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Department of Meteorology and Oceanography

Final Report

ON THE UNDULAR BORE

Stanley J. Jacobs

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ABSTRACT

A study is made of the growth and development of the weak or undular bore. Small amplitude theory is used and the solution is made valid over the time range of interest through a variant of the method of multiple scales. The solution implies the validity of Rayleigh's result for the speed of the bore. Computed wave heights are in qualitative agreement with observations.

1. INTRODUCTION

The purpose of the present work is to study the growth and development of the weak or undular bore. This phenomenon occurs as the wave of transition between uniform streams of slightly different depth. By contrast to the strong bore, for which the transition region is narrow and vigorously turbulent, the undular bore consists of a smooth rise of water level to a crest followed by a series of undulations of gradually diminishing amplitude. Waves of depression are seldom seen, but waves of elevation as described above are a common feature on rivers subject to tidal forcing.

We will be concerned here with bores propagating into water at rest. Observations (Favre, 1935; see also Keulegan and Patterson, 1940, Sturtevant, 1965) show that in this situation the properties of the first few waves behind the crest vary slowly in time, if at all, and these waves are similar to conoidal waves. No breaking occurs if the ratio of the constant depths behind and ahead of the transition region is less than 1.28. In addition, it is found that the classical discontinuity relations (Rayleigh, 1914; Lamb, 1932, art. 187) are obeyed provided that data pertaining to the uniform streams is used in the calculation.

It might be thought useful to base a theoretical treatment on the assumption of steady flow in a suitably moving reference frame, but the observations argue against this. For, for a steady state to exist, energy transported into the transition region must be dissipated, and in the absence of breaking the dissipation must take place in boundary layers at the channel bottom and sides and perhaps at the surface; and a short calculation, based on Rayleigh's result for the rate of energy flux and on the assumption of no appreciable convection of vorticity away from the boundaries indicates that, except for very large times, there is a net accumulation of energy in the transition region. It follows that a time-dependent theory is needed.

Meyer, in a recent paper (Meyer, 1967), makes roughly the same criticism of steady-state theories. He neglects viscous effects entirely and shows that for small amplitude waves the motion is characterized by the magnitude of a dimensionless parameter $\tau = (\epsilon^3 g/H)^{1/2}\Delta t$, where H is the mean depth, ϵH a measure of the wave amplitude, g the gravitational constant, and Δt the time scale of the motion. He finds that for $\tau \ll 1$ linear theory is valid, for $\tau \doteq 1$ the waves are approximately conoidal, and for $\tau \gg 1$ the waves are similar to solitary waves. His analysis is valid only near the head wave, and consequently a complete solution valid over the entire flow domain is not obtained.

The aim of the present investigation is to supply such a solution for a properly posed initial value problem. A variant of the method of multiple scales is employed, and as a result the solution is not valid for all time; it is believed, however, that it is useful over the time range in which conoidal waves develop. The main deductions of Meyer's work are confirmed and reasonable agreement with observations is obtained. However, one important goal is not achieved, namely an estimate for the value of ϵ at which breaking occurs. This is an object for future study.

2. FORMULATION

Let (x^*,z^*) be rectangular coordinates with the x^*-axis horizontal and with the z^*-axis vertically upwards. Let $z^* = \zeta^*(x^*,t^*)$, where t^* denotes time, be the equation of the free surface, $z^* = -H$ the equation of the rigid bottom, and g the gravitational constant. We assume two-dimensional irrotational flow with the velocity $\vec{v}^* = (u^*,w^*)$ given by the gradient of a potential φ^*, and scale the variables according to

$$\zeta^* = H\zeta \;, \quad \vec{v}^* = (gH)^{1/2}\vec{v} \;, \quad \varphi^* = (gH^3)^{1/2}\varphi \;,$$
$$(x^*,z^*) = H(x,z) \;, \quad \text{and} \quad t^* = (H/g)^{1/2}t \;. \tag{1}$$

