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Published by the Industrial Engineering Department of The University of Michigan

Mailing Address: The University of Michigan
Department of Industrial Engineering
231 West Engineering Building
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Ann Arbor, Michigan

"PERIODIC REPORT OF INDUSTRIAL ENGINEERING RESEARCH"
at The University of Michigan

This Periodic Report of Industrial Engineering Research is offered as a medium for the rapid publication of articles of industrial engineering research-in-progress and of bibliographies with potential use in such research. The justification for this new publication is based on a number of apparent needs. Faculty and students have frequently expressed concern for their difficulty in keeping informed about the current activities of their colleagues. It is hoped that they will find the report helpful in reducing this difficulty. The active researcher, it is hoped, will use the report freely as a means for inviting suggestions on work which may be still in progress. A means of easy communication should stimulate others to undertake research in areas of their own interest. The availability of a medium for publication of unfinished work should encourage preparation of progress reports and thus furnish the writer with early experience in recording his results for publication.

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Ann Arbor, Michigan

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A PROBLEM IN ESTIMATING A SAMPLE PARAMETER

By R. L. Patterson

Consider a denumerable population $\mathcal{N}$, composed of two subpopulations $\mathcal{W}$ and $\mathcal{B}$, $\mathcal{N} = \mathcal{W} \cup \mathcal{B}$. A set of observations are made on a sample of size N from $\mathcal{N}$, which consist of "decisions" as to whether or not each element came from $\mathcal{W}$ or $\mathcal{B}$. The observations are subject to the following two types of errors.

1. An element drawn from $\mathcal{N}$ is believed to have come from $\mathcal{B}$ when it came from $\mathcal{W}$.
2. An element is drawn from $\mathcal{N}$ and is believed to have come from $\mathcal{W}$ when it actually came from $\mathcal{B}$.

The problem is to make accurate statements about the fraction of the sample of size N which came from $\mathcal{W}$ so that further inferences may be made concerning $\mathcal{N}, \mathcal{W}, \mathcal{B}$. The purpose of this paper is to suggest a method of describing what is known about this fraction in a way which is useful for making further inferences about the ratio of the sizes of $\mathcal{W}$ and $\mathcal{B}$.

Sampling problems of this nature abound in the social sciences. For example, an estimate is made of the number of voters who will vote for L.B.J. in the next presidential election. A poll is conducted. Some of the individuals who are included in the sample state that they are going to vote for L.B.J. and then they either will not vote at all or else will vote for the opponent. Some of the individuals say that they will not vote for L.B.J. and when election day rolls around, they will vote for L.B.J. Before the prediction of the percentage of the vote expected to go to L.B.J. is made, this sample must be adjusted to compensate for these two types of errors. There are various methods available for adjusting the fraction of the sample in the above example who say they will vote for L.B.J. to bring it closer to the fraction in the sample who actually do vote for L.B.J. on election day. Some of these techniques involve selecting sets of population characteristics (Bi,..,Bk) for which

(a) no mistake will be made in deciding whether or not an individual possesses characteristic Bi, and

(b) each Bi is correlated to the way in which the individual will actually vote on election day.

Other examples of extracting samples which are subject to the errors (1) and (2) can easily be drawn from the fields of product merchandising, industrial quality-control, and reliability analysis. To cite one more example, consider the problem of sampling batches of items to determine the fraction of the lot defective. On the basis of this sample, a series of decisions may be made which involves large sums of money. Suppose that the inspector mistakenly calls some of the good items "defective" and calls some of the defective items "good".

The result is that the sample fraction defective is in error. What are the likely consequences? In this example, unlike the previous one, it may be undesirable or even impossible to attempt to perform a correlation analysis between some set of "indicators" which are related to whether or not an item is defective, and which can positively be observed to be either "yes" or "no".

In fact, it can be argued that almost no observation is compeletely free of error. In practice, the accepted methodology is to

(a) ignore the slight probability of "calling a spade a heart or a heart a spade" if it is low enough, or

(b) estimate the fraction of the pupulation which contains property "α" by statistical techniques such as observing additional variables Bi..,Bk for which the probability of misidentification may be ignored, and correlating the Bi with α.

There are, however, a set of probability distributions which, if known and tabulated, could aid in estimating the fraction of a population which possesses a particular set of characteristics. They might be more useful in certain situations than other statistical techniques of parameter estimation.

These distributions will be identified by first hypothesizing a sampling situation and then asking a series of questions related to the sampling problem.

Consider two denumerable populations $\mathcal{W}$ and $\mathcal{B}$. $\mathcal{W} \cup \mathcal{B} = \mathcal{N}$. Extract samples of size W and B from $\mathcal{W}$ and $\mathcal{B}$, respectively. Let W + B = N.

Of the sample of size W, T are believed to come from $\mathcal{W}$ and W-T are believed to come from $\mathcal{B}$. Of the sample of size B, F are believed to come from $\mathcal{W}$ and B-F are believed to come from $\mathcal{B}$. Suppose there exist probability distributions

$$P\ (T = t)\ ,\ t = 0,1,\ldots,W \quad \left(\text{e.g. } P(T = t) = \binom{W}{t} P_1^{\,t}\ (1-p_1)^{W-t}\right)$$

$$\text{and } Q(F = f),\ f = 0,1,\ldots,B \quad \left(\text{e.g. } Q(F=f) = \binom{B}{f} P_2^{\,f}\ (1-p_2)^{B-f}\right)$$

which describe these errors of identification.*

Let D be the number of elements which are "detected" to have come from $\mathcal{W}$. The D = T + F and D has a probability distribution

$$R\ (D = d) = \sum_{t+F=d} P\ (T = t)\ Q(F = d - t \mid T = t).$$

---

* P_1 is the probability that an element of W is believed to come from $\mathcal{W}$, P_2 is the probability that an element of B is believed to come from $\mathcal{B}$.

A series of questions may now be asked.

1. How many items from $\mathcal{W}$ and how many from $\mathcal{B}$ must be drawn to make D = d? i.e., what is the joint distribution of W and B implied by D = d?

2. How many items in all must be drawn to make D = d? i.e., if N = W + B, how large must N be to make D = d?

3. What is the difference in the conclusions if
   (a) a probability distribution of the proportion of B's to W's is assumed.
   (b) a certain proportion of B's to W's is assumed. Is it necessary to postulate (a) or (b) to answer questions 1 and 2?

4. Given a fixed sample size N= n and D = d, what is the distribution of W?

5. What fraction of $\mathcal{N}$ is $\mathcal{W}$? (Limit of a Bernoulli sequence)

For example, the answer to question (2) is that N follows the negative binominal distribution providing

(a) P(T = W) = 1
(b) P(F = 0) = 1
(c) A certain proportion of the elements of   are assumed to consist of elements from .

