THE UNIVERSITY OF MICHIGAN

COLLEGE OF ENGINEERING
Department of Meteorology and Oceanography

Final Report

AGEOSTROPHIC STABILITY OF DIVERGENT JETS

Stanley J. Jacobs

ORA Project 07344

under contract with:

DEPARTMENT OF THE NAVY
OFFICE OF NAVAL RESEARCH
CONTRACT NO. Nonr-1224(55)
WASHINGTON, D.C.

administered through:

OFFICE OF RESEARCH ADMINISTRATION ANN ARBOR

June 1969

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

ABSTRACT

The hydrodynamic stability of divergent jets is studied under the condition of small but nonzero Rossby number. Small disturbances to the basic state consist of fast waves of inertio-gravity type and slow waves with speed comparable to that of the basic current. It is shown that the fast waves are stable and do not amplify. It is shown also that, for a velocity profile resembling that of the Gulf Stream, the slow waves are destabilized by the presence of nonzero Rossby number. For very shallow jets it is found that the complex wave velocity is small and that, in general, disturbances will be stable unless the jet has a countercurrent.

1. INTRODUCTION

This paper is devoted to a reexamination of the hydrodynamic stability of divergent jets with special reference to the stability of the Gulf Stream. Despite the many recent studies of this problem there are a number of unresolved questions, even in situations for which the quasi-geostrophic theory would seem to apply. In addition, for the Gulf Stream problem, there is a legitimate doubt that the quasi-geostrophic theory is valid even as a low order approximation. Since intuition alone is unlikely to suffice in resolving these matters, an analytical treatment is presented.

The first question to be discussed is that of the stability of inertio-gravity waves. The fact that these waves are unimportant is exploited in all of the models used for simplifying the primitive equations; the reason why they are unimportant is an open question, at least as far as formal mathematical proof is concerned. It will be shown below that, when the Rossby number is small, the class of perturbations corresponding to inertio-gravity waves is stable and will never amplify, at least under barotropic conditions. This result provides a formal justification for ignoring inertio-gravity waves.

Next, we will consider the effect of small but nonzero Rossby number in modifying the stability characteristics of quasi-geostrophic disturbances to a velocity profile resembling that of the Gulf Stream. There are a number of reasons for doubting the validity of quasi-geostrophic theory as applied to this problem; nevertheless, such treatments have been made (Stern, 1961, Lipps, 1963) and it is of interest to calculate the effects due to finite Rossby number. The result of the calculation is to prove wrong Lipps' speculation that the reason his computed growth rates are higher than observed (c.f. Stommel, 1965, p. 196; also, Gulf Stream Summaries) is due to his neglect of higher order Rossby number effects. The Rossby number effect, for the profile studied, in fact proves to be destabilizing.

Finally, we will consider a situation in which the Rossby number is small but the fluid is still divergent. This appears to be more applicable to the Gulf Stream problem than the usual quasi-geostrophic theory. The conclusion reached for this model is that the existence of a countercurrent is necessary for instability.

2. FORMULATION

We consider a two-layer fluid on the β-plane, the density ρ of each layer being constant. Let x, y, and z measure distance to the east, the north, and the vertical, and let subscripts 1 and 2 refer to the upper and lower layers, respectively. The layers are of finite depth, with depths $D_\alpha (\alpha=1,2)$ in the absence of motion. We take the upper boundary of the fluid to be the free surface $z = D_1 + D_2 + (\Delta\rho/\rho_2)\eta_1$, where $\Delta\rho = \rho_2 - \rho_1$, the interface between the layers to be $z = D_2 + \eta_2 - (\rho_1/\rho_2)\eta_1$, and the lower boundary to be the rigid plane $z = 0$. Also, we assume shallow water theory to be valid and denote by $\vec{q}_\alpha$ the horizontal velocity and by D/Dt_α the material derivative,

$$\frac{D}{Dt_\alpha} = (\frac{\partial}{\partial t} + \vec{q}_\alpha \cdot \nabla).$$

The equations of motion are then

$$\frac{D\vec{q}_\alpha}{Dt_\alpha} + f\hat{k} \times \vec{q}_\alpha + g'\nabla\eta_\alpha = 0, \text{ (no sum convention)} \tag{1}$$

$$\frac{Dh_\alpha}{Dt_\alpha} + h_\alpha \nabla\cdot\vec{q}_\alpha = 0, \tag{2}$$

where the depths h_1 and h_2 are given by

$$h_1 = D_1 + \eta_1 - \eta_2, \quad h_2 = D_2 + \eta_2 - (\rho_1/\rho_2)\eta_1, \tag{3}$$

and where $f = f_o + \beta y$ is the Coriolis parameter and $g' = g\Delta\rho/\rho_2$ is the reduced gravity.

Our aim is to study the stability of small perturbations to the flow

$$\vec{q}_1 = (VU(y/L),\ 0),\ \vec{q}_2 = 0,$$

V being a characteristic velocity and L a characteristic length. Two approximations will be made at the outset. First, we will neglect entirely motions in the lower layer and thus eliminate potential energy conversion as a source of instability. This is justified in the quasi-geostrophic case, provided that a source of kinetic energy is present and that $D_1 \leq D_2$ (Pedloskey, 1965, Sec. 4). We assume that it is true also for the ageostrophic case. Secondly, we restrict our attention to horizontal length scales so small that $L^2 \ll V/\beta$. It is then permissable to neglect βy next to f_o provided that we simulate the trapping effect due to the earth's curvature (Jacobs, 1967). This will be achieved here by supposing the fluid to be confined between walls at $y = LA$, $y = LB$, $B > A$.