Now set $\zeta = \epsilon\eta$, $\varphi = \epsilon\psi$, with $\epsilon \ll 1$. Then $\vec{v} = (u,w) = \epsilon\nabla\psi$ is the dimensionless velocity, $h = 1 + \epsilon\eta$ is the dimensionless depth, and, with surface tension and viscous effects neglected, ψ and η are determined by

$$\nabla^2\psi = 0 \quad \text{for} \quad -1 < z < \epsilon\eta \;, \tag{2.a}$$

$$\psi_z = 0 \quad \text{at} \quad z = -1 \;, \tag{2.b}$$

$$\left.\begin{aligned} \eta_t - \psi_z - \epsilon\nabla\psi\cdot\nabla\eta &= 0 \;, \\ \psi_t + \eta + \frac{1}{2}\epsilon\nabla\psi\cdot\nabla\psi &= 0 \;, \end{aligned}\right\} \text{at} \quad z = \epsilon\eta \;, \tag{2.c}$$

and initial conditions.

The initial conditions are such that

$$\eta(x,0) \to -\frac{1}{2}\,\mathrm{sgn}x \quad , \qquad \psi_x(x,z,0) \to \frac{1}{2}\,\beta(1 - \mathrm{sgn}x) \quad ,$$

as $|x| \to \infty$, where β is a constant of order unity. It is convenient to introduce new variables X and Ψ through

$$\Psi \;=\; \psi - \frac{1}{2}\,\beta x + \frac{1}{8}\,\epsilon\beta^2 t \quad , \qquad X \;=\; x - \frac{1}{2}\,\epsilon\beta t \quad , \tag{3}$$

in order that Fourier transforms of the dependent variables have only algebraic singularities. The equations (2) are invariant under the transformation (3), and

$$\eta(X,0) \to -\frac{1}{2}\,\mathrm{sgn}X \quad , \qquad \Psi_X(X,z,0) \to -\frac{1}{2}\,\beta\,\mathrm{sgn}X \tag{4}$$

as $|X| \to \infty$. Thus, if k is the Fourier transform variable, the transforms of η and Ψ are singular at the origin like k^{-1} and k^{-2}, respectively, but are otherwise expected to be ordinary rather than generalized functions.

3. ANALYSIS

Transferring the upper boundary conditions to $z = 0$ yields

$$\left.\begin{aligned} \Psi_z &= \eta_t + \epsilon(\eta\Psi_X)_X + O(\epsilon^2) \quad , \\ -\Psi_t &= \eta + \epsilon(\eta\Psi_{tz} + \frac{1}{2}\nabla\Psi\cdot\nabla\Psi) + O(\epsilon^2) \quad , \end{aligned}\right\} \text{ at } \quad z = 0 \quad , \qquad (5)$$

and we write

$$\eta(X,t) = i \sum_s \int_{-\infty}^{\infty} a_k^s \, e^{i(kX+s\omega t)} \, k^{-1} dk \quad , \qquad (6.a)$$

and

$$\Psi(X,z,t) = - \sum_s s \int_{-\infty}^{\infty} a_k^s \, \frac{\cosh[k(z+1)]}{\omega k \cosh k} \, e^{i(kX+s\omega t)} \, dk \quad , \qquad (6.b)$$

where s is a sign parameter, equal to +1 and -1 in the summation, $a_k^s = a^s(k,t)$, $\omega = \omega(k) = \Omega(k,1)$, and

$$\Omega(k,h) = k(\tanh kh/k)^{1/2} \quad , \qquad (7)$$

with positive square root. Laplace's equation and the boundary condition at $z = 1$ are satisfied by (6), and (5) and the initial conditions determine a_k^s. Substitution of (6) into (5) gives the equations obeyed by a_k^s,

$$\dot{a}_k^s = \frac{1}{2} \epsilon k \sum_{s_1,s_2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{k_1^{-1} F_{k_1k_2k}^{s_1s_2s} \, a_{k_1}^{s_1} \, a_{k_2}^{s_2} \, e^{it(s_1\omega_1+s_2\omega_2-s\omega)} \delta(k_1 + k_2 - k)\}$$

