q_1q_2}{9q_0^2} - \frac{35}{81}\left(\frac{q_1}{q_0}\right)^3 \right] \\
&\text{etc.}
\end{aligned}\right\} \qquad (79)
$$

Next, one lets

$$\phi(\eta)\ \frac{d\eta}{d\tau} = \sum_{m=o}^{\infty} h_m\, \tau^{\frac{1}{3}(m-2)} \qquad (80)$$

Then, by Cauchy's theorem of residues and the relation

$$\oint^{(0^+)} \frac{\phi(\eta)}{\tau^{\frac{1}{3}(1+m)}}\, d\eta = \oint^{(0^+,0^+,0^+)} \frac{\phi(\eta)}{\tau^{\frac{1}{3}(1+m)}}\, \frac{d\eta}{d\tau}\, d\tau$$

$$= \oint^{(0^+,0^+,0^+)} h_m\, \frac{d\tau}{\tau} = 6\,\pi i\, h_m \qquad (81)$$

the coefficient h_m is seen to be equal to one third of the coefficient of η^{-1} in the expression $\phi(\eta)\,\tau^{-\frac{1}{3}(m+1)}$. If Equations (74) and (76) are used, then h_m is the coefficient of η^m in the expression

$$\frac{1}{3}\left[\ell_0 + \ell_1\eta + \ell_2\eta^2 + \cdots\right]\left[q_0 + q_1\eta + q_2\eta^2 + \cdots\right]^{-\frac{1}{3}(m+1)}$$

so that

$$\left.\begin{aligned}
h_0 &= \frac{1}{3} A_2 b_0' \\
h_1 &= b_0'^2\left[\frac{A_3}{6} + \frac{A_2}{2}\left(\frac{RB_1}{1+R}\right)\right] \\
h_2 &= b_0'^3\left[-\frac{A_2}{80}\left(\frac{RB_1}{1+R}\right)^2 + \frac{23A_3}{120}\left(\frac{RB_1}{1+R}\right) + \frac{3A_4}{20} - \frac{A_3^2}{16A_2}\right] \\
h_3 &= b_0'^4\left[-\frac{A_2}{720}\left(\frac{RB_1}{1+R}\right)^3 + \frac{13A_3}{240}\left(\frac{RB_1}{1+R}\right)^2 + \frac{79A_4}{540}\left(\frac{RB_1}{1+R}\right)\right. \\
&\quad \left. - \frac{11A_3^2}{144A_2}\left(\frac{RB_1}{1+R}\right) + \frac{7A_5}{135} - \frac{7A_3A_4}{108A_2} + \frac{91A_3^3}{3888A_2^2} - \frac{2A_2^3}{243(1+R)}\right]
\end{aligned}\right\} \quad (82)$$

etc.

The integral (73) can now be integrated in terms of τ, that is

$$f_\eta(\eta,\xi) = \int_0^\tau e^{-\frac{1}{\xi^3}\tau} \sum_{m=0}^{\infty} h_m \tau^{\frac{1}{3}(m-2)} \, d\tau$$

$$= \sum_{m=0}^{\infty} h_m \Gamma_\tau\left(\frac{1+m}{3}\right) \xi^{m+1} \quad (83)$$

in which Γ_τ are incomplete gamma functions; the value of η corresponding to a given τ can be obtained from Equation (78).

By putting $\xi = 1$ in Equation (83) one has

$$f_\eta(\eta) = \sum_{m=o}^{\infty} h_m \Gamma_\tau \left(\frac{1+m}{3}\right) \tag{84}$$

after applying boundary condition (49), Equation (84) gives

$$1 = \sum_{m=o}^{\infty} h_m \Gamma \left(\frac{1+m}{3}\right) \tag{85}$$

This relation together with Equation (65) is used to determine the two unknown constants A_2 and B_1 for given values of the parameters A, B, R, and S.

It is of course possible to determine A_2 and B_1 in an analytical form from the two relations Equation (65) and Equation (85) if only a finite number of terms in the relations are considered. However, since b_m in Equation (65) and h_m in Equation (85) are rather complicated functions of A_2 and B_1, the expression for general values of the parameters is much too cumbersome to obtain, and a trial and error procedure would be necessary to determine A_2 and B_1. For a large Schmidt number, Equation (62) indicates that the series in the denominator of Equation (65) will converge rather rapidly. But for a small Schmidt number, the series in Equation (65) is a divergent asymptotic series, therefore, only the first few converging terms should be considered in the determination of A_2 and B_1.[13] Equation (65) can now be written as

$$B = 3[g(\infty) -1] / \sum_{m=o}^{M} b_m \Gamma \left(\frac{1+m}{3}\right) \tag{86}$$

for the determination of A_2 and B_1, where M is an integer depending mostly on the Schmidt number. By a trial and error procedure, A_2 and B_1

can be obtained from Equation (85) and Equation (86) for given values of the parameters A, B, R, and S.

After values of A_2 and B_1 have been determined, the concentration distribution of the coating and the velocity profile can be obtained from Equation (64) and Equation (84), respectively. The coefficient of the skin friction can then be calculated from Equation (50), in which $f_{\eta\eta}(0) \equiv A_2$.

The series of Equation (78) and Equations (84) and (85) which involve the coefficients b_m' and h_m are in general divergent. These divergences arise because the integral in Equation (70) does not involve a large parameter and because the expansion used for $\phi(\eta)$ has a small radius of convergence about $\eta = 0$. The sum of a divergent series can not be obtained directly. However, since the sum of a divergent series is the finite numerical value of the convergent expression from which the divergent series is derived,[14] it is possible, by a suitable transformation of the series, to obtain an asymptotic series which sums to the correct value. In the present study the transformation due to Euler is used.

To understand Euler's method for summing a divergent series, one can consider the function

$$X(x) = (1+x)^{-1} \tag{87}$$

This function when expanded about x = 0 takes the form

$$X(x) = 1 - x + x^2 - x^3 + --- \tag{88}$$

It is seen that when $x = 1$ the original function (87) has a value of $\frac{1}{2}$, but the expansion (88) gives no value for $x \geq 1$ because it is a divergent series. Euler's transformation changes the divergent series (88) into a convergent series which gives a proper representation for large values of x.

The principle underlying Euler's transformation[15,16] is that of analytic continuation. If a power series

$$P(x) = \sum_{n=0}^{\infty} \alpha_n x^{n+1} \tag{89}$$

is convergent for only sufficiently small values of x, then by the transformation

$$\frac{x}{1+x} = y \quad , \quad x = \frac{y}{1-y} \tag{90}$$

Equation (89) expanding in powers of y becomes

$$P(x) = \sum_{n=0}^{\infty} \beta_n y^{n+1} \tag{91}$$

in which

$$\beta_0 = \alpha_0, \; --- \; , \beta_n = \alpha_0 + \binom{n}{1} \alpha_1 + \binom{n}{2} \alpha_2 + --- + \alpha_n$$

$$\binom{n}{m} = \frac{n(n-1)(n-2) \; --- \; (n-m+1)}{m!}$$

The series in Equation (91) is valid for sufficiently small values of y. The transformation (90) indicates that a small value of y can correspond to a large value of x. Therefore Equation (91) can represent the sum of

Equation (89) when x is large. For the special case when $x = 1$, $y = \frac{1}{2}$, one has

$$P(1) = \sum_{n=o}^{\infty} \alpha_n = \sum_{n=o}^{\infty} \beta_n 2^{-(n+1)} \tag{92}$$

This relation (92) is the one used by Euler. By applying Euler's transformation (92) to the series (88), one finds from the transformed series that when $x = 1$

$$X(1) = \frac{1}{2} + 0 + 0 + --- + 0$$

Thus Euler's transformation eliminates a singularity which does not belong to the function itself but which was introduced by the method of expansion.

