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Final Report

NONLINEAR INTERACTIONS IN ROTATING STRATIFIED FLOW

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TABLE OF CONTENTS

SUMMARY

Nonlinear interactions in a rotating, stratified fluid with a free surface are considered. The field variables are expanded in terms of eigenfunctions of the linearized problem, with coefficients dependent on time. Orthogonality relations are developed to obtain evolution equations for the coefficients, and these are simplified through use of the method of averaging, for weakly nonlinear motions.

The geostrophic mode, which alone possesses potential vorticity, is found to obey the quasi-geostrophic equations, even though wave modes are present. The phases of the wave modes are affected by the presence of the geostrophic mode, with frequency splitting for steady geostrophic flow, but there is no energy transfer between the geostrophic mode and the wave modes. Resonant interactions between waves are found to occur for a resonant triad consisting of two external waves and one internal wave. When the wave vectors of the external waves are colinear, the internal wave generated by the interaction has a frequency very close to the inertial frequency, but exceeding it slightly. The growth rate for inertial motions generated in this manner is comparatively slow.

INTRODUCTION

In recent years extensive interest has been shown in the effect of nonlinearity on wave motions, both for wave motions in the sea (Hasselmann, 1966) and in other branches of physics as well. The interest stems from the fact that nonlinearity, though small, may have an important cumulative effect over long periods of time. In particular, for oceanic motions, the nonlinearity serves to modify the energy spectrum of external gravity waves and perhaps of internal waves.

Past attention in this area has been largely confined to periodic waves, either with a continuous or discrete spectrum. In both cases, the point of interest is that secondary waves, generated by the nonlinear interaction, may stay in phase with a forcing wave, thereby allowing a continuous transfer of energy. This phenomenon is called a resonant interaction, and when it takes place the effect of nonlinearity is significant.

The purpose of the present study is to develop a formalism for treating nonlinear interaction of nonperiodic waves, such as would occur in a bounded basin. The particular case treated is that of the motion of a rotating, stratified fluid with a free surface on the β-plane. In the linear case, solutions can be obtained, in principle, by expanding in terms of normal modes. Solutions of this type, for fluids confined entirely by a rigid surface, have been obtained by Siegmann and Howard (Howard, 1968). The expansion procedure can be modified to accommodate the nonlinear case by allowing the coefficients in the expansion to be functions of time. It is then quite easy to set up the interaction equations.

A particular point of interest concerns the geostrophic mode, which is an equilibrium solution of the linear equations with β effect neglected. Perhaps not surprisingly, the geostrophic mode, with nonlinearity and β effect included, proves to obey the quasi-geostrophic equations. Also of interest is the generation of internal waves due to nonlinear interaction of external waves. It is found in the case of motions periodic in the horizontal that motions with a frequency slightly exceeding the inertial frequency are generated by the interaction of colinear external waves. However, the growth rate of such inertial motions is comparatively slow.

We do not attempt in this study the development of a statistical theory, and consequently the results are not directly applicable to situations of goephysical interest. For this reason, no attempts have been made to compare predictions of the theory with observations.

2. FORMULATION

Consider an inviscid stratified fluid of constant mean depth H on the rotating earth. Let $\vec{x} = (x,y,z)$ denote the position vector, with x measuring distance to the east, y to the north, and z vertically upwards, and let $\vec{v} = (u,v,w)$ be the particle velocity. The upper boundary of the fluid is a free surface at $z = \eta(x,y,t)$, with $\eta = 0$ in the absence of motion. External forcing and surface tension will be neglected. Let $\bar{\rho}(z)$ be a basic density distribution, and let the bouyancy b and the gauge pressure p be defined by

$$b = -g(\rho - \bar{\rho}(z))/\rho_o, \tag{1}$$

$$p = \wp + g\int_o^z \bar{\rho}(z')dz', \tag{2}$$

where ρ and $\wp$ denote the density and pressure, and $\rho_o = \bar{\rho}(0)$. The vertical unit vector is denoted by $\hat{m}$. In the β-plane and Boussineq approximations, with Coriolis parameter $f = f_o + \beta y$ and Brunt-Väisälä frequency

$$N(z) = \left(-\frac{g}{\rho_o}\frac{d\bar{\rho}}{dz}\right)^{1/2}, \tag{3}$$

the equations of motion are

$$\nabla \cdot \vec{v} = 0, \tag{4}$$

$$\frac{D\vec{v}}{Dt} + f\,\hat{m} \times \vec{v} + \frac{1}{\rho_o}\nabla p = b\,\hat{m}, \tag{5}$$

$$\frac{Db}{Dt} + N^2 W = 0, \tag{6}$$

for $-H < z < \eta$, with boundary conditions

$$w(x,y,-H,t) = 0 \tag{7}$$

and

$$\frac{D}{Dt}(z - \eta) = 0, \qquad p = g\int_o^\eta \bar{\rho}(z)dz, \tag{8}$$

at $z = \eta$.

It is convenient to scale the variables and deal with nondimensional equations, but there is a difficulty in interpreting the results. For any scaling there will arise nondimensional parameters

α^* = ratio of time scale to an internal wave period,
β^* = ratio of residence time to a Rossby wave period,
γ^* = ratio of time scale to an external gravity wave period,
δ^* = ratio of vertical length scale to horizontal length scale,
ε^* = ratio of time scale to residence time,
r^* = ratio of time scale to an inertial period.

In general, it is incorrect to make approximations based on the magnitude of these parameters and to expect such approximations to be valid for all conceivable scales of motion. The only exception involves ε^*, a measure of nonlinearity, which we anticipate is small for all scales of motion. The scaling to be used here is appropriate for external gravity waves with a horizontal length scale equal to the depth of the fluid. In this scaling $\gamma^* = \delta^* = 1$ and all the other parameters are small, but approximations will be based only on the condition $\varepsilon^* \ll 1$.