$$\times \, dk_1 dk_2 + O(\epsilon^2) \quad , \qquad (8)$$

where an overdot indicates differentiation with respect to time, s_1 and s_2 are additional sign parameters, $\omega_\nu = \omega(k_\nu)$, and the interaction coefficients are

$$F^{s_1 s_2 s}_{k_1 k_2 k} = s \frac{\omega}{k}[\omega_1^2 + \omega_2^2 + s_1 s_2(\omega_1\omega_2 - \frac{k_1 k_2}{\omega_1\omega_2})] - (s_1 \frac{\omega_1}{k_1} + s_2 \frac{\omega_2}{k_2}) \quad . \qquad (9)$$

The linear solution, obtained by setting $\epsilon = 0$ in (8), is $a^s_k = a^s(k,0)$. As might be expected, a perturbation series solution leads to secular terms. For initial disturbances of finite extent $a^s(0,0) = 0$ and the secularity occurs at a high order in the expansion. Here, however,

$$a^s(0,0) = (1 - s\beta)/4\pi \quad , \qquad (10)$$

which follows from (4) and the rule

$$\lim_{N\to\infty} \frac{e^{iNg(k)}}{k} = i\pi \,\mathrm{sgn}[g'(0)]e^{iNg(0)}\delta(k) \quad , \qquad (11)$$

and the integrands in (8) are singular at $k_1 = 0$. Application of (11) yields

$$k_1^{-1}\, e^{it(s_1\omega_1+s_2\omega_2-s\omega)} \to i\pi \,\mathrm{sgn}[s_1 - s_2\omega']e^{it(s_2-s)\omega}\delta(k_1)$$

as $t \to \infty$ with k fixed, and the terms with $s_2 = s$ give rise to secularities in the first iteration on the linear solution. In wave interaction language this is a resonance, and the linear solution is valid only in the time range $\epsilon t \ll 1$.

In order to extend the time interval in which the lowest order approximation is useful we retain in the equations of this approximation the small terms which give rise to secularities. The other small terms will be neglected. We note that the expression

$$\{F^{s_1ss}_{k_1k_2k}\, a^{s_1}_{k_1}\, a^{s}_{k_2} - F^{s_1ss}_{okk}\, a^{s_1}_{o}\, a^{s}_{k}\}$$

is $o(k_1)$ at $k_1 = 0$ and therefore innocuous. The terms with $s_2 \neq s$ also do not give rise to secularities. Retaining only the secular terms on the right side of (8), and noting that these terms may be taken out of the integrals, we obtain as the equations for the lowest order approximation

$$\dot{a}^s_k = -\frac{1}{2}\, i\epsilon\pi k(\sigma a^s_o I^s + \tau a^{-s}_o J^s) a^s_k \quad , \tag{12}$$

where, with an obvious notation,

$$I^s(k,t) = is\, \pi^{-1} \int_{k_1+k_2=k} k_1^{-1}\, e^{-ist(\omega-\omega_1-\omega_2)} \quad , \tag{13.a}$$

$$J^s(k,t) = -\, is\, \pi^{-1} \int_{k_1+k_2=k} k_1^{-1}\, e^{-ist(\omega+\omega_1-\omega_2)} \quad , \tag{13.b}$$

and

$$\sigma(k) = 2 + k\,\mathrm{sech}^2 k/\omega \quad , \qquad \tau(k) = 2 - k\,\mathrm{sech}^2 k/\omega \quad . \tag{14}$$