In the evaluation of a divergent series, Euler's transformation can be repeatedly applied until a convergent expression is obtained, and the transformation can be started at any term of the original or the transformed series. However, if only a finite number of terms is used in summing the series, too many repeated transformations will reduce the accuracy of the sum because the convergence of these first few terms will be slowed down by repeated transformation.

V. RESULTS AND DISCUSSION

The University of Michigan IBM 7090 computer was used to perform the calculations needed. In the course of computing a given set of A_2 and B_1 with the given parameters A, B, R, and S, Equations (53) and (54) were employed directly to calculate B_n and A_n. However Equations (62), (79), and (82) were not used to calculate b_m, b_m', and h_m; instead the recursion relations used in obtaining them were programmed to avoid possible errors in transfering to the machine.

Case I. Viscoelastic Liquids With Homogeneous Properties (g=1 Throughout The Flow Region) Flowing Past The Wedge.

For this case the coefficients B_n are all zero, since g=1. To determine the correct value of A_2, eight terms of the series (85) are considered. It was necessary to transform this divergnet series twice using Euler's transformation. The first transformation was started at the very first term of the series, and the second transformation at the second term of the transformed series. It is noted from the boundary layer equation of motion that when $R = 0$ and $A/B = 1$, the non-Newtonian phenomena will not be observed, that is the flow pattern of this case is the same as the one due to a Newtonian liquid. The results obtained for various values of A and B are shown in Figure 1 and Figure 2.

Figure 1 shows the relation between $f_{\eta\eta}(0)$ and the ratio A/B for $B = 0.10 - 0.60$ when the parameter R is zero. Since $f_{\eta\eta}(0)$ is the slope of the velocity profile at the body, it is legitimate to say that a larger value of $f_{\eta\eta}(0)$ implies a thinner displacement thickness.

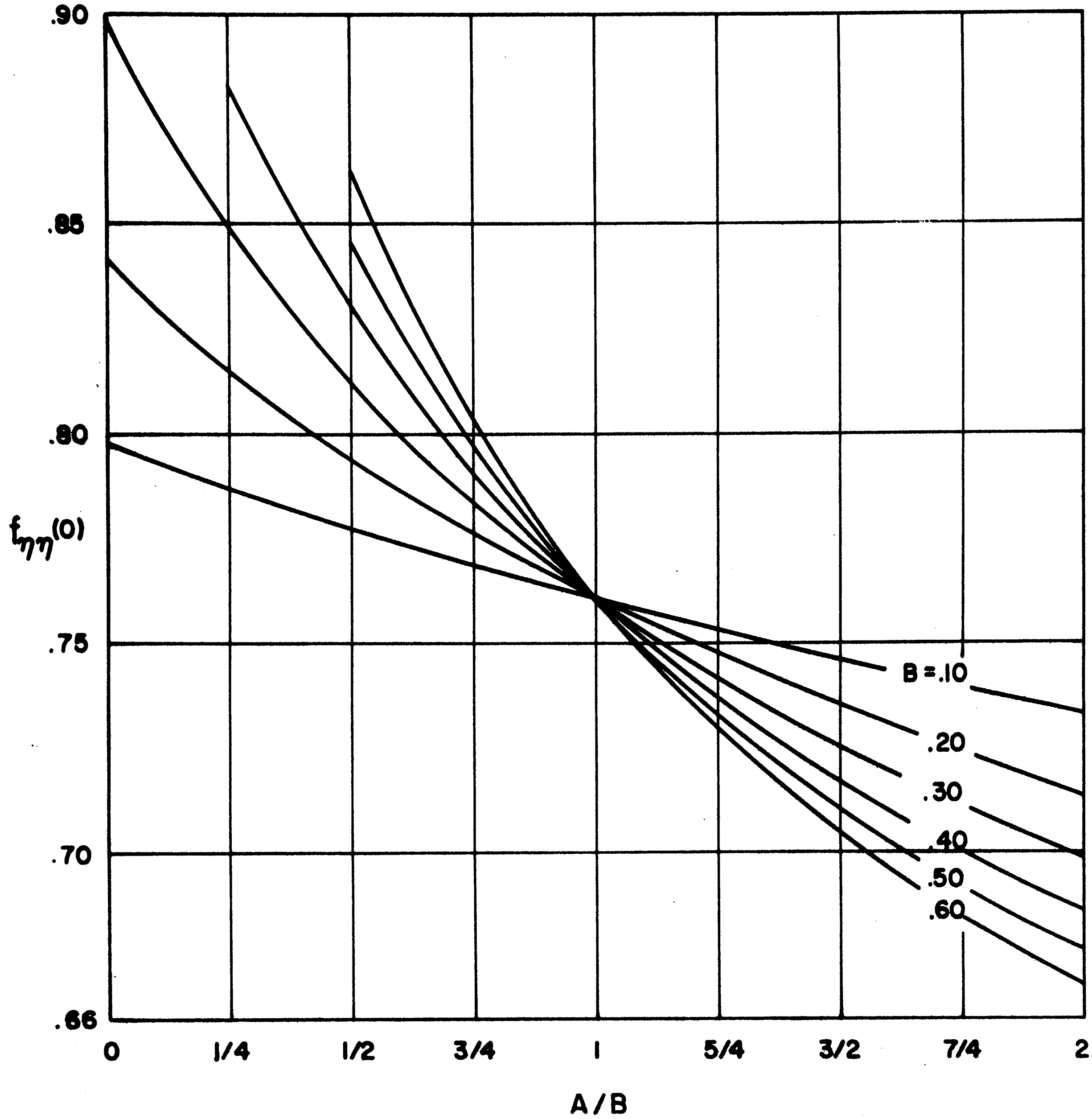

Figure 1. Effect of the Parameters A and B on the Slope of the Velocity Profile at the Body When R = 0.