With this in mind, we let V be a characteristic velocity, N_m the maximum value of N, and introduce dimensionless variables through the scaling

$$\vec{x} = H\,\vec{x}^*, \qquad t = (H/g)^{1/2}\,t^*, \qquad \vec{v} = V\,\vec{v}^*,$$

$$\eta = V(H/g)^{1/2}\,\eta^*, \qquad p = V(gH)^{1/2}\,P^*, \qquad b = V\,N_m b^*,$$

$$N = N_m\,N^*, \qquad \bar{\rho} = \rho_o\,\bar{\rho}^*. \tag{9}$$

With this scaling, the nondimensional parameters are

$$\alpha^* = (H/g)^{1/2}\,N_m, \qquad \beta^* = \beta\,H^2/V,$$

$$\varepsilon^* = V/(gH)^{1/2}, \qquad r^* = (H/g)^{1/2}\,f_o, \tag{10}$$

and have the meanings discussed above. Omitting asterisks, the boundary conditions at $z = \varepsilon\eta$ are

$$w = \frac{\partial \eta}{\partial t} + \varepsilon\,\vec{v}\cdot\nabla\eta, \qquad \varepsilon p = \int_o^{\varepsilon\eta} \bar{\rho}(z)dz, \tag{11}$$

where the nondimensional density distribution satisfies

$$\frac{d\bar{\rho}}{dz} = -\alpha^2 N^2, \qquad \bar{\rho}(0) = 1. \tag{12}$$

For ε small, we can transfer the second of these to the level $z = 0$, obtaining

$$\eta = p\,[1 + \varepsilon\,(\frac{\partial p}{\partial z} + \frac{1}{2}\alpha^2 N^2 p)] + O(\varepsilon^2) \tag{13}$$

at $z = 0$. Transferring the kinematic condition to $z = 0$ and eliminating η yields

$$\frac{\partial p}{\partial t} - w = -\varepsilon\,[\nabla_1 \cdot (p\vec{u}) + w\,(\frac{\partial p}{\partial z} + \alpha^2 N^2 p) + p\,\frac{\partial^2 p}{\partial z\,\partial t}] + O(\varepsilon^2)$$

$$\equiv \varepsilon P + O(\varepsilon^2), \tag{14}$$

where

$$\nabla_1 = \nabla - \hat{m}\frac{\partial}{\partial z}, \qquad \vec{u} = \vec{v} - \hat{m}\,w. \tag{15}$$

The lower boundary condition is

$$w(x,y,-1,t) = 0, \tag{16}$$

and the other equations governing the flow become

$$\nabla \cdot \vec{v} = 0, \tag{17}$$

$$\frac{\partial \vec{v}}{\partial t} + r\,\hat{m} \times \vec{v} + \nabla p - \alpha\, b\, \hat{m} = -\varepsilon\{\hat{m}\,\beta\, y \times \vec{v} + (\vec{v}\cdot\nabla)\,\vec{v}\} \equiv \varepsilon\,\hat{U}, \tag{18}$$

$$\frac{\partial b}{\partial t} + \alpha N^2 w = -\varepsilon\,\vec{v}\cdot\nabla b \equiv \varepsilon B. \tag{19}$$

In addition, we assume either that the fluid is confined in a closed region bounded by a vertical wall or that the motion is periodic in the horizontal. Obviously, the latter case connot be precisely correct, since there must be refraction of waves due to the sphericity of the earth. This will be neglected here.

We now introduce the vectors

$$\theta = (\vec{v}, b, p), \qquad \Theta = (\vec{U}, B, P). \tag{20}$$

Then the above equations define an initial value problem for Θ, with weak nonlinearity as expressed by the presence of $\varepsilon\ \Theta$. For the linear problem, eigensolutions can be obtained by assuming the time dependence of the form $\exp(-i\ \sigma\ t)$, and the initial value problem can be solved by expanding Θ in terms of the corresponding eigenfunctions. This expansion can also be employed for the nonlinear problem, but the coefficients in the eigenfunction expansions must be allowed to depend on t. Of course, Θ must also be expanded in terms of the eigenfunctions, and the resulting equations are quite complicated. Nevertheless, there are many advantages to the use of this interaction representation, as will be seen shortly.

3. AN EIGENFUNCTION EXPANSION

The eigenvalue problem is described by

$$\nabla \cdot \vec{v} = 0, \tag{21}$$

$$-i \sigma \vec{v} + r \hat{m} \times \vec{v} + \nabla p = \alpha b \hat{m}, \tag{22}$$

$$-i \sigma b + \alpha N^2 w = 0, \tag{23}$$

for $-1 < z < 0$, with

$$w = 0 \text{ at } z = -1, \qquad w = -i \sigma p \text{ at } z = 0. \tag{24}$$

In addition, the flow is either periodic in the horizontal or has vanishing normal component of velocity at vertical walls. In what follows, the symbol V will stand either for the volume of a periodic cell or for the total volume of the fluid, the symbol S for the upper boundary, and the symbol R for the vertical walls in the second of the cases mentioned above.

The geostrophic mode, for which $\sigma = 0$, must be present in general. Denoting this mode by subscript g, we have

$$r \vec{v}_g = \hat{m} \times \nabla \psi, \qquad \alpha b_g = \frac{\partial \psi}{\partial z}, \tag{25}$$

where

$$\psi \equiv p_g. \tag{26}$$

The function ψ is undetermined at this stage, other than being periodic in the first of the cases mentioned above or being such as to make the normal component of $\vec{v}_g$ vanish at R, in the second of the cases. For arbitrary σ, elimination of p between the horizontal momentum equations and substitution from the continuity and energy equations leads to the result

$$\sigma \{\hat{m} \cdot \nabla \times \vec{v} + \frac{r}{\alpha} \frac{\partial}{\partial z} (b/N^2)\} = 0. \tag{27}$$

Also, integration of the continuity equation over the horizontal area of V and substitution from the energy equation yields

$$\sigma \{ \iint b \, d A\} = 0, \tag{28}$$

and elementary operations provide the additional relations

$$\sigma\,\{b\} = 0 \text{ at } z = -1, \qquad \sigma\,\{b + \alpha^2 N^2 p\} = 0 \text{ at } z = 0. \qquad (29)$$

Consequently, the bracketed quantities assume nonzero values only for the geostrophic mode. In particular, identifying the term in (27) as the potential vorticity, we see that the geostrophic mode alone possesses potential vorticity.