It is convenient at this point to scale the variables. Omitting subscripts, which are now superflous since $\vec{q}_2 = \eta_2 = 0$ by assumption, we define nondimensional variables by

$$(x,y) = L(x^*,y^*),\ t = (L/V)t^*,\ \vec{q} = V\ \vec{q}^*,\ \eta = (f_o VL/g')\eta^*,\ h = Dh^*. \quad (4)$$

The nondimensional equations obtained through use of this scaling are, with asterisks omitted,

$$\epsilon \frac{D\vec{q}}{Dt} + \hat{k} \times \vec{q} + \nabla\eta = 0, \quad (5)$$

and

$$\frac{Dh}{Dt} + h\nabla\cdot\vec{q} = 0, \quad (6)$$

where

$$h = 1 + \epsilon\gamma^2\eta. \tag{7}$$

The nondimensional parameters are the Rossby number,

$$\epsilon = V/f_o L, \tag{8.a}$$

and a nondimensional radius of deformation γ, given by

$$\gamma = f_o L/(g'D)^{1/2}. \tag{8.b}$$

If we now take

$$\vec{q} = (U(y) + u,\ v),\ \eta = \Phi + \varphi, \tag{9}$$

where $\Phi' = -U$, and neglect products of the perturbation quantities, we obtain the linear equations

$$\epsilon(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})(u,v) + (-\zeta v, u) + (\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}) = 0, \tag{10}$$

and

$$\epsilon\gamma^2(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})\varphi + \frac{\partial(Hu)}{\partial x} + \frac{\partial(Hv)}{\partial y} = 0, \tag{11}$$

where H, the unperturbed depth of the layer, and ζ, the unperturbed total vorticity, are given by

$$H = 1 + \epsilon\gamma^2\Phi,\ \zeta = 1 - \epsilon U'. \tag{12}$$

As is usual in stability problems (see, however, Case, 1960), we bypass the initial value problem and instead separate variables according to

$$F(x,y,t) = \tilde{F}(y)e^{ik(x-ct)},\ k \geq 0,\ c = c_r + ic_i, \tag{13}$$

where F is any of u, v, or φ. Equations (10) and (11) then become a homogeneous system of ordinary differential equations with the homogeneous boundary conditions $\tilde{v}(A) = \tilde{v}(B) = 0$, and c plays the role of an eigenvalue. The values of c for which

$$\int_A^B (|\tilde{u}|^2 + |\tilde{v}|^2 + \gamma^2|\varphi|^2)dy \neq 0$$

will be called the spectrum for this system. and if $c_i > 0$ the motion is unstable.

Invoking (13) and eliminating $\tilde{u}$, we obtain two equations for the two remaining unknowns. With the definitions

$$w = U(y)-c, \quad \tilde{\psi} = \tilde{v}H/ik, \quad q = H(\zeta-\epsilon^2k^2w^2)^{-1}, \quad Q = (H-\epsilon^2\gamma^2w^2)^{-1}, \tag{14}$$

and with the tildes omitted, we have

$$q(\epsilon w\varphi' - \varphi) + \psi = 0, \tag{15}$$

$$Q(\epsilon w\psi' + \zeta\psi) - \varphi = 0, \tag{16}$$

to be solved subject to $\psi(A) = \psi(B) = 0$. Alternate formulations in terms of a single equation for a single unknown are

$$(Q\psi')' - (\frac{k^2}{H} + \frac{\gamma^2Q\zeta}{H} - \frac{(Q\zeta)'}{\epsilon w})\psi = 0, \tag{17}$$

or

$$(q\varphi')' - (k^2q + \gamma^2 + q'/\epsilon w)\varphi = 0, \tag{18}$$

the latter equation being subject ot $\varphi = \epsilon w\varphi'$ at $y = A$ and $y = B$.

We note that if $|c| = O(1)$ as $\epsilon \to 0$, then, in the limit, $\varphi = \psi$ and (17) becomes the quasi-geostrophic equation

$$w\psi'' - (k^2w + U'' - \gamma^2c)\psi = 0, \tag{19}$$

for which one can show that c lies in a certain circle Γ in the c plane. If U_1 and U_2 are constants such that

$$U_1 \leq U(y) \leq U_2, \quad U_2 > 0,$$

then the center of Γ is at $c = 1/2\ (U_2 + b)$ and its radius is $1/2\ (U_2 - b)$, where $b = U_1$ if $U_1 \leq 0$ and $b = 0$ otherwise. We will refer to Γ a number of times in what follows.

3. LOCATION OF THE SPECTRUM

It would be desirable to proceed without making any further approximations. This is impractical, however, and we will be content instead to consider only the case $\epsilon \ll 1$. If we can show that for unstable waves $|c| = O(1)$ as $\epsilon \to 0$, then the quasi-geostrophic theory is a valid first approximation and may be improved upon by carrying out a perturbation expansion in powers of ϵ. Accordingly, it is desirable to show that for unstable waves c lies in the circle Γ, and this is the object of the present section.