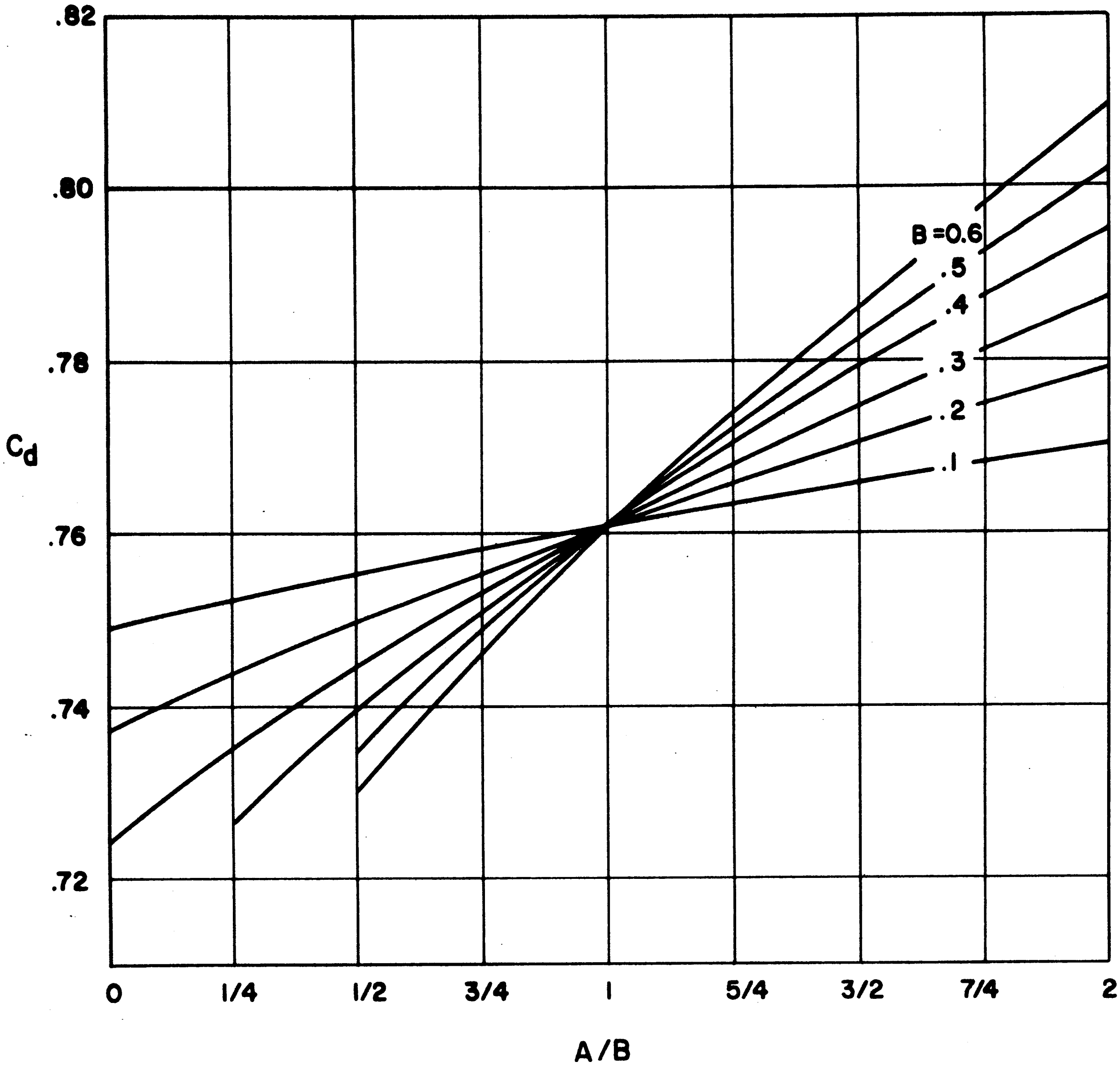

Figure 2. Effect of the Parameters A and B on the Frictional Coefficient C_d when $R = 0$.

Thus, Figure 1 shows that the displacement thickness increases with increasing A/B for a given B. For the case of a Newtonian liquid, that is when $A/B = 1$, the obtained result $f_{\eta\eta}(0) = 0.761$ is in very good agreement with Hartree's[17] result (0.758). This implies that the method used is probably quite accurate.

Figure 2 shows the relation between the coefficient C_d of the skin friction and the ratio A/B for $R = 0$.

In the Newtonian flow problem, it is well known that the frictional coefficient C_d is linearly proportional to $f_{\eta\eta}(0)$ under the conditions of the boundary layer assumption. But the results obtained here for viscoelastic liquids do not show such a simple relation. While $f_{\eta\eta}(0)$ decreases with increasing A/B as shown in Figure 1, Figure 2 indicates that C_d increases with increasing A/B. Therefore the skin friction of a viscoelastic liquid past the body is rather strongly affected by the values of the material constants λ_1, μ_1, etc., and a displacement thickness thinner than that in a Newtonian fluid does not necessarily imply a larger frictional force for viscoelastic liquids.

The general expressions for normal stresses in terms of given A and B given by Equations (14), (16) and (17) show a rather complex dependency on the seven material constants. For the special case which has been shown to predict the general form of some experimentally observed relations between steady state and oscillatory phenomena suggested by Williams and Bird[18], that is when

$$\mu_1 = \lambda_1 , \quad \nu_1 = \frac{2}{3}\lambda_1 , \quad \mu_0 = 0 , \quad \mu_2 = \lambda_2 , \quad \nu_2 = \frac{2}{3}\lambda_2$$

Equations (14) - (17) become, after similarity transformation,

$$\tau_{xx} = -2\,\tau_{yy} = -2\,\tau_{zz} = \frac{4}{3} E^3 \mu\, f_{\eta\eta}^2 (\lambda_1 - \lambda_2)/(1 + \frac{2}{3} \lambda_1^2 E^3 f_{\eta\eta}^2)$$

$$\tau_{xy} = E^{3/2} \mu\, f_{\eta\eta}(1 + \frac{2}{3} \lambda_1 \lambda_2 E^3 f_{\eta\eta}^2)/(1 + \frac{2}{3} \lambda_1^2 E^3 f_{\eta\eta}^2)$$

Thus, for pseudoplastic fluids $(A/B < 1)$, τ_{xx} is a tensile stress while τ_{yy} and τ_{zz} are compressive stresses; for dilatant fluids $(A/B > 1)$, τ_{xx} becomes a compressive stress while τ_{yy} and τ_{zz} become tensile stresses. Hence, in order to have a steady, two dimensional, incompressible viscoelastic flow, depending upon whether the fluid is a pseudoplastic or a dilatant liquid, it is necessary to apply a compressive or a tensile stress normal to the plane of the flow.

The effect of the parameter R on either $f_{\eta\eta}(0)$ or the frictional coefficient C_d can be obtained from the results given in Figure 1 and Figure 2 by a simple modification. It is seen that if the characteristic viscosity used in forming the dimensionless quantities is taken to be $\eta_0(1 + R)$, the equation of motion then obtained is independent of R. Thus, for given values of the parameters A, B, and R, the corresponding $f_{\eta\eta}(0)$ and C_d can be found, using Figure 1 and Figure 2, from the relations

$$\left.\begin{aligned} f_{\eta\eta}(\eta = 0, R) &= f_{\eta\eta}(\eta = 0, R = 0) / [1 + R]^{1/2} \\ C_d(R) &= [1 + R]^{1/2} C_d(R = 0) \end{aligned}\right\} \quad (93)$$

respectively, for given values of A and B.

Case II. Newtonian Solvents Flowing Past The Coated Wedge

Here the case is considered when the Schmidt number is larger than zero and $g(\infty)$ in Equation (86) is equal to zero.

Figures 3 - 5 show the relationship between the frictional coefficient C_d and the Schmidt number S for $B = 0.3$, $R = 0.00$-0.08 with the ratio $A/B = 3/4$, 1, 3/2 respectively. The results obtained here indicate that the frictional coefficient C_d will increase or decrease from the corresponding value of the homogeneous viscoelastic flow with concentration c_0 and approaches a limit when the Schmidt number increases. It will next be shown that this limit is the frictional coefficient of a Newtonian liquid past the wedge.