We next determine an orthogonality relation. Let σ_n and σ_m be eigenvalues, with eigenfunctions Θ_n and Θ_m, and let an asterisk denote the complex conjugate. It is easily shown that

$$i(\sigma_m^* - \sigma_n)\,(\vec{v}_n \cdot \vec{v}_m^* + b_n\, b_m^*/N^2) + \nabla \cdot (p_n\, \vec{v}_m^* + p_m^*\, \vec{v}_n) = 0, \qquad (30)$$

and integration over V and use of the boundary conditions leads to

$$(\sigma_m^* - \sigma_n)\,(\Theta_m, \Theta_n) = 0, \qquad (31)$$

where

$$(\Theta_m, \Theta_n) = \iiint_V (\vec{v}_n \cdot \vec{v}_m^* + b_n\, b_m^*/N^2)\,dV + \iint_S p_n\, p_m^*\, dA, \qquad (32)$$

and may be considered to be an inner product.

We will call any mode with $\sigma \neq 0$ a wave mode. Putting $\sigma_m = 0$ in (31), so that $\Theta_m = \Theta_g$, we find that the geostrophic mode is orthogonal to all wave modes. Putting $m = n$ and noting that (Θ_n, Θ_n) is positive definite, we find that the σ's are pure real. Finally, we have the orthogonality condition

$$(\Theta_m, \Theta_n) \quad 0, \quad m \neq n \qquad (33)$$

Actually, (31) implies orthogonality only if the eigenvalues are different for different modes, and the possibility does exist that an eigenvalue σ may possess more than one eigenfunction. If this occurs, we use the Schmidt orthogonalization process to insure the validity of (33).

From now on, the subscripts on σ will be reserved for wave modes only. Assuming the completeness of the eigenfunctions generated by the above eigenvalue problem, we expand Θ in the series

$$\Theta = \Theta_g(\vec{x}, t) + \sum_n A_n(t)\, \Theta_n(\vec{x})\, e^{-i\sigma_n t} \qquad (34)$$

in which we now allow Θ_g to depend on time but still satisfy the geostrophic relations given in (25).

To compute the initial values for Θ_g and the A's, we note that if $\Theta = \Theta_I$, a known function, at $t = 0$, the conditions

$$\vec{v}_I = \vec{v}_g + \sum_n A_n \vec{v}_n$$

$$b_I = b_g + \sum_n A_n b_n$$

$$p_I = \psi + \sum_n A_n p_n \qquad \text{at } z = 0 \qquad (35)$$

must be satisfied at $t = 0$. To find $A_n(0)$, we compute the inner product (Θ_m, Θ_I) for some m. Invoking the orthogonality condition, we readily obtain the initial condition

$$A_n(0) = (\Theta_n, \Theta_I) \div (\Theta_n, \Theta_n). \qquad (36)$$

To obtain an initial value for Θ_g, we compute the potential vorticity for Θ_I. Since the potential vorticity vanishes for each of the wave modes, this yields the equation

$$\hat{m} \cdot \nabla \times \vec{v}_I + \frac{r}{\alpha} \frac{\partial}{\partial z}(b_I/N^2) = \hat{m} \cdot \nabla \times \vec{v}_g + \frac{r}{\alpha} \frac{\partial}{\partial z}(b_g/N^2). \qquad (37)$$

In similar manner, using the conditions implied by (28) and (29), we obtain

$$b_g = b_I \text{ at } z = -1, \qquad b_g + \alpha N^2 \psi = b_I + \alpha N^2 p_I \text{ at } z = 0, \qquad (38)$$

and

$$\iint b_g\, dA = \iint b_I\, dA. \qquad (39)$$

Together with equation (25), these yield

$$\nabla_1^2 \psi + \frac{r^2}{\alpha^2} \frac{\partial}{\partial z}\left(\frac{1}{N^2} \frac{\partial \psi}{\partial z}\right) = r\, [\hat{m} \cdot \nabla \times \vec{v}_I + \frac{r}{\alpha} \frac{\partial}{\partial z}(b_I/N^2)], \qquad (40)$$

to be solved subject to

$$\frac{\partial \psi}{\partial z} = \alpha\, b_I \quad \text{at } z = -1$$

$$\frac{\partial \psi}{\partial z} + \alpha^2 N^2 \psi = \alpha(b_I + \alpha N^2 p_I) \quad \text{at } z = 0 \tag{41}$$

the lateral boundary conditions, and

$$\frac{\partial}{\partial z} \iint \psi \, d A = \alpha \iint b_I \, dA. \tag{42}$$

It is of interest to show why the last equation is needed. In the periodic case the condition (42) is satisfied identically, but if the fluid is confined between rigid walls (42) is not identically satisfied and the solution for $\psi(\vec{x}, 0)$ is apparently not unique. This is because the requirement that the normal conponent of $\vec{v}_g$ vanishes at R implies only that $\psi = f(z)$ on R, for arbitrary $f(z)$. To determine $f(z)$ (42) must be used. The simplest method appears to be as follows. Let $\psi = \psi_1 + \psi_2$, where ψ_1 satisfies (40) and (41) and vanishes on R, while ψ_2 satisfies the homogeneous form of (40) and (41) and assumes the value $f(z)$ on R. Both ψ_1 and ψ_2 are uniquely determined, as is easily shown, and ψ_1 does not depend on $f(z)$. In place of (42), we use

$$\iint \nabla_1^2 \psi \, d A = r \iint \hat{m} \cdot \nabla x \, \vec{v}_I \, d A, \tag{43}$$

