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THEORY OF PLASMAS, II

LINEAR OSCILLATIONS IN RELATIVISTIC PLASMAS

by

K. Imre

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CONTENTS

ABSTRACT

The linear oscillations in a hot plasma which is representable by the relativistic Vlasov equation with the self-consistent fields are investigated. The method which is used by Bernstein in the nonrelativistic case is generalized to obtain the formal solution of the linearized problem. Particular attention is given to the case when the unperturbed distribution function is of the Maxwell-Boltzmann-Jüttner type (i.e., the relativistic equilibrium distribution) in which case the integrations involving the velocity space are carried out explicitly. The dispersion equation is derived and studied to some extent, considering the spatial dispersions explicitly in some special cases of interest. The ordinary and extraordinary modes, and the magnetohydrodynamic waves are investigated when the propagation vector is along the unperturbed magnetic field. The asymptotic expansions are developed corresponding to the dispersion relations of the cases considered, and they are shown to be in agreement with the results of previous studies in their respective order of approximations. It is found that circularly polarized transverse waves propagating along the unperturbed magnetic field are evanescent if $\nu^2 > 1 - \Omega^2 / \omega^2$, where ν is the index of refraction (kc/ω) and Ω is the gyrofrequency. In the absence of the external field the cut-off frequency is found to be a monotonically decreasing function of the temperature.

I

INTRODUCTION

In the recent literature, the problems involving the hot ionized gases have increasingly attracted the plasma investigators. The relativistic Vlasov equation together with Maxwell's field equations have been used in most of these approaches. Since the particle-particle and particle-photon correlations are ignored completely in this model, the validity and the applicability of this representation are somewhat restricted. The extent to which this imposes limitations has not yet been made evident in the literature. However, leaving these questions unanswered, in this work we shall assume that the above-mentioned model can properly represent the system of interest to some extent. Furthermore, to study the oscillatory behavior of the plasma a linearized theory will be employed.

In Section II we give the basic formulation of the mathematical problem, mainly for the purpose of introducing the notation. Also some discussion involving the moment equations is presented.

The linearization procedure is introduced in Section III, and the formal solution of the system of equations is obtained using the integral transform technique, which is the direct generalization of Bernstein's well-known procedure to the relativistic case.

Particular attention is given to a special case in which the unperturbed distribution is of the Maxwell-Boltzmann-Jüttner (MBJ) type.

The study of the dispersive phenomena is considered in Section V. The results of the earlier investigations are examined. Transverse oscillations propagating along a constant magnetic field are considered, and the dispersion relation corresponding to the relativistic magnetohydrodynamic waves is derived.

A short discussion of the results deduced in the context of the present formalism is given in Section VI. It is pointed out that in the absence of the external field the cut-off frequency decreases monotonically with increasing temperatures. The longitudinal mode possesses the same cut-off frequency as that of the transverse mode.

II

BASIC EQUATIONS

1. General Formulation

The relativistic collisionless Boltzmann-Vlasov equation for a species of type N is given as.*

$$Df_N \equiv u_\mu \frac{\partial f_N}{\partial x_\mu} + \frac{e_N}{m_N c} F_{k\mu} u_\mu \frac{\partial f_N}{\partial u_k} = 0 \ . \tag{1}$$

The summation convention both for Greek (1 to 4) and Latin (1 to 3) indices are used here. The reduced velocity, u_μ, and the field tensor, $F_{\nu\mu}$, are defined as

$$u_k = v_k \gamma \ , \quad u_4 = i\,c\gamma$$

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} = \left(1 + \frac{u^2}{c^2}\right)^{1/2} \qquad (V^2 = V_k V_k)$$

$$F_{ij} = \epsilon_{ijk} H_k \ , \quad F_{4j} = -F_{j4} = i\,E_j \ . \tag{2}$$

We denote by m_N and e_N ($=Z_N e$) the rest mass and the electric charge of the species of type N, respectively. In what follows we shall suppress the suffix N when no ambiguity arises. The fourth component of the space-time is chosen as $x_4 = ict$.

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* Only Cartesian tensors will be used throughout this work; no distinction will be necessary between the covariant and the contravariant components.

In addition to equation (1) we have Maxwell's field equations

$$\frac{\partial}{\partial x_\mu} F_{\nu\mu} = \frac{4\pi}{c} J_\nu$$

$$= \frac{4\pi}{c} \sum e \int \frac{d^3u}{\gamma} u_\nu f$$

$$\epsilon_{\sigma\mu\nu\lambda} \frac{\partial}{\partial x_\mu} F_{\lambda\nu} = 0 \ . \qquad (3)$$

Here the summation is understood to be extended over all species in the plasma. Henceforward, we shall use the symbol $\sum$ always in this meaning, unless it is otherwise specified. The symbol $\epsilon_{\sigma\lambda\mu\nu}$ denotes the completely antisymmetric (Levi-Civitá) tensor density in the 4-space.

The last of equations (3) implies that there exists a vector potential, A_μ, such that

$$F_{\nu\mu} = \frac{\partial A_\mu}{\partial x_\nu} - \frac{\partial A_\nu}{\partial x_\mu} \ . \qquad (A_4 = i\,\phi)$$

Now, with the Lorentz condition

$$\frac{\partial A_\mu}{\partial x_\mu} = 0 \ ,$$

equation (3) can be written as

$$\Box^2 A_\mu = -\frac{4\pi}{c} J_\mu \ . \qquad \left(\Box^2 = \frac{\partial^2}{\partial x_\mu \partial x_\mu}\right) \qquad (4)$$

2. Moment Equations

Let us multiply (1) by an arbitrary function $\frac{1}{\gamma}\, g(x_\mu, u_k)$ and integrate in the $\underline{u}$ space.

$$\frac{\partial}{\partial x_\mu} \int \frac{d^3u}{\gamma}\, u_\mu f g = \int \frac{d^3u}{\gamma}\, f D g \quad . \tag{5}$$

Here we have used the relation

$$F_{k\mu} \frac{\partial}{\partial u_k} \left(\frac{u_\mu}{\gamma}\right) = 0 \ .$$

In particular, for g = m, equation (5) gives the continuity equation*

$$\frac{\partial}{\partial x_\mu} j_\mu = 0 \ , \tag{6}$$

where the mass current j_μ is defined as

$$j_\mu = m \int \frac{d^3u}{\gamma}\, u_\mu f \ . \tag{7}$$

For $g = m u_\nu$, one obtains after some manipulations

$$\frac{\partial}{\partial x_\mu} \mathbf{P}_{\nu\mu} - \frac{e}{mc} F_{\nu\mu}\, j_\mu = 0 \ ,$$

where the energy-momentum tensor,

$$\mathbf{P}_{\nu\mu} = m \int \frac{d^3u}{\gamma}\, u_\nu u_\mu f \ . \tag{8}$$

Now, with the aid of equation (3), we have

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*We assume the rest mass, m, to be a constant.

$$\frac{4\pi}{c} F_{\nu\mu} J_{\mu} = F_{\nu\mu} \frac{\partial}{\partial x_{\rho}} F_{\mu\rho}$$

$$= \frac{\partial}{\partial x_{\rho}} F^2_{\nu\rho} - \frac{1}{4} \frac{\partial}{\partial x_{\nu}} (\mathrm{tr}\, \underline{\underline{F}}^2) \ .$$

Thus, introducing the traceless, symmetric tensor, (known as the electromagnetic stress tensor),

$$S_{\mu\nu} = -\frac{1}{4\pi} \left[F^2_{\mu\nu} - \frac{1}{4} (\mathrm{tr}\, \underline{\underline{F}}^2) \delta_{\mu\nu} \right] \tag{9}$$

we obtain the well-known energy and momentum conservation equation

$$\frac{\partial}{\partial x_{\mu}} \left[\mathbf{P}_{\nu\mu} + S_{\nu\mu} \right] = 0 \ . \tag{10}$$

The five equations given by equations (6) and (10) are far from being sufficient to provide a closed system of equations. This difficulty customarily is relaxed by introducing a set of systematic approximations which leads to the relativistic magnetohydrodynamic equations. To do this let us introduce the quantities

$$n \equiv \int f\, d^3u \left(\equiv \frac{1}{imc} j_4 \right)$$

$$\lambda_{\mu} \equiv j_{\mu} (-j_{\alpha} j_{\alpha})^{-1/2} = j_{\mu} \frac{\sqrt{1+\lambda^2}}{nmc}$$

$$\tau_{\mu\nu} \equiv \delta_{\mu\nu} + \lambda_{\mu}\lambda_{\nu} \ ,$$

so that we have the relations

$$\lambda_\mu \lambda_\mu = -1$$

$$\lambda_\mu \tau_{\mu\nu} = \tau_{\nu\mu} \lambda_\mu = 0$$

$$\tau_{\mu\alpha} \tau_{\alpha\nu} = \tau_{\mu\nu} \; .$$

Now the energy-momentum tensor may be decomposed (cf. [1])

$$P_{\mu\nu} = w \lambda_\mu \lambda_\nu + \frac{1}{c} \lambda_\mu q_\nu + \frac{1}{c} q_\mu \lambda_\nu + \Psi_{\mu\nu} \; ,$$

where

$$w = \lambda_\mu P_{\mu\nu} \lambda_\nu \; .$$

is the invariant energy density,

$$q_\mu \equiv -c \, \tau_{\mu\alpha} P_{\alpha\beta} \lambda_\beta$$

is the heat flow vector, and

$$\Psi_{\mu\nu} \equiv \tau_{\mu\alpha} P_{\alpha\beta} \tau_{\beta\nu}$$

is the stress tensor. We observe that

$$\lambda_\mu q_\mu = 0 \; , \qquad \lambda_\mu \Psi_{\mu\nu} = 0 \; .$$

The energy-momentum tensor corresponding to a "perfect" gas may then be obtained by assuming: (i) $q_\mu = 0$, and (ii) $\Psi_{\mu\nu} = p \, \tau_{\mu\nu}$ (cf. [2, 3, 4]) so that

$$P_{\mu\nu} = (w + p) \lambda_\mu \lambda_\nu + p \, \delta_{\mu\nu} \; ,$$

where p is the invariant pressure

$$p = \frac{1}{3} \mathcal{T}_{\mu\nu} \mathbf{P}_{\mu\nu} \quad .$$

The terminology introduced above may be justified as follows. First, let us consider the invariant energy density

$$w = \lambda_\mu \mathbf{P}_{\mu\nu} \lambda_\nu = m \int \frac{d^3u}{\gamma} \lambda_\mu u_\mu \lambda_\nu u_\nu f \quad .$$

Since w is an invariant we can choose $\lambda_j = 0, (\lambda_4 = i)$ without losing any generality. Thus

$$w = \int d^3u \; \mathcal{E} \; f$$

where $\mathcal{E} = mc^2 \gamma$ is the total energy.

Similarly the heat flow vector introduced above may be written

$$q_j = -i c \mathbf{P}_{j4} = \int d^3u \; \mathcal{E} \; v_j f \qquad (\gamma v_j = u_j)$$

where we set $\lambda_\mu = (0, i)$. This means that q_j is to be measured with respect to a frame moving with the instantaneous average velocity of the system. We note that if $f^{(0)}$ is isotropic in $\underline{u}$ then $q_j = 0$. Finally the tensor $\Psi_{\mu\nu}$ introduced above reduces to

$$\Psi_{jk} = \mathbf{P}_{jk} = \int d^3u \, (mu_j) \, v_k \, f$$

when measured with respect to a frame which moves with the instantaneous average velocity of the plasma. Obviously Ψ_{jk} is diagonal when f is isotropic. The invariant pressure reads then (in this frame)

$$p = \frac{1}{3} \Psi_{jj} = \frac{m}{3} \int \frac{d^3u}{\gamma} u^2 f .$$

In Section IV we shall show that these conditions are satisfied for the equilibrium distribution (cf. equations (9), (10), (11)).

It is customary to introduce the proper (invariant) mass density

$$\rho^o = n^o m = \frac{nm}{\sqrt{1+\lambda^2}} ,$$

and the internal energy per unit rest mass

$$\varepsilon \equiv \frac{w - \rho^o c^2}{\rho^o} ,$$

so that we get

$$\frac{\partial}{\partial x_\mu} (\rho^o \lambda_\mu) = 0$$

$$\frac{\partial}{\partial x_\mu} (P_{\nu\mu} + S_{\nu\mu}) = 0$$

where

$$P_{\nu\mu} = \rho^o (\varepsilon + c^2 + p/\rho^o) \lambda_\nu \lambda_\mu + p \ \delta_{\nu\mu} .$$

The latter set is known as the relativistic magnetohydrodynamic equations. Clearly, these equations still are not complete. However, they may be closed by an equation of state, relating ε . p and ρ^o, and with some additional considerations concerning the electrical conductivity properties of the medium [4, 5]. Harris [4] has shown that for the case of infinite conductivity the well-known adiabatic law is satisfied.