For a large Schmidt number, the diffusion layer is much thinner than the Prandtl boundary layer. Thus the boundary layer may be divided into two regions, the first being a region of constant concentration far from the boundary, the second, a region of rapidly changing concentration in the immediate vicinity of the coated surface. For the case considered here the concentration of the first region is $g(\infty) = 0$, and the governing differential equation for the region is

$$f_\eta^2 - 2 f f_{\eta\eta} - 1 = 3 f_{\eta\eta\eta}$$

The governing equations of the thin diffusion layer are Equation (46) and Equation (47). Since the diffusion term of Equation (47) is comparable to the convective term and $f_{\eta\eta}(\eta)$ is expected to be order one in the diffusion layer, the transformation

$$\zeta = S^{1/3} \eta \quad , \quad F = S^{2/3} f \tag{94}$$

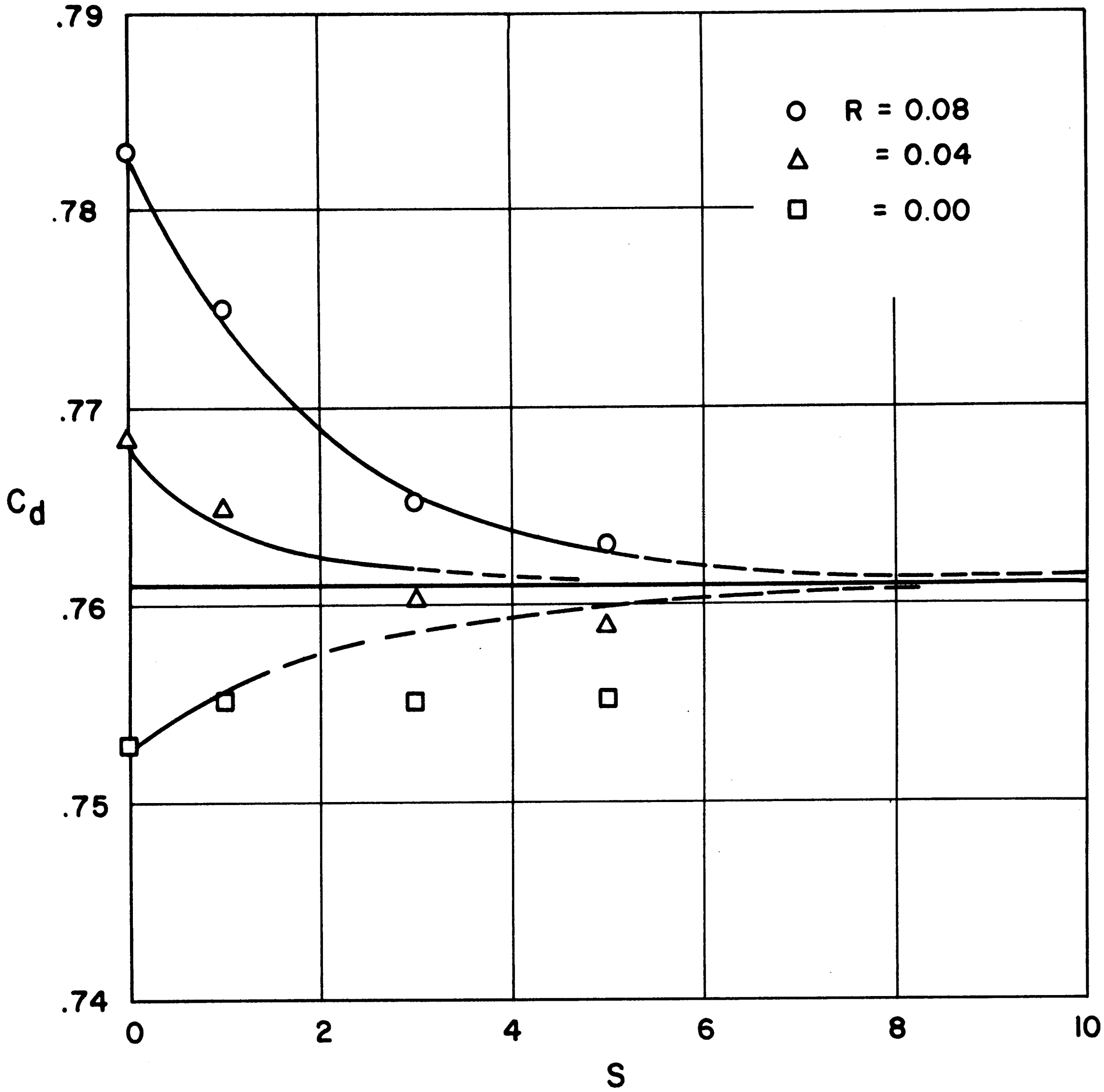

Figure 3. Relationship Between the Schmidt Number S and the Frictional Coefficient C_d for psedoplastic coatings with $3B = 4A = 0.9$.

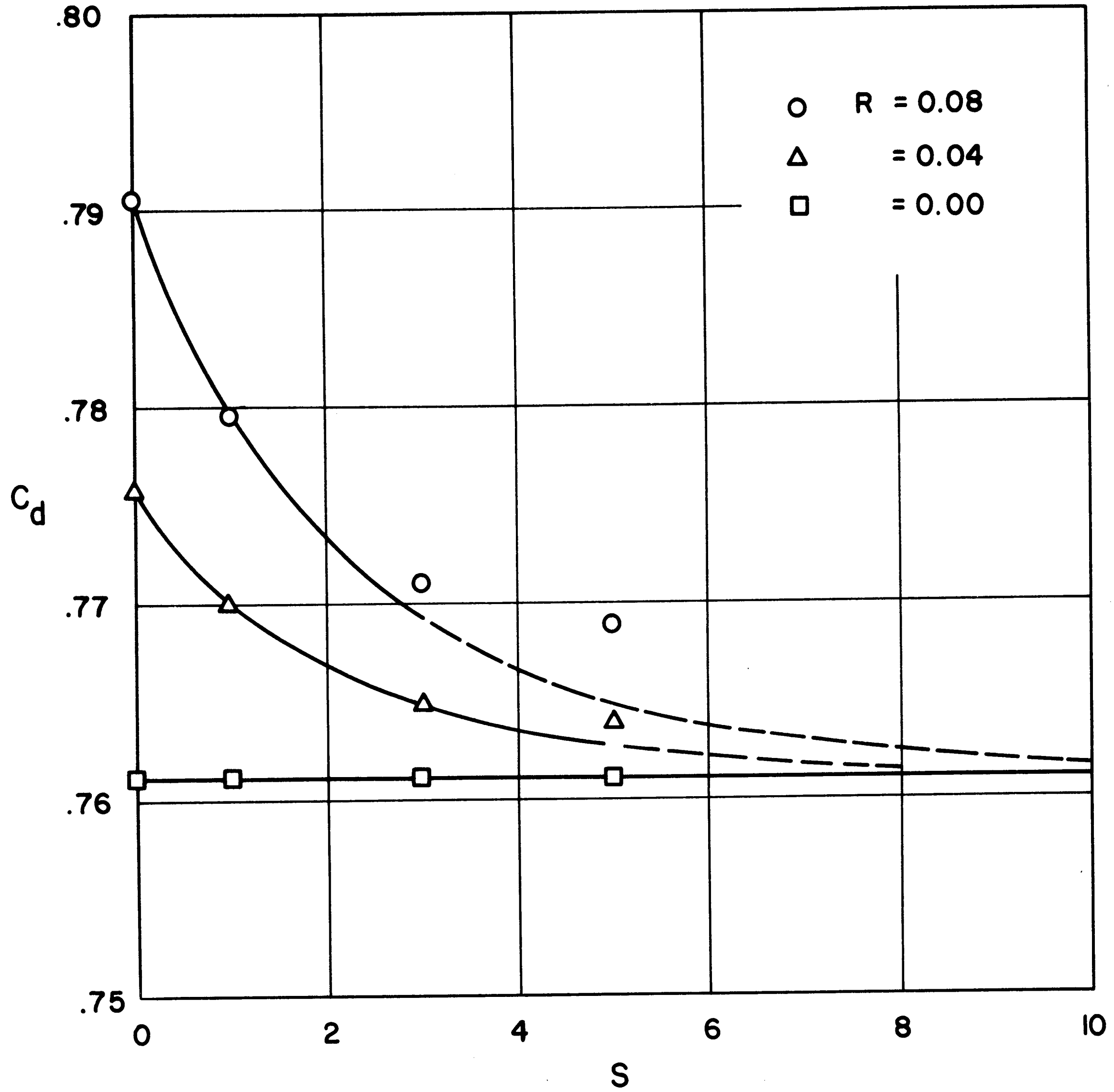

Figure 4. Relationship Between the Schmidt Number S and the Frictional Coefficient C_d When A = B.