It should be noted that it is untrue that all the eigenvalues lie in Γ. For example, if $U(y) \equiv 0$, the spectrum consists of the points $c = 0$, the geostrophic mode, $\epsilon^2\gamma^2c^2 = 1$, the Kelvin waves, and $\epsilon^2\gamma^2c^2 = 1 + k^{-2}[\gamma^2 + n^2\pi^2/(B-A)^2]$, $n = 1,2,\ldots$, the Poincaré waves. In general, even for $U(y) \neq 0$, we must anticipate the existence of eigenvalues c such that $\epsilon|c| \neq 0$ in the limit $\epsilon \to 0$. This is because the solutions of the initial value problem defined by (10) and (11) must during part of their temporal history have a time scale of order ϵ^{-1} in order that all the initial conditions be satisfied, a fact which is reflected in the modal analysis. The question, then, is whether the fast waves have imaginary parts.

We start by showing that for unstable waves $\epsilon|c|$ is bounded. Let

$$\varphi = H^{-1/4}e^{\pm i\lambda(y)}\rho(y), \quad \psi = H^{1/4}e^{\pm i\lambda(y)}\tau(y), \tag{20}$$

where

$$\lambda(y) = -\epsilon k\gamma \int_A^y H^{-1/2}\, w\, dy. \tag{21}$$

Substituting into (15) and (16), we obtain

$$\epsilon^2 w^2(\gamma^2\rho \mp ik\gamma\tau) + \epsilon w H^{1/2}(\tau' + \frac{1}{4}\frac{H'}{H}\tau) + H^{1/2}(\zeta\tau - H^{1/2}\rho) = 0, \qquad (22)$$

and

$$\pm\, \epsilon^2 w^2(\gamma^2\rho \mp ik\gamma\tau) + i\gamma\epsilon w H^{1/2}(\rho' - \frac{1}{4}\frac{H'}{H}\rho) + i\gamma(\zeta\tau - H^{1/2}\rho) = 0. \qquad (23)$$

For $\epsilon|c|$ large we may expand ρ and τ in inverse powers of $\epsilon|c|$, the result being that in the lowest order approximation

$$\gamma\rho = \pm\, ik\tau, \quad \rho' = \tau' = 0. \qquad (24)$$

If ψ is identically zero, so are φ and u. Otherwise, picking the solution for ψ which vanishes at y = A, we have

$$\psi = H^{1/4}\sin\lambda + O(\epsilon^{-1}|c|^{-1}) \qquad (25)$$

as $\epsilon|c| \to \infty$. We note that this asymptotic integration is uniformly valid in the c plane, i.e., there are no turning points. In the limit as $\epsilon|c| \to \infty$

$$\epsilon k\gamma \int_A^B H^{-1/2} w\, dy = \pm\, N\pi,\ N \text{ integral}, \qquad (26)$$

and c is real. Conversely, if c has an imaginary part, $\epsilon|c|$ must be bounded.

We turn now to the case where $\epsilon|c|$ tends to a positive constant as $\epsilon \to 0$. Let $\sigma(\epsilon) = \epsilon c$. We will show that if $\sigma(0) = \sigma_o \neq 0$ and if ϵ is sufficiently small, then σ is expressible in a convergent power series of the form

$$\sigma = \sigma_o + \epsilon\sigma_1 + \epsilon^2\sigma_2 + \ldots, \qquad (27)$$

and that all of the coefficients σ_ν ($\nu = 0,1,2,\ldots$) are real. This result, together with the above one concerning unbounded σ, serves to establish that if c has an imaginary part then $\epsilon|c| \to 0$ as $\epsilon \to 0$.

With $\sigma = \epsilon c$, we have from (15) and (16)

$$(\epsilon U-\sigma)\varphi' - \varphi + H^{-1}(\zeta - k^2(\epsilon U-\sigma)^2)\psi = 0, \tag{28}$$

$$(\epsilon U-\sigma)\psi' + \zeta\psi - (H - \gamma^2(\epsilon U-\sigma^2)\varphi = 0, \tag{29}$$

with $\psi(A) = \psi(B) = 0$, and we note that if $\sigma \neq \epsilon U$ throughout $A \leq y \leq B$, i.e., if c is not real and in the range of U, (28) and (29) have no singular points. Thus, for eigenvalues σ whose absolute value is bounded away from zero, the solution of (28) and (29) satisfying (say) $\psi(A) = 0$ and $\varphi(A) = \sigma$ is an analytic function of ϵ and σ, the other parameters being held constant. The eigenvalue relation is then

$$\psi(B) = F(\epsilon,\sigma) = 0. \tag{30}$$

where $F(\epsilon,\sigma)$ is an analytic function of ϵ and σ. It follows that σ is a continuous function of ϵ.

Now if $\epsilon = 0$,

$$\psi(B) = F(0,\sigma) = (\gamma^2\sigma^2-1)\sin[\mu(B-A)]/\mu, \tag{31}$$

where

$$\mu = \{k^2\gamma^2\sigma^2 - (k^2 + \gamma^2)\}^{1/2} \tag{32}$$

and where $\psi(A) = 0$, $\varphi(A) = \sigma$. have been imposed. The eigenvalues σ_o satisfying $F(0,\sigma_o) = 0$ are given by $\gamma^2\sigma_o^2 = 1$ and $\mu(\sigma_o) = n\pi/(B-A)$, n=1,2,..., and are the same as if U = 0. We note that $F_\sigma(0,\sigma_o) \neq 0$. Therefore, by a theorem of Weirstrass,