The linear plasma oscillations may be studied by these macroscopic equations. However, here we shall employ the Vlasov equation directly to obtain the necessary relationships. Although additional assumptions concerning the unperturbed state (and the initial perturbations) will be assumed, no phenomenological specific requirement will be adopted explicitly.

III

THE FORMAL SOLUTION OF THE LINEARIZED SYSTEM

Small amplitude oscillations may be studied by the usual perturbation technique employed on equation (1). Let us introduce

$$f = f^{(0)} + f^{(1)}$$

$$F_{\mu\nu} = F^{(0)}_{\mu\nu} + F^{(1)}_{\mu\nu}$$

$$F^{(0)}_{kj} = \text{constant}, \quad F^{(0)}_{4j} = 0 \ . \tag{11}$$

The zeroth order equation, now can be written as

$$u_\mu \frac{\partial f^{(0)}}{\partial x_\mu} + \frac{e}{mc} F^{(0)}_{kj} u_j \frac{\partial f^{(0)}}{\partial u_k} = 0 \ . \tag{12}$$

In this case equation (12) may be solved by Lagrange's method. The corresponding characteristic equations are given as

$$\frac{dx_\mu}{ds} = u_\mu \ , \qquad \frac{du_k}{ds} = \frac{e}{mc} F^{(0)}_{kj} u_j \ . \tag{13}$$

The latter may be solved easily. Introducing the quantities

$$\Omega \equiv \frac{eH^{(0)}}{mc}$$

$$\underline{\underline{b}} \equiv \frac{1}{H^{(0)}} \underline{\underline{F}}^{(0)} \ , \tag{14}$$

and using the properties

$$b_{ij}^{2} = h_i h_j - \delta_{ij} \quad , \quad \left(h_i \equiv \frac{1}{H^{(0)}} H_i^{(0)}\right)$$

$$b_{ij}^{3} = - b_{ij} \quad , \tag{15}$$

one can write the solution as

$$\underline{u} = \left\{ \exp \left[\Omega \, s \, \underline{\underline{b}} \right] \right\} \cdot \underline{\eta} \; .$$

Here

$$\exp \left[\Omega \, s \, \underline{\underline{b}} \right] = \sum_{n=0}^{\infty} \frac{(\Omega \, s)^n}{n!} \underline{\underline{b}}^n$$

$$= \underline{\underline{I}} + \sin \Omega \, s \, \underline{\underline{b}} + (1 - \cos \Omega \, s) \underline{\underline{b}}^2 \; ,$$

where

$$\underline{\underline{b}}^{o} = \underline{\underline{I}} = (\delta_{jk}) \; . \tag{16}$$

Then, the unperturbed orbit equations are found to be

$$u_\mu = \left[\delta_{\mu\nu} + \sin \Omega \, s \, b_{\mu\nu} + (1 - \cos \Omega s) \, b_{\mu\nu}^{2} \right] \cdot \eta_\nu$$

$$x_\mu = \left[s \, \delta_{\mu\nu} + \frac{1 - \cos \Omega s}{\Omega} b_{\mu\nu} + \frac{\Omega s - \sin \Omega s}{\Omega} b_{\mu\nu}^{2} \right] \eta_\nu + \xi_\mu \; . \tag{17}$$

Inversely, we have

$$\eta_\mu = u_\nu \left[\delta_{\nu\mu} + \sin \Omega s \, b_{\nu\mu} + (1 - \cos \Omega s) \, b_{\nu\mu}^{2} \right]$$

$$\xi_\mu = x_\mu - u_\nu \left[s \, \delta_{\nu\mu} + \frac{1 - \cos \Omega \, s}{\Omega} b_{\nu\mu} + \frac{\Omega s - \sin \Omega s}{\Omega} b_{\nu\mu}^{2} \right] . \tag{18}$$

In the above formulation, it is understood that

$$b_{\mu 4} = b_{4\mu} = b^2_{\mu 4} = b^2_{4\mu} = 0 \quad .$$

Therefore, the general solution of the unperturbed equation is

$$f^{(0)}(x_\mu, u_k) = g\left(\xi_k(s),\ \eta_k(s)\right) \quad ,$$

where g is an arbitrary function of its arguments, and s is the proper time:

$$s = \frac{t - t_o}{\gamma} = \frac{x_4 - x_4^o}{u_4} \quad .$$

In particular if $f^{(0)}$ is space-time independent, i.e. $\frac{\partial f^{(0)}}{\partial x_\mu} = 0$, then we simply have

$$f^{(0)} = g\,(\underline{u} \cdot \underline{h}\ ,\ u) \quad . \tag{19}$$

A special distribution of considerable interest, which satisfies the latter condition is the relativistic Maxwell-Boltzmann (Jüttner [6]) distribution[2]

$$f^{(0)}_{MBJ} = C\ e^{-\beta\gamma} \tag{20}$$

which will be given more attention in our future discussions.

The first-order equation is found to be

$$u_\mu \frac{\partial f^{(1)}}{\partial x_\mu} + \Omega\, b_{kj}\, u_j \frac{\partial f^{(1)}}{\partial u_k} = -\frac{e}{mc} F^{(1)}_{k\mu}\, u_\mu \frac{\partial f^{(0)}}{\partial u_k} \tag{21}$$

which now can be written in the integral form as usual:

$$f^{(1)}(x_\mu, u_j) - f^{(1)}\left(\xi_\mu\left(\frac{x_4}{u_4}\right),\ \eta_j\left(\frac{x_4}{u_4}\right)\right) = -\int_0^{x_4/u_4} \phi\left(\xi_\mu(s'),\ \eta_\nu(s')\right) ds' \quad , \tag{22}$$

where

$$\phi(x_\mu, u_\nu) \equiv \frac{e}{mc} F^{(1)}_{k\mu} u_\mu \frac{\partial f^{(0)}}{\partial u_k} .$$

It is convenient to introduce the matrices*

$$R_{\mu\nu}(s) = \delta_{\mu\nu} - \sin\Omega s \, b_{\mu\nu} + (1-\cos\Omega s)\, b^2_{\mu\nu}$$

$$G_{\mu\nu}(s) = s\delta_{\mu\nu} - \frac{1-\cos\Omega s}{\Omega} b_{\mu\nu} + \frac{\Omega s - \sin\Omega s}{\Omega} b^2_{\mu\nu} \tag{23}$$

so that the orbit equations become

$$\begin{cases} u_\mu = \eta_\nu R_{\nu\mu} \\ x_\mu = \eta_\nu G_{\nu\mu} + \xi_\mu \end{cases}$$

$$\begin{cases} \eta_\mu = R_{\mu\nu} u_\nu \\ \xi_\mu = x_\mu - G_{\mu\nu} u_\nu . \end{cases}$$

Hence

$$\phi(\xi_\mu, \eta_\nu) = \frac{e}{mc} F^{(1)}_{k\mu} (x_\lambda - G_{\lambda\nu} u_\nu) R_{\mu\rho} u_\rho \frac{\partial f^{(0)}}{\partial \eta_k} ,$$

where

$$\frac{\partial f^{(0)}}{\partial \eta_k} = \frac{\partial f^{(0)}}{\partial x_\mu} G_{k\mu} + \frac{\partial f^{(0)}}{\partial u_j} R_{kj} .$$

In what follows we shall restrict ourselves to the case in which

$$\frac{\partial f^{(0)}}{\partial x_\mu} = 0 , \implies b_{kj} u_j \frac{\partial f^{(0)}}{\partial u_k} = 0.$$

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*Several properties of R_{ij} and G_{ij} (and also $M_{jk}(s) = \int_0^s G_{jk}(s')\, ds' = \int_0^s \int_0^{s'} R_{jk}(s'')\, ds''\, ds'$) are given in [7].

Thus *

$$f^{(1)} = -\frac{e}{mc} u_\nu \frac{\partial f^{(0)}}{\partial u_j} \int_0^{x_4/u_4} F^{(1)}_{k\mu} (x_\lambda - G_{\lambda\rho} u_\rho) R_{\mu\nu} R_{kj} \, ds \ . \qquad (24)$$

In this special case it is convenient to introduce the Fourier transformation in the space-time variables:

$$X^+(k_\mu) = \int_{(+)} e^{-ik_\mu x_\mu} X^{(1)}(x_\mu) \, d^4x \ . \qquad (25)$$

The (+) sign indicates that the x_4 integration is to be restricted to the positive x_4 range. Clearly this corresponds to the Laplace transform with respect to time, with a transform parameter $p = -k_4 c$.

Equation (24), then, will transform into

$$f^+ = -\frac{e}{mc} u_\nu \frac{\partial f^{(0)}}{\partial u_j} F^+_{k\mu}(k_\rho) \int_0^\infty e^{-y(s)} R_{\mu\nu}(s) R_{kj}(s) \, ds \ , \qquad (26)$$

where

$$y(s) = i k_\rho u_\sigma G_{\rho\sigma}(s)$$

$$= is\,\underline{k} \cdot \underline{u} + ps\gamma - i\frac{1-\cos\Omega s}{\Omega}(\underline{u} \cdot \underline{h} \times \underline{k}) + i\frac{\Omega s - \sin\Omega s}{\Omega}\left[(\underline{u} \times \underline{h}) \cdot (\underline{h} \times \underline{k})\right]$$

Let us note that if $f^{(0)}$ is isotropic, i.e. $\frac{\partial f^{(0)}}{\partial u_j} \propto u_j$, then all terms in the summation containing F^+_{kj} vanish due to the antisymmetric property of the field tensor. Thus in this case we simply have

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*Hereafter we shall suppress the terms involving the initial perturbations explicitly, and then take them into account subsequently as a whole whenever needed.

$$f^{+} = -\frac{e}{mc}\, u_4 \frac{\partial f^{(0)}}{\partial u_j}\, F^{+}_{k4} \int_0^{\infty} e^{-y(s)}\, R_{kj}(s)\, ds\ .$$

Now, using (26), we shall compute the electric charge current J^{+}_{ρ}, which will enable us to eliminate the particle variables, i.e. f^{+}, through Maxwell's equations.

$$J^{+}_{\rho} = \sum e \int f^{+} u_{\rho} \frac{d^3u}{\gamma}$$

$$= i\, \sigma_{\rho k \mu}\, F^{+}_{k\mu},$$

where the "generalized conductivity tensor" is defined as

$$\sigma_{\rho j \mu} = i \sum \frac{e^2}{mc} \int \frac{d^3u}{\gamma} \frac{\partial f^{(0)}}{\partial u_k}\, u_{\nu} u_{\rho} \int_0^{\infty} e^{-y} R_{jk} R_{\mu\nu}\, ds\ . \tag{27}$$

The usual conductivity tensor is related to the one introduced above as *

$$\sigma_{\rho\tau} = \frac{k_{\nu}}{k_4}\, \delta^{\nu\tau}_{\mu j}\, \sigma_{\rho j \mu}$$

so that

$$J_{\rho} = \sigma_{\rho\tau} E_{\tau}\ . \qquad (E_4 \equiv i F_{44} \equiv 0)$$

We note that in the case when $f^{(0)}$ is isotropic in $\underline{u}$, this reduces to

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*The Kronecker delta $\delta^{\mu_1 \ldots \mu_n}_{\nu_1 \ldots \nu_n}$ may be defined as

$$\delta^{\mu_1 \ldots \mu_n}_{\nu_1 \ldots \nu_n} = \text{Det}\left(\delta^{\mu_p}_{\nu_q}\right) = \begin{vmatrix} \delta^{\mu_1}_{\nu_1} & \cdots & \delta^{\mu_1}_{\nu_n} \\ \vdots & & \vdots \\ \delta^{\mu_n}_{\nu_1} & \cdots & \delta^{\mu_n}_{\nu_n} \end{vmatrix}\ .$$

Also it is understood that $\delta^{\alpha}_{j} = 0$ if $\alpha = 4$ and $\delta^{\alpha}_{j} = \delta^{k}_{j} = \delta_{kj}$ when $\alpha = k(=1, 2, 3)$.

$$\sigma_{ij} = \sigma_{ij4} = -\sum \frac{e^2}{m} \int d^3u \, \frac{\partial f^{(0)}}{\partial u_k} \, u_i \int_0^{\infty} e^{-y} R_{jk} \, ds$$

(cf. for instance, [9]).