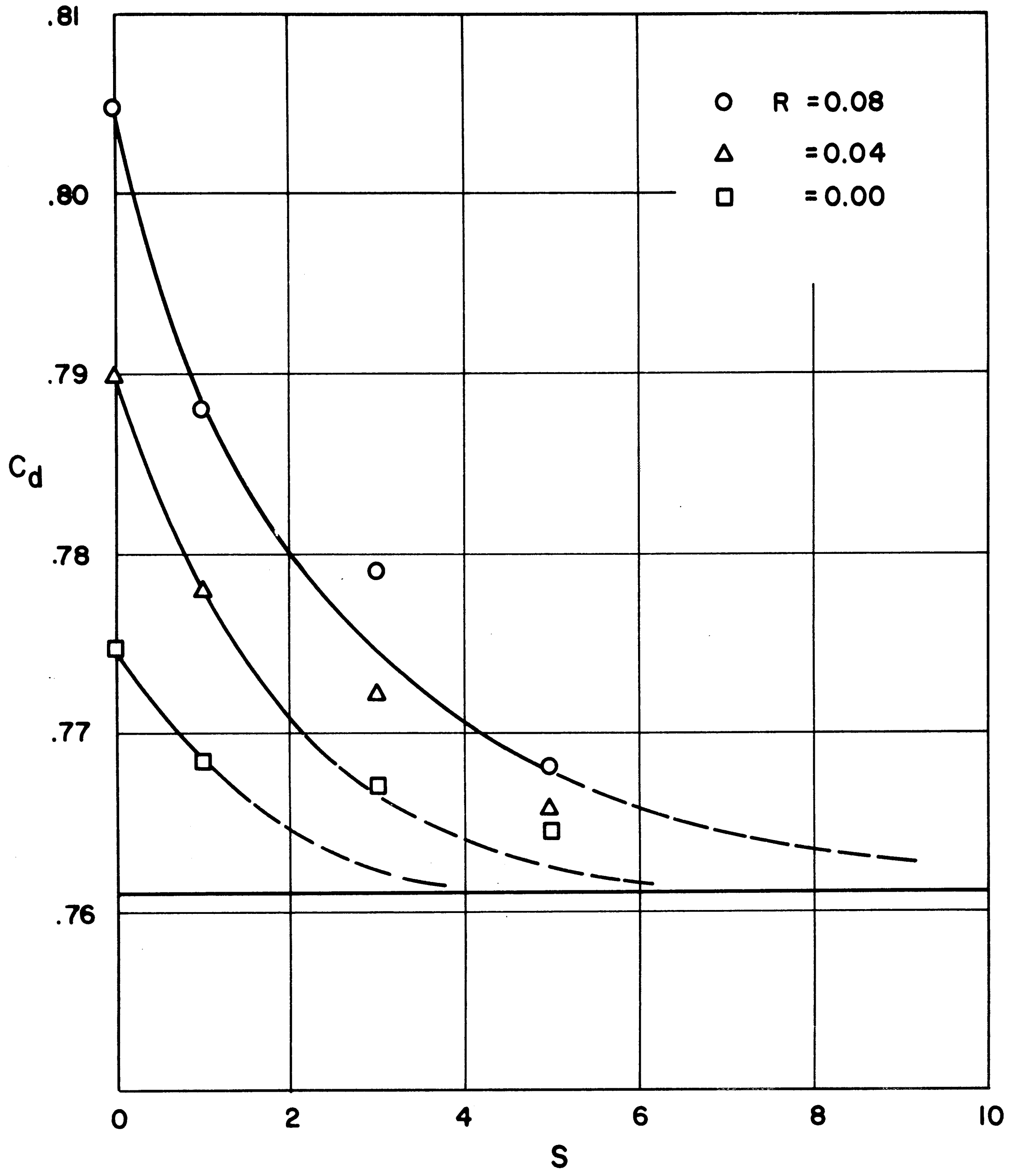

Figure 5. Relationship Between the Schmidt Number S and the Frictional Coefficient C_d for Dilatant Coatings with $3B = 2A = 0.9$.

should be chosen for a large Schmidt number. By the transformation (94), Equations (46) and (47) become

$$\frac{d}{d\zeta}\ [F_{\zeta\zeta}\ (1 + Rg)(1 + Ag^2F_{\zeta\zeta}^2)/(1 + Bg^2F_{\zeta\zeta}^2)] \;=\; O(S^{-1/3}) \qquad (95)$$

$$-\frac{2}{3}\ F\ g_\zeta \;=\; g_{\zeta\zeta} \qquad (96)$$

Integrating Equation (95) once, one has

$$F_{\zeta\zeta}(1 + Rg)(1 + Ag^2F_{\zeta\zeta}^2)/(1 + Bg^2F_{\zeta\zeta}^2) = \text{const.} \equiv \tau_o \qquad (97)$$

$$\text{for} \quad S \rightarrow \infty$$

This shows that the thin diffusion layer has a constant shear stress τ_o . From Equation (96) it is known that g decays rapidly and approaches zero as $\zeta \rightarrow \infty$, thus Equation (97) can be written as

$$F_{\zeta\zeta} = \tau_o \quad \text{as} \quad \zeta \rightarrow \infty \quad \text{and} \quad S \rightarrow \infty \qquad (98)$$

Integration of Equation (98) yields

$$F_\zeta = \tau_o \zeta + \text{const.} \quad , \text{ for } S \rightarrow \infty$$

$$F = \frac{1}{2}\,\tau_o \zeta^2 + \text{const.}\ \ \zeta + (\text{const.})_2, \text{ for } \ S \rightarrow \infty \qquad (99)$$

To match the solutions of the two regions it is required that[13]

$$f_{\eta\eta}\ (0) \;=\; F_{\zeta\zeta}\ (\infty)$$

This implies that τ_o is the dimensionless shear stress at the body for Nowtonian flow past the wedge. Furthermore Equation (99) satisfies the matching conditions

$$f\ (0) \;=\; f_\eta(0) \;=\; 0$$

Hence as the Schmidt number approaches infinity, the frictional coefficient C_d has to approach that of the Newtonian case.

As mentioned in the previous chapter the method utilized in this study does not give the correct numerical result for a large Schmidt number. It is unlikely in fact that any of the standard methods can be directly applied. However the above analysis shows that the frictional coefficient C_d will approach that of a Newtonian fluid as the Schmidt number becomes large. Hence based on the calculated results, the curves in Figures 3 - 5 can be extended smoothly to approach the Newtonian limit. Due to the fact that only the finite numbers of terms are used to obtain the results, some of the calculated results shown in Figure 3 - 5 are away from the expected curves. However the deviation in all cases is less than one percent of the total C_d .

Figures 6 - 8 show the relation between the frictional coefficient C_d and the ratio of parameters A and B for Schmidt number $S = 0$, 1, 3, and 5 when $B = 0.3$ and $R = 0.00$, 0.04, and 0.08. For $S = 0$ and 1 the calculated results are probably quite good, but when the Schmidt number becomes large, the deviation increases due to the method of solution. The curve shown in these figures have been adjusted according to Figures 3 - 5 for Schmidt numbers greater than 3.