The writer has recently been working on an Arms Control Verifications Requirements project in which it was necessary to estimate the number of nuclear tests which had been conducted in a time interval (0,Td) which extended up to the instant that the dth "detection" occurred. A "detection" could occur in two ways: (a) a hypothetical inspection system could detect a test, if it occured, with a constant probability p. (b) a stream of "false alarms" occur according to some "known" distribution, e.g., a Poisson distribution with parameter β. The answer to this question is that, under the above assumptions, the number, k, of tests conducted up to the dth detection is distributed according to the negative binomial distribution when β=0. If β>0, the problem becomes somewhat more difficult. Dr. Wyman Richardson and the writer showed that when β>0, the number of tests conducted in the interval $(0, T_d)$ was distributed according to the distribution Q(k/d) =

$$\frac{\sum\limits_{D \geq d} \sum\limits_{i=0}^{d} P_i \left(\gamma_{D-i,k} - \gamma_{D-i,\,k-1} \right)}{\sum\limits_{i=0}^{d} P_i}$$

where

$$\gamma_{ik} = \binom{k}{i} p^i (1-p)^{i-i} \qquad \text{and}$$

$$P_j = \frac{e^{-\beta} B^j}{j!}$$

To recapitulate, assume that a population N is composed of two subpopulations W and B. A sample of size N is drawn from N, and D elements are "detected" to come from W and N - D from B. Instead of knowing the value of W, the number of elements in N which were drawn from W exactly, the number is known only to within a probability distribution. There are two approaches to the problem of making inferences about N and W.

1. Make observations on the sample drawn from N for which the error of observation if "virtually" zero, and from these observations, use correlation and regression analysis to make point estimates of the fraction of the sample, which came from W. With this point estimate in hand and its associated "confidence" level, inferences are then drawn regarding the composition of N.

2. Assuming the error distributions describing the likelihood of misidentifying elements from W and B, derive the distribution of the number of elements in the sample which came from W. Proceed to make inferences about the composition of N based upon this distribution. If there is no possibility of misidentifying elements from W and B, then the distribution degenerates to a "point" distribution as in the case of an inspector drawing k "defectives" out of a sample of size N and assumes that he can always state the fraction K/N perfectly. In this example the subpopulation B is "defectives" and W is "nondefectives".

It is suggested that the distributions mentioned earlier in the paper be derived where possible and tables be tabulated for a variety of cases. Such information, if available, would serve to generate a complete probability distribution that a sample fraction assumes a given value in situations where it is suspected that the sample fraction may be in error.

The populations W and B need not necessarily be denumerable. For instance, a machine may be searching a length of wire, a sheet of metal, or a volume of space in order to discover the occurrence of a given phenomenon. After it has searched a space until the dth "detection" has been made, the question is asked, "how many occurrences of the phenomenon has the machine really observed?"

SOME FURTHER RESULTS TO PALM'S OVERFLOW PROBELM*

By Kenneth R. Eaton Jr.
Gary S. Beckerman

A telephone switching network whereby an incoming call tests to determine if a line is free and, if not, switches into another line is an example of the behavior characteristic of what Palm refers to as an overflow problem. The distribution of the calls which overflow one line constitutes the distribution of the input to the next line. Because, in a large system, the overflow behavior may be repeated for many lines, this paper will discuss the distribution in further detail.

More abstractly the overflow problem consists of an input to n ordered (numbered) serving stations. The input tests each station sequentially until it finds the lowest numbered free station where it may be serviced. Therefore, the input may be considered in two parts; the traffic serviced by the first m stations and the "overflow" which is the input serviced by the remaining n-m stations. Obviously then the input distribution to the ith station is the same as the overflow distribution from the i-1st station. It is this distribution that is of interest because knowing it, the mathematically difficult problem of a n-stage system may be reduced to the n single stage systems which are comparatively simple to solve.

Interest developed in the overflow problem when the authors began a study of power & free conveyor systems. Here a part arrives at the first of n ordered stations and overflows if it is busy. This process continues until the part is serviced or overflows the system. The authors were interested in describing the input distribution to each station after the first and the distribution of parts which were lost to the system (i.e. those parts which overflowed the last station). Although the conveyor system under study included storage space in front of each station, Palm's results provided an insight to a method of approach for solving the new problem.

Let us assume that calls arrive to the first of m stations at the instants $\tau_1, \tau_2, \ldots, \tau_n, \ldots$ and denote the interarrival times for the arrival distribution of this station as:

$$\theta_n = \tau_n - \tau_{n-1} \quad (\tau_0 \equiv 0;\ n=1,2,\ldots)$$

* This paper is an extensive revision by Mr. Eaton of work originally performed by Mr. Beckerman and Mr. Eaton. Listings of the computer programs used may be obtained by writing Mr. Eaton, Department of Industrial Engineering, University of Michigan, Ann Arbor, Michigan.

Also, let us assume, as Takács[1] does, that the θ_n are identically distributed, independent, positive random variables whose distribution function is:

$$F(x) = P\{\theta_n \leq x\}$$

Denoting the Laplace-Stieltjes transform of F(x) as Ø(s) we have:

$$\emptyset(s) = \int_0^\infty e^{-sx}\, dF(x)$$

If we now define t_{ri} as the epoch of arrival of the ith demand serviced in the overflow group, the time intervals $t_{r,i+1}$ for i=1,2,... are independent, identically distributed random variables whose distribution function Takács denotes as $G_r(t)$. In 1959 Takács[2] showed that this $G_r(t)$ can be represented by the integral recurrence relation:

$$G_r(t) = \int_0^t [e^{-bu} + (1-e^{-bu})] G_r(t-u)\, dG_{r-1}(u)$$

Furthermore, he noted that the Laplace-Stieltjes transform:

$$\gamma_r(s) = \int_0^\infty e^{-st} dG_r(t)$$

may be represented by a recurrence relation of the form:

$$\gamma_r(s) = \frac{\gamma_{r-1}(s+b)}{1-\gamma_{r-1}(s)+\gamma_{r-1}(s+b)} \qquad r=1,2,\ldots$$

Continuing Takács proved that $\gamma_r(s)$ could also be expressed as:

$$\gamma_r(s) = \frac{\sum_{j=0}^{r} \binom{r}{j} \prod_{i=0}^{j-1} \left(\frac{1-\emptyset(s+i\mu)}{\emptyset(s+i\mu)}\right)}{\sum_{j=0}^{r+1} \binom{r+1}{j} \prod_{i=0}^{j-1} \left(\frac{1-\emptyset(s+i\mu)}{\emptyset(s+i\mu)}\right)}$$

Thus, the Laplace-Stieltjes transform of the interarrival distribution for any station can be expressed in terms involving only the Laplace-Stieltjes transform of the arrival distribution to the first station. In order then to find $G_r(t)$, one must solve for the inverse Laplace-Stieltjes transform of $\gamma_r(s)$. In most cases this is a difficult, if not impossible, problem.

Palm [3] has also studied the overflow problem and found an integral recurrence relation for $G_r(t)$ where now $G_r(t) \geq$ P(overflow time -t) = 1-P(overflow time <t) and

$$G_r(t) = G_{r-1}(t) - \int_0^t (1-e^{-bu})[1-G_r(t-u)]\, dG_{r-1}(u)$$

However, Takács' form for $G_r(t)$ can be shown to be a reformulation of the above formula and is easier to derive. Khintchine [4] shows, as does Palm, that if a simple[5] stream of calls enters line L_1, then there enters into any line $L_r (r > 1)$ a stream of the type P (stationary, orderly, and with limited aftereffects).[6]

Defining $G_r^*(s) = \int_0^\infty e^{-st} G_r(t)dt$ Palm obtains a recurrence relation for the Laplace transform of $G_r(t)$ as:

$$G_r^*(s) = \frac{G^*_{r-1}(s)}{1+sG_{r-1}(s) - (s+1)G_{r-1}(s+1)}$$

It can be shown that Takács' recurrence relation follows from this. Therefore, knowing $G_0^*(s)$ [the Laplace transform of the arrival distribution to the first station] we can solve successively for the Laplace transforms of higher orders of r.