Substituting the current computed above in equation (3) we get

$$ik_{\mu} F^{+}_{\rho\mu} - \frac{4\pi}{c} i \sigma_{\rho j \mu} F^{+}_{j\mu} = \zeta_{\rho} \quad . \tag{28}$$

Here ζ_{ρ} is an explicit function of the initial perturbations and the unperturbed parameters only, and contains all the terms of that nature which have been suppressed in our equations. Furthermore, we observe that it satisfies the relation

$$k_{\rho} \zeta_{\rho} = 0 \; ,$$

for

$$k_{\mu} k_{\rho} F^{+}_{\rho\mu} = 0 \qquad \text{and} \qquad k_{\mu} J_{\mu} = 0 \; .$$

Hence only three of the four equations given in (28) are independent. We shall use the curl equation, i.e. the second of (3), in order to eliminate the spatial components of the field tensor

$$F^{+}_{\mu\nu} = \delta^{\lambda j}_{\mu\nu} \frac{k_{\lambda}}{k_4} F^{+}_{4j} \; . \tag{29}$$

Substituting in (28) we obtain

$$\left[k_{\nu} \, \delta^{\lambda k}_{\nu i} \frac{k_{\lambda}}{k_4} - \frac{4\pi}{c} \sigma_{ij\mu} \, \delta^{\lambda k}_{\mu j} \frac{k_{\lambda}}{k_4} \right] E^{+}_{k} = \zeta_i$$

which may be written in the form

$$T_{ik} E_u^+ = -\frac{p}{k^2 c} \xi_i = \bar{\xi}_i ,$$

where

$$T_{ij} = S_{ij} + \frac{p^2}{k^2 c^2} \epsilon_{ij} \quad . \tag{30}$$

Here we substituted $k^2 = k_j k_j$ and $k_4 c = -p$. The projection tensor S_{ij}, and the "dielectric" tensor ϵ_{ij} are given as

$$S_{ij} = \delta_{ij} - \hat{k}_i \hat{k}_j \qquad (k_i = k \hat{k}_i)$$

$$\epsilon_{ij} = \delta_{ij} + \frac{4\pi}{p} \sigma_{ij} , \tag{31}$$

where σ_{ij} is the conductivity tensor introduced earlier.

Therefore the formal solution of the field equation is obtained as

$$F^+_{\mu\nu} = i \, \delta^{\lambda j}_{\mu\nu} \frac{k_\lambda}{k_4} T^{-1}_{jk} \bar{\xi}_k \quad . \qquad (T_{ik} T^{-1}_{kj} = \delta_{ij})$$

If this solution is used to eliminate the field variables in (26), one obtains the formal solution for the transformed distribution function, which enables one to compute the average values of the physical properties of interest in terms of the initial values and the unperturbed parameters explicitly.

Alternatively, one can deduce an integral equation for the particle variables by eliminating the field variables. The latter form is found to be more convenient in dealing with some particular aspects of the problem. For this reason we shall give it as an appendix.

IV

A SPECIAL CASE

In this section we shall give special attention to the case in which the unperturbed distribution function is of the Maxwell-Boltzmann-Jüttner (MBJ) type [2, 6], namely*

$$f^{(0)}_{MBJ} = \frac{n}{4\pi c^3} \frac{\beta}{K_2(\beta)} e^{-\beta\gamma} , \tag{32}$$

where $\beta = mc^2/\Theta$, Θ is temperature in ergs, $K_2(\beta)$ is the modified Bessel function of the second kind, and

$$n = \int f^{(0)} d^3u .$$

Then the first order average of the property g, i.e.

$$n \langle g \rangle^{(1)} = \int f^{(1)} g \frac{d^3u}{\gamma}$$

may be written in the transformed space as follows:

$$n \langle g \rangle^{+} = \int f^{+} g \frac{d^3u}{\gamma}$$

$$= -\frac{e}{m} E^{+}_{k} \int d^3u \frac{\partial f^{(0)}}{\partial u_j} g \int_0^{\infty} e^{-y(s)} R_{kj} ds .$$

Changing the order of the integrations and integrating by parts, one obtains

$$n \langle g \rangle^{+} = -\frac{e}{m} E^{+}_{k} \int_0^{\infty} \frac{\beta\, ds}{\beta + ps} \int d^3u \; e^{-y} f^{(0)} \left[i k_j G_{kj} g - R_{kj} \frac{\partial g}{\partial u_j} \right] , \tag{33}$$

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*In the form of f_{MBJ} given above we have selected the spatial components of λ_μ equal to zero [2]. The general case can be obtained by a transformation without much difficulty.

where as before $p = -k_4 c$ and $y = i\,\underline{k}\cdot\underline{\underline{G}}\cdot\underline{u} + \rho\gamma s$. In deriving equation (33) we have used the relation

$$\left(1+\frac{ps}{\beta}\right)\int d^3u\; e^{-y}\,\frac{\partial f^{(0)}}{\partial u_j}\,g = \int d^3u\; e^{-y}\,f^{(0)}\left(i\,k_\ell G_{\ell j}\,g - \frac{\partial g}{\partial u_j}\right).$$

The integral*

$$\frac{1}{4\pi c^3}\int_{\underline{u}} \frac{d^3u}{\gamma}\, e^{\frac{i}{c}\,a_\mu a_\mu} = \frac{K_1\left(\sqrt{a_\mu a_\mu}\right)}{\sqrt{a_\mu a_\mu}}$$

and its derivatives are useful in carrying out the velocity integrations corresponding to the first-order moments. For instance the first-order mass current (which is obtained by setting $g = m u_\mu$) is found to be

$$j_k^+ = en\,E_j^+\,\frac{\beta^2}{K_2(\beta)}\int_0^\infty ds\left[R_{jk}\,\frac{K_2(\varpi)}{\varpi^2} - c^2\,G_{j\ell}k_\ell k_q G_{qk}\,\frac{K_3(\varpi)}{\varpi^3}\right]$$

$$j_4^+ = \frac{\rho^+}{ic} = enc\,E_j^+ k_\ell\,\frac{\beta^2}{K_2(\beta)}\int_0^\infty ds\,(\beta + sp)\,G_{j\ell}\,\frac{K_3(\varpi)}{\varpi^3}$$

where

$$\varpi^2 = (\beta + sp)^2 + 2c^2\,\underline{k}\cdot\underline{\underline{M}}\cdot\underline{k}\;. \qquad \left(\underline{\underline{M}} = \int_0^s \underline{\underline{G}}(s')ds'\right) \tag{34}$$

*Strictly speaking, a singular term, proportional to $\delta(a_\mu a_\mu)$, should be added to the right-hand side. However, we shall ignore this term for the time being. In the next section, while discussing the spatial dispersions, we shall give more attention to the contributions due to this improper part.

It can be shown without much difficulty that the continuity equation $k_\mu j^+_\mu = 0$ is satisfied by the form derived above.

The energy-momentum tensor may be computed similarly; however it will not be given here.

Let us note that the nonrelativistic limit may be carried out easily by using the relations

$$\lim_{\beta \to \infty} \frac{\beta}{\varpi} = 1 \quad , \quad \lim_{\beta \to \infty} (\beta - \varpi) = - sp - \frac{\Theta}{m} \underline{k} \cdot \underline{\underline{M}} \cdot \underline{k} \; .$$

$$\lim_{\beta \to \infty} \frac{K_n(\varpi)}{K_2(\beta)} = \lim_{\beta \to \infty} e^{\beta - \varpi} = \exp\left[- sp - \frac{\Theta}{m} \underline{k} \cdot \underline{\underline{M}} \cdot \underline{k} \right] \; . \tag{35}$$

Thus it can be seen readily that equation (34) approaches the results obtained in the nonrelativistic analysis (see for instance [7]).

We also note that it is possible to introduce the diffusion and the mobility tensors, as discussed in [7], in such a way that

$$j^+_k = - i \, D^+_{kj} \, k_j \, \rho^+ + n m \, \kappa^+_{kj} \, E^{+}_j \tag{36}$$

where

$$\kappa^+_{kj} = \frac{e}{m} \frac{\beta^2}{K_2(\beta)} \int_0^\infty R_{jk} \frac{K_2(\varpi)}{\varpi^2} \, ds$$

$$D^+_{kj} = \frac{\Theta}{m} \frac{\frac{\beta^3}{K_2(\beta)} \int_0^\infty G_{jk} \frac{K_3(\varpi)}{\varpi^3} \, \bar{\rho} \, ds}{\rho^+}$$

$$\rho^{+} \equiv \frac{\beta^2}{K_2(\beta)} \int_0^{\infty} (\beta + sp) \frac{K_3(\varpi)}{\varpi^3} \bar{\rho} \, ds \quad . \tag{37}$$

The electric current density is computed as usual

$$J_{\mu} = \sum \frac{e}{m} j_{\mu}$$

so that one obtains

$$J_k^{+} = \sigma_{kj} E_j^{+} \tag{38}$$

where

$$\sigma_{kj} = \sum \frac{e^2 n}{m} \frac{\beta^2}{K_2(\beta)} \int_0^{\infty} ds \left[R_{jk} \frac{K_2(\varpi)}{(\varpi)^2} - \frac{\Theta}{m} G_{j\ell} k_{\ell} k_q G_{qk} \beta \frac{K_3(\varpi)}{\varpi^3} \right] .$$

This result is in agreement with Trubnikov [8], and in the nonrelativistic limit reduces to Mower's conductivity tensor (cf. for instance, [7]). For the electric charge density we have

$$Q^{+} = \sum \frac{e}{m} \rho^{+}$$

$$= - i E_j^{+} k_{\ell} \sum \frac{e^2 n}{m} \frac{\beta^2}{K_2(\beta)} \int_0^{\infty} (\beta + sp) G_{j\ell} \frac{K_3(\varpi)}{\varpi^3} ds \quad . \tag{39}$$

We shall return to these relations later in connection with the study of the dispersion relations.

In closing this section we give some of the zero order properties corresponding to the MBJ distribution.

The stress tensor (in a reference system which moves with the local velocity of the plasma)

$$\Psi_{jk}^{(0)} = \int \frac{d^3u}{\gamma} \; m\, u_j u_k \, f^{(0)} = p\, \delta_{jk}$$

$$p^{(0)} = n \cdot \Theta = \frac{nmc^2}{\beta} \quad . \tag{40}$$

The energy density

$$w^{(0)} = \mathbf{P}_{44} = mc^2 \int \frac{d^3u}{\gamma} \; \gamma^2 \, f^{(0)}$$

$$= nmc^2 \left(\frac{K_3}{K_2} - \frac{1}{\beta} \right) = nmc^2 \, \frac{K_3}{K_2} - p^{(0)} \quad . \tag{41}$$

Thus in an arbitrary (Galilean) frame the energy-momentum tensor can be written immediately

$$\mathbf{P}_{\mu\nu} = nmc^2 \, \frac{K_3(\beta)}{K_2(\beta)} \, \lambda_\mu \lambda_\nu + p \, \delta_{\mu\nu} \quad . \tag{42}$$

(cf. [2]). (We see that these properties satisfy the conditions imposed in Section II for the perfect fluid).

V

RELATIVISTIC PLASMA OSCILLATIONS

It follows from (30) that, assuming $\mathfrak{f}_i$ is analytic, the dispersion relation for the linear plasma oscillations is

$$\text{Det}\left(S_{ij} + \frac{p^2}{k^2c^2}\epsilon_{ij}\right) = 0 \quad . \tag{43}$$

This may be written in the form

$$\epsilon^{\ell} + \frac{p^2}{k^2c^2}\left[\epsilon^{\ell}\cdot \text{tr}\,(\underline{\underline{\epsilon}}) + (\epsilon^2)^{\ell}\right] + \frac{p^4}{k^4c^4}\,\text{Det}\,(\underline{\underline{\epsilon}}) = 0 \,, \tag{44}$$

where

$$\epsilon^{\ell} = \underline{\hat{k}}\cdot\underline{\underline{\epsilon}}\cdot\underline{\hat{k}} \quad , \qquad (\epsilon^2)^{\ell} = \underline{\hat{k}}\cdot\underline{\underline{\epsilon}}\cdot\underline{\underline{\epsilon}}\cdot\underline{\hat{k}} \,,$$

and

$$\text{tr}\,(\underline{\underline{\epsilon}}) = \epsilon_{jj} \quad .$$

We note that in the nonrelativistic limit (i. e. $c \longrightarrow \infty$) this equation reduces to the well-known longitudinal (Landau-Vlasov) dispersion relation

$$0 = \epsilon^{\ell} = 1 + \frac{4\pi}{p}\,\underline{\hat{k}}\cdot\underline{\underline{\sigma}}\cdot\underline{\hat{k}} \quad .$$

We shall return to this point later.

The biquadratic equation (44) may be solved as

$$\frac{k^2c^2}{p^2} = -\left[\frac{\text{tr}\,\underline{\underline{\epsilon}}}{2} - \frac{(\epsilon^2)^{\ell}}{2\epsilon^{\ell}}\right] \mp \sqrt{\left[\frac{\text{tr}\,\underline{\underline{\epsilon}}}{2} - \frac{(\epsilon^2)^{\ell}}{2\epsilon^{\ell}}\right]^2 - \frac{\text{Det}\,(\underline{\underline{\epsilon}})}{\epsilon^{\ell}}} \tag{45}$$

when $\epsilon^{\ell} \neq 0$. The two possible modes indicated by $\mp$ correspond to the ordinary and extraordinary waves.

As is well known, the transverse and the longitudinal waves are usually coupled. Due to this fact, in the general case the study of the dispersion relation given above is somewhat complicated. However, there exist some special cases of interest for which a decoupling can be established, so that equation (45) reduces to the purely transverse dispersion relationship. These cases will be studied in the following subsections.