which is derived by integrating (40) over the horizontal area of V and substituting from (42). Since ψ_1 may be regarded as a known function of $\vec{x}$, (43) implies that

$$\iint \nabla_1^2 \psi_2 \, d A = F(z), \tag{44}$$

where $F(z)$ is a known function of z. We now let λ_j and $h_j(z)$ be the eigenvalues and eigenfunctions satisfying

$$-\frac{r^2}{\alpha^2} \frac{d}{dz} \left(\frac{1}{N^2} \frac{dh}{dz}\right) = \lambda\, h(z), \tag{45}$$

with

$$h'(-1) = h'(0) + \alpha^2 N^2 (0)\, h(0) = 0. \tag{46}$$

The eigenvalues are pure real and positive, and the eigenfunctions form a complete set (Courant and Hilbert, 1953, Chapter V). Expanding f(z) and ψ_2 in the series

$$f = \sum_{j=1}^{\infty} f_j h_j(z), \qquad \psi_2 = \sum_{j=1}^{\infty} f_j \varphi_j(x, y) h_j(z), \tag{47}$$

we see that for the determination of ψ_2 the terms φ_j must satisfy

$$\nabla_1^2 \varphi_j = \lambda_j \varphi_j \tag{48}$$

with $\varphi_j = 1$ on R. Then, since

$$F(z) = \sum_{j=1}^{\infty} f_j \left(\iint \nabla_1^2 \varphi_j \, dA\right) h_j(z), \tag{49}$$

we determine f_j by expanding the known function F(z) in terms of the eigenfunctions h_i and equating coefficients.

To complete the proof of the validity of this method, we must show that the area integrals in (49) do not vanish. To prove this, we note that the maximum principle for elliptic equations (Courant and Hilbert, 1962, Chapter IV) implies that $\varphi_j(x,y) \geq 0$ everywhere. Hence

$$\iint \nabla_1^2 \varphi_j \, dA = \lambda_j \iint \varphi_j \, dA \geq 0,$$

and we can make this an inequality by using the fact that $\varphi_j = 1$ on R and a continuity argument. Consequently, the above procedure for determining f(z) is valid.

We turn now to the solution of the initial value problem. Substituting the expansion (34) into the equations defining the initial value problem, we obtain

$$\frac{\partial \vec{v}_g}{\partial t} + \sum_n \dot{A}_n e^{-i\sigma_n t} \vec{v}_n = \varepsilon \vec{U}, \tag{50}$$

$$\frac{\partial b_g}{\partial t} + \sum_n \dot{A}_n e^{-i\sigma_n t} b_n = \varepsilon B, \tag{51}$$

for $-1 < z < 0$, and

$$\frac{\partial\psi}{\partial t} + \sum_n \dot{A}_n \, e^{-i\sigma_n t} \, p_n = \varepsilon P \quad \text{at } z = 0. \tag{52}$$

Invoking the orthogonality condition, the coefficients $A_n(t)$ are found to obey the equations

$$\dot{A}_n = \varepsilon \, e^{i\sigma_n t} \, (\Theta_n, \Theta) \div (\Theta_n, \Theta_n). \tag{53}$$

Also, carrying out a calculation similar to that involved in determining $\psi(\vec{x}, 0)$, we find that $\psi(\vec{x},t)$ solves the partial differential equation

$$\frac{\partial}{\partial t} \{\nabla_1^2 \psi + \frac{r^2}{\alpha^2} \frac{\partial}{\partial z} (\frac{1}{N^2} \frac{\partial\psi}{\partial z})\} = \varepsilon r \, [\hat{m} \cdot \nabla \times \vec{U} + \frac{r}{\alpha} \frac{\partial}{\partial z} (B/N^2)],$$

subject to

$$\frac{\partial^2\psi}{\partial t \, \partial z} = \alpha \, \varepsilon \, B \qquad \text{at } z = -1 \tag{55}$$

$$\frac{\partial}{\partial t} (\frac{\partial\psi}{\partial z} + \alpha^2 N^2 \psi) = \alpha \, \varepsilon \, (B + \alpha N^2 P) \quad \text{at } z = 0, \tag{56}$$

$$\frac{\partial^2}{\partial t \, \partial z} \iint \psi \, d A = \alpha \, \varepsilon \iint B \, d A, \tag{57}$$

and the lateral boundary conditions. If $\vec{U}$, B, and P were known functions, the initial value problem is solved by integration of the above equations. However, Θ is actually a function of Θ and therefore must also be expanded in terms of Θ_g and Θ_n. The procedure use here thus replaces the original set of partial equations with an infinite set of ordinary differential equations, for A_n, and a partial differential equation for ψ.

Substitution of the expansion (34) into the definition of Θ yields

$$\Theta = G_{gg} [\Theta_g] + \sum_i G_{gw}^{(i)} [\Theta_g, \Theta_i] \, A_i \, e^{-i\sigma_i t}$$

$$+ \sum_{i,j} G_{ww}^{(ij)} [\Theta_i, \Theta_j] \, A_i \, A_j \, e^{-i(\sigma_i + \sigma_j)t}, \tag{58}$$

where the subscript gg denotes the interaction of the geostrophic mode with itself, gw the interaction of the geostrophic mode with the wave modes, and ww the interaction of the wave modes with themselves. If we let X denote $(\Theta_g, A_1, A_2, \ldots)$, then X satisfies an equation of the form

$$L \dot{X} = \varepsilon F(X,t), \tag{59}$$

where F is an almost periodic function of t and L is a linear operator with a bounded inverse. The component of F corresponding to the evolution of the geostrophic mode has Fourier exponents 0, $-\sigma_i$, $(-\sigma_i - \sigma_j)$, and the component of F corresponding to evolution of the wave mode n has Fourier exponents σ_n, $(\sigma_n - \sigma_i)$, and $(\sigma_n - \sigma_i - \sigma_j)$.