$$F(\epsilon,\sigma) = [(\sigma-\sigma_o)F_\sigma(0,\sigma_o) + r(\epsilon)]\ E(\epsilon,\sigma-\sigma_o), \tag{33}$$

where $r(\epsilon)$ and $E(\epsilon,\sigma-\sigma_o)$ are analytic in a neighborhood of 0 and (0,0), respectively, and where $r(0) = 0$, $E(0,0) = 1$. As $\sigma-\sigma_o = O(1)$ as $\epsilon \to 0$, $E \neq 0$ for small ϵ. Therefore, since $F(\epsilon,\sigma) = 0$,

$$\sigma = \sigma_o - r(\epsilon)/F_\sigma(0,\sigma_o), \tag{34}$$

and $\sigma(\epsilon)$ is a convergent power series in ϵ in a neighborhood of $\epsilon = 0$.

We now need to show that all the coefficients in the expansion are real, and for this purpose it is convenient to cast the problem in integral equation form. Consider the following rewritten form of (18),

$$(q\varphi')' - (k^2q+q'/\epsilon w) = \gamma^2\varphi, \tag{35}$$

with $\varphi = \epsilon w\varphi'$ at $y = A$ and $y = B$. The existence of an eigensolution of (35) with the right side replaced by zero is equivalent to the existence of an eigensolution for the original system with γ set equal to zero where it appears explicitly in $Q = (H-\epsilon^2\gamma^2w^2)^{-1}$. From (15) and (16), we see that under this condition $\psi \equiv 0$ implies $\varphi \equiv 0$ which in turn would imply $u \equiv 0$. If ψ is not identically zero, it solves

$$\left(\frac{\psi'}{H}\right)' = \left(\frac{k^2}{H} - (\zeta/H)'/\epsilon w\right)\psi = 0,$$

from which it follows that c lies in Γ. Conversely, if c does not lie in Γ, the operator on the left side of (35) has a bounded inverse, and (35) is equivalent to the integral equation

$$\varphi(y) = \gamma^2 \int_A^B G(y,y';\ \epsilon,\sigma)\varphi(y')dy'. \tag{36}$$

The Green's function G solves

$$(qG')' - (k^2 q + q'/\epsilon w)G = \delta(y-y'), \tag{37}$$

with $G = \epsilon w G'$ at $y = A$ and $y = B$. It is symmetric in y and y' and is real for neutral waves. Furthermore, since c being outside Γ implies c is not in the range of U, G is analytic in ϵ and σ, and all derivatives with respect to these parameters are real for real σ.

We now apply the usual Rayleigh-Schroedinger perturbation theory. Let

$$\varphi = \varphi_o(y) + \epsilon\varphi_1(y) + \dots, \tag{38}$$

$$\sigma = \sigma_o + \epsilon\sigma_1 + \dots, \tag{27 bis}$$

and substitute into (37). Equating powers of ϵ, we obtain

$$\varphi_o(y) - \gamma^2 \int_A^B G(y,y';\, 0,\sigma_o)\, \varphi_o(y')\, dy' = 0, \tag{39.a}$$

$$\varphi_1(y) - \gamma^2 \int_A^B G(y,y';\, 0,\sigma_o)\, \varphi_1(y')\, dy' =$$

$$= \gamma^2 \int_A^B (G_\epsilon + \sigma_1 G_\sigma)_{\substack{\epsilon=0\\ \sigma=\sigma_o}} \varphi_o(y')\, dy', \tag{39.b}$$

etc. Note that σ_ν $(\nu = 0,1,2,\dots)$ appears in the νth equation, and that for $\nu \geq 1$ it appears linearly. Note also that G and all its derivatives are real for $\epsilon = 0$, $\sigma = \sigma_o$, and that the operator on the left side of (39) is Hermitian. Therefore, by the Fredholm alternative theorem,

$$(\varphi_o,\chi_\nu) \equiv \int_A^B \bar{\varphi}_o\, \chi_\nu\, dy = 0, \quad \nu = 1,2,\dots, \tag{40}$$

where $\chi_\nu(\nu = 1,2,\dots)$ is the right side of (39.b) and of subsequent equations, and an overbar denotes complex conjugation. This condition, together with the normalizations $(\varphi_o,\varphi_o) = 1$, $(\varphi_o,\varphi_\nu) = 0$, $\nu \geq 1$, determines a unique solution of (39).

Applying (40) to the right side of (39.b), we find that σ_1 is real. This, together with the normalization, implies that φ_1 is real, and proceeding on in this way we find that all coefficients in the expansion (27) are real. It follows that if c has an imaginary part, $\epsilon|c| \to 0$ as $\epsilon \to 0$.