Further simplification is possible, for, for the purpose at hand, I^s and J^s may be replaced by their limits as $t \to \infty$, as is done when the method of multiple scales is used (c.f. Benney and Saffman, 1966). In a naive application of the method of multiple scales we would replace I^s and J^s by 1, the limit as $t \to \infty$ with k fixed. But, though $J^s \to 1$ as $t \to \infty$, uniformly in k, the limit process for I^s is not uniform; $I^s(k,\infty) = 1$ for $k \neq 0$, $I^s(0,t) = 0$. It can be shown, however, that as $t \to \infty$ $I^s \to \tilde{I}^s(k,t)$, uniformly in k, where $\tilde{I}^s(k,t)$ is a tabulated function. We note also that a^+_o and a^-_o are independent of t, since the right side of (12) vanishes when $k = 0$, so a^s_o may be replaced by $(1 - s\beta)/4\pi$, from (10). Thus the equations (12) are linear and the solution is

$$a^s(k,t) = a^s(k,0) \exp\{- \frac{1}{8} i\epsilon k[\tau(1 + s\beta)t + \sigma(1 - s\beta) \int_0^t \tilde{I}^s \, dt]\} \quad . \tag{15}$$

Substitution back into (6) then gives the solution for η and Ψ.

To obtain $\tilde{I}^s$ we need consider only $I \equiv I^-$ and $\tilde{I} \equiv \tilde{I}^-$, since $I^+ = (I^-)^*$, the complex conjugate. Set $k_1 = k(\lambda + \frac{1}{2})$ in the integral defining I. After a little algebra we get

$$I = - i(\mathrm{sgn}k)\pi^{-1} \int_0^\infty (\lambda^2 - \tfrac{1}{4})^{-1} e^{it\rho(k,\lambda)} d\lambda \quad , \tag{16}$$

where

$$\rho(k,\lambda) = \omega(k) - \omega[k(\tfrac{1}{2} + \lambda)] - \omega[k(\tfrac{1}{2} - \lambda)] \quad . \tag{17}$$

$\tilde{I}$ is the right side of (16) with ρ replaced by $\tilde{\rho}$, the sum of the first four terms of an expansion of ρ in powers of k. We have

$$\tilde{\rho} = \frac{1}{2} k^3(\lambda^2 - \tfrac{1}{4}) \tag{18}$$

and we want to show that $E = I - \tilde{I} \to 0$ as $t \to \infty$, uniformly in k.

Evidently both I and $\tilde{I}$ approach 1 as $t \to \infty$ in limit A, with k fixed, so $E \to 0$ in this limit. To treat limit B, $t \to \infty$, $k \to 0$, with k^3t bounded, we write

$$E = - i(\mathrm{sgn}k)\pi^{-1} \left\{ \int_0^n \frac{e^{it\rho} - e^{it\tilde{\rho}}}{\lambda^2 - \frac{1}{4}} d\lambda + \int_n^\infty \frac{e^{it\rho} - e^{it\tilde{\rho}}}{\lambda^2 - \frac{1}{4}} d\lambda \right\} \tag{19}$$

where n is greater than 1/2. We note that for any finite λ the integrands are o(1) in limit B, and that

$$\rho - \tilde{\rho} = \frac{1}{24} k^s \omega^{(V)} (\Theta k)(\lambda^2 - \frac{1}{4})(\lambda^2 + \frac{3}{4}) \quad , \tag{20}$$

$0 \leq \Theta \leq 1$, which follows from the remainder formula for Taylor series.

Now the second integral is o(1) in limit B, by dominated convergence. As regards the first integral, we note that since both ρ and $\tilde{\rho}$ vanish at $\lambda = \frac{1}{2}$,

$$\left|\int_0^n\right| \leq \int_0^n \left|\frac{e^{it\rho} - e^{it\tilde{\rho}}}{\lambda^2 - \frac{1}{4}}\right| d\lambda \leq t \int_0^n \left|\frac{\rho - \tilde{\rho}}{\lambda^2 - \frac{1}{4}}\right| d\lambda \quad ,$$

upon use of the inequality $|e^{i\Theta_1} - e^{i\Theta_2}| \leq |\Theta_1 - \Theta_2|$, and substitution from (20) shows that the integral is $O(|k|^5 t)$ which is $O(t^{-2/3})$ in limit B. Thus $E \to 0$ as $t \to \infty$, uniformly in k.