From the results obtained in this case one can conclude that for dilatant coatings the frictional coefficient C_d will decrease with increasing Schmidt number and approaches the frictional coefficient of the Newtonian case when the Schmidt number approaches infinity. If the coating is a pseudoplastic material, the frictional coefficient in general will increase and approaches that of the Newtonian case as the Schmidt number increases;

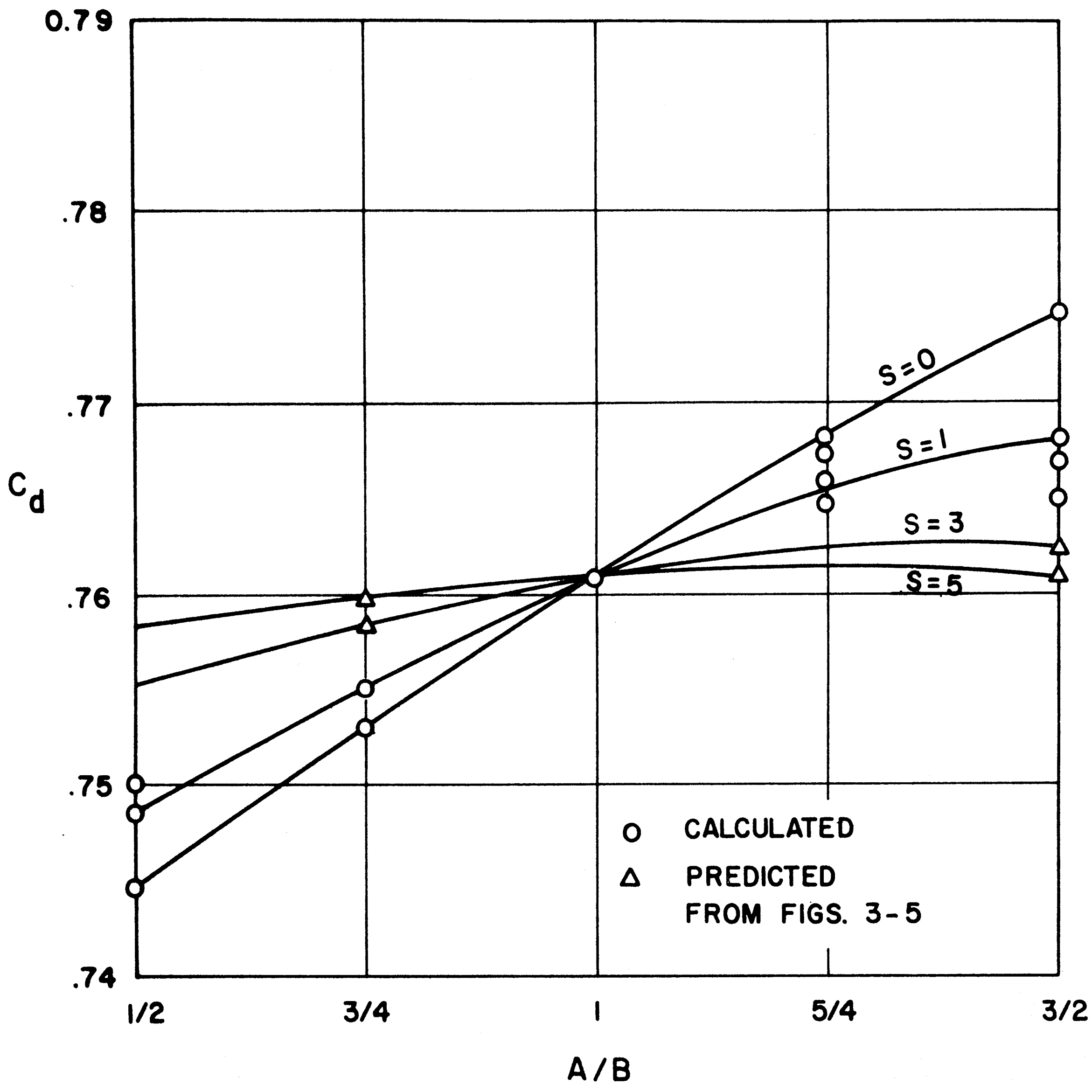

Figure 6. Relationship Between the Frictional Coefficient C_d and the Parameter A for the Schmidt Number S = 0, 1, 3, and 5 When B = 0.3 and R = 0.00.

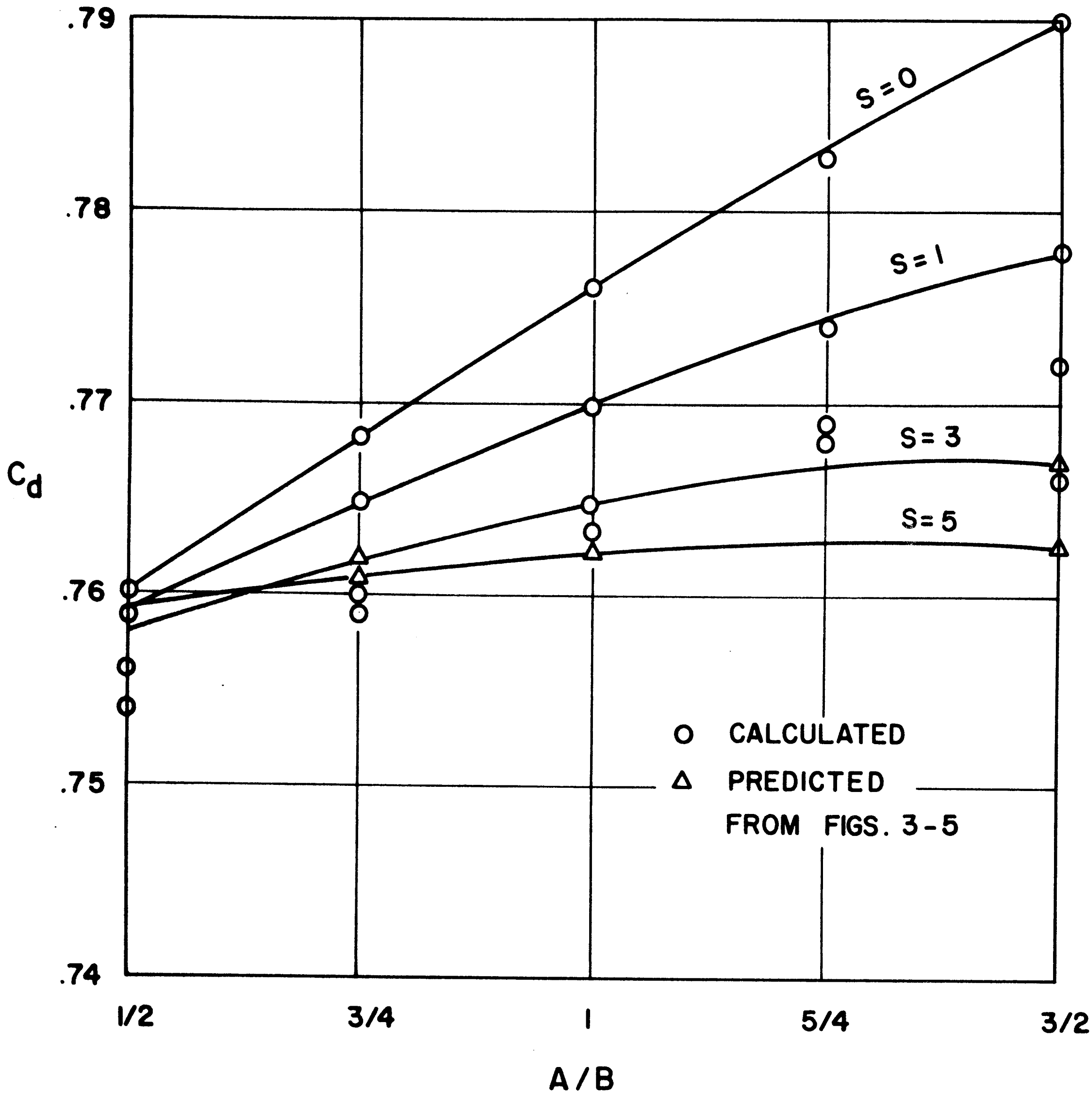

Figure 7. Relationship Between the Frictional Coefficient C_d and the Parameter A for S = 0, 1, 3, and 5 When B = 0.3 and R = 0.04.