Our last result, that for small ϵ c lies in Γ if it has an imaginary part, is implicit in the above result. For $c_i \neq 0$, let

$$\psi = (wHe^{\epsilon\gamma^2 c \int_A^y H^{-1}dy})\theta. \tag{41}$$

Under this transformation, Eq. (17) becomes

$$(Qw^2H^2\theta')' - w^2(k^2H - \gamma^2c/w)\theta = (2\epsilon\gamma^2cw^2QH)\theta' \equiv M(y)\theta' \tag{42}$$

$\theta(A) = \theta(B) = 0$. Assume that c is outside Γ. We can then show that (42), with zero on the right-hand side, has no eigensolution. Hence (42) is equivalent to the integral equation

$$\theta(y) = \int_A^B M(y')\,\tilde{G}(y,y')\,\frac{\partial\theta}{\partial y'}\,dy' = -\int_A^B K(y,y')\theta(y')dy', \tag{43}$$

where

$$K(y,y') = \frac{\partial}{\partial y'}\,[M(y')\tilde{G}(y,y')]. \tag{44}$$

Applying the Schwarz inequality, we have

$$|\theta(y)|^2 \leq \int_A^B|\theta(y')|^2dy' \int_A^B |K(y,y')|^2dy', \tag{45}$$

and integration over y and division by (θ,θ) yields

$$1 \leq \int_A^B \int_A^B |K(y,y')|^2\, dydy', \tag{46}$$

which is a necessary condition for an eigensolution to exist. Now $|K| \to 0$ as $\epsilon \to 0$ because $\tilde{G}$ is bounded and $M \to 0$ as $\epsilon \to 0$ for $c_i \neq 0$. Hence for small ϵ (46) cannot be satisfied, and this implies that c must lie in Γ.

4. MARGINALLY STABLE WAVES

We anticipate the results of the next section in stating here that marginally stable waves, i.e., neutral waves which are the limit as $c_i \to 0+$ of unstable waves, occur only for c in the range of U. A necessary condition for the existence of these waves can be obtained by multiplying equation (17) by $\bar{\psi}$, the complex conjugate of ψ, the conjugate of (17) by ψ, subtracting, and then integrating. The result is

$$\mathrm{Im}\int_A^B \frac{(Q\zeta)'}{\epsilon w} |\psi|^2 dy = \mathrm{Im}\int_A^B Q[|\psi'|^2 + \frac{\gamma^2\zeta}{H} |\psi|^2]dy. \tag{47}$$

If we take the limit as $c_i \to 0$ from above with c_r in the range of U, we obtain

$$\sum_{y_c} \frac{|\psi_c|^2}{|U_c'|} (Q\zeta)_c' = 0. \tag{48}$$

Here y_c solves $c = U(y_c)$, and the subscript c denotes the value of a function evaluated at $y = y_c$. Since $w_c = 0$ by definition, we have upon recalling that $Q = (H-\epsilon^2\gamma^2w^2)^{-1}$,

$$\sum_{y_c} \frac{|\psi_c|^2}{|U_c'|} (\frac{\zeta}{H})_c' = 0. \tag{49}$$

The quantity (ζ/H) is the potential vorticity of the basic state, and (49) is a natural generalization of the results obtained through use of the quasi-geostrophic theory.

For monotonic profiles, $U' \neq 0$ and there is only one critical layer at which $c = U$. Also, in this case, $\psi_c \neq 0$. To show that $\psi_c \neq 0$, we note that the exponents of ψ relative to the point $y = y_c$ are 0 and 1, and that if $\psi_c = 0$

$$\psi = (y-y_c)\ P(y) \tag{50}$$

throughout [A,B], where P(y) is analytic in y and can be taken to be real. Let y_1 be the boundary at which $U(y_1) < U_c$, and for definiteness we take $U' > 0$ so that $y_1 = A$. Multiply (17) by ψ and integrate between y_1 and y_c. After an integration by parts, we obtain

$$\int_{y_1}^{y_c} [Q(\psi')^2 + (\frac{k^2}{H} + \frac{\gamma^2 \zeta}{H} Q - \frac{(Q\zeta)'}{\epsilon w})\psi^2]dy = 0, \tag{51}$$

the integral being convergent because of (50). After a number of integrations by parts, (51) can be put into the form

$$\int_{y_1}^{y_c} \{Q[\psi' + \frac{\zeta QH-1}{\epsilon wQH} \psi]^2 + [\frac{k^2}{H} + \frac{\gamma^2}{H^2} \frac{c}{c-U}]\psi^2\}dy = 0. \tag{52}$$

For ϵ sufficiently small, $Q > 0$ on $[y_1,y_c]$, and $c/(c-U) = U_c/(U_c-U)$ is non-negative in this interval. Consequently, (52) cannot be satisfied and therefore $\psi_c \neq 0$. It follows that for monotonic profiles y_c must be the point at which the potential vorticity gradient vanishes, and $c = U_c$.

For nonmonotonic profiles it can be shown that, provided the profile has not more than two interior extrema, with $U > 0$ at the maximum and $U < 0$ at the minimum, c cannot equal U at an extremum. It can also be shown that $\psi_c \neq 0$. The argument is similar to that given above. However, even for symmetric profiles, it is not true in general that the points y_c coincide with maxima or minima of the potential vorticity, since the potential vorticity is not symmetric. This limits the usefulness of (49) in finding marginally stable waves. On the other hand, since we anticipate the existence of a neutral stability curve and since (49) holds for marginally stable waves, the potential vorticity gradient must vanish somewhere in the flow domain if the motion is unstable, and this is the major result of this section.