1. Zero External Field Case

Here we shall consider the case in which the external field is zero, i.e. $\Omega = 0$. Then we have $R_{\mu\nu} \longrightarrow \delta_{\mu\nu}$ and $G_{\mu\nu} \longrightarrow s\,\delta_{\mu\nu}$. Furthermore, if we assume that the unperturbed distribution function is isotropic, i.e. $\frac{\partial f^{(0)}}{\partial u_k} \propto u_k$, the conductivity tensor becomes

$$\sigma_{ij} = -\sum \frac{e^2}{m} \int d^3u \; \frac{u_i \frac{\partial f^{(0)}}{\partial u_j}}{i\, k_s u_s + p\,\gamma} . \tag{46}$$

It may be readily seen that in this case the longitudinal and the transverse parts are completely decoupled, so that the dielectric tensor

$$\epsilon_{ij} = \epsilon^{\ell}\, \hat{k}_i \hat{k}_j + \epsilon^t\, S_{ij} ,$$

where $\epsilon^{\ell} = \underline{\hat{k}} \cdot \underline{\underline{\epsilon}} \cdot \underline{\hat{k}}$, and $\epsilon^t = \frac{1}{2}\, S_{ij}\epsilon_{ij}$.

Thus the dispersion relation becomes

$$\mathrm{Det}\left[S_{ij} + \frac{p^2}{k^2c^2}\,\epsilon_{ij} \right] = \frac{p^2}{k^2c^2}\left(1 + \frac{p^2}{k^2c^2}\,\epsilon^t\right)^2 \epsilon^{\ell} = 0$$

so that we get the familiar relations

$$\epsilon^{\ell} = 1 + \frac{4\pi}{p}\,\sigma^{\ell} = 0 \tag{47}$$

for the longitudinal oscillations, and

$$1 + \frac{p^2}{k^2c^2}\,\epsilon^t = 1 + \frac{p^2}{k^2c^2}\left(1 + \frac{4\pi}{p}\,\sigma^t\right) = 0 \tag{48}$$

for the transverse oscillations.

These relationships have already been examined to some extent in the literature [11, 13, 14] . We shall study them in some detail.

We have

$$\sigma^{\ell} = \underline{\hat{k}} \cdot \underline{\underline{\sigma}} \cdot \underline{\hat{k}} = i \sum \frac{e^2}{m} \int d^3u \; \frac{(u_z)^2 \frac{1}{u} \frac{df^{(0)}}{du}}{k\,u_z - i p\,\gamma} ,$$

and

$$\sigma^t \equiv \frac{1}{2}\,\underline{\underline{S}} : \underline{\underline{\sigma}} = \frac{1}{2}\left(\mathrm{tr}\,\underline{\underline{\sigma}} - \sigma^{\ell}\right)$$

$$= i \sum \frac{e^2}{m} \int d^3u \; \frac{(u_x)^2 \frac{1}{u} \frac{df^{(0)}}{du}}{k\,u_z - i p\,\gamma} . \tag{49}$$

In the above integrations, we have selected the u_z-axis along $\underline{k}$.

1.1. Longitudinal Oscillations

We shall restrict ourselves merely to the study of the solutions which behave as outgoing waves at $+\infty$. In this case one substitutes $p = \tau - i\omega$ and considers the limit as $\tau \longrightarrow 0+$, so that*

*In the general case, the factor $i\pi$ in the last term is to be replaced by an arbitrary function of ω and k, cf. [15].

$$\frac{1}{ku_z - ip\gamma} \doteq \frac{\mathcal{P}}{ku_z - \omega\gamma} + i\pi \quad \delta(ku_z - \omega\gamma) \quad . \tag{50}$$

The symbol $\mathcal{P}$ indicates that the first term is to be interpreted as a principal value. The contribution of the last term, which gives rise to the spatial dispersion, will be studied next. That is, considering the imaginary part of $\frac{4\pi}{p} \sigma^{\ell}$ on the imaginary axis of the complex p plane (as $\tau \rightarrow 0$), we get

$$\mathrm{Im}\left(\frac{4\pi}{p}\sigma^{\ell}\right) = -\theta(\nu^2 - 1)\frac{8\pi^3}{\nu^3}\frac{c^3}{\omega^2}\sum\frac{e^2}{m}\int_U^{\infty} du \frac{df^{(0)}}{du}\gamma^2 . \tag{51}$$

Here ν is the index of refraction $(=\frac{kc}{\omega})$, $\theta(x)$ is the Heaviside's step function, and U is the "reduced" phase velocity, that is

$$U = \frac{\omega}{k}\Gamma \; , \qquad \Gamma = \sqrt{1+\frac{U^2}{c^2}} = \frac{1}{\sqrt{1-\frac{\omega^2/k^2}{c^2}}} \; .$$

The step function in equation (51) indicates that (as is pointed out by the former investigators [11]) no spatial dispersion occurs for the waves propagating faster than the speed of light. However, the nonrelativistic analysis, which gives (cf. for instance, [15])

$$\mathrm{Im}\left(\frac{4\pi}{p}\sigma^{\ell}\right) = \frac{8\pi^3}{\nu^3}\frac{c^3}{\omega^2}\sum\frac{e^2}{m} f^{(0)}\left(\frac{\omega}{k}\right)$$

cannot predict this point, unless the distribution function is required to vanish outside a sphere of radius c.

In particular when $f^{(0)}$ is of the MBJ type, as discussed in the previous section, one finds

$$\mathrm{Im}\left(\frac{4\pi}{p}\,\sigma\ell\right) = \frac{\pi}{2\nu^3}\sum\frac{\omega_p^2}{\omega^2}\,\frac{e^{-\beta}}{\beta K_2(\beta)}\left(2+2\beta\Gamma+\beta^2\Gamma^2\right)\theta(\nu^2-1), \tag{52}$$

where the plasma frequency

$$\omega_p^2 = \frac{4\pi n e^2}{m} \; . \tag{53}$$

Silin [14] studied the ultrarelativistic limit, assuming $f^{(0)} = C\,e^{-\beta u/c}$. (This corresponds to replacing γ by u/c, i.e. $c \longrightarrow 0$). His result can be obtained by ignoring the terms in equation (52) of the order of β^2, which leads to

$$\mathrm{Im}\left(\frac{4\pi}{p}\,\sigma\ell\right) = \frac{\pi}{2\nu^3}\sum\frac{\omega_p^2}{\omega^2}\,\beta + \Theta(\beta^2)\,, \qquad (\nu^2 > 1)\;. \quad \mathrm{UR} \tag{54}$$

In the nonrelativistic limit, i.e. $\beta >> 1$, one obtains

$$\mathrm{Im}\left(\frac{4\pi}{p}\,\sigma\ell\right) = \sqrt{\frac{\pi}{2}}\,\frac{\Gamma^2}{\nu^3}\sum\frac{\omega_p^2}{\omega^2}\,\beta^{3/2}\,e^{-\beta(\Gamma-1)}\;. \quad \mathrm{NR} \tag{55}$$

This, when $\nu >> 1$, reduces to the result corresponding to the Maxwellian distribution

$$\mathrm{Im}\left(\frac{4\pi}{p}\,\sigma\ell\right) = \sqrt{\frac{\pi}{2}}\,\frac{1}{\nu^3}\sum\frac{\omega_p^2}{\omega^2}\,\beta^{3/2}\,e^{-\beta/2\nu^2}\,, \tag{56}$$

which leads to the well-known Landau damping [16] .

Now, we consider the real part of $\frac{4\pi}{p}\sigma^{\ell}$ on the imaginary axis of the complex p plane (i.e. the principal value integral)

$$\mathrm{Re}\,\frac{4\pi}{p}\sigma^{\ell} = -\frac{(4\pi)^2}{k^2}\sum\frac{e^2}{m}\int_0^{\infty} du\,\gamma u\,\frac{df^{(0)}}{du}\left[1-\frac{c\gamma}{\nu u}\tanh^{-1}\frac{\nu u}{c\gamma}\right]. \tag{57}$$

Again if $f^{(0)}$ is of the MBJ type one finds

$$\mathrm{Re}\,\frac{4\pi}{p}\sigma^{\ell} = \sum\frac{\omega_p^2}{\omega^2}\frac{\beta}{\nu^2}\left[1-\frac{1}{\nu^2 K_2(\beta)}\int_0^{\infty}\left(\cosh^2 x+\frac{2\cosh x}{\beta}+\frac{2}{\beta^2}\right)\frac{e^{-\beta\cosh x}dx}{1+\frac{1-\nu^2}{\nu^2}\cosh^2 x}\right]. \tag{58}$$

When $\nu = 1$, one can carry out the above integration

$$\mathrm{Re}\,\frac{4\pi}{p}\sigma^{\ell} = -\sum\frac{\omega_p^2}{\omega^2}\left[\frac{2}{\beta}\frac{K_o(\beta)}{K_2(\beta)}+\frac{K_1(\beta)}{K_2(\beta)}\right]. \tag{59}$$

Alternatively, one may use the relations derived in Section IV. In the absence of the external field ($\Omega = 0$), equation (37) reduces to

$$\sigma_{ij} = \sigma^{\ell}\,\hat{k}_i\hat{k}_j + \sigma^t S_{ij} \tag{60}$$

where

$$\frac{4\pi}{p}\sigma^{\ell} = \sum\omega_p^2\frac{\beta^2}{K_2(\beta)}\int_0^{\infty}\frac{K_3(\varpi)}{\varpi^3}s(\beta+sp)\,ds$$

$$\frac{4\pi}{p}\sigma^{t} = \sum\frac{\omega_p^2}{p}\frac{\beta^2}{K_2(\beta)}\int_0^{\infty}\frac{K_2(\varpi)}{\varpi^2}ds$$

$$\varpi^2 = (\beta+ps)^2+k^2c^2s^2 \qquad \left(M_{jk}\longrightarrow\frac{s^2}{2}\delta_{jk}\ \text{ for } \Omega = 0\right). \tag{61}$$

The longitudinal dispersion relation then is obtained by setting the first of equation (61) equal to -1 when $|\nu| < 1$. We have deduced an asymptotic expansion for the integration involved;

$$1 \sim \sum \frac{\omega_p^2}{\omega^2} \sum_{n=0,1,\ldots} \frac{K_{n-1}(\beta)}{K_2(\beta)} \frac{(-1)^n}{\beta^n} \left[(1-\nu^2)^n a_n - (1-\nu^2)^{n-1} b_n \right]$$

where

$$p = -i\omega$$

$$a_n = \frac{(2n+2)!}{2^{n+1}(n+1)!} \quad , \quad b_n = \frac{(2n)!}{2^{n-1}(n-1)!} \quad , \quad (b_o = 0) \ . \qquad (62)$$

We see that for $\nu = 1$ we obtain

$$1 = \sum \frac{\omega_p^2}{\omega^2} \left(\frac{K_1(\beta)}{K_2(\beta)} + \frac{2}{\beta} \frac{K_o(\beta)}{K_2(\beta)} \right)$$

which is the same as the result obtained from equation (58) for this case.

For the cold plasma, i.e. $\beta \longrightarrow \infty$, equation (62) reduces to the well-known Langmuir and Tonks equation

$$\omega^2 = \sum \omega_p^2 \quad ,$$

and to the first order it gives the following result which is first derived by Clemmow and Willson [11],

$$\omega^2 \sim \sum \omega_p^2 \left[1 + \frac{3}{\beta} \left(\nu^2 - \frac{5}{6} \right) \right] + \mathcal{O}\left(\frac{1-\nu^2}{\beta^2} \right) . \qquad (63)$$

In deducing equation (63) we have used the asymptotic expansion of the Bessel functions

$$\frac{K_n(\beta)}{K_m(\beta)} \sim 1 + \frac{n^2-m^2}{2\beta} + \frac{(n^2-m^2)(n^2-m^2-2)}{2!\,(2\beta)^2} + \mathcal{O}\left(\frac{1}{\beta^3}\right) .$$

The ultrarelativistic limit as considered by Silin [14] can be rederived either from equation (58) by replacing $\gamma \longrightarrow u/c$, or from equation (61) by setting $K_3(\varpi) \sim 8/\varpi^3$. The result is

$$\nu^2 = \sum \frac{\omega_p^2}{\omega^2} \beta \left(\frac{1}{\nu} \tanh^{-1} \nu - 1 \right) , \qquad \nu^2 < 1 , \quad \text{UR} . \tag{64}$$

For the case $|\nu| << 1$, approximating successively one obtains

$$\frac{3}{5} \nu^2 \cong 1 - \frac{1}{3} \sum \frac{\omega_p^2}{\omega^2} \beta$$

which indicates that no longitudinal waves can propagate in an ultrarelativistic plasma with frequencies $\omega^2 < \frac{1}{3} \sum \omega_p^2 \beta$.