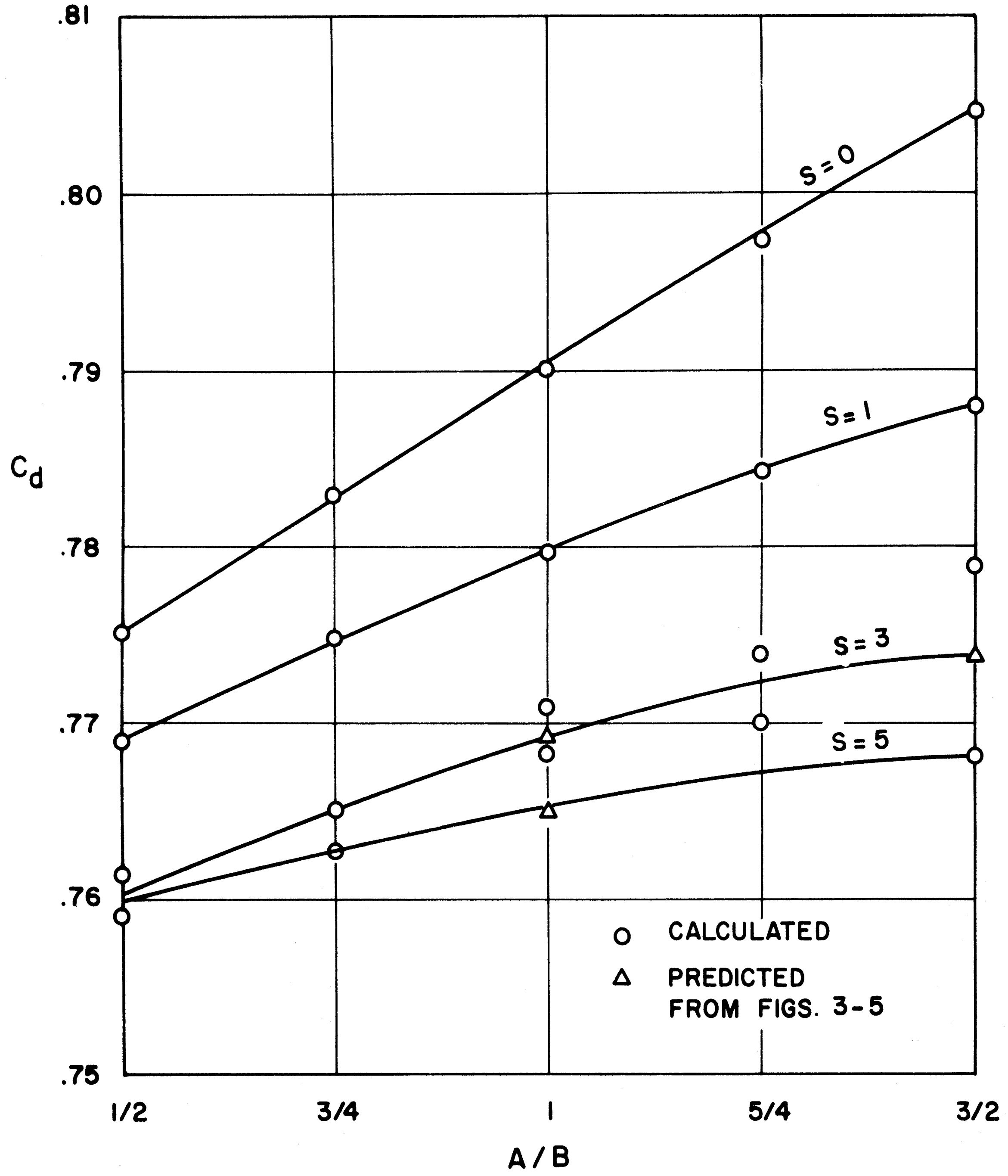

Figure 8. Relationship Between the Frictional Coefficient C_d and the Parameter A for S = 0, 1, 3, and 5 When B = 0.3 and R = 0.08.

however for some of the highly pseudoplastic coatings the frictional coefficient will decrease first and then increase to approach the Newtonian limit as the Schmidt number increases.

Case III. Viscoelastic Liquids Flowing Past The Coated Wedge

The case where $g(0) \neq g(\infty) \neq 0$ has not been studied in detail because the results for this case can be predicted qualitatively from the results obtained in Case I and Case II.

If the external flow now considered is a solution of the coating with the dimensionless concentration $g(\infty)$, it is expected that the frictional coefficient will increase or decrease from that of the homogeneous viscoelastic flow with concentration c_0 as the Schmidt number increases, and will approach a limit as the Schmidt number becomes very large. However the limit now is the frictional coefficient of the viscoelastic liquid with $g(\infty)$ flowing past the non-coated wedge as can be shown by an analysis similar to that of the previous section. This limit can be obtained from Figure 2 and Equation (93) by a suitable choice of the values of R, A, and B because $R \equiv \gamma c_0$, $A \equiv \alpha c_0^2 E^3$, $B \equiv \beta c_0^2 E^3$. Since the values for C_d for $S = 0$ and $S \rightarrow \infty$ can be obtained from Figure 2 and Equation (93), the relationship between C_d and S can then be predicted at least qualitatively for the case considered.

Now, Figure 2 and Equation (93) indicate that for dilatant fluids if $g(\infty) < 1$, the skin friction will decrease with increasing Schmidt number; on the other hand, if $g(\infty) > 1$, C_d increases with increasing Schmidt number. For pseudoplastic fluids, depending on the values of R, A, and B, the frictional coefficient C_d may either decrease or increase with increasing Schmidt number when $g(\infty) \lesseqgtr 1$.

The velocity distribution of the flow was obtained for Case I and Case II from Equations (78) and (84) by calculating η and $f_\eta(\eta)$ for a given value of τ. Similarly, the concentration distribution of the viscoelastic material can be obtained from Equations (59) and (64). Figure 9 shows the velocity profile of the homogeneous viscoelastic flow when $B = 0.3$ and $R = 0.00$. It indicates that the general form of the velocity distribution of the viscoelastic flows is very similar to that of the Newtonian flow ($^A/B = 1$). If the viscoelastic liquids having $0 \leq \frac{A}{B} \leq 2$, their velocity profiles of the flow will fall between the two curves of $\frac{A}{B} = 0$ and 2 shown in Figure 9.

Figure 10 shows the velocity profile and the concentration distribution of the non-homogeneous viscoelastic flow when $B = 0.3$, $R = 0.08$ and the Schmidt number $S = 1$. It is seen that the velocity profiles of the liquids with $\frac{A}{B} = \frac{1}{2}$ and $^3/2$ are both very similar to each other and the deviation from that of the Newtonian flow is small. The concentration distributions shown in Figure 10 indicate that the thickness of the diffusion layers is almost independent of the material properties and has the same order of magnitude as that of the velocity profiles when the Schmidt number equals to one. The thickness of the diffusion layer will of course decrease as the Schmidt number increases. But the comparison of Figures 9 and 10 shows that the order of magnitude of the boundary layer thickness is relatively insensitive to the Schmidt number.