It is easily shown that

$$\tilde{I} \equiv \tilde{I}^- = \operatorname{erf}[e^{i\pi/4} t^{1/2}(k/2)^{3/2}] \tag{21}$$

with a branch cut in the k plane from k = 0 to k = ∞ in the lower half plane. Also,

$$\Lambda(k,t) \equiv \Lambda^-(k,t) = \int_0^t \tilde{I}^-(k,t')dt' = \frac{8}{k^3} \{(\frac{1}{2} i + \frac{k^3 t}{8}) \tilde{I}^-(k,t)$$
$$+ \frac{e^{-i\pi/4}}{\sqrt{\pi}} t^{1/2}(k/2)^{3/2} \exp(-ik^3 t/8)\} \quad , \tag{22}$$

so the a_k^s are expressible in terms of tabulated functions. For future reference we note that $\Lambda(-k,t) = \Lambda^*(k,t)$, $\Lambda(0,t) = 0$, and $\Lambda \to t$ in limit A, $t \to \infty$ with k fixed.

Now let $b^s(k) = a^s(k,0)$ and let

$$\Theta_s = kx + s\omega t - \epsilon k[\frac{1}{2}\beta t + \frac{1}{8}\tau(1 + s\beta)t + \frac{1}{8}\sigma(1 - s\beta)\Lambda(-sk,t)] \quad . \tag{23}$$

Then, from (3), (6), and the preceding analysis,

$$\eta = i \sum_s \int_{-\infty}^{\infty} b^s(k)\, e^{i\Theta_s} k^{-1} dk \quad , \tag{24.a}$$

and, apart from a function of t,

$$\psi = \frac{1}{2}\beta x - \sum_s s \int_{-\infty}^{\infty} b^s(k)\, e^{i\Theta_s} \frac{\cosh[k(z+1)]}{\omega k \cosh k} dk \quad , \tag{24.b}$$

and this is the final form for the solution.

4. DEDUCTIONS FROM THE THEORY

It is instructive to consider the linear theory, obtained by setting $\epsilon = 0$, and this will be done first. For large $|x|$ and t asymptotic approximations are readily obtained. Contributions come from the singularities at $k = 0$ and from the stationary points of Θ_s, with $\epsilon = 0$. Except for the case $|x| \doteq t$, which will be discussed subsequently, the stationary points are well separated from $k = 0$ and from each other, contributions from the singularities are dominant, and we obtain

$$\eta \sim -\frac{1}{2} , \quad U \sim 0 , \qquad \text{for} \quad x > t ; \tag{25.a}$$

$$\eta \sim \frac{1}{2}\beta , \quad U \sim \frac{1}{2}\epsilon(1+\beta) , \qquad \text{for} \quad |x| < t ; \tag{25.b}$$

$$\eta \sim \frac{1}{2} , \quad U \sim \epsilon\beta \qquad \text{for} \quad x < -t . \tag{25.c}$$

For $|x| \doteq t$ we use the method of Chester et al. (1957) with an adaptation to take account of the singularities at $k = 0$. For x positive the result for η is

$$\eta \sim -\frac{1}{4}(1-\beta) + \pi[b^{-}(m) + b^{-}(-m)][\int_0^{\alpha} Ai(-\lambda)d\lambda - \frac{1}{6}] , \tag{26}$$

where

$$m = [2(1 - x/t)]^{1/2} , \quad \alpha = (t - x)(2/t)^{1/3} , \tag{27}$$

and some approximations have been made based on m being small. A further approximation,

$$b^-(\pm m) \doteq b^-(0) = (1 + \beta)/4\pi ,$$

yields

$$\eta \sim \frac{1}{2}\beta + \frac{1}{2}(1 + \beta)[\int_0^\alpha Ai(-\lambda)d\lambda - \frac{2}{3}] . \tag{28}$$

Obviously ψ and $u = \psi_x$ can be found in the same way, and corresponding results can be found for $x \doteq -t$.

For α large and negative η reduces to -1/2, in agreement with (25.a); for α positive the integral in (28) is oscillatory. It attains a maximum at $\alpha = 2.3$ with value 0.94, a minimum at $\alpha = 4.1$ with value 0.48, and continues to oscillate with increasing α, the maxima and minima converging to 2/3 as $\alpha \to \infty$. The absolute maximum is at the first crest, and the local wavelength is proportional to $t^{1/3}$.