5. EFFECT OF ROSSBY NUMBER ON STABILITY CHARACTERISTICS

As mentioned at the start of Section 3, a perturbation analysis can be carried out to improve upon the results of the quasi-geostrophic theory. Some of the labor involved in this procedure can be circumvented through use of the following device, which is due to Howard (1963). Consider the functional

$$I[\varphi] = \{\frac{q\varphi^2}{\epsilon w}\Big|_A^B - \int_A^B \{q[(\varphi')^2 + k^2\varphi^2] + q'\varphi^2/\epsilon w\}dy\} \div \int_A^B \varphi^2 dy. \qquad (53)$$

It is readily verified that $I[\varphi] = \gamma^2$ and that I is stationary in the sense of the calculus of variations (though not an extremum), from which it follows that the derivative of I with respect to any parameter does not involve contributions coming from the dependence of φ on that parameter. Letting n be any of ϵ, k, or γ, we have

$$\frac{\partial\gamma^2}{\partial n} = \frac{\partial I}{\partial n} + \frac{\partial I}{\partial c}\frac{\partial c}{\partial n}, \qquad (54)$$

from which $\frac{\partial c}{\partial n}$ can be calculated. It should be noted that if c is real but not in the range of U, $\frac{\partial c}{\partial n}$ is real and the neutral wave in question is not a marginally stable wave.

Now, from (54), we obtain

$$\left(\frac{\partial c}{\partial \epsilon}\right)_{\epsilon=0} = \{\int_A^B \{q_1[(\varphi_o')^2 + k^2\varphi_o^2] + \frac{q_2'}{2w_o}\varphi_o^2\}dy$$

$$+ w_o(\varphi_o')^2\Big|_A^B\} \div \int_A^B [-\frac{q_1'}{w_o^2}]dy, \qquad (55)$$

where φ_o is the solution of the quasi-geostrophic Eq. (19), $w_o = U(y)-c_o$, c_o

being the zero order wave speed, and $q_\nu = \partial^\nu q/\partial\epsilon^\nu)_{\epsilon=0}$. This provides the first order term in a perturbation series for c.

From the definition of q, we have

$$q_1 = U' + \gamma^2\Phi, \quad q_2 = 2[(U')^2 + \gamma^2\Phi U' + k^2(w_o)^2]. \tag{56}$$

If the velocity profile U(y) is even in y and if A = -B, $\varphi_o(y)$ is either even or odd in y, and $(\varphi_o')^2$ and φ_o^2 are even. Then q_1 is odd, q_2 is even, the numerator of (55) integrates to zero while the denominator is nonzero. Hence, for a symmetric flow $c = c_o + O(\epsilon^2)$, i.e., there is no first order Rossby number effect on c.

The velocity profile which will be treated here is not symmetric and hence there will be a first order Rossby number effect. The flow is

$$U(y) = \cos \pi y + \frac{\sqrt{3}}{2}, \quad A = -7/6, \ B = 5/6, \tag{57}$$

and is plotted in Figure 1. A countercurrent is included, since (57) is meant to model the mean velocity of the Gulf Stream. The marginally stable waves corresponding to the above profile (with ϵ=0) are readily calculated using standard methods, and these will be used in the formula for $\partial c/\partial\epsilon$. In this way we find the perturbation off the neutral stability curve due to the Rossby number.

Of the two possible modes solving (19) with ϵ=0, the more unstable symmetric mode is given by

$$\varphi_o = \cos\left[\frac{\pi}{2}\left(y + \frac{1}{6}\right)\right], \quad k = \frac{\sqrt{3}}{2}\pi, \quad c_o = \frac{\sqrt{3}}{2}\pi^2/(\pi^2+\gamma^2) + i0+. \tag{58}$$