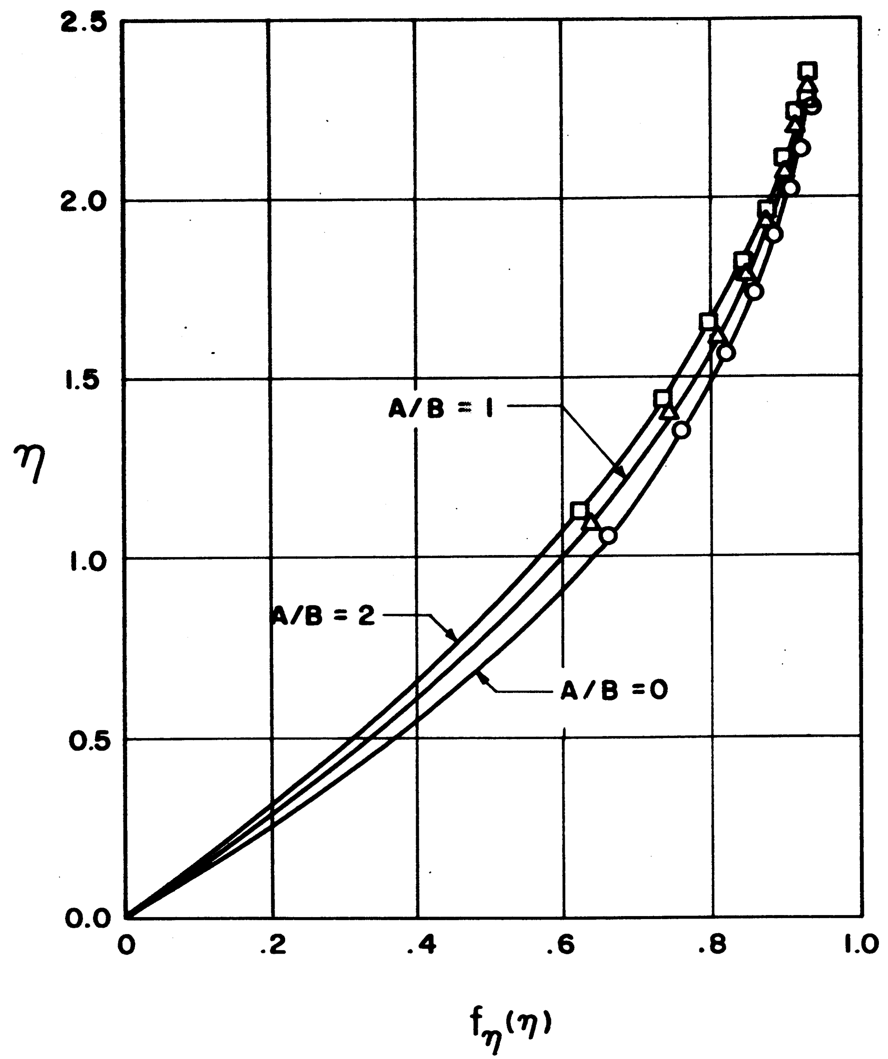

Figure 9. Velocity Distribution of the Homogeneous Viscoelastic Flow with B = 0.3 and R = 0.00.

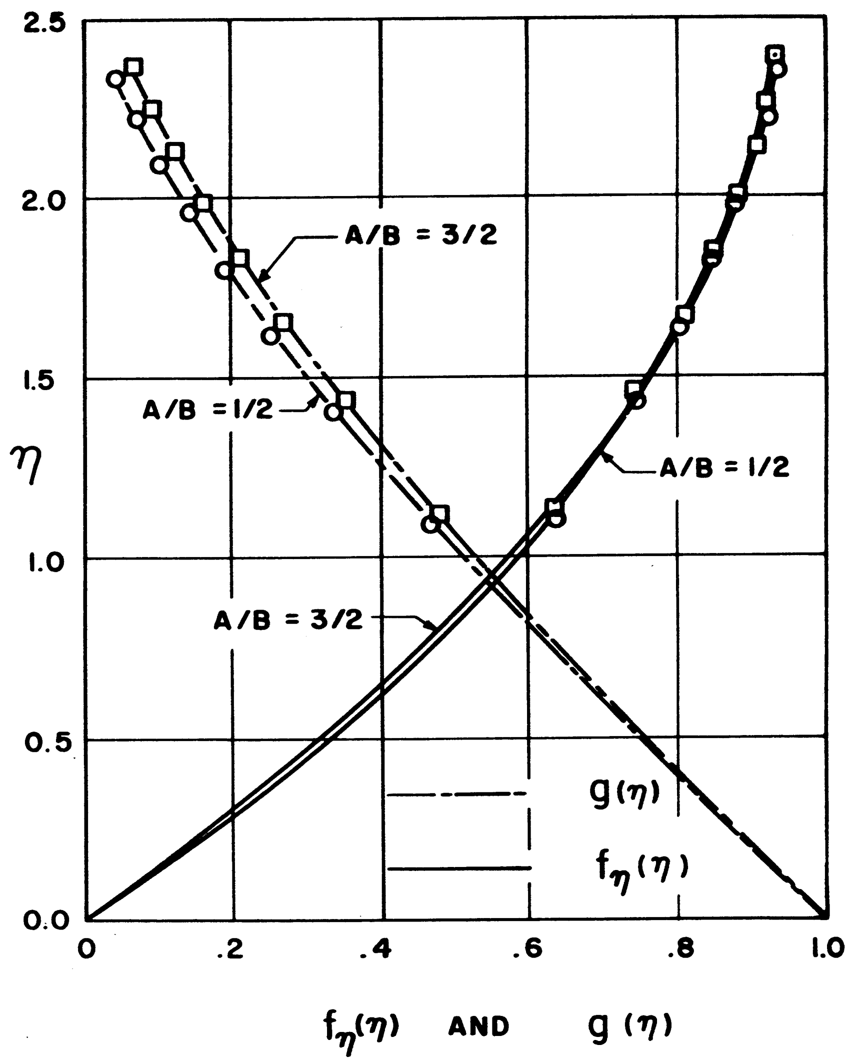

Figure 10. Velocity Profile and Concentration Distribution of the Non-homogeneous Viscoelastic Flow with S = 1, B = 0.3 and R = 0.08.

VI. CONCLUDING REMARKS

In this study the constitutive equations of Oldroyd show that in order to have a two dimensional viscoelastic flow a normal stress perpendicular to the flow is necessary. Depending on the values of the material constants this normal stress can be either a tensile or a compressive stress, and is a function of the rate of deformation.

The results obtained for a two dimensional viscoelastic flow past the wedge of 90 degrees show that:

(i) The thinner displacement thickness does not necessarily imply a larger frictional force.

(ii) For a homogeneous viscoelastic flow, the frictional force increases as the degree of dilatancy of the material increases, and decreases with increasing degree of pseudoplasticity of the material.

(iii) For a non-homogeneous viscoelastic flow with given material constants, depending on whether the material is pseudoplastic or dilatant and the ratio of the material concentration of outer flow and the concentration at the body, the frictional coefficient will decrease or increase from that of the homogeneous flow with the concentration at the body as the Schmidt number increases, and will approach a limit when the Schmidt number becomes very large. This limit is the frictional coefficient of the homogeneous flow with the concentration of the outer flow.

From the results obtained in this study, one can see that if the frictional force should be reduced by applying a soluble coating, then the coating must have the following properties: (a) highly pseudoplastic material, (b) the viscosity is equal to or less than that of the solvent, (c) a small Schmidt number.

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