In contrast to Takács' general solution, Palm solves a special case of the overflow problem wherein the input has a negative exponential distribution. His results follow.

Defining $B_r(s)$ for r= -1,0,1,... as the polynomial of power r+1:

$$B_r(s) = \lambda^{r+1} + \sum_{k=0}^{r} \binom{r+1}{k} s(s+1)(s+2)\ldots(s+r-k)\lambda^k$$

where $\sum_{k=0}^{-1}$ is defined as zero and, consequently, $B_{-1}(s) = \lambda^0 = 1$, Palm proves that these polynomials are related by the formula:

$$B_r(s) = sB_{r-1}(s+1) + \lambda B_{r-1}(s) \qquad r=0,1,\ldots$$

He then proves that these polynomials describe the Laplace transform of $G_r(t)$:

$$G_r^*(s) = \frac{B_{r-1}(s+1)}{B_r(s)}$$

It follows from the definition then, that the polynomial $B_r(t)$ of order r+1 has positive coefficients and the coefficient of s^{r+1} is 1. Therefore, all real roots, s_{ri}, of this polynomial are negative and can be written:

$$B_r(s) = (s+a_{r0})(s+a_{r1})\ldots(s+a_{rr})$$

where $a_{ri} \equiv -s_{ri}$ and $a_{ri} > 0$.

Consequently, the denominator of $G_r^*(s)$ can be factored into the form:

$$G_r^*(s) = \frac{B_{r-1}(s+1)}{B_r(s)} = \frac{C_{ro}}{s+a_{r0}} + \frac{C_{r1}}{s+a_{r1}} + \ldots + \frac{C_{rr}}{s+a_{rr}}$$

Now knowing that:

$$L(\nu e^{-\rho t}) = \frac{\nu}{s+\rho}$$

and

$$G_r^*(s) = \sum_{k=0}^{r} \frac{C_{rk}}{s+a_{rk}}$$

Palm shows that:

$$G_r(t) = \sum_{k=0}^{r} C_{rk} e^{-(a_{rk})t}$$

Essentially then Palm has solved the overflow problem under the assumption that the input is negative exponentially distributed. He then derives a polynomial which simplifies the computation for inverting the Laplace transforms of the interarrival distributions.

The remainder of this paper is devoted to two different approaches to solving a simple two station overflow problem. First, Palm's results are solved with the successful application of the University of Michigan Computing Center's programs for finding the roots of polynomial equations and the coefficients of partial fraction expansions. And secondly, a simulation is developed and tested against the known results.

Solution of two station case using Palm's results.

It was assumed for this case that the input stream or arrival process was governed by the negative exponential distribution. Therefore,

$$G_0(t) = e^{-\lambda t} \qquad t \geq 0$$

and $B_0(s) = \lambda + s$

Now, the polynomial of degree 1 can be determined to be:

$$B_1(s) = s^2 + (2\lambda + 1)s + \lambda^2$$

which yields the Laplace transform of $G_1(t)$.

$$G_1{}^*(s) = \frac{B_0(s+1)}{B_1(s)} = \frac{\lambda + (s+1)}{s^2 + (2\lambda+1)s + \lambda^2} = \frac{s + (\lambda+1)}{s^2 + (2\lambda+1)s + \lambda^2}$$

In order to obtain the partial fraction expansion, the denominator of $G_1{}^*(s)$ is factored to give:

$$s = -(\lambda + \tfrac{1}{2}) \pm \sqrt{\lambda + \tfrac{1}{4}}$$

Letting a_{11} equal the root with positive coefficient we find:

$$a_{11} = \lambda + \frac{1}{2} - \sqrt{\lambda + \frac{1}{4}}$$

$$a_{12} = \lambda + \frac{1}{2} + \sqrt{\lambda + \frac{1}{4}}$$

Therefore, by partial fraction expansion:

$$G_1{}^*(s) = \frac{s + (\lambda+1)}{(s+a_{11})(s+a_{12})} = \frac{C_{11}}{s+a_{11}} + \frac{C_{12}}{s+a_{12}}$$

Where C_{11} and C_{12} can be found by equating coefficients. Solving we find:

$$C_{11} = \frac{1}{2} + \frac{1}{4\sqrt{\lambda + \frac{1}{4}}}$$

$$C_{12} = \frac{1}{2} - \frac{1}{4\sqrt{\lambda + \frac{1}{4}}}$$

Finally then:

$$G_1{}^*(s) = \frac{\frac{1}{2} + \frac{1}{4\sqrt{1+\frac{1}{4}}}}{s+\lambda+\frac{1}{2}-\sqrt{\lambda+\frac{1}{4}}} \quad \frac{+\frac{1}{2} - \frac{1}{4\sqrt{1+\frac{1}{4}}}}{s+\lambda+\frac{1}{2}+\sqrt{\lambda+\frac{1}{4}}}$$

Because both of these expressions contain only first order powers of s, the inverse Laplace transform is of the form:

$$G_1(t) = C_{11}e^{-a_{11}t}+C_{12}e^{-a_{12}t}$$

or

$$G_1(t) = \left(\frac{1}{2} + \frac{1}{4\sqrt{\lambda+\frac{1}{4}}}\right)e^{-(\lambda+\frac{1}{2}-\sqrt{\lambda+\frac{1}{4}})} + \left(\frac{1}{2} - \frac{1}{4\sqrt{\lambda+\frac{1}{4}}}\right)e^{-(\lambda+\frac{1}{2}+\sqrt{\lambda+\frac{1}{4}})}$$

This then is the overflow distribution from the first station.

Although the mathematical manipulations performed in obtaining the final values were not difficult, they were lengthy. For this reason the authors decided to use the computer facilities at the University to find the roots of the denominator and the partial fraction expansion of $G_2{}^*(s)$.

For the second station we have as before:

$$B_2(s) = s^3+(3\lambda+3)s^2+(3\lambda^2+3\lambda+2)s+\lambda^3$$

and

$$G_2{}^*(s) = \frac{B_1(s+1)}{B_2(s)} = \frac{s^2+(2\lambda+3)s+(\lambda^2+2\lambda+2)}{s^3+(3\lambda+3)s^2+(3\lambda^2+3\lambda+2)s+\lambda^3}$$

At this time various subroutines from the Computing Center Library were used to compute the roots of the denominator and the partial fraction expansion of $G_2{}^*(s)$.

To find the cube roots of the denominator of $G_2{}^*(s)$ the subroutine:

M = ZER2. (N,A,R)

was used where:

N = the degree of the polynomial in integer mode

A = a vector containing the real and imaginary coefficients of X^i $(i = n,n-1,\ldots,1)$

R = a vector containing the real and imaginary roots of the equation

M = an error indicator such that m=1 means execution OK and m = 2,3,4 means an error has occured.