Now, it is obvious that an ordinary perturbation expansion in powers of ε will fail due to the presence of secular terms, which may also be called resonant interactions. For the geostrophic mode, the resonant interaction is due to a gg interaction and a ww interaction involving all modes i and j such that $\sigma_i + \sigma_j = 0$. For the wave mode n, the resonant interactions are a gw interaction involving the modes i such that $\sigma_i = \sigma_n$ and ww interactions involving modes i and j such that $\sigma_i + \sigma_j = \sigma_n$. The resonant interactions will cause the $O(\varepsilon)$ term in an ordinary perturbation serier to grow like a power of t. To avoid this possibility, we use the method of averaging, which has been justified both for ordinary differential equations (Bogoliubov and Mitropolsky, 1961) and for partial differential equations of elliptic or parabolic type (Khasminskii, 1963). In this method, the $O(\varepsilon^0)$ term in a perturbation expansion must satisfy

$$L \dot{X} = \varepsilon \bar{F}(X), \tag{60}$$

where

$$\bar{F}(X) = \lim_{T\to\infty} \frac{1}{T} \int_0^T F(X,t)dt, \tag{61}$$

with integration over t where it appears explicitly. Thus in the averaged equation for the evolution of X only the terms giving rise to resonant interactions appear on the right side.

Turning first to the evolution of the geostrophic mode, we see that equation (57) has the form

$$\frac{\partial^2}{\partial t\, \partial z} \iint \psi \, d\, A = -\alpha\, \varepsilon \, \{ \iint \vec{v}_g \cdot \nabla b_g \, d\, A + \sum_i [\iint \vec{v}_i \cdot \nabla b_g$$

$$+ \vec{v}_g \cdot \nabla b_i]\, A_i \, e^{-i\sigma_i t} + \frac{1}{2} \sum_{i,j} [\iint (\vec{v}_i \cdot \nabla b_j$$

$$+ \vec{v}_j \cdot \nabla b_i)] A_i A_j e^{-i(\sigma_i + \sigma_j)t} \}. \qquad (62)$$

Since $\vec{v}$ is solenoidal, it can be moved inside the gradient operator and the horizontal divergence integrates to zero, by virtue of the lateral boundary conditions. The vertical velocity $w_g = 0$, and consequently there is no gg interaction in this equation. There is a gw interaction, but it does not contribute to a secularity. The integral occuring in the ww terms takes the form

$$\iint \nabla \cdot (\vec{v}_i b_j + \vec{v}_j b_i) dA = \frac{\partial}{\partial z} \iint (w_i b_j + w_j b_i) dA, \qquad (63)$$

and substitution from (23) yields

$$\frac{\partial}{\partial z} \iint (w_i b_j + w_j b_i) dA = \frac{i}{\alpha} (\sigma_i + \sigma_j) \frac{\partial}{\partial z} N^{-2} \iint b_i b_j \, dA. \qquad (64)$$

This vanishes when $(\sigma_i + \sigma_j) = 0$, and consequently in this equation there is no resonant interaction. Therefore, the averaged equation is simply (62) with zero on the right side.

The calculation for the other equations governing the evolution of the geostrophic mode is more complicated, but the ultimate result is quite simple. The ww terms either vanish identically or have $(\sigma_i + \sigma_j)$ as a factor as in equation (64). Consequently, the averaged equations involve only gg interactions. Defining

$$\frac{D}{Dt}\Big)_g = \frac{\partial}{\partial t} + \varepsilon \vec{v}_g \cdot \nabla_1, \qquad (65)$$

it follows from use of the method of averaging that the evolution of the geostrophic mode is governed by

$$\frac{D}{Dt}\Big)_g \{ \nabla_1^2 \psi + r \beta y + \frac{r^2}{\alpha^2} \frac{\partial}{\partial z} (\frac{1}{N^2} \frac{\partial \psi}{\partial z}) \} = 0, \qquad (66)$$

to be solved subject to

$$\frac{\partial^2}{\partial t \partial z} \iint \psi \, dA = 0, \qquad (67)$$

$$\frac{D}{Dt}\Big)_g \frac{\partial \psi}{\partial z} = 0 \quad \text{at } z = -1 \qquad (68)$$

$$\frac{D}{Dt}\Big)_g \left(\frac{\partial \psi}{\partial z} + \alpha^2 N^2 \psi\right) = 0 \qquad \text{at } z = 0, \tag{69}$$

and the lateral boundary conditions. These are simply the quasi-geostrophic equations! Hence the quasi-geostrophic equations can be derived, even when wave modes are present, simply by requiring that a perturbation expansion remain uniformly valid in time.

In discussing evolution of the wave modes, we must consider the inner product (Θ_n, Θ) occurring on the right side of (53). Substitution of (34) into the definition of this inner produce leads to complicated expansions which may be simplified by considering only those terms which have nonzero average in the sense of equation (61). For equation (53), the gg terms have zero average, the gw terms have a nonzero average involving wave modes i such that $\sigma_n = \sigma_i$, and the ww terms have a nonzero average involving modes i and j such that $\sigma_i + \sigma_j = \sigma_n$. The resulting averaged equation is

$$\dot{A}_n = \varepsilon \{ \sum_i I_{in} A_i + i \sigma_n \sum_{i,j} H_{ij-n} A_i A_j \} \div (\Theta_n, \Theta_n), \tag{70}$$

where I_{in} and H_{ij-n} are functionals of Θ_g and of the eigenfunctions for the wave modes.