Evidently the linear solution is in qualitative agreement with observations. Quantitatively the agreement is not so good. Rayleigh's formula for the speed of the head wave would give, upon substitution from (25),

$$c_R = \{[1 + \frac{1}{2}\epsilon\beta][1 + \frac{1}{4}\epsilon(\beta - 1)][1 - \frac{1}{2}\epsilon]^{-1}\}^{1/2} = 1 + \epsilon(1 + 3\beta)/8 + O(\epsilon^2) \tag{29.a}$$

for the speed of the wave progressing to the right, and

$$c_L = \{[1 + \frac{1}{2}\epsilon\beta][1 + \frac{1}{4}\epsilon(\beta + 1)][1 + \frac{1}{2}\epsilon]^{-1}\}^{1/2} - \epsilon\beta = 1 - \epsilon(1 + 5\beta)/8 + O(\epsilon^2) \tag{29.b}$$

for the speed of the wave progressing to the left, and this result is known to be in agreement with observations; the linear theory, by contrast, gives

1 ($(gH)^{1/2}$ in dimensional units) for the speeds c_R and c_L. Also, though it has not been measured, the dispersion relation for waves far from the head waves would no doubt prove to be

$$\omega_R = \Omega(k, 1 + \frac{1}{2}\epsilon\beta) + \frac{1}{2}\epsilon(1 + \beta)k$$

$$= \omega(k) + \frac{1}{2}\epsilon(1 + \beta)k + \frac{1}{4}\epsilon\beta k^2 \operatorname{sech}^2 k/\omega + O(\epsilon^2) \qquad (30.a)$$

for the wave progressing to the right, and for the other wave

$$\omega_L = \omega(k) - \frac{1}{2}\epsilon(1 + \beta)k + \frac{1}{4}\epsilon\beta k^2 \operatorname{sech}^2 k/\omega + O(\epsilon^2) \quad , \qquad (30.b)$$

since the mean depth in $|x| < t$ is $1 + \frac{1}{2}\epsilon\beta$ and the mean velocity is $\frac{1}{2}\epsilon(1 + \beta)$. Finally, the wave amplitude is underestimated by the linear theory. The ratio r, of the elevation of the first crest above the greater of the uniform depths to the depth difference, is 0.27, from (25) and (28). Values of r as low as 0.3 are reported in the literature, but the range of values is between 0.3 and 1.

The reasons for the failure of the linear theory are these. Far from the lines $|x| = t$ the mean depth and mean horizontal velocity are not 1 and 0, as is assumed in the calculation of the frequency ω. This induces a secularity in the first iteration on the linear solution, the secularity being of the type which can be suppressed through use of the Stokes expansion. Near the head waves, on the other hand, the local wave number is proportional to $t^{-1/3}$, the amplitude to ϵ, and the quantity (wave amplitude) $\div$ (wave number)^2, which must be small for the Stokes expansion to be useful, is proportional to $\epsilon t^{2/3}$ and is not small for sufficiently large t. It is this lack of validity of the

Stokes expansion which makes necessary the elaborate treatment of the integral I and the distinction between limits A and B. Limit A, $t \to \infty$ with k fixed, pertains to the Stokes expansion, and limit B, $t \to \infty$ with k^3t bounded, pertains to the flow near $|x| = t$ where the Stokes expansion is invalid.