It is seen that equation (64) blows up for $\nu = 1$. However, an examination of equation (59) shows that the phase velocity corresponding to $\nu = 1$ is finite. This observation indicates the limitation of Silin's ultrarelativistic result in the vicinity of $\nu = 1$. One may write

$$\omega^2 \approx \sum \omega_p^2 \beta \left(\frac{1}{2} - \ln \beta\right) , \qquad \begin{pmatrix} \nu = 1 \\ \beta \ll 1 \end{pmatrix} \text{UR} .$$

1.2. Transverse Oscillations

Similarly, one can perform the angular integrations for the transverse conductivity which leads to the results

$$\mathrm{Im}\left(\frac{4\pi}{p}\sigma^t\right) = -\frac{4\pi^3}{\omega k}\sum\frac{e^2}{m}\int_U^\infty du\,\frac{df^{(0)}}{du}\,u^2\left[1-\frac{\omega^2\gamma^2}{k^2u^2}\right]\theta(\nu^2-1)\ ,$$

$$\mathrm{Re}\left(\frac{4\pi}{p}\sigma^t\right) = \frac{8\pi^2}{\omega k}\sum\frac{e^2}{m}\int_0^\infty du\,\frac{df^{(0)}}{du}\,u^2$$

$$\cdot\left[\frac{\omega}{k}\frac{\gamma}{u} + \left(1-\frac{\omega^2}{k^2}\frac{\gamma^2}{u^2}\right)\tanh^{-1}\frac{ku}{\omega\gamma}\right]\ . \tag{65}$$

In particular if $f^{(0)}$ is MBJ we obtain

$$\mathrm{Im}\left(\frac{4\pi}{p}\sigma^t\right) = \frac{\pi}{2}\sum\frac{\omega_p^2}{\omega kc}\,\frac{1+\beta\Gamma}{\beta K_2(\beta)}\,\frac{e^{-\beta\Gamma}}{\Gamma^2}\,\theta(\nu^2-1)\ . \tag{66}$$

In the ultrarelativistic limit one obtains Silin's result [14]

$$\mathrm{Im}\left(\frac{4\pi}{p}\sigma^t\right) = \frac{\pi}{2}\sum\frac{\omega_p^2}{\omega kc}\,\frac{\beta}{\Gamma^2} + \mathcal{O}(\beta^3)\ ,\qquad (\nu^2<1)\ .\quad \mathrm{UR}\ . \tag{67}$$

The nonrelativistic case is obtained by setting $\beta >> 1$,

$$\mathrm{Im}\left(\frac{4\pi}{p}\sigma^t\right) = \sqrt{\frac{\pi}{2}}\sum\frac{\omega_p^2}{\omega kc}\,\frac{\sqrt{\beta}}{\Gamma}\,e^{-\beta(\Gamma-1)}$$

or for $\nu >> 1$,

$$\mathrm{Im}\left(\frac{4\pi}{p}\sigma^t\right) \approx \sqrt{\frac{\pi}{2}}\sum\frac{\omega_p^2}{\omega^2}\,e^{-\beta/2\nu^2}\,\sqrt{\frac{\beta}{\nu^2}}\ ,\quad \mathrm{NR}. \tag{68}$$

We found it more convenient to work with the representation developed in Section IV when dealing with the $\mathrm{Re}\,(\frac{4\pi}{p}\sigma^t)$ when $f^{(0)}$ is the MBJ type. As is indicated by equations (48) and (60), the dispersion relation for transverse waves when $\nu^2 < 1$ becomes

$$1 = -\frac{p}{k^2c^2+p^2}\sum \omega_p^2 \frac{\beta^2}{K_2(\beta)}\int_0^\infty \frac{K_2(\varpi)}{\varpi^2}\,ds$$

$$\varpi^2 = (\beta+ps)^2 + k^2c^2s^2 \quad . \tag{69}$$

An asymptotic representation of this relationship may be deduced without much difficulty

$$\nu^2 \sim 1 - \sum \frac{\omega_p^2}{\omega^2}\sum_{i=0,1,2,\ldots}(-1)^i \frac{K_{i-1}(\beta)}{K_2(\beta)}\left(\frac{1-\nu^2}{\beta}\right)^i \frac{(2i)!}{2^i\, i!} \tag{70}$$

where again we set $p = -i\omega$. It is seen that for cold plasma, i.e. $\beta \longrightarrow \infty$, one obtains

$$\nu^2 = 1 - \sum \frac{\omega_p^2}{\omega^2}$$

which is the well-known Langmuir dispersion relation. If we ignore terms of the order of $\left(\frac{1-\nu^2}{\beta}\right)^2$ we find*

$$\nu^2 \sim \frac{1 - \sum \frac{\omega_p^2}{\omega^2}\left(1 - \frac{5}{2\beta}\right)}{1 + \sum \frac{\omega_p^2}{\omega^2}\frac{1}{\beta}} \quad . \tag{71}$$

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*Approximating successively one gets

$$\nu^2 \sim 1 - \sum \frac{\omega_p^2}{\omega^2}\sqrt{1 - \frac{3}{\beta}} \quad .$$

Thus one may write $k^2c^2 = \omega^2 - \hat{\omega}_p^2$ where

$$\hat{\omega}_p^2 = \sum \frac{4\pi ne^2}{\hat{m}}\ , \qquad \hat{m} = \frac{m}{\sqrt{1 - \frac{\langle v^2\rangle}{c^2}}}\ , \qquad \langle v^2\rangle = \frac{3\,\Theta}{m} \ .$$

The latter is in agreement with Buneman's result [13]. The ultrarelativistic limit can either be calculated from equation (65) by setting $\gamma \longrightarrow u/c$ (cf. Silin [14]) or from equation (60) in the limit of $\beta \ll 1$. The result is

$$\nu^2 = 1 - \sum \frac{\omega_p^2}{\omega^2} \frac{\beta}{2\nu^2} \left[1 + \frac{\nu^2 - 1}{\nu} \tanh^{-1} \nu \right] \quad \text{UR} \ . \qquad (72)$$

For $\nu \ll 1$ one can obtain by successive approximation*

$$\frac{6}{5} \nu^2 = 1 - \frac{2}{3} \sum \frac{\omega_p^2}{\omega^2} \beta \ .$$

2. Constant External Magnetic Field Case

In this subsection first we shall assume that the unperturbed distribution function is of the MBJ type, so that the conductivity tensor given in equation (38) can be used directly. The second term of this tensor contributes the major difficulty in the solving of the determinantal dispersion equation. However, if we restrict ourselves to the case in which the propagation vector is along the unperturbed magnetic field, then again it is possible to decouple the transversal and the longitudinal modes. The latter is found to be exactly the same as studied in the previous subsection, and hence will be omitted here.

Then, assuming $\underline{k} \times \hat{\underline{h}} = 0$ which implies that $\underline{\underline{R}} \cdot \underline{k} = \frac{1}{s} \underline{\underline{G}} \cdot \underline{k} = \frac{2}{s^2} \underline{\underline{M}} \cdot \underline{k} = \underline{k}$, we get

* Our formula differs from the one given by Silin by a factor of 2/3 .

$$\epsilon_{ij} = \delta_{ij} + \sum \frac{\omega_p^2}{p} \frac{\beta^2}{K_2(\beta)} \int_0^\infty ds \left[R_{ji} \frac{K_2(\varpi)}{\varpi^2} - c^2 k_i k_j s^2 \frac{K_3(\varpi)}{\varpi^3} \right]$$

where

$$\varpi^2 = (\beta + sp)^2 + c^2 k^2 s^2 \ . \tag{73}$$

We see that $\underline{k} \cdot \underline{\underline{\epsilon}} \cdot \underline{\underline{S}} = \underline{\underline{S}} \cdot \underline{\underline{\epsilon}} \cdot \underline{k} = 0$ so that

$$\epsilon_{ij} = \epsilon^{\ell} \hat{k}_i \hat{k}_j + \epsilon_{ij}^t$$

where

$$\epsilon^{\ell} = 1 + \sum \frac{\omega_p^2}{p} \frac{\beta^2}{K_2(\beta)} \int_0^\infty ds \left[\frac{K_2(\varpi)}{\varpi^2} - c^2 k^2 s^2 \frac{K_3(\varpi)}{\varpi^3} \right]$$

$$= 1 + \sum \omega_p^2 \frac{\beta^2}{K_2(\beta)} \int_0^\infty ds \frac{K_3(\varpi)}{\varpi^3} s(\beta + ps)$$

(cf. equation (61)), and

$$\epsilon_{ij}^t = S_{ik} \epsilon_{k\ell} S_{\ell j}$$

$$= S_{ij} + \sum \frac{\omega_p^2}{p} \frac{\beta^2}{K^2(\beta)} \int_0^\infty ds \, (R_{ji} - \hat{k}_i \hat{k}_j) \frac{K_2(\varpi)}{\varpi^2}$$

$$= S_{ij} \left(1 + \sum \frac{\omega_p^2}{p} \frac{\beta^2}{K_2(\beta)} \int_0^\infty ds \, \cos \Omega s \frac{K_2(\varpi)}{\varpi^2} \right)$$

$$+ \epsilon_{ij\ell} k_\ell \sum \frac{\omega_p^2}{p} \frac{\beta^2}{K_2(\beta)} \int_0^\infty ds \, \sin \Omega \, s \frac{K_2(\varpi)}{\varpi^2}$$

(Observe that the latter is in agreement with equation (60) for $\Omega = 0$).

It is a straightforward task to show that

$$\text{Det}\,(A\,S_{ij} + B\,\hat{k}_i\hat{k}_j + C\,\epsilon_{ij\ell}\,\hat{k}_\ell) = B\,(A^2 + C^2) = B(A + i\,C)\,(A - i\,C)\ .$$

In our problem the first factor corresponds to the longitudinal dispersion relation ($B = \epsilon^\ell$) which has already been studied. The last two factors represent the two (ordinary, extraordinary) modes of the transversal oscillations:

$$\frac{k^2c^2}{k^2c^2+p^2}\,(A\mp iC) = 1 + \frac{p}{k^2c^2+p^2}\sum \omega_p^2\,\frac{\beta^2}{K_2(\beta)}\int_0^\infty ds\,e^{\mp i\Omega s}\,\frac{K_2(\varpi)}{\varpi^2}$$

$$= 0 \tag{74}$$

(cf. equation (69) for $\Omega = 0$).

If we consider the nonrelativistic limit [cf. equation (35)], we get

$$1 + \frac{p}{k^2c^2+p^2}\sum \omega_p^2\int_0^\infty ds\;e^{-ps\mp i\Omega s-\frac{\Theta}{m}\frac{k^2s^2}{2}} = 0\ .$$

For cold plasma, i.e. $\Theta = 0$, one obtains

$$\frac{k^2c^2}{\omega^2} = 1 - \sum\frac{\omega_p^2}{\omega^2}\,\frac{1}{1\pm\frac{\Omega}{\omega}} \tag{75}$$

which is the well-known transversal dispersion relationship ($p = \mp\, i\omega$).

We have developed an asymptotic expansion for the above dispersion relation. The result is* ($p = \mp\, i\omega$)

---

*Because of the divergent character of the involved summations, extra care should be exercised in dealing with the above representation.

$$\nu^2 \sim 1 + \sum \frac{\omega_p^2}{\omega^2} \sum_{i,j=0,1,\ldots} (-1)^i \left(\frac{1-\nu^2}{\beta}\right)^i C_i^j \left(\mp \frac{\omega}{\Omega}\right)^{j+2i+1} \frac{K_{2+i+j}(\beta)}{K_2(\beta)}$$

or

$$\nu^2 \sim 1 - \sum \frac{\omega_p^2}{\omega^2} \sum_{i,j=0,1,\ldots} (-1)^i \left(\frac{1-\nu^2}{\beta}\right)^i C_i^j \left(\mp \frac{\Omega}{\omega}\right)^j \frac{K_{i+j-1}(\beta)}{K_2(\beta)} \qquad (76)$$

where

$$C_i^j = \frac{(j+2i)!}{2^i\, i!\, j!} \quad .$$

It is observed that the second of equations (76) reduces to equation (70) for $\Omega = 0$. Both equations become equation (75) when $\beta \longrightarrow \infty$ (cold plasma). To the first order in β we obtain

$$\frac{k^2c^2}{\omega^2} = \frac{1 - \sum \frac{\omega_p^2}{\omega^2} \frac{1}{1 \pm \Omega/\omega}\left[1 - \frac{5}{2\beta}\frac{1}{1 \pm \Omega/\omega}\right]}{1 + \sum \frac{\omega_p^2}{\omega^2} \frac{1}{\beta(1 \pm \frac{\Omega}{\omega})^3}} \quad . \text{ NR} \qquad (77)$$

In deriving the latter, we have made use of the asymptotic expansion of the Bessel functions involved.