For the particular example in this paper:

$$A(0) = 1 \quad A(2) = 3\lambda+3 \quad A(4) = 3\lambda^2+3\lambda+2 \quad A(6) = \lambda^3$$

$$A(1) = A(3) = A(5) = A(7) = 0$$

Next, a subroutine which solves simultaneous equations was used to find the partial fraction expansion. For this particular problem we wish to determine roots r_1, r_2, r_3 which are negative. Letting

$$a_1 = -r_1 \quad a_2 = -r_2 \quad a_3 = -r_3$$

we desire that:

$$\frac{A}{s+a_1} + \frac{B}{s+a_2} + \frac{C}{s+a_3} = \frac{s^2+ps+q}{(s+a_1)(s+a_2)(s+a_3)}$$

where $p = 2\lambda+3$ and $q = \lambda^2+2\lambda+2$.

Now finding a common denominator and equating coefficients we obtain:

$$A+B+C = 1$$

$$(a_2+a_3)A + (a_1+a_3)B + (a_1+a_2)C = p$$

$$a_2a_3A + a_1a_3B + a_1a_2C = q$$

which can be expressed in matrix notation as:

$$\begin{Vmatrix} 1 & 1 & 1 \\ a_2+a_3 & a_1+a_3 & a_1+a_2 \\ a_2\ a_3 & a_1\ a_3 & a_1\ a_2 \end{Vmatrix} \begin{Vmatrix} A \\ B \\ C \end{Vmatrix} = \begin{Vmatrix} 1 \\ p \\ q \end{Vmatrix}$$

To solve the simultaneous equations, the subroutine,

X = GJRDT. (N,M,A,D)

was used, where:

N = the number of rows

M = the number of columns including the right had side

A = the first element of the matrix of coefficients (A(1,1))

X = an erros return where X=1 means execution OK and X=2,3,... indicates an error.

D = the value of the determinent

The input format required is of the form:

$$N = 3 \qquad \left[\begin{array}{c|c} a_{ij} & \begin{array}{c} 1 \\ p \\ q \end{array} \end{array}\right]$$

$$M=4=N+1$$

and the output is of the form:

$$N \qquad \left[\begin{array}{c|c} (a_{ij})^{-1} & \begin{array}{c} A \\ B \\ C \end{array} \end{array}\right]$$

$$M=N+1$$

Thus, a simple program incorporating the two subroutines solved for both the roots of the denominator and the partial fraction expansion of $G_2^*(s)$. Inverting the final form of $G_2^*(s)$ we obtain:

$$G_2(t) = Ae^{-a_1 t} + Be^{-a_2 t} + Ce^{-a_3 t}$$

The following pages contain listings of the coefficients of $G_1(t)$ and $G_2(t)$ for different values of λ with $\mu = 1$.

Values of $G_1(t)$ for Various λ's

$\lambda = 1/6 \quad G_1(t) = .123e^{-1.312t} + .887e^{-.021t}$

$\lambda = 1/5 \quad G_1(t) = .127e^{-1.37t} + .827e^{-.029t}$

$\lambda = 1/4 \quad G_1(t) = .146e^{-1.46t} + .853e^{-.043t}$

$\lambda = 1/3 \quad G_1(t) + .173e^{-1.60t} + .827e^{-.070t}$

$\lambda = 2/5 \quad G_1(t) = .190e^{-1.71t} + .810e^{-.094t}$

$\lambda = 1/2 \quad G_1(t) = .211e^{-1.86t} + .784e^{-.134t}$

$\lambda = 3/5 \quad G_1(t) = .229e^{-2.02t} + .771e^{-.178t}$

$\lambda = 2/3 \quad G_1(t) = .239e^{-2.12t} + .761e^{-.209t}$

$$\lambda = 3/4 \quad G_1(t) = .250e^{-2.25t} + .750e^{-.250t}$$

$$\lambda = 4/5 \quad G_1(t) = .256e^{-2.32t} + .744e^{-.295t}$$

$$\lambda = 5/6 \quad G_1(t) = .260e^{-2.37t} + .740e^{-.292t}$$

Values of $G_2(t)$ for Various λ's

$$\lambda = 1/6 \quad G_2(t) = .055e^{-2.44t} + .028e^{-1.05t} + .916e^{-.002t}$$

$$\lambda = 1/5 \quad G_2(t) = .064e^{-2.52t} + .037e^{-1.07t} + .900e^{-.003t}$$

$$\lambda = 1/4 \quad G_2(t) = .072e^{-2.64t} + .050e^{-1.11t} + .878e^{-.005t}$$

$$\lambda = 1/3 \quad G_2(t) = .084e^{-2.82t} + .071e^{-1.16t} + .844e^{-.001t}$$

$$\lambda = 2/5 \quad G_2(t) = .095e^{-2.97t} + .086e^{-1.22t} + .819e^{-.018t}$$

$$\lambda = 1/2 \quad G_2(t) = .106e^{-3.17t} + .107e^{-1.30t} + .786e^{-.030t}$$

$$\lambda = 3/5 \quad G_2(t) = .116e^{-3.37t} + .126e^{-1.38t} + .758e^{-.046t}$$

$$\lambda = 2/3 \quad G_2(t) = .122e^{-3.50t} + .137e^{-1.44t} + .741e^{-.059t}$$

$$\lambda = 3/4 \quad G_2(t) = .128e^{-3.66t} + .149e^{-1.51t} + .722e^{-.076t}$$

$$\lambda = 4/5 \quad G_2(t) = .132e^{-3.75t} + .156e^{-1.56t} + .712e^{-.087t}$$

$$\lambda = 5/6 \quad G_2(t) = .134e^{-3.81t} + .160e^{-1.59t} + .706e^{-.095t}$$

After the successful compilation of the constants for $G_1(t)$ and $G_2(t)$, the authors could accurately evaluate a conveyor simulator which was to be used for further research. The simulator was designed to include storage before each service station. However, it had to be validated before any results could be accepted. Setting the storage at zero level, the simulator provided a model of the two station no-storage case. Consequently its results for the overflow distribution could be validated against the known distribution through suitable statistical tests. The simulation was used to study the particular case where the values of $\lambda = 1/2$ and $\mu_1 = \mu_2 = 1$ were used.

In order to obtain what the authors considered to be statistical equilibrium, a run of 16,000 time units was simulated. Actually four different runs of 4000 time units were made. However, because the starting point for each run was chosen "at random" by the IBM 7090, the results could be combined and considered as one run. Approximately 18 minutes were required to simulate 16,000 time units.

The 16,000 simulationed time units gave rise to 5393 trials for station 1 and 1280 trials for station 2. These number of trials were considered to be sufficiently large to meaningfully apply any statistical test.

The results from the computer simulation were used to determine the cumulative frequency of the interarrival distribution. In other words, the number of trials for which a success occurred in ≤t, time, was divided by the total number of trials. These values and those obtained from the solution of the analytic equations for stations 1 and 2 are given in tables 1 and 2, (Pages 16 and 17). Also, the difference between the two values is given.

The Kolomogorov-Smirnov goodness of fit statistical test was made. This statistic compares the maximum deviation between the actual and theoretical curves with a tabulated maximum. Acceptance of the hypothesis that both curves come from the same distribution occurs if and only if the calculated maximum deviation is less than the tabulated maximum.