We return now to (24) and carry out an asymptotic analysis similar to that given for the solution of the linear problem. Consider first the case $(1 - |x|/t)$ not too small. At $k = 0$

$$\frac{d}{dk}\Theta_+ = x + t[1 - \epsilon(1 + 5\beta)/8] \quad ,$$
$$\frac{d}{dk}\Theta_- = x - t[1 + \epsilon(1 + 3\beta)/8] \quad , \tag{31}$$

from the definition of τ and the fact that $\Lambda(0,t) = 0$, and the stationary points are well separated from $k = 0$ and may be calculated by replacing Λ by t, its value in limit A. It follows that except near the lines $|x| = t$

$$\eta \sim -\frac{1}{4}\{(1 - \beta)\ \text{sgn}[x + t(1 - \epsilon(1 + 5\beta)/8)] + (1 + \beta)\ \text{sgn}[x - t(1 + 3\beta)/8)]\} \tag{32}$$

plus contributions from the stationary points of

$$kx + t[\omega - \frac{1}{2}\epsilon(1 + \beta)k + \frac{1}{4}\epsilon k^2 \text{sech}^2 k/\omega]$$

and (33)

$$kx - t[\omega + \frac{1}{2}\epsilon(1 + \beta)k + \frac{1}{4}\epsilon k^2 \text{sech}^2 k/\omega]$$

The speeds c_R and c_L given by (32) and the local frequencies deduced from (33) are in agreement with (29) and (30). This serves as a check on the calculation, and in addition indicates that there is a net flux of energy into the transition region for a wave of elevation and out of it for a wave of depression, by Rayleigh's theory.

An asymptotic evaluation for $|x| \doteq t$ can be made, using the methods which led to equation (26), when $\epsilon t^{2/3} \ll 1$. The major result is that when $x \doteq t$ the wave amplitude has a fractional rate of increase proportional to $(1 + \beta)$, and when $x \doteq -t$ a fractional rate of increase proportional to $(\beta - 1)$. Thus, at least initially, the energy flux into the transition region causes a change in amplitude of the waves, a growth of the wave amplitude for a wave of elevation and a decay for a wave of depression. The factors $(1 + \beta)$ and $(\beta - 1)$ are consistent with (32), which serves as another check.

When $\epsilon t^{2/3} \geq 1$ the asymptotic expansion of Chester _et al._ is no longer useful, and for this case numerical evaluation of (24) is desirable. The results of a number of these integrations are given in Table 1. For case I the initial form for η is

$$\eta(x,0) = -\frac{1}{2}\tanh(\pi x/2)$$

and the motion is started from rest. For case II the same initial form for η is used, but the initial distribution for ψ is chosen to make $\overset{+}{b}(k) = 0$, corresponding to a wave in one direction only. The quantity r was defined previously. The depths of the uniform streams ahead of and behind the transition region are denoted by h_1 and h_2, respectively, and s is the horizontal distance from the crest of maximum elevation to the following crest, divided by h_1. A typical wave profile is shown in Figure 1. The forerunner ahead of the main crest is to be noted. This is apparently not observed and is not found in Peregrine's recent numerical solution of the long wave equations (Peregrine, 1966); its existence here must be attributed to some deficiency in the theory, the origin of which is unknown.

TABLE 1

RESULTS

			Nonlinear Theory		Linear Theory	
Case	h_2/h_1	t	r	s	r	s
I	1.143	50	0.34	8.0	0.27	9.9
I	1.143	100	0.46	11.4	0.27	12.5
II	1.133	50	0.45	8.6	0.27	10.6
II	1.133	100	0.50	10.7	0.27	13.4

One set of observations, made by Bazin (1865), is quoted in the article by Keulegan and Patterson. For the depth ratios in Table 1 Bazin finds values of r ranging from 0.3 to 0.5, the scatter being due to the fact that measurements were made at different points in the channel. Bazin's results agree nicely with the theory. Favre's observations, on the other hand, give a larger value for r, 0.87 for his run 22, with depth ratio 1.14. Favre also measured s, and for this experiment found it to be 9.4. It is believed that the discrepancy between the theory and Favre's value for r is due to the relatively small value of t used here; t = 100, for example, corresponds to the head wave having progressed only 10 meters down Favre's 74-meter channel, for the depth of water used in his run 22.