It may appear paradoxical that a gw resonant interaction takes place, since the geostrophic mode evolves by itself. However, this interaction leads only to phase modulation of the wave modes. The interaction coefficient for the gw interaction is

$$I_{in} = \iiint_R \{(\vec{v}_n^* \times \vec{v}_i) \cdot (\vec{\omega}_g \times \hat{m} \beta y) + \vec{v}_g \cdot (\vec{\omega}_i \times \vec{v}_n^*) +$$

$$(\vec{v}_i b_g + \vec{v}_g b_i) \cdot \nabla (b_n^*/N^2)\} dV - \iint_S p_n^* \{[2 \nabla_1 \cdot$$

$$(\vec{v}_g p_i + \vec{u}_i \psi) + \psi(\frac{\partial w_i}{\partial z} - i \sigma_i^3 p_i)]\} dA, \tag{71}$$

for $\sigma_i = \sigma_n$, and $I_{in} = 0$ otherwise. Here

$$\vec{\omega} = \nabla \times \vec{v} \tag{72}$$

is the vorticity and $\vec{u}$ is the horizontal velocity defined earlier. It can be shown that I_{in} is skew-Hermitian,

$$I_{in}^* = -I_{ni}, \tag{73}$$

and this implies that if there are no ww resonant interactions and if the geostrophic mode is independent of time, the gw interaction serves only to modify the frequencies of the wave modes. In the general case, we temporarily normalize the eigenfunctions so that $(\Theta_n, \Theta_n) = 1$ and the energy of any wave mode n is proportional to $|A_n|^2$. The rate of change of this energy due to the gw interaction is

$$\frac{d}{dt}(|A_n|^2)_{gw} = \sum_i (I_{in} A_i A_n^* + I_{in}^* A_i^* A_n), \tag{74}$$

and the sum of (74) over all modes having a common frequency σ_n vanishes, by virtue of (73). Consequently, there is no net energy transfer between the geostrophic mode and the wave modes.

The ww interaction coefficient is more easily written in terms of a symmetric expression H_{ijn}. By a change of n to -n we mean changing Θ_n to Θ_n^* and σ_n to $-\sigma_n$. Also, let P(i,j,n) denote cyclic permutation over (i,j,n). Then H_{ijn} vanishes for $\sigma_i + \sigma_j + \sigma_n \neq 0$, and otherwise is given by

$$\begin{aligned} H_{ijn} = \frac{1}{2} P(i,j,n) \{ & \frac{1}{3} (\sigma_i^2 + \sigma_j^2 + \sigma_i \sigma_j) \iint_S (p_i p_j p_n) dA \\ & - i \iiint_R [\frac{1}{\sigma_n} \frac{\partial w_n}{\partial z} \vec{u}_i \cdot \vec{u}_j + \frac{1}{\sigma_i} w_i \vec{u}_n \cdot \nabla_1 w_j \\ & + \frac{1}{\sigma_j} w_j \vec{u}_n \cdot \nabla_1 w_i \} dV. \end{aligned} \tag{75}$$

4. INTERACTIONS FOR PERIODIC WAVES

To solve the eigenvalue problem, we use the method of separation of variables. Let

$$u = -\frac{\varphi'}{k^2}(i\,\sigma\,\nabla_1 F + r\,\hat{m} \times \nabla_1 F), \qquad w = -i\,\sigma\,\varphi\,F,$$

$$b = -\alpha\,N^2\,\varphi\,F, \qquad p = [(\sigma^2 - r^2)/k^2]\varphi' F, \tag{76}$$

where $F = F(x,y)$, $\varphi = \varphi(z)$, and k is a separation constant. Then F solves

$$(\nabla_1^2 + k^2)F = 0, \tag{77}$$

with F either periodic or with $\vec{u}$ having vanishing normal component at R, and φ solves

$$\varphi''(z) + (\lambda h - k^2)F = 0, \tag{78}$$

$$\varphi(-1) = \lambda\varphi(0) - \alpha^2\varphi'(0) = 0, \tag{79}$$

where

$$\lambda = k^2\alpha^2/(\sigma^2 - r^2), \qquad h(z) = N^2(z) - (r/\alpha)^2. \tag{80}$$

Substitution of (76) into (75) gives the interaction coefficient, and solution of the eigenvalue problem determines whether resonant ww interactions can take place. Some interesting effects are produced by the presence of walls, particularly the interaction of Kelvin waves (Saylor, 1970), but we will limit ourselves to the study of periodic waves. Let $F = \exp(i\,\vec{k}\cdot\vec{x})$, where $\vec{k}$ is a horizontal wave vector. Then (77) is solved, with $k = |\vec{k}|$, and we are left with the problem of finding the vertical eigenfunctions and evaluating the interaction coefficients.

It is seen that any subscript i for a wave mode must denote four numbers, the two components of $\vec{k}$, the number of zeros of φ in $[-1,0]$, and the sign of σ. Denote dependence on $\vec{k}$ by a subscript, and let a superscript s be a positive or negative integer, with $|s|$ being the number of zeros of φ. The sign convention is that $\text{sgn}\,\sigma = \text{sgn}\,s$, with σ invariant under the change $\vec{k} \to -\vec{k}$. In addition, φ is invariant under a change of sign of either $\vec{k}$ or s, and hence

$$A^{-s}_{-\vec{k}} = (A^{s}_{\vec{k}})^* \tag{81}$$

is a reality condition. Letting subscript i denote $\vec{k}_i$, s_i, and Δ the area of a periodic cell, we have

$$H_{ij-n} = \Delta \, T^{s_i s_j -s_n}_{\vec{k}_i \vec{k}_j -\vec{k}_n} \tag{82}$$

where

$$T^{s_i s_j s_n}_{\vec{k}_i \vec{k}_j \vec{k}_n} = 0 \qquad \vec{k}_i + \vec{k}_j + \vec{k}_n \neq 0, \quad \sigma_i + \sigma_j + \sigma_n \neq 0,$$

$$= \frac{1}{2}P(i,j,n) \; \{ \frac{1}{3}(\sigma_i^2 + \sigma_j^2 + \sigma_i \, \sigma_j)\varphi_i(0) \; \varphi_n(0) -$$

$$-\int_{-1}^{0} [\frac{\varphi_i \varphi_j \varphi_n'}{k_n^2} \; (\sigma_i \, \sigma_n \, \vec{k}_i \cdot \vec{k}_n + \sigma_j \, \sigma_n \, \vec{k}_j \cdot \vec{k}_n) +$$