According to the Kolomogorov-Smirnov statistic we then

1) reject the hypothesis that the sample came from the theoretical population if:

$$D_{5393} = \max_x |F_n(x) - F(x)| > \frac{1.63}{\sqrt{n}} = .02218$$

at the .01 level for station 1.

2) reject the hypothesis that the sample came from the theoretical population if:

$$D_{1280} = \max_x |F_n(x) - F(x)| > \frac{1.63}{\sqrt{n}} = .04557$$

at the .01 level for station 2.

The maximum deviations which occurred can be read from tables 1 and 2. They are .0045 and .0227 for station 1 and station 2 respectively. Therefore, the hypothesis that the values from the simulation results came from the same distribution as the theoretical results is not rejected at the .01 level.

This paper has presented the results of an analytic solution to Palm's overflow problem using an electronic computer to calculate the necessary coefficients of the overflow distribution and of a simulation of the same problem. Because a statistical test showed that the simulations results accurately portrayed the system under study, further research could now be performed on systems which do not yield to analytic solutions attempts.

TABLE I

TIME BETWEEN OVERFLOWS - STATION 1

$\lvert\varepsilon\rvert$	$G_1(t)$	
	Palm	Simulation
.0016	.72290	.7245
.0024	.60863	.6062
.0025	.52862	.5311
.0011	.46176	.4637
.0002	.40376	.4040
.0013	.35311	.3544
.0027	.30882	.3115
.0003	.27009	.2698
.0003	.23622	.2359
*.0045	.20650	.2021
.0022	.18069	.1785
.0019	.15803	.1561
.0025	.13821	.1357
.0020	.12088	.1189
*.0045	.10572	.1012
.0032	.09246	.0892
.0015	.08086	.0794
.0002	.07072	.0709
.0004	.06185	.0623
.0028	.05410	.0569
.0015	.04731	.0488
.0016	.04138	.0430
.0013	.03619	.0375
.0006	.03165	.0323
.0001	.02768	.0278
.0007	.02421	.0235
.0006	.02117	.0206
.0009	.01852	.0176
.0016	.01620	.0146
.0005	.01416	.0137
.0004	.01239	.0119
.0000	.01083	.0108
.0001	.00948	.0096
.0002	.00829	.0081
.0001	.00725	.0074
.0002	.00634	.0065
.0002	.00554	.0057
.0001	.00485	.0050
.0004	.00424	.0046
.0003	.00371	.0040
.0005	.00324	.0037
.0002	.00284	.0026
.0003	.00248	.0022

TABLE II

TIME BETWEEN OVERFLOWS - STATION 2

$\lvert\varepsilon\rvert$	$G_2(t)$	
	Palm	Simulation
.0082	.79638	.7891
.0061	.74836	.7422
.0022	.72052	.7227
.0031	.69771	.7008
.0009	.67668	.6758
.0004	.65657	.6562
.0035	.63713	.6406
.0019	.61829	.6164
0	.60002	.6000
.0034	.58228	.5789
.0057	.56507	.5594
.0031	.54837	.5453
.0002	.53217	.5320
.0016	.51644	.5148
.0059	.50118	.4953
.0036	.48636	.4828
.0040	.47199	.4680
.0057	.45804	.4523
.0114	.44450	.4359
.0111	.43137	.4203
.0061	.41862	.4125
.0085	.40625	.3977
.0122	.39424	.3820
.0146	.38259	.3680
.0158	.37128	.3555
.0173	.36031	.3430
.0161	.34966	.3336
.0151	.33932	.3242
.0191	.32930	.3102
.0173	.31956	.3023
.0156	.31012	.2945
.0143	.30095	.2867
.0147	.29206	.2773
.0147	.28343	.2687
.0142	.27505	.2609
.0130	.26692	.2539
.0129	.25903	.2461
.0139	.25138	.2375
.0182	.24395	.2258
.0172	.23674	.2195
*.0227	.22974	.2070
.0207	.22295	.2023
.0204	.21636	.1960
.0212	.20997	.1898
.0171	.20376	.1867
.0172	.19774	.1805
.0200	.10180	.1719
.0206	.18623	.1656
.0205	.18072	.1602
.0176	.17538	.1578
.0155	.17020	.1547
.0183	.16517	.1469
.0181	.16029	.1422
.0188	.15555	.1367
.0918	.15095	.1312
.0176	.14649	.1289
*.0227	.14216	.1195
.0216	.13796	.1164
.0214	.13388	.1125
.0190	.12992	.1109
.0183	.12609	.1078
.0177	.12236	.1047
.0203	.11874	.0984
.0184	.11523	.0969
.0181	.11183	.0937
.0163	.10852	.0922
.0147	.10532	.0906
.0139	.10220	.0883

TABLE II (con't)

$\|\varepsilon\|$	$G_2(t)$ Palm	$G_2(t)$ Simulation
.0140	.09918	.0852
.0158	.09625	.0805
.0161	.09341	.0773
.0148	.09065	.0758

FOOT NOTE REFERENCE

1. Takács, pp. 174-188
2. Riordan, pp. 37-38
3. Khintchine, pp. 82-89
4. Khintchine, pp. 89-95
5. Khintchine, pp. 11-12
6. Khintchine, pp. 44-48

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2. Riordan, J., Stochastic Service Systems, John Wiley and Sons, Inc. New York, 1962.

3. Siegel, S., Nonparametric Statistics For The Behavioral Sciences, McGraw-Hill Book Company, Inc., 1956.

4. Takács, L., Introduction to the Theory of Queues, Oxford University Press, New York, 1962.

THE IMPACT OF COMPUTING SYSTEMS AND SIMULATION ON TOP-LEVEL MANAGEMENT DECISION PROCESSES

By J. B. Neuhardt

(I) Introduction

Originally, this paper was to have been a study of the impact of the 1975 computer system on administration control systems, with special emphasis on multivariate sequential sampling. As it became apparent that the latter could never be practical without computers, a simple control system was modelled. One approach to optimization of the system would have been to set partial derivatives equal to zero, and search for minimization of expected costs. This analytical approach was not taken, but instead the expected cost model was explored on a digital computer. The relative ease with which the pseudo-optimum conditions became apparent was rather startling (as opposed to the straight analytical approach). The idea of replacing analytical studies with simulation is certainly not new. Many authors, including G. Morgenthaler (Ref. 2), note that simulation is one possible alternative when analytical models become unwieldy.

The basis direction of this paper then changed toward answering: What types of simulation languages and computer systems would increase the use of such tools by top management directly? Would this direct use partially solve the problems of insufficient support of top management in O.R. studies?

(II) Past Problems in Achieving Results from Operations Research Studies- One Partial Answer.

It has been written that some of the reasons that operations research studies fail to materialize into significant results include:

(1) Inability of the O.R. team members to obtain clear statements of the problems to be solved, leading eventually to the study of the wrong variables,

(2) Less than full indorsement of top management due to poor communications, or simply to management change.

In addition, future Operations Research efforts may falter in many organizations simply because there are not enough specialists in this area to go around.

One suggestion seems obvious: give top management a tool which it can use directly; a tool which gives the ability to deal with complex situations, arrives at answers relatively quickly, and requires no extensive background in refined Operations Research techniques to use.