The magnetohydrodynamic (Alfvén) waves may be studied by the assumption that $\left|\frac{\omega}{\Omega}\right| \ll 1$, so that the ion dynamic is particularly important. Then for a binary, initially neutral, singly ionized plasma, ignoring the terms of the order of $(\frac{\omega}{\Omega})^3$ in the first of equations (76), one obtains

$$\nu^2 - 1 = \frac{4\pi c^2}{H_o^2}\left[\rho_+ \frac{K_3(\beta_+)}{K_2(\beta_+)} + \rho_- \frac{K_3(\beta_-)}{K_2(\beta_-)}\right]. \qquad (77)$$

To the first order in $1/\beta$ we have

$$\nu^2 - 1 \sim \frac{c^2}{a^2}\left(1 + \frac{5}{2}\frac{p}{c^2}\right)$$

where p is the total pressure (zero order), and a is the Alfvén speed

$$a = \frac{H_o}{\sqrt{4\pi(\rho_+ + \rho_-)}} .$$

Alternatively equation (77) can be written as

$$\frac{k^2c^2}{\omega^2} = 1 + \frac{4\pi}{H_o^2}\left(w^{(0)} + p^{(0)}\right) \qquad (78)$$

where $w^{(0)}$ is the zero order total energy density. In deriving the latter form we have used equation (41) (cf. Harris [4]).

Next we shall consider the spatial dispersion for the case discussed above. However, we shall reformulate the problem without specifying the unperturbed distribution function explicitly. To do so let us consider the conductivity tensor σ_{ij} assuming (i) $f^{(0)}$ is isotropic in $\underline{u}$, and (ii) $\underline{k} \times \hat{\underline{h}} = 0$. Then

$$\sigma_{ij} = -\sum \frac{e^2}{m}\int d^3u \,\frac{1}{u}\frac{df^{(0)}}{du}\, u_i\, u_k \int_0^\infty e^{-ik_\mu u_\mu s} R_{jk}\, ds . \qquad (79)$$

For the sake of simplicity, we select $\hat{\underline{k}}$ (and $\hat{\underline{h}}$) along the z-axis. Clearly in this case, due to the azimuthal symmetry, one has

$$\sigma_{11} = \sigma_{22}, \ \sigma_{12} = -\sigma_{21}, \ \sigma_{13} = \sigma_{31} = \sigma_{23} = \sigma_{32} = 0.$$

Then the transverse dispersion equation can be computed easily from (45), using the relations

$$\frac{\mathrm{tr}\ \underline{\underline{\epsilon}}}{2} - \frac{(\epsilon^2)^{\ell}}{2\epsilon^{\ell}} = 1 + \frac{4\pi}{p}\sigma_{11}$$

$$\frac{\mathrm{Det}(\underline{\underline{\epsilon}})}{\epsilon^{\ell}} = (1 + \frac{4\pi}{p}\sigma_{11})^2 + (\frac{4\pi}{p}\sigma_{21})^2 .$$

Thus one obtains

$$1 + \frac{k^2c^2}{p^2} + \frac{4\pi}{p^2}(\sigma_{11} \pm i\sigma_{21}) = 0 . \qquad (80)$$

(cf. equation (74)). Here, the upper and lower signs correspond to the extra-ordinary and the ordinary modes, respectively.* Let us consider the real

---

*It should be kept in mind that Ω is defined as an algebraic quantity depending on the sign of the electric charge of the corresponding species, e.g., for electrons $\Omega < 0$.

and imaginary parts of

$$\frac{4\pi}{p}\sigma^{t}_{\pm} \equiv \frac{4\pi}{p}(\sigma_{11} \pm i\,\sigma_{21}) \tag{81}$$

on the imaginary axis of the complex p plane. After performing the angular integrations in $\underline{u}$-space, one obtains

$$\mathrm{Im}\,\frac{4\pi}{p}\sigma^{t}_{\pm} = -\frac{4\pi^3}{\omega k}\sum\frac{e^2}{m}\,\theta\,(k^2c^2+\Omega^2-\omega^2)\int_{U_1}^{U_2} du\,\frac{df^{(0)}}{du}\,u^2 \cdot\left[1-\left(\frac{\omega\gamma\pm\Omega}{ku}\right)^2\right],$$

$$\mathrm{Re}\,\frac{4\pi}{p}\sigma^{t}_{\pm} = \frac{8\pi^2}{\omega k}\sum\frac{e^2}{m}\int_{0}^{\infty} du\,\frac{df^{(0)}}{du}\,u^2 \cdot\left\{\frac{\omega\gamma\pm\Omega}{ku}+\frac{1}{2}\left[1-\left(\frac{\omega\gamma\pm\Omega}{ku}\right)^2\right]\ln\left|\frac{\omega\gamma\pm\Omega+ku}{\omega\gamma\pm\Omega-ku}\right|\right\}. \tag{82}$$

(cf., equation (65)). In deriving the latter we have used the well-known improper integral

$$i\int_0^{\infty} e^{-ixs}\,ds = \frac{\mathcal{P}}{x} + i\pi\,\delta(x)\,.$$

The limits of the u-integration in the first of equations (82) are determined from the condition

$$y^{\mp} = \left| \frac{\gamma \mp \Omega/\omega}{\nu\, u/c} \right| \leq 1. \tag{83}$$

Again the upper and lower signs correspond to the extraordinary and ordinary modes, respectively.

The imaginary part of $\frac{4\pi}{p}\,\sigma_{\pm}$ vanishes identically on any interval of u for which $y^{\mp} > 1$. In other words, in this case there is no damping (or instability).*

Equation (83) is satisfied by the ordinary mode (i.e., $y^{+} \leq 1$) provided $\nu > 1$ and $U_1 \leq u < \infty$, where

$$U_1 = c \left| \frac{|\nu\Omega/\omega| - \sqrt{\nu^2 - 1 + \Omega^2/\omega^2}}{\nu^2 - 1} \right| . \tag{84}$$

On the other hand for the extraordinary mode we have the following cases:

$y^{-} \leq 1$ whenever

(i) $\nu \geq 1$ and $U_1 \leq u < \infty$

(ii) $\nu < 1$ and $U_1 \leq u \leq U_2$

---

*Strictly speaking this is the only case when a true dispersion relationship exists [15].

where

$$U_2 = c\,\frac{|\nu\,\Omega/\omega| + \sqrt{\nu^2 - 1 + \Omega^2/\omega^2}}{1 - \nu^2}\,. \tag{85}$$

Let us note that

$$\lim_{\nu \to 1} U_1 = \frac{c}{2}\left|\frac{1 - \Omega^2/\omega^2}{\Omega/\omega}\right|. \tag{86}$$

(cf. Figure 1)

In the case when $f^{(0)}$ in the MBJ distribution the u-integration involved in the first of (83) can be carried out explicitly ($\nu^2 > 1 - \Omega^2/\omega^2$)

$$\mathrm{Im}\,\frac{4\pi}{p}\,\sigma^t_{\pm} = \frac{\pi}{2\nu^3}\sum \frac{\omega_p^2}{\omega^2}\,\frac{e^{-\beta\gamma}}{\beta K_2(\beta)}\left\{\nu^2 - 1 + \beta\left[\gamma(\nu^2 - 1) \pm \frac{\Omega}{\omega}\right]\right.$$

$$\left. + \frac{\beta^2}{2}\left[\gamma^2(\nu^2 - 1) \pm 2\gamma\,\frac{\Omega}{\omega} - \nu^2 - \frac{\Omega^2}{\omega^2}\right]\right\}\Bigg|_{U_2}^{U_1}\,. \tag{87}$$

Let us consider the case $\nu > 1$

$$\mathrm{Im}\,\frac{4\pi}{p}\,\sigma^t_{\pm} = \frac{\pi}{2\nu^3}\sum \frac{\omega_p^2}{\omega^2}\,\frac{\exp\left[-\beta\,\frac{\nu\sqrt{\nu^2 - 1 + \Omega^2/\omega^2} - |\Omega/\omega|}{\nu^2 - 1}\right]}{\beta K_2(\beta)}\,.$$

$$\cdot\left\{\nu^2 - 1 + \beta\left[\nu\sqrt{\nu^2 - 1 + \Omega^2/\omega^2} - 2|\Omega/\omega|\,\theta(\mp\Omega/\omega)\right] - \right.$$

$$\left. - 2\beta^2\,\frac{\nu\,|\Omega/\omega|\sqrt{\nu^2 - 1 + \Omega^2/\omega^2} - \Omega^2/\omega^2}{\nu^2 - 1}\,\theta(\mp\Omega/\omega)\right\} \tag{88}$$

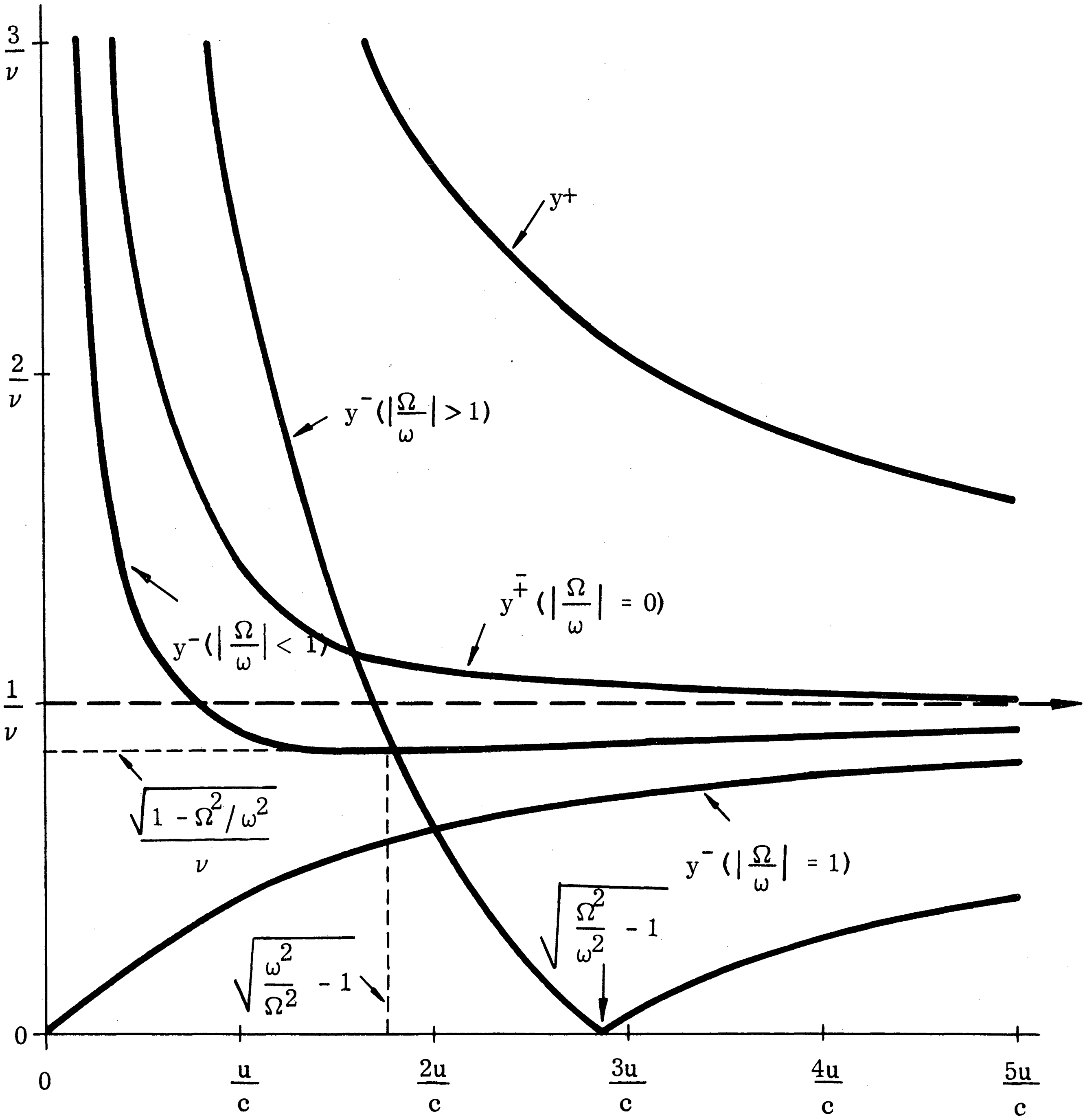
3/ν
2/ν
1/ν
0
y+
$y^-(|\frac{\Omega}{\omega}| > 1)$
$y^{\mp}(|\frac{\Omega}{\omega}| = 0)$
$y^-(|\frac{\Omega}{\omega}| < 1)$
$\frac{\sqrt{1 - \Omega^2/\omega^2}}{\nu}$
$y^-(|\frac{\Omega}{\omega}| = 1)$
$\sqrt{\frac{\Omega^2}{\omega^2} - 1}$
$\sqrt{\frac{\omega^2}{\Omega^2} - 1}$
0
u/c
2u/c
3u/c
4u/c
5u/c

FIGURE 1

Clearly, for $\Omega = 0$ equation (88) reduces to equation (66). In the nonrelativistic limit for $\nu^2 >> 1$ and $\Omega^2/\omega^2 << 1$ we obtain

$$\text{Im} \frac{4\pi}{p} \sigma^{t}_{\pm} \approx \sqrt{\frac{\pi}{2}} \sum \frac{\omega_p^2}{\omega^2} \sqrt{\frac{\beta}{\nu^2}} \exp \left[-\beta \frac{(1 - |\Omega/\omega|)^2}{2\nu^2} \right] . \quad \text{NR} \qquad (89)$$

We have also ignored the terms of the order of $\frac{\beta}{\nu^2} \left| \frac{\Omega}{\omega} \right|$ in deriving the latter. Equation (81) is to be compared with equation (68).