$$\varphi_i' \, \varphi_j' \, \varphi_n' \, (\sigma_i \, \sigma_j \, \frac{\vec{k}_i \cdot \vec{k}_j}{k_i^2 \, k_j^2} + \frac{2}{3} \, \frac{r^2}{k_i^2 \, k_j^2 \, k_n^2} \, |\vec{k}_i \times \vec{k}_j|^2) +$$

$$i \, r \, \hat{m} \cdot (\vec{k}_i - \vec{k}_j)(\sigma_i - \sigma_j) \, (\frac{\varphi_i' \, \varphi_j' \, \varphi_n'}{k_i^2 \, k_j^2} - \frac{\varphi_i \varphi_j \varphi_n'}{k_n^2})] \, dz \},$$

$$\tag{83}$$

otherwise.

Returning to the eigenvalue problem for φ, we assume that the unscaled Brunt-Väisälä frequency exceeds f_o for all z. Then h(z) is positive, and

$$\lambda = D[\varphi]/H[\varphi], \tag{84}$$

where D and H are the positive forms

$$D[\varphi] = \int_{-1}^{0} \{\varphi'^2 + k^2 \varphi^2\} \, dz,$$

$$H[\varphi] = \varphi^2(0)/\alpha^2 + \int_{-1}^{0} h \, \varphi^2 \, dz. \qquad (85)$$

It is convenient to regard λ as the eigenvalue with σ to be determined by the first of the relations (80). It is then seen that the eigenvalue problem is characterized by a variational principle, with D/H the Rayleigh quotient. The largest value of $|\sigma|$, corresponding to the least value of λ, is the frequency of the external mode, and the other frequencies are for the internal modes.

The following results can be obtained without the aid of an explicit solution.

(i) From (84),

$$|\sigma| \geq r, \text{ with equality for } k = 0. \qquad (86)$$

(ii) The phase velocity $c = |\sigma|/k$ and the group velocity $U = d\,|\sigma|/dk$ satisfy

$$\frac{dc}{dk} < -r^2/ck^3, \qquad U < c, \qquad 0 < U, \text{ for } k \neq 0. \qquad (87)$$

Since D/H is an extremum when φ is an eigenfunction, the derivative of D/H with respect to any parameter involves only the explicit appearance of the parameter. Therefore, λ is an increasing function of k, which implies the first inequality, the first implies the second, and λ/k^2 is a decreasing function of k which implies the third.

(iii) For the external mode

$$|\sigma| \geq \{r^2 + k \tanh k \, [1 + \alpha^2 \int_{-1}^{0} h \, \frac{\sinh^2[k(1+z)]}{\sinh^2 k} \, dz]\}^{\frac{1}{2}} \qquad (88)$$

This is obtained by taking $\varphi = \sinh[k(1+z)]$ as a trial function.

(iv) For the internal modes,

$$|\sigma| < \left(\frac{\alpha^2 k^2 + r^2 \Pi^2}{k^2 + \Pi^2}\right)^{\frac{1}{2}} \leq \alpha. \quad (89)$$

If h is replaced by its maximum value, $(1 - r^2/\alpha^2)$, the eigenvalues are decreased. The altered eigenvalues can be found by integrating (78) and using a graphical solution to solve a transcendental equation. The values of the altered eigenvalues imply the first inequality of (89), which implies the second.

(v) For the external mode, there exists a k_* such that

$$\frac{dU}{dk} < 0 \text{ for } k > k_*. \quad (90)$$

Equation (88) implies $\lambda/k^2 \ll 1$ when $(k \tanh k) \gg \alpha^2$. Then $\varphi = \sinh [k(1+z)]$ is the first term in an asymptotic expansion, and this implies (90). For small α, $k_* \ll 1$.

For a resonant three-wave external mode interaction to take place, the equations

$$\sigma^1_{\vec{k}_n} = \sigma^1_{\vec{k}_i} \pm \sigma^1_{\vec{k}_j}, \quad \vec{k}_n = \vec{k}_i \pm \vec{k}_j, \quad (91)$$

must be satisfied simultaneously. Using equations (87) and (90), we can rule out this possibility if the wave numbers are sufficiently large. There is no possibility of an interaction involving two internal modes and one external mode, if the wave number of the external mode is sufficiently large, as can be seen by use of (88) and (89). Interactions between three internal modes are possible, but have not been investigated here. Finally, as regards interactions between one internal mode and two external modes, it can be seen that the interaction

$$\sigma^{|s|}_{k_n} = \sigma^1_{\vec{k}_i} \pm \sigma^1_{\vec{k}_j}, \quad \vec{k}_n = \vec{k}_i \pm \vec{k}_j, \quad |s| > 1, \quad (92)$$

is impossible for the sum interaction, if k_i and k_j are sufficiently large, but is possible for the difference interaction if $k_i > k_j$, and if $(k_i - k_j)$ is sufficiently small.

This last possibility has been investigated for no rotation, $r = 0$, both for a two-layer model (Ball, 1964) and more generally (Thorpe, 1966). It is of interest to study the effect of rotation for this interaction. We note first that $(k_i - k_j)$ must be very small, since in general $\alpha \ll 1$ and the internal modes have frequencies which do not exceed α. If k_i and k_j are co-linear, then k_n is also very small, and the frequencies of the internal modes so generated is very close to the inertial frequency, r, but exceeding it slightly. If the angle between the wave vectors of the two external modes is increased, then k_n increases and for fixed s the frequency of any internal mode which can be created by this mechanism also increases.