(III) Future Computing Systems-Present Simulation Language

No attempt will be made to extrapolate speeds, memory sizes, etc., of future generation computers. It is merely noted that the future management simulation language would probably require the following type of computing system:

(a) Parallel processing, to insure access of perhaps several management inquiries in addition to fulfilling lower priority daily data processing requirements,

(b) input devices located near the inquirer to facilitate processing,

(c) high processing speeds relative to in-out speeds (characteristicly scientific rather than commercial which stresses mass in-out data processing) for large scale simulation.

These elements represent a minimum consideration, and are unfortunately geared to present state-of-the-art computing techniques.

The only system simulation language known to the writer is the "General Purpose Systems Simulation Program", written by G. Gordon. Time did not permit becoming operationally acquainted with this language, but it is interesting that Mr. Gordon, in discussing the program's application, states "the program involves compromises and it seldom meets exactly the requirements of the users." This would undoubtedly be true of any general purpose program, but would not deter an avid operations "researcher" in its use. However, difficulty in the use might discourage top management in direct utilization of the program, (if the reader has ever tried to operate a program by simply following directions in a general program write-up, imagine a company president being given such an item for immediate use).

The answer lies partially in specific rather than general programs, written possibly by company personnel for use by that company's top management, rather than a general program written by a computer organization for all its users. The use of such specific programs might only require a basic knowledge in probability theory, descriptive statistics and possibly inferential statistics.

A specific problem is now considered.

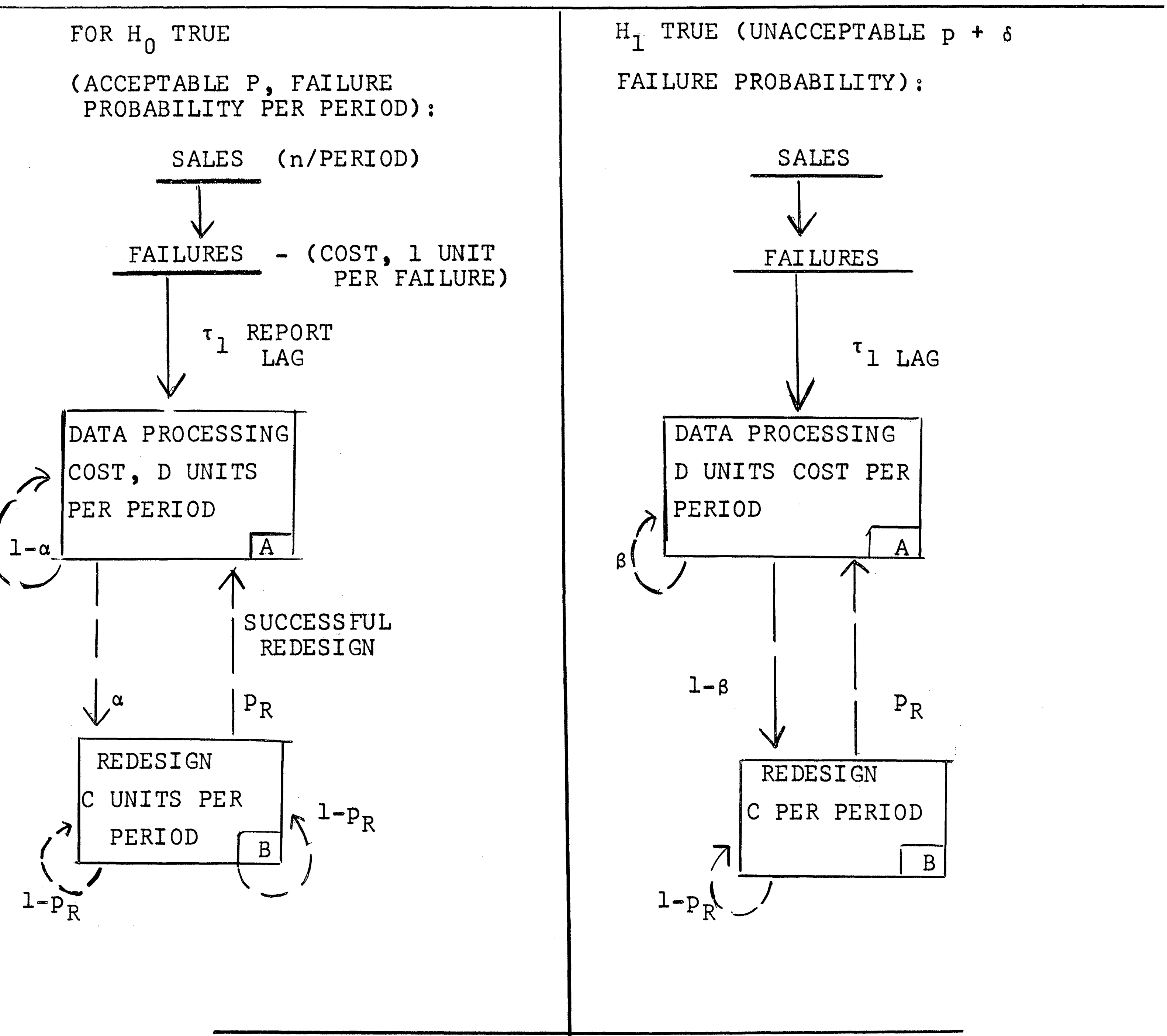
(IV) Example of Problem, Suggested Extensions

(A) Problem Statement (Figure 1)

Suppose "n" units of a given product are sold each period, and this product is under continual use after it is sold. Let P_{ij} be the probability that any product sold in the "i^{th}" period will fail in the "j^{th}" period of use. For simplicity, it is assumed that P_{ij} is not dependent on i, the period of sales. It is assumed that the failures are independent in the statistical sense. Upon failing, this fact is reported to

FIGURE 1

FLOW DIAGRAM OF WARRANTY PROBLEM



EXPECTED FAILURE COST PER PERIOD = np

EXPECTED FAILURE COST PER PERIOD BEFORE REDESIGN = n(p+δ) data processing and analysis, with a τ_1 period lag. It is assumed that all failures occuring in a period have the same lag time in data processing. The data processing and analysis section tests the following hypothesis, H_0, with the associated alternate, H_1, upon receipt of failure data:

$$H_0: \quad \hat{p}_j = p$$

$$H_1: \quad \hat{p}_j = p + \delta \quad (\delta > 0)$$

where p is a tolerable failure probability, and $p + \delta$ is considered excessive. Let α, β be the associated type I, II errors, respectively, of the test.

It is assumed that sales, n, is constant from period to period. If H_0 is accepted, data processing continues to monitor failures, performing independent tests from period to period. If H_0 is rejected, it is assumed that an engineering redesign is necessary. It is further assumed that the probability that engineering redesign will result in success (with the new failure probability, p) in one period is p_R, and this probability is constant from period to period while redesign efforts are in progress (a most unrealistic assumption). It is assumed that p_R is fixed whether H_0 is true or not. Let:

D = cost of data processing per period,
C = cost of engineering redesign per period,
1 = cost of one product failing.

Figure 1 is a flow diagram of the possible states under H_0 and H_1, and transition probabilities are noted. For instance, under H_1, the probability of remaining in state "A" , data processing, upon receipt of failure data is β , the probability of accepting a false hypothesis.