In evaluating the integrals in (55) we will regard c_o as being the limit of an

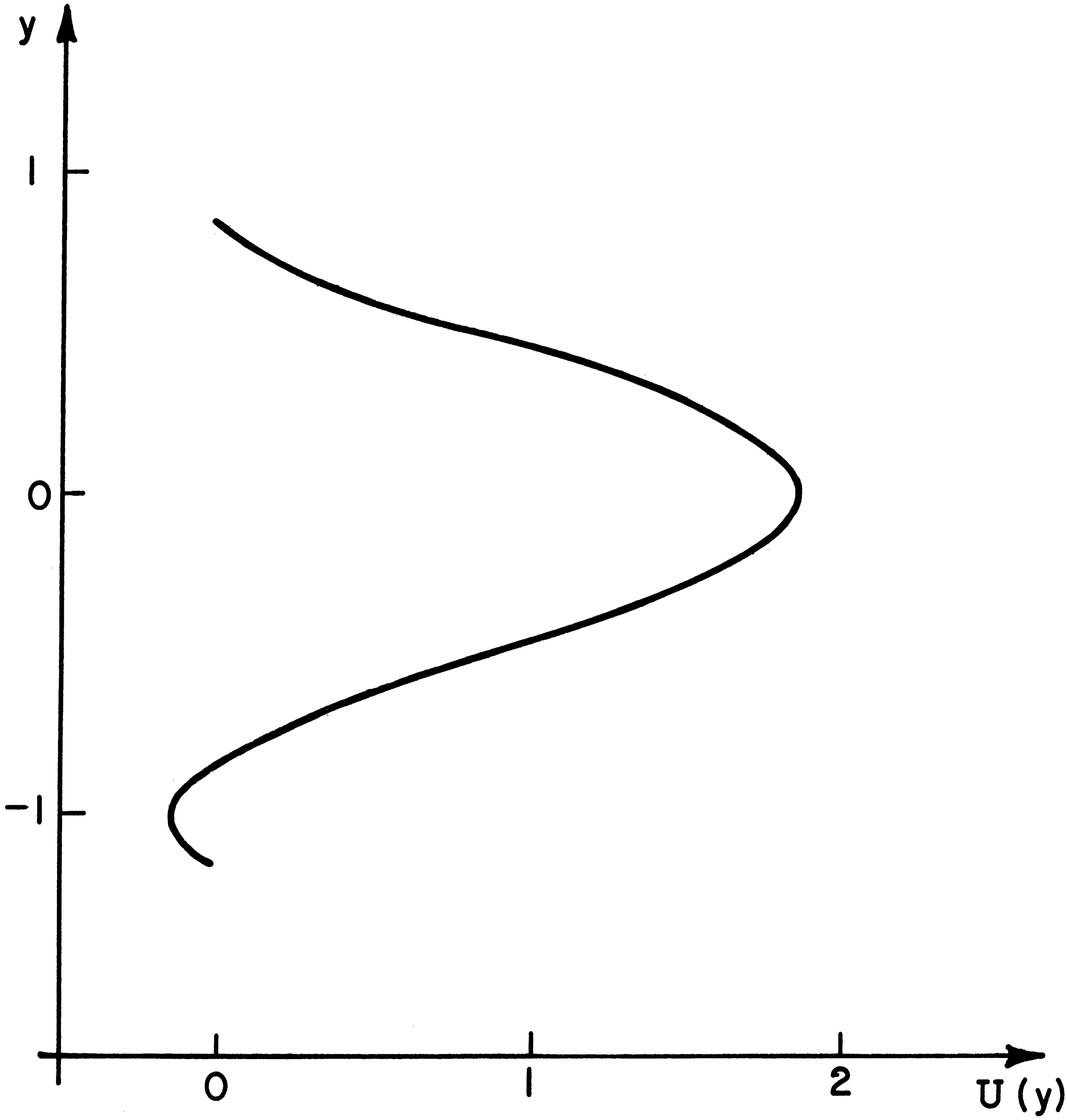

Figure 1. Velocity profile.

unstable wave, as indicated by the last equation of (58). These integrals, with one exception, can be put into the form

$$\int_{\alpha}^{\alpha+2\pi} R(\cos \theta, \sin \theta) d\theta$$

where R is a rational function, and hence can be evaluated as contour integrals. The exception is due to a term linear in y because of the presence of Φ. This is treated by expanding $(y + \frac{1}{6})$ in a Fourier sine series in the interval $-1 \leq (y + \frac{1}{6}) \leq 1$ and evaluating each integral by converting it into a contour integral. The resulting infinite series can be summed exactly.

The details of this calculation will be omitted and we present only the final result. With

$$\Gamma = \frac{\sqrt{3}}{2} \frac{\gamma^2}{\pi^2+\gamma^2}$$

and with

$$F(\Gamma) = 2 \frac{1 - \frac{\sqrt{3}}{2}\Gamma}{(1-\Gamma^2)^{1/2}} \log \left[\frac{1 - \frac{\sqrt{3}}{2}\Gamma + \frac{1}{2}(1-\Gamma^2)^{1/2}}{\frac{\sqrt{3}}{2} - \Gamma}\right] + \log [\sqrt{3} - 2\Gamma], \quad (59)$$

we obtain after a number of nontrivial integrations

$$\mathrm{Im}\left(\frac{\partial c}{\partial \epsilon}\right)_{\epsilon=0} = \frac{2\pi\Gamma(1-\Gamma^2)}{7-4\sqrt{3}\,\Gamma} \left\{ \frac{\sqrt{3}}{2} \left[\frac{1 - \frac{\sqrt{3}}{2}\Gamma}{(1-\Gamma^2)^{1/2}} + (1-\Gamma^2)^{1/2} \right.\right.$$

$$\left.\left. + \Gamma \cos^{-1}(-\Gamma)\right] - \Gamma \frac{1 - \frac{\sqrt{3}}{2}\Gamma}{(1-\Gamma^2)^{1/2}} [1 + \pi/2\sqrt{3} - F(\Gamma)]\right\}. \quad (60)$$

We note that for $\gamma = 0$ ($\Gamma = 0$) the right side of (60) vanishes. This is to be expected, since in this case there should be no Rossby number effect. The formula is inaccurate in the limit $\gamma \to \infty$ ($\Gamma \to \frac{\sqrt{3}\,2}{2}$) because the product $\epsilon\gamma^2$ has been treated as being small; accordingly, the calculation is valid only when $\epsilon\gamma^2 \ll 1$.

The result is given in Table I. As can be seen, for this flow the Rossby number effect is destabilizing in the sense that a marginally stable configuration for zero Rossby number is unstable for finite Rossby number. This is a surprising result; it had previously been felt that Rossby number effects would be stabilizing. Further calculations based on different profiles have shown that the Rossby number effect may in some cases be stabilizing, in some cases destabilizing. We therefore can come to no definite conclusion regarding this matter except to say that in the present case for a profile resembling that of the Gulf Stream in many ways, the first correction to quasi-geostrophic theory indicates a destabilizing Rossby number effect.