Although the other cases of interest can be studied in a similar manner, a detailed discussion will not concern us here. However, it might be of some interest to note that the ultrarelativistic limit also can be considered as was done in the case of the absence of the external field. One finds that the effect of the magnetic field does not appear until to the second order in β (rather in $\left| \frac{\beta^2 \Omega/\omega}{\nu^2 - 1} \right|$).

VI

DISCUSSION AND CONCLUSIONS

In this report we have attempted the study of the linear oscillations in a hot plasma. It was assumed that the system can be represented by the relativistic Vlasov equation coupled with the Maxwell's field equations.

The formal solution of the linearized problem was deduced employing the integral transform technique which was originally used by Bernstein for the nonrelativistic case.

Giving particular attention to the spatial dispersion, we examined the dispersion equation for some special cases in which the longitudinal and transversal modes can be decoupled. We developed asymptotic expansions for the cases studied when the unperturbed distribution function is of the MBJ type.

The longitudinal oscillations were found to be undamped* when $\omega^2 > k^2c^2$; this is in agreement with the result of the former investigations. It was shown that unattenuated, circularly polarized, transverse waves propagating along the unperturbed magnetic field can exist provided $\omega^2 > k^2c^2 + \Omega^2_A$, where the suffix A represents the species which has the largest $|e_A/m_A|$ ratio.

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*Here, it should be pointed out that the collisions, which are ignored in the framework of the present theory, can provide an additional damping mechanism which is different from the one discussed above.

For example, for a simple electron gas, ignoring the terms of the order of $1/\beta^2$, the above condition may be written as follows (cf. (77))

$$\frac{k^2c^2}{\omega^2} = \frac{1 - \frac{\omega_p^2}{\omega^2} \frac{1}{1 \pm \Omega/\omega} \left(1 - \frac{5}{2\beta} \frac{1}{1 \pm \Omega/\omega}\right)}{1 + \frac{1}{\beta} \frac{\omega_p^2}{\omega^2} \left(\frac{1}{1 \pm \Omega/\omega}\right)^3}$$

$$< 1 - \frac{\Omega^2}{\omega^2} ,$$

which reduces to the quadratic form

$$\frac{\Omega^2}{\omega_p^2} X^2 - \left(1 - \frac{1}{\beta}\right) X + \frac{1}{2\beta} < 0 ,$$

where

$$X = 1 \pm \Omega/\omega.$$

An examination of the latter indicates that in order for the above inequality to hold one must have

$$\frac{\Omega^2}{\omega_p^2} \equiv \frac{H_o^2}{4\pi \, nm \, c^2} < \frac{\beta}{2} \left(1 - \frac{1}{\beta}\right)^2,$$

or, approximately, $(\beta >> 1)$

$$H_o < \left(\frac{10^5\, n\beta}{2}\right)^{1/2} (1 - 1/\beta)\,, \text{ (in c.g.s.)}.$$

An estimate of magnitude of the right-hand side shows that the latter condition is satisfied for almost all plasmas of practical interest. Moreover, one finds that (i) If

$$\frac{\omega_p^2}{\Omega^2} < 1 + \frac{3}{2\beta}\,,$$

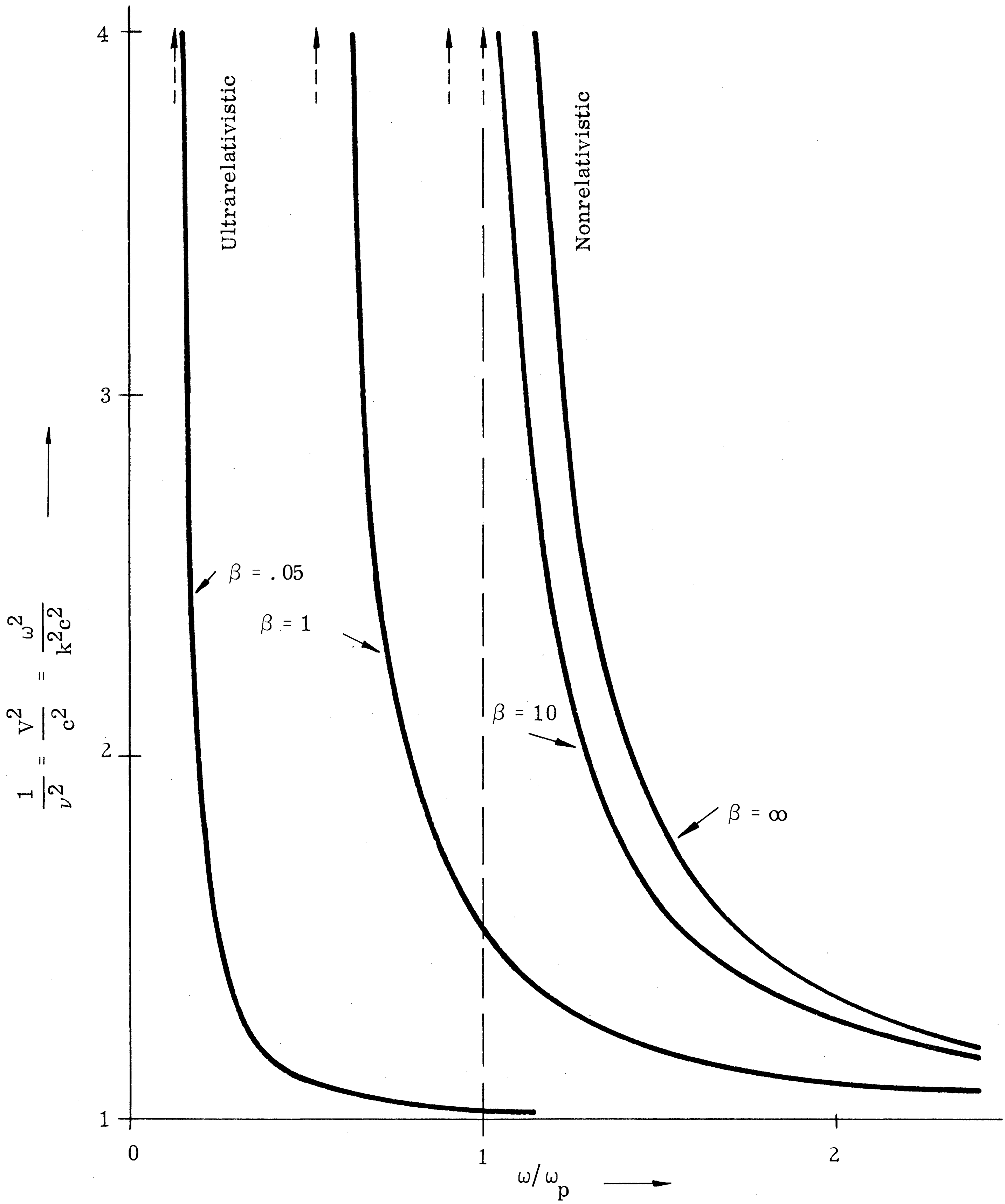
there is no frequency region for which the ordinary mode is unattenuated, (ii) If $\frac{\omega_p^2}{\Omega^2} < 1 + \frac{3}{2\beta}$, the extraordinary mode is undamped for frequencies

$$\left| \frac{\omega}{\Omega} \right| > 1 + \frac{1}{2\beta}.$$

However, when $\frac{\omega_p^2}{\Omega^2} > 1 + \frac{3}{2\beta}$, the undamped frequency region is bounded. We note that the resonance frequency, viz. $\omega = |\Omega|$, lies in the damped region.

The magnetohydrodynamic waves may be examined in a similar manner. One finds that there is no undamped frequency range in this case.

The results obtained may be further illustrated by the following sketches. Assuming $f^{(0)}$ is MBJ distribution, the transverse and the longitudinal waves in a simple electron gas are considered in Figures 2 and 3, respectively, when the external field is absent. In Figure 4 the cut-off frequency, ω_o, is plotted



ARL-TR-60-274, Part II

FIGURE 2

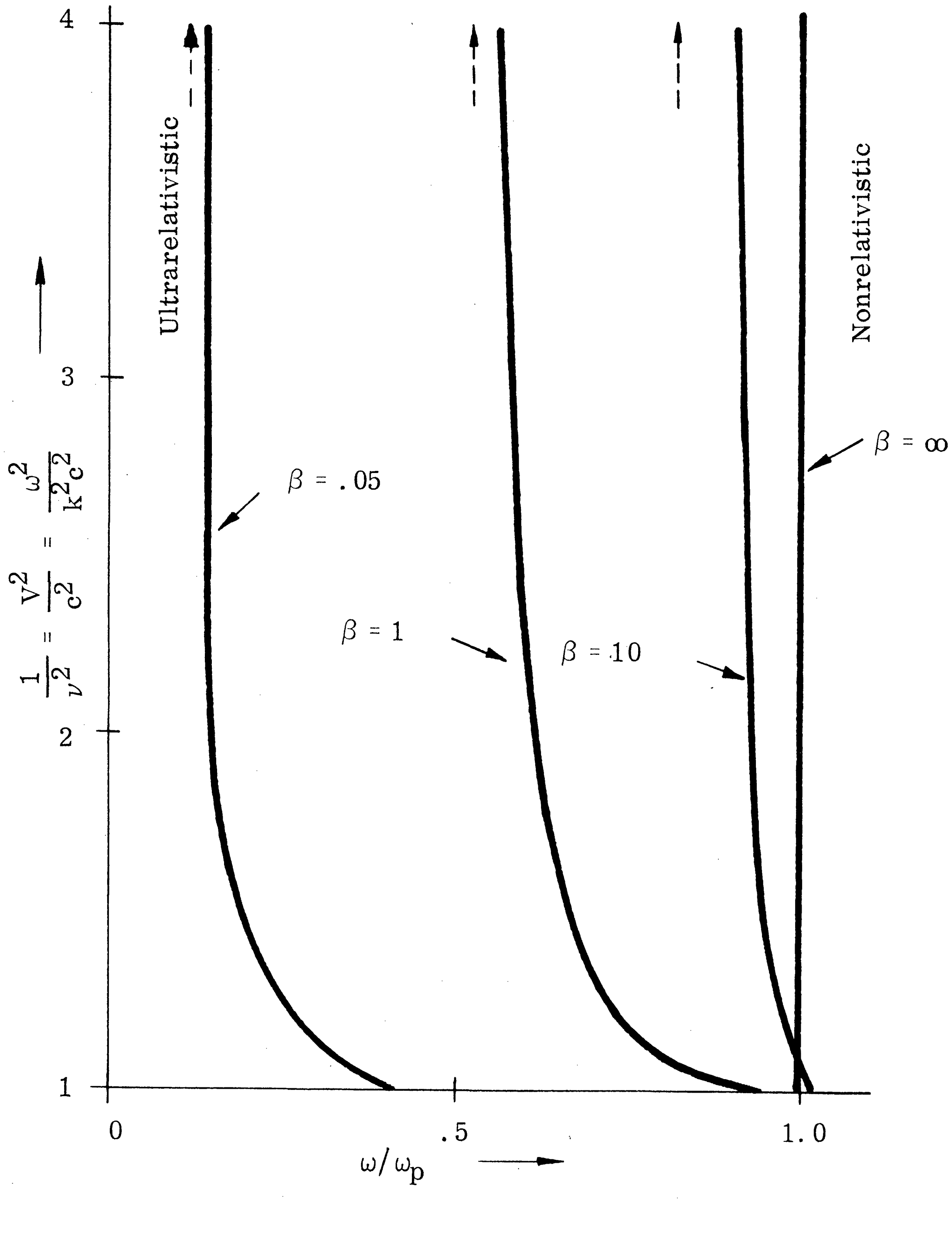

4
3
2
1
0
.5
1.0
$\frac{1}{\nu^2} = \frac{V^2}{c^2} = \frac{\omega^2}{k^2c^2}$
ω/ω_p
Ultrarelativistic
Nonrelativistic
β = .05
β = 1
β = 10
β = ∞