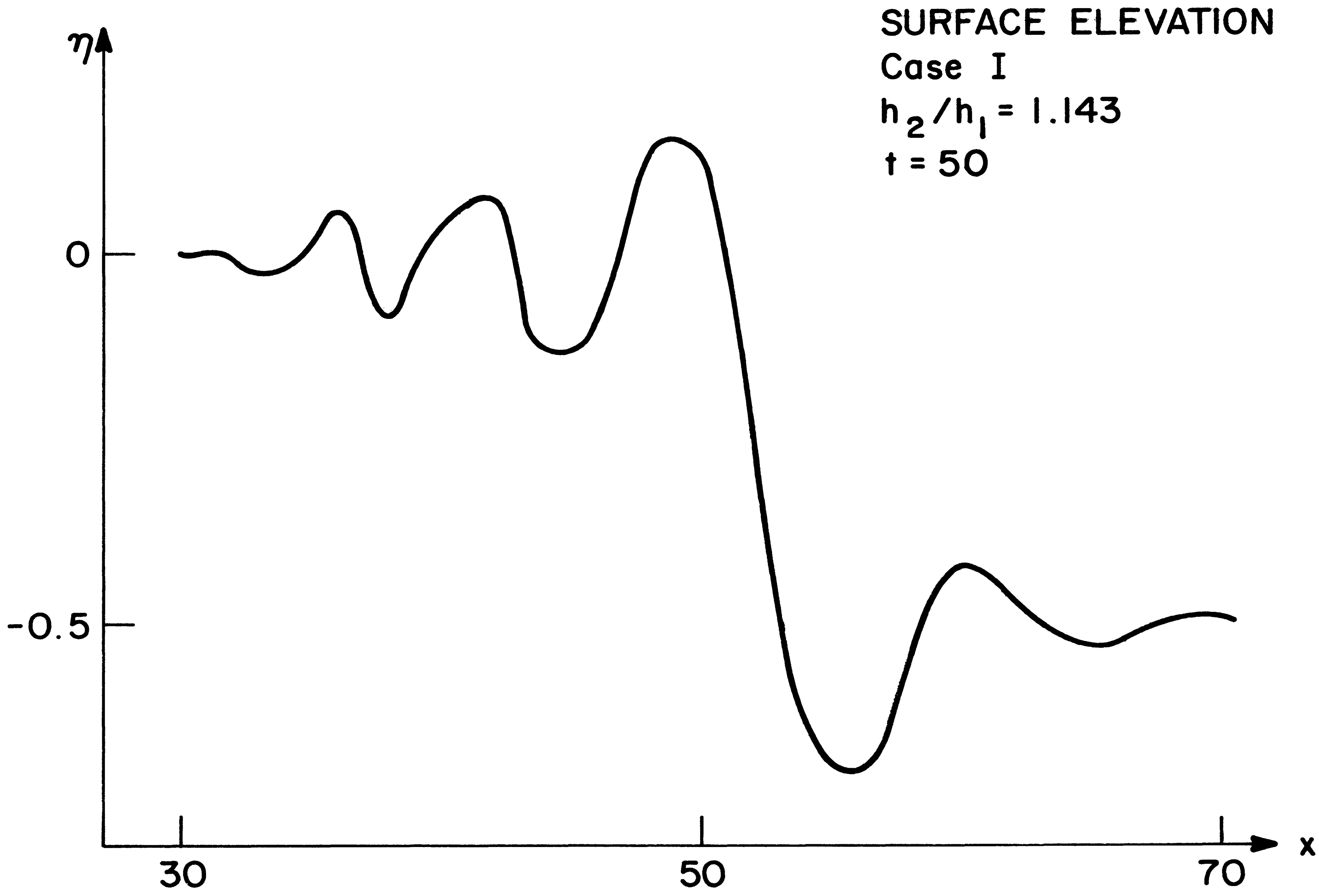

Figure 1. Surface elevation.

Numerical evaluations of (24) were also made for depth ratios exceeding 1.28, the ratio at which breaking should commence. The wave amplitudes for this case are very large, with $r > 4$ for $t = 100$, and the wave profile is wildly oscillatory. Whether this is to be interpreted as indicating breaking or merely application of the theory beyond its range of validity is a moot point.

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