(B) Analytical Model of Expected Costs

Let: $\overline{AB}$ = expected number of periods in making the transition from state A to B. $T = \overline{ABA} = \overline{AB} + \overline{BA} = \sum_{k=0}^{\infty} \alpha(k+1)(1-\alpha)^k + \sum_{m=0}^{\infty}(m+1)(1-p_R)^m p_R$

$$= 1/\alpha + 1/p_R$$

For H_0 true: E_0 = Expected Cost per period $= D + 1/T\ (C/p_R$

For H_1 true: $E_1 = 1/T[DT + C/p_R + n\delta \sum_{j=1}^{T} j] = D + \dfrac{C}{p_R T} + \dfrac{n\delta\ (T+1)}{2}$

Further assumptions on cost relations:

$$B = e^{-\lambda_\beta D}, \alpha = e^{-\lambda_\alpha D}, 1-p_R = e^{-\lambda_p C}, \ (\lambda_\beta, \lambda_\alpha, \lambda_p \text{ Constants})$$

This is the assumed relationship of data processing costs, D, and the errors α, β, while larger expenditures in redesign per period, C, increases the chances of successful redesign.

The above cost assumptions result in the following expected cost functions:

$$E_0 = D + \frac{1}{e^{\lambda_\alpha D}\left(1-e^{-\lambda_p C}\right)-1}\left[\frac{C}{1-e^{-\lambda_p C}}\right]$$

$$E_1 = D + \frac{C}{T\left(1-e^{-\lambda_p C}\right)} + \frac{n\delta(T+1)}{2} \quad \text{Where: } T=\tau_1+\left(1-e^{-\lambda_\beta D}\right)^{-1}+\left(1-e^{-\lambda_p C}\right)^{-1}$$

It is desired to minimize the expected cost, E, where:

$E = E_0$ (probability that H_0 is true) + E_1 (probability that H_1 is true).

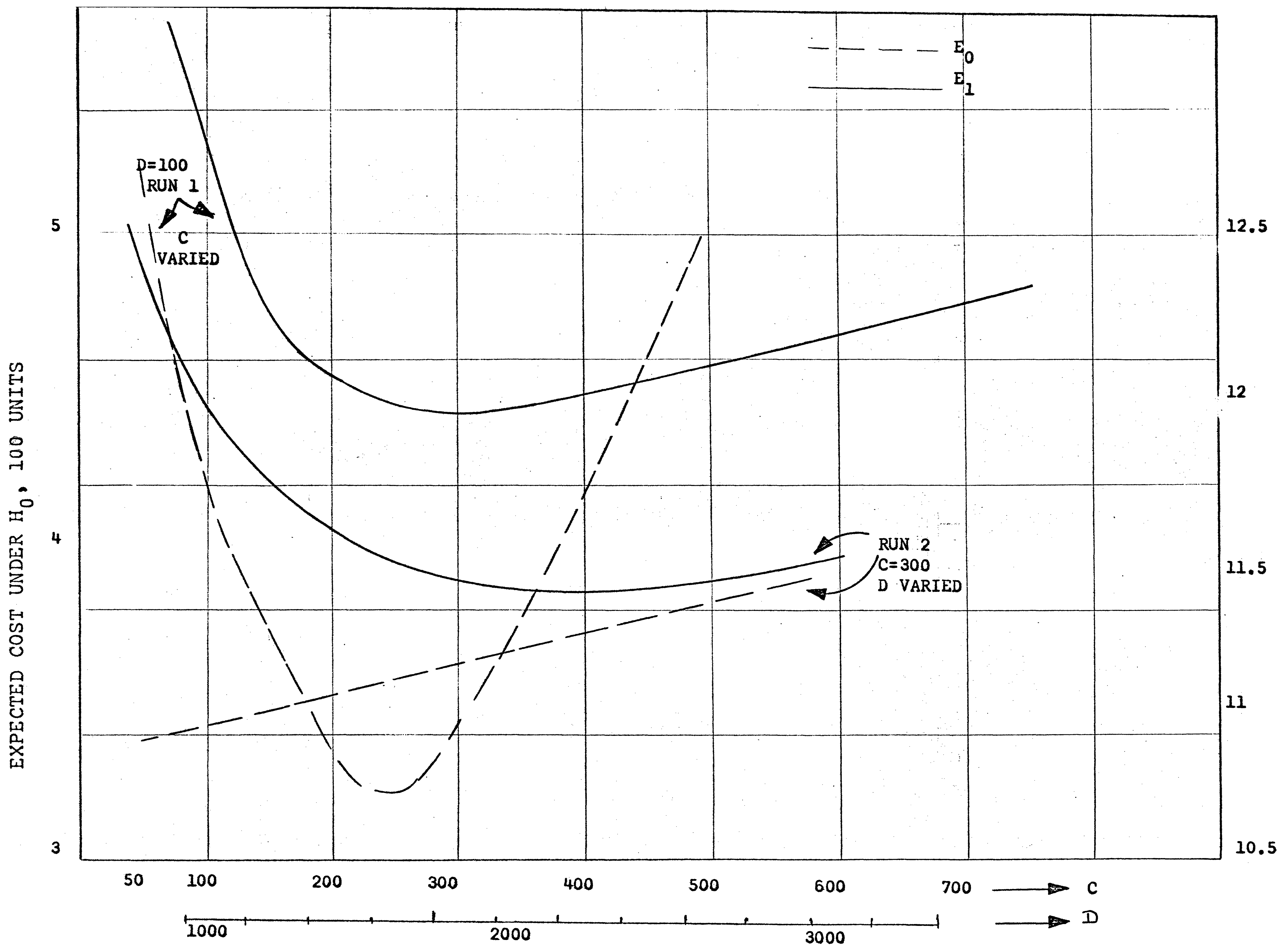
Actually, it was found that E_0 and E_1 differed considerably, and for practical purposes E_1 was the variable to concentrate on.

(C) Computer Solution

After spending some time studying the analytical expression for expected cost, and contemplating that model for more complicated models, it was decided to analyze the model on a digital computer. A "one variable at-a-time" method to seek an optimum was used. In a matter of two man hours and \$20 computer costs, relationships of the variables seemed evident with respect to cost, and a "flat" region of expected cost was apparent (Figure 2). The computer program, written in basic machine language, was simple, yet the relative ease with which the expected cost region was studied seemed remarkable to the writer. Several items were of immediate interest in the computer runs (Figure 2). A relatively constant expected cost region existed over rather considerable ranges in expenditures, C and D, (Figure 2). It is far better to overspend rather than underspend, due to the slope of E_1 above and below the apparent optimum. It is noted that data processing and analysis costs are about 6 to 1 relative to redesign costs D, in the area of optimum E_1. One other run was made, increasing the time lag to data processing. This quantity seemed rather insignificant relative to C and D expenditures.

(D) Remarks

Many of the assumptions made, point to the impracticality of this specific model. The practical aspect is the inexpensive study via computer runs, which might have implications in more complex models. An extremely valuable piece of information would be obtained if the expected cost curves were relatively flat in even complex models.