TABLE I

$\frac{\partial c_i}{\partial \epsilon}$ vs. Γ

Γ	$\frac{\partial c_i}{\partial \epsilon}$
.0	.0
.087	.038
.173	.086
.260	.144
.346	.212
.520	.379
.693	.581
.866	.782

6. SMALL ROSSBY NUMBER DIVERGENT FLOW

The obvious difficulty in applying quasi-geostrophic theory to the Gulf Stream problem is that the Rossby number, based on a relative vorticity of 0.4 or 0.5 x 10^{-4} $\sec^{-1}$ (Stommel, op. cit.) is not particularly small. Not so obvious but equally important is the fact that the slope of the interface must also be small if the fluid is to be nondivergent in the lowest approximation. In the present paper this slope is the quantity $\epsilon\gamma^2$, and the conditions for quasi-geostrophic theory to be valid are $\epsilon \ll 1$, $\epsilon\gamma^2 \ll 1$. The first condition is satisfied marginally, the second not at all, for based on the data in Stommel's book $\epsilon\gamma^2 \approx 1$ if the 10°C isotherm is taken to be the interface. This can be seen either by taking as the characteristic depth D the depth of the 10°C isotherm at the midpoint of the stream or else by directly computing the slope of the isotherm.

It is pertinent to mention at this point that Stern (1961) discusses the case $\epsilon \ll 1$, $\epsilon\gamma^2 \ll 1$, $\gamma^2 \gg 1$. It is difficult to see how this could apply to the Gulf Stream, but one of Stern's conclusions, namely that an increase in γ^2 is stabilizing, turns out in the sequel to be true.

The foregoing remarks imply that any low Rossby number theory must be modified by taking the quantity $\Delta = \epsilon\gamma^2$ to be of order unity even though ϵ is small. Turning to Eq. (18), we find that in the limit $\epsilon \to 0$, $\gamma^2 \to \infty$, with $\Delta \approx 1$,

$$(H\varphi')' + [\frac{\Delta c}{\epsilon(U-c)} - \frac{(HU')'}{U-c} - k^2H]\varphi = 0, \qquad (61)$$

with $\varphi = 0$ at $y = A$, $y = B$. We note that unless c is of order ϵ (61) is a singular perturbation type of equation which can be treated through use of the W.K.B. method. This treatment, omitted here, reveals that in this case c must be real. The remaining possibility is that c is of order ϵ, say $c = \epsilon\lambda$, and we see that the joint limits $\epsilon \to 0$, $\gamma^2 \to \infty$ imply that the complex wave speed is small.

Now, if U does not vanish anywhere in $A < y < B$, we may neglect $c = \epsilon\lambda$ next to U, and (61) is a regular self-adjoint Sturm-Liouville type of equation with real eigenvalues λ. In this case the perturbations are nonamplifying. However, if $U = 0$ at some point $y = y_o$, a neglect of $\epsilon\lambda$ next to U is not uniformly valid. In this case with $\lambda = \lambda_r + i\lambda_i$, $\lambda_i > 0$ we have

$$\lim_{\epsilon \to 0} \frac{1}{U-\epsilon\lambda} = P\left(\frac{1}{U}\right) + \frac{i\pi}{|U_o'|}\,\delta(y-y_o), \tag{62}$$

where P denotes principal value, and (61) becomes

$$(H\varphi')' + \left[\frac{\Delta\lambda-(HU')'}{U} - k^2H\right]\varphi = 0,\ y \neq y_o, \tag{63}$$

where $\varphi(A) = \varphi(B) = 0$, and

$$\varphi\Big|_{y_o-}^{y_o+} = 0, \quad \varphi'\Big|_{y_o-}^{y_o+} = \frac{i\pi\varphi_o}{H_o|U_o'|}\,[(H_oU_o')' - \Delta\lambda]. \tag{64}$$

We will not pursue the implications of this last set of equations except to remark that it is highly probable that complex eigenvalues result. The main points of interest are that in the model discussed in this section, which appears to be more relevant to the Gulf Stream than the usual quasi-geostrophic model, the complex wave speeds are small and the motion is stable unless the velocity profile has a zero. The fact that the existence of a countercurrent plays a role in the stability problem may appear bizarre, but is explained by the fact that a critical layer is in general necessary for instability and for very small wave speeds such a layer can exist only if the unperturbed velocity profile has one or more zeros.

7. CONCLUDING REMARKS

When the theory of Section 5 is applied to a profile resembling that of the Gulf Stream it is found that first-order Rossby number effects are destabilizing. Since growth rates predicted by the quasi-geostrophic theory are already too high, one must conclude that the quasi-geostophic theory is completely invalid for studying the stability of the Gulf Stream. The theory of Section 6, which takes account of the divergence of the Stream, is more applicable and gives results in better qualitative agreement with observations. Nevertheless, the asymptotic analysis is rather delicate, and a numerical study is necessary to answer the question of whether the theory of hydrodynamic stability is relevant at all to the problem of Gulf Stream meanders. Such a study is now in progress.

8. REFERENCES

Case, K. M. 1960. Stability of inviscid plane couette flow. Phys. Fluids, 3, p. 143.

Gulf Stream Summaries, U. S. Naval Oceanographic Office.

Howard, L. N. 1963. Neutral curves and stability boundaries in stratified flow. J. Fluid Mech., 16, p. 333.

Jacobs, S. J. 1967. An asympototic solution of the tidal equations. J. Fluid Mech., 20, p. 417.

Lipps, F. B. 1963. Stability of jets in a divergent barotropic fluid. J. Atmos. Sci., 20, p. 120.

Pedloskey, J. 1965. The stability of currents in the atmosphere and the ocean: Part II. J. Atmos. Sci., 21, p. 342.

Stern, M. 1961. The stability of thermoclinic jets. Tellus 13, p. 503.

Stommel, H. 1965. The Gulf Stream. University of California Press.