FIGURE 3

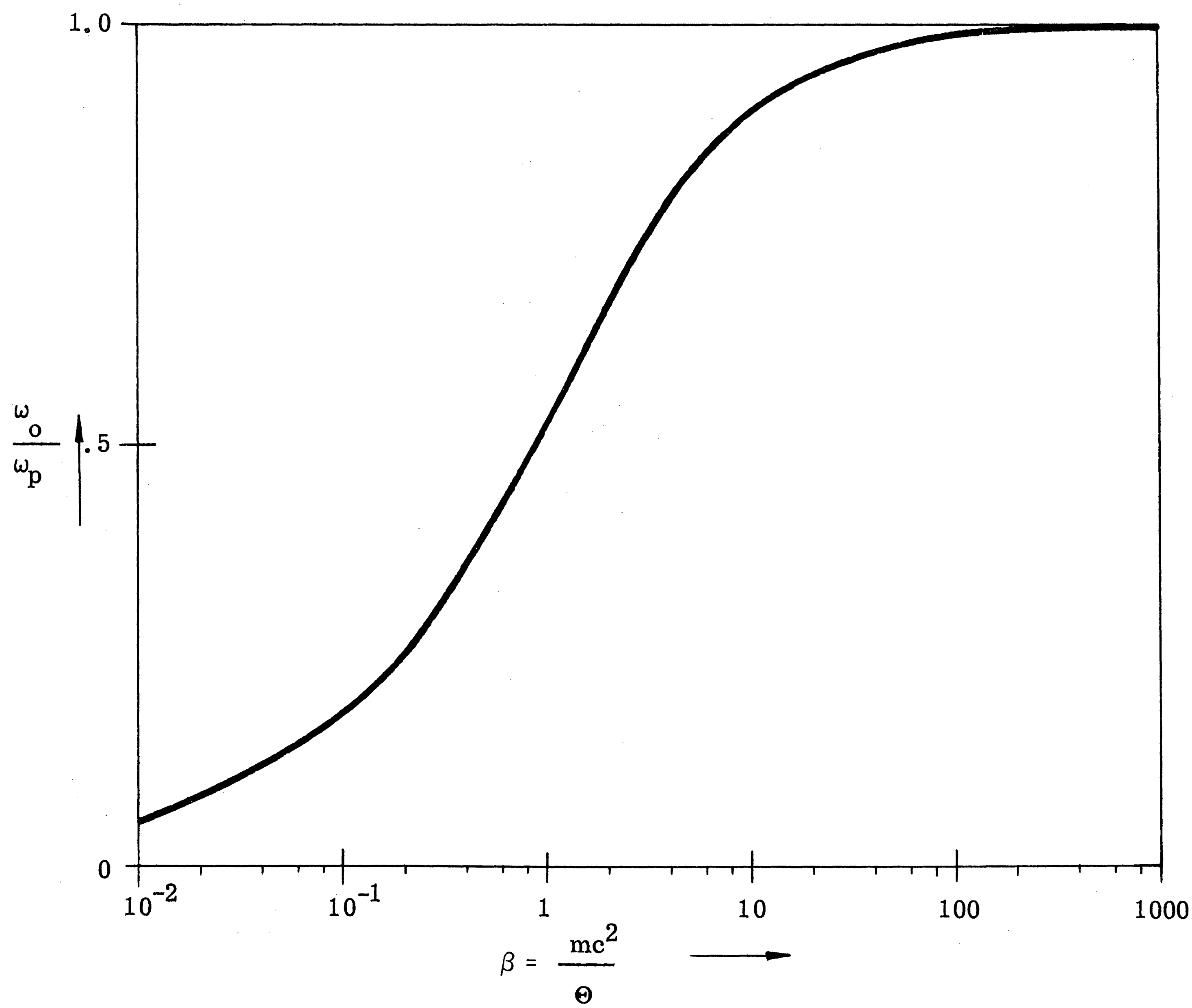

1.0
.5
0
$\frac{\omega_o}{\omega_p}$
10^{-2}
10^{-1}
1
10
100
1000
$\beta = \frac{mc^2}{\Theta}$

FIGURE 4

vs. β, using

$$\left(\frac{\omega_o}{\omega_p}\right)^2 = \frac{\beta^2}{K_2(\beta)} \int_{\beta}^{\infty} \frac{K_2(X)}{X^2}\, dX$$

$$\approx 1 - \frac{5}{2\beta} + \frac{55}{8\beta^2}, \qquad \beta >> 1, \qquad \text{NR}$$

$$\approx \beta/3, \qquad \beta << 1, \qquad \text{UR}.$$

APPENDIX A

The Integral Equation for f^+

It was previously pointed out that instead of eliminating the distribution function through the Maxwell's equation as is done in Section III, one can eliminate the field variables in equation (26) to obtain an integral equation for f^+. To do this one substitutes equation (29) in the Maxwell's equation

$$ik_\mu F^+_{\nu\mu} = \frac{4\pi}{c} J^+_\nu = ik_\mu \delta^{\lambda j}_{\nu\mu} \frac{k_\lambda}{k_4} F^+_{4j} = \frac{4\pi}{c} \sum e \int \frac{d^3u}{\gamma} u_\nu f^+ .$$

Only the first three of the above four equations are needed for our purposes

$$\frac{4\pi}{c} J^+_i = \frac{i}{k_4} F^+_{4j} (k_j k_i - k^2_\mu \delta_{ij}) . \qquad (k^2_\mu \equiv k_\mu k_\mu)$$

Using the relation

$$\frac{-1}{k^2_\mu k^2_4} (k_i k_\ell - k^2_\mu \delta_{i\ell})(k_\ell k_j + k^2_4 \delta_{\ell j}) = \delta_{ij}$$

one obtains

$$F^+_{4i} = \frac{i}{k_4 k^2_\mu} \frac{4\pi}{c} (k_i k_j + k^2_4 \delta_{ij}) J^+_j ,$$

and hence substituting back in equation (29)

$$F^+_{\nu\mu} = \delta^{\lambda j}_{\nu\mu} \frac{k_\lambda}{k^2_4 k^2_\alpha} \frac{4\pi}{c} i (k_j k_\ell + k^2_4 \delta_{j\ell}) J^+_\ell$$

The latter is used to eliminate the field variables in equation (26) to obtain the integral equation

$$f^{+} = i \frac{4\pi e}{mc^2} u_\nu \frac{\partial f^{(0)}}{\partial u_j} \frac{k_\lambda}{k_4^2 k_\alpha^2} \delta^{\lambda \ell}_{\mu k} (k_\ell k_p + k_4^2 \delta_{\ell p}) \int_0^\infty e^{-y} R_{\mu\nu} R_{kj} \, ds$$

$$\sum e \int \frac{d^3u}{\gamma} u_p f^{+} . \qquad (90)$$

Now if we multiply the above equation by $e \, u_q \frac{d^3u}{\gamma}$ and integrate over $\underline{u}$ space, then sum over all species, we obtain

$$\left[\delta_{pq} - \frac{4\pi}{c} \frac{k_\lambda}{k_4^2 k_\alpha^2} \delta^{\lambda\ell}_{\mu k} (k_\ell k_p + k_4^2 \delta_{\ell p}) \sigma_{qk\mu} \right] \left[\sum e \int \frac{d^3u}{\gamma} u_q f^{(0)} \right]$$

$$= \left[\delta_{pq} - \frac{4\pi}{k_4 c} \frac{1}{k_\alpha^2} (k_\ell k_p + k_4^2 \delta_{\ell p}) \sigma_{q\ell} \right] J_q$$

$$= \text{Initial Conditions} = \zeta_p .$$

Here we added to the right-hand side the initial conditions which were being suppressed.

The solubility condition then reads

$$\text{Det} \left[\delta_{ij} - \frac{4\pi}{k_4 c} \frac{1}{k_\alpha^2} (k_i k_\ell + k_4^2 \delta_{i\ell}) \sigma_{j\ell} \right] = 0 .$$

Since $\text{Det}(\underline{\underline{A}} \cdot \underline{\underline{B}}) = \text{Det}(\underline{\underline{A}}) \cdot \text{Det}(\underline{\underline{B}})$ we can write

$$\mathrm{Det}\left[(k_\alpha^2\,\delta_{ij} - k_ik_j) - \frac{4\pi}{k_4c}\,k_4^2\,\sigma_{ij}\right] = 0$$

$$= \mathrm{Det}\left[k^2\,(\delta_{ij} - \hat{k}_i\hat{k}_j) + k_4^2\,(\delta_{ij} - \frac{4\pi}{k_4c}\,\sigma_{ij})\right]$$

or

$$\mathrm{Det}\left[S_{ij} + \frac{p^2}{k^2c^2}\,\epsilon_{ij}\right] = 0$$

where again we put

$$S_{ij} = \delta_{ij} - \hat{k}_i\hat{k}_j$$

$$\epsilon_{ij} = \delta_{ij} + \frac{4\pi}{p}\,\sigma_{ij}$$

$$k_4c = -p\ , \quad k_j = k\hat{k}_j$$

[cf. equations (30) and (31)] .

In particular, if $f^{(0)}$ $(\equiv n\,\hat{f}^{(0)})$ is isotropic then $u_\nu \dfrac{\partial\hat{f}^{(0)}}{\partial u_j}$ becomes symmetric for $\nu = 1, 2, 3$; thus the pair (μ, k) is also symmetric for $\mu = 1, 2, 3$. Remembering the antisymmetric property of $\delta^{\lambda\ell}_{\mu k}$, equation (90) can be written for this case as

$$f^+ = \omega_p^2\,\gamma\,\frac{\partial\hat{f}^{(0)}}{\partial u_j}\left[\frac{S_{i\ell}}{1-\nu^2} + \hat{k}_i\hat{k}_\ell\right]\frac{1}{p}\int_0^\infty e^{-y}\,R_{ij}\,ds\,\sum e\int\frac{d^3u}{\gamma}\,u_\ell\,f^+$$

where we set $\nu^2 = -\dfrac{k^2c^2}{p^2}$.

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The University of Michigan, Ann Arbor, Michigan
THEORY OF PLASMAS, II - LINEAR OSCILLATIONS IN RELATIVISTIC PLASMAS
K. Imre

Unclassified

Radiation Laboratory Report No. 2756-1-F, February 1961, 56pp. Aeronautical Research Laboratory, Air Force Research Division (ARDC) Contract No. AF 33(616)-5585. Unclassified Report.

The linear oscillations in a hot plasma which is representable by the relativistic Vlasov equation with the self-consistent fields are investigated. The method which is used by Bernstein in the nonrelativistic case is generalized to obtain the formal solution of the linearized problem. Particular attention is given to the case when the unperturbed distribution function is of the Maxwell-Boltzmann-Jüttner type (i.e., the relativistic equilibrium distribution) in which case the integrations involving the velocity space are carried out explicitly. The dispersion equation is derived and studied to some extent, considering the spatial dispersions explicitly in some special cases of interest. The ordinary and extraordinary modes, and the magnetohydrodynamic waves are investigated when the propagation vector is along the unperturbed magnetic field. The asymptotic expansions are developed corresponding to the dispersion relations of the cases considered, and they are shown to be in agreement with the results of previous studies in their respective order of approximations. It is found that circularly polarized transverse waves propagating along the unperturbed magnetic field are evanescent if $\nu^2 > 1 - \Omega^2/\omega^2$, where ν is the index of refraction (kc/ω) and Ω is the gyrofrequency. In the absence of the external field the cut-off frequency is found to be a monotonically decreasing function of the temperature.

1. Theory of Plasmas
2. Linear Oscillations in Relativistic Plasmas
3. Aeronautical Research Laboratory, Air Force Research Division (ARDC)
4. Aeronautical Research Laboratory Contract No. AF 33(616)-5585

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The linear oscillations in a hot plasma which is representable by the relativistic Vlasov equation with the self-consistent fields are investigated. The method which is used by Bernstein in the nonrelativistic case is generalized to obtain the formal solution of the linearized problem. Particular attention is given to the case when the unperturbed distribution function is of the Maxwell-Boltzmann-Jüttner type (i.e., the relativistic equilibrium distribution) in which case the integrations involving the velocity space are carried out explicitly. The dispersion equation is derived and studied to some extent, considering the spatial dispersions explicitly in some special cases of interest. The ordinary and extraordinary modes, and the magnetohydrodynamic waves are investigated when the propagation vector is along the unperturbed magnetic field. The asymptotic expansions are developed corresponding to the dispersion relations of the cases considered, and they are shown to be in agreement with the results of previous studies in their respective order of approximations. It is found that circularly polarized transverse waves propagating along the unperturbed magnetic field are evanescent if $\nu^2 > 1 - \Omega^2/\omega^2$, where ν is the index of refraction (kc/ω) and Ω is the gyrofrequency. In the absence of the external field the cut-off frequency is found to be a monotonically decreasing function of the temperature.

1. Theory of Plasmas
2. Linear Oscillations in Relativistic Plasmas
3. Aeronautical Research Laboratory, Air Force Research Division (ARDC)
4. Aeronautical Research Laboratory Contract No. AF 33(616)-5585

The University of Michigan, Ann Arbor, Michigan
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