The following case serves as an example. Consider two external modes with wave vectors $\vec{k}_o$ and $\vec{k}_1$, $k_o > k_1$, and let $A^1_{-\vec{k}} = 0$, so that these vectors are the directions of wave propagation. Also, normalize the vertical eigen-functions by taking

$$D[\varphi] = (k/\sigma)^2. \tag{93}$$

With this normalization, $(\Theta_n, \Theta_n) = 2\Delta$, and $p^1_{\vec{k}}(0) = 1$, so that the A^1's are effectively the scaled amplitudes of surface gravity waves. Letting

$$A_i = A^{|s_i|}_{\vec{k}_i}, \quad \sigma_i = \sigma^{|s_i|}_{\vec{k}_i}, \quad |s_o| = |s_1| = 1, \tag{94}$$

and

$$T = T^{1|s_2|-1}_{\vec{k}_1 \vec{k}_2 - \vec{k}_o} \tag{95}$$

the interaction equations are found to reduce to

$$\dot{A}_o = i\,\varepsilon\,\sigma_o\,T\,A_1\,A_2,$$

$$\dot{A}_1 = i\,\varepsilon\,\sigma_1\,T^*\,A_o\,A_2^*,$$

$$\dot{A}_2 = i\,\varepsilon\,\sigma_2\,T^*\,A_o\,A_1^*, \tag{96}$$

where

$$\vec{k}_1 + \vec{k}_2 = \vec{k}_o, \qquad \sigma_1 + \sigma_2 = \sigma_o. \tag{97}$$

Equation (97) was solved as follows. First, adopting the tractible but unrealistic model of constant N, the eigenvalue problem (78) was solved using a perturbation approach, for small r and α. Next, values of r, α, $\vec{k}_0$, $|s_2|$, and χ, the angle between $\vec{k}_0$ and $\vec{k}_1$, were chosen. Then (97), regarded an an equation for determining k_1, was solved numerically. The procedure was repeated for different values of χ and of the other parameters. As anticipated, the calculation showed a continuous increase of σ_2 with χ. Comparison with a seperate calculation for r = 0 showed that the effect of the earth's rotation is felt only for fairly small values of χ, that is, when the wave vectors $\vec{k}_0$ and $\vec{k}_1$ are colinear or almost so. When $\chi = 0°$, $\sigma_2 \doteq r$, as anticipated; when $|\chi| > 15°$, very little effect of r is seen.

Returning to (96), it can be seen that if A_0 and A_2 are infinitesimal, then A_1 is slowly varying, and the equations for A_0 and A_2 are effectively linear with constant coefficients. The solutions prove to be neutrally stable. However, if A_1 and A_2 are initially infinitesimal, they grow exponentially with a growth rate

$$R = (\sigma_1/\sigma_2)^{\frac{1}{2}} Q, \tag{98}$$

where

$$Q = \varepsilon \sigma_2 |A_0 T|.$$

This is in accord with a general result (Hasselman, 1967a) and indicates that a short external wave tends to lose energy to a longer external wave and an internal wave. From the numerical calculations, Q, a measure of the initial growth rate for the internal mode, proves to increase monotonically with χ, for small χ, and hence is favorable to the growth of short internal waves with frequencies large compared to the inertial frequency. The exponential growth rate, on the other hand, has a sharp local maximum at $\chi = 0$. For a variety of conditions typical for the oceanic case, the e-folding time for the growth of internal waves with inertial frequency is of the order of a few days.

5. CONCLUDING REMARKS

For application to realistic situations, the work reported above must be modified in a number of ways. The effect of variable depth must be treated and, under certain circumstances, it is necessary to provide a realistic treatment of refraction due to the sphericity of the earth. When such effects are included, the low frequency oscillations are strongly affected, and the treatment of the geostrophic mode is somewhat different. Also, it would be desirable to include higher order nonlinear effects. These modifications could be made in the context of the formalism used here, but it would be extremely difficult to solve the linear eigenvalue problem.

Much more serious is the problem of constructing a statistical theory. For motions in an unbounded fluid, with constant mean depth, this can be accomplished formally either by introducing a Gaussian approximation for the Fourier coefficients and passing to the limit of a continuous spectrum (Hasselmann, 1966) or by a formal multiple time approach in which a continuous spectrum is assumed from the start (Davidson, 1967). An important point is that the statistical theories which have appeared to date are based on the assumption that the field variables are homogeneous random functions of the horizontal spatial coordinates. This assumption is built into the multiple time approach, and appears in Hasselmann's work also, particularly in his proof of the approximately Gaussian character of the wave amplitudes in the linear case (Hasselmann, 1967b). If the field variables are not homogeneous random functions, the multiple time approach must be modified considerably or some other method must be found to effect a closure of the moment equations.

The difficulty is that for motions in bounded basins, and in particular for the longer waves, the assumption of statistical homogeneity is untenable. Though statistical homogeneity is not needed to prove the Gaussian character of linear wave fields, or rather of the wave amplitudes, one does need a mixing condition which expresses the asymptotic independence of field variables at two different points for large separation between these points (Volkonskii and Rozanov, 1959). It is hard to see how such a condition could be satisfied for long waves in a bounded basin, when the wavelength is comparable to the dimension of the basin. A similar problem arises for systematic changes in depth. For such cases, creation of a statistical theory is an important and difficult problem, as yet unsolved.

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13. ABSTRACT

Nonlinear interactions in a rotating, stratified fluid with a free surface are considered. The field variables are expanded in terms of eigenfunctions of the linearized problem, with coefficients dependent on time. Orthogonality relations are developed to obtain evolution equations for the coefficients, and these are simplified through use of the method of averaging for weakly nonlinear motions.

The geostrophic mode, which alone possesses potential vorticity, is found to obey the quasi-geostrophic equations, even though wave modes are present. The phases of the wave modes are affected by the presence of the geostrophic mode, with frequency splitting for steady geostrophic flow, but there is no energy transfer between the geostrophic mode and the wave modes. Resonant interactions between waves are found to occur for a resonant triad consisting of two external waves and one interval wave. When the wave vectors of the external waves are colinear, the internal wave generated by the interaction has a frequency very close to the inertial frequency, but exceeding it slightly. The growth rate for inertial motions generated in this manner is comparatively slow.

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