EXPECTED COST UNDER H_0, 100 UNITS
3
4
5
10.5
11
11.5
12
12.5
E_0
E_1
D=100
RUN 1
C
VARIED
RUN 2
C=300
D VARIED
50
100
200
300
400
500
600
700
C
1000
2000
3000
D

Practical extensions of this model might include the probability of failure changing with time, several products tested simultaneously with a restriction on redesign expenditure (above and below), sequential and multivariate sequential testing of failure rates, and inclusion of more than one possibility of product improvement (production, material etc.).

(V) Problem Extension Through Simulation Language

When a special purpose simulation language is considered to enable the study of the warranty problem with most of the elements included, and the simulation is mere parameter study, top management has not played a part in model construction. For instance, with reference to the warranty problem as stated above, a simulation program could be written which would select failures randomly, and with given alpha, beta and p_R values, simulate the problem and log costs per period. The executive could study costs as the expenditures and even the functional relationships of errors with expenditures were changed. However, in this role, the executive is not utilizing his experience in fundamental relationships of certain variables or, as previously stated, he is not playing an active role in the construction of the model. How, then, can this role be made easier, so that it does not consume a prohibitive amount of the executive's time? A general purpose simulator is probably too complicated and unwieldy to use, while a specific simulator relegates the executive to studying parameter effects in a given model.

It is suggested that a compromise might exist in a language written specifically for company executives which would be flexible enough to allow the executive the study of many problems by constructing his own simulation models, yet operationally simple. This task is obviously a formidable one, and the following represents a first attempt:

Given the four types of activities:

(A) Random function generator; For random variables $x_1, x_2, \ldots\ldots x_n$, and associated distribution functions $F_1, F_2, \ldots\ldots, F_n$, generate sample $\hat{F}_1, \hat{F}_2, \ldots\ldots, \hat{F}_n$, possibly in independent periods in time, with associated costs $C(\hat{F}_1), \ldots, C(\hat{F}_n)$.

(B) Storage Activity; Ability to store quantities for time τ, at cost $C(\tau)$.

(C) Decision Activity;

(D) Action activity; Function, which hopefully affects the F_i, with associated costs per period.

Transition probabilities from activity are needed, which may be fixed, or functions of costs, or expenditures in specific activities.

With these building blocks, an extension of the warranty problem is indicated in figure 3. The random variable is the failure probability at time t, possibly multivariate, with associated cost to the company. This is followed by a time lag and then to a decision activity which tests the typothesis that the failure level is below some preassigned amount. With probabilities p_{00} or p_{01}, the problem is considered not significant or it warrants further analysis, respectively. Under further analysis, the decision is made to refer the problem to one or more action activities, which in turn have costs associated with them, and certain probabilities of achieving results.

It is conceivable that this model could be read automatically in its flow diagram form, by a computer. The problems facing an executive might involve questioned elimination of the decision process for instance, and submit the problem to all action groups, or it might be desired to investigate the effects of spending more money in decreasing the time lag of failure reporting. Perhaps raw data could be fed to the action groups, and allow them to perform their own analyses. These are all questions that involve more than parameter study, and questions of this type could lead to basic model changes quickly and efficiently by the executive.

It is hoped that a similar language could be applied to, say, a production control problem or similar inventory and distribution systems. The above suggestions concerning a simulation language certainly in no way exhaust possibilities, but the complex flow diagram resulting from just four elements might indicate that further language complication would render the approach useless as an active tool for top management.

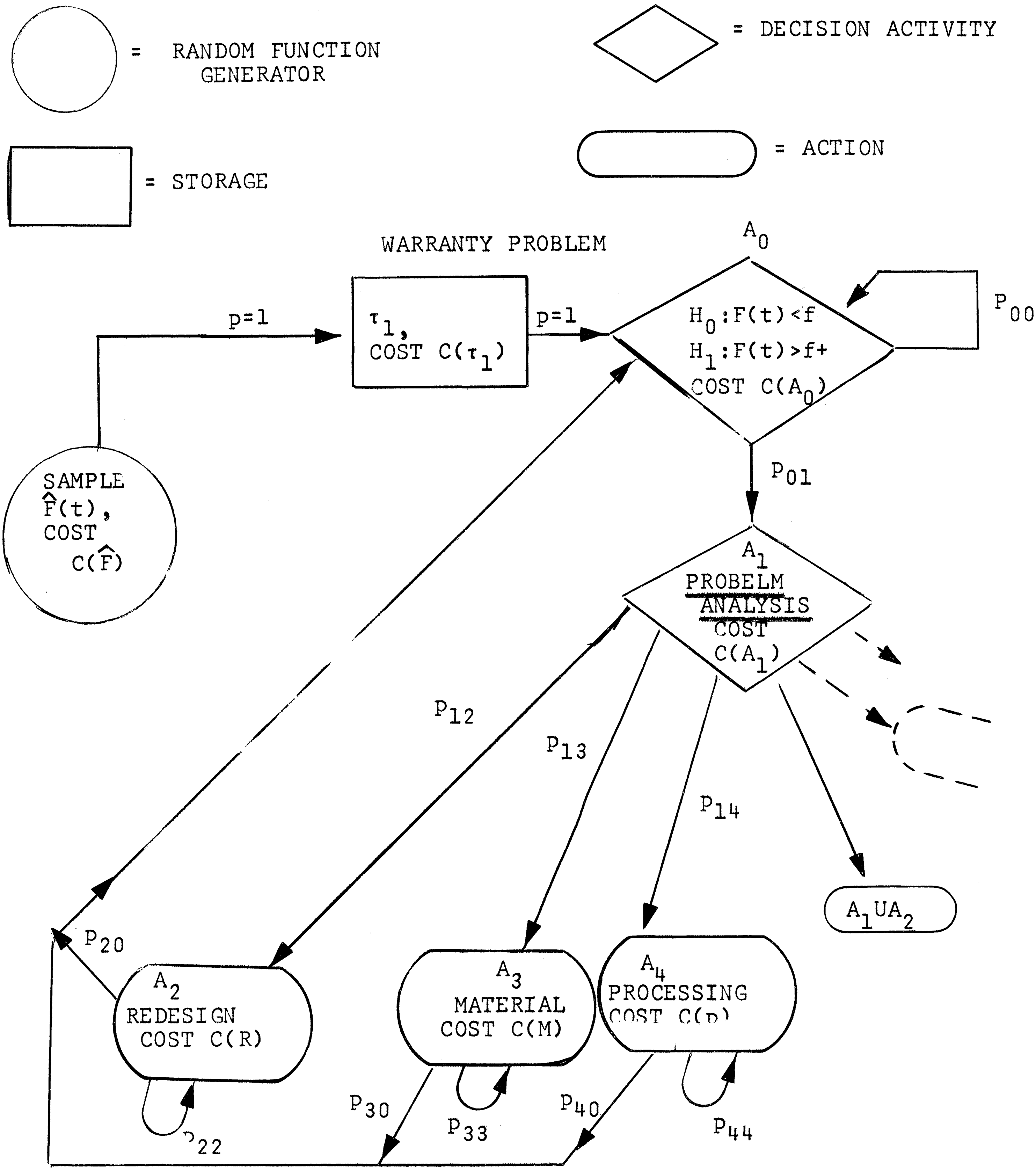
Nothing has been said about data display. Experience has shown that graphic outputs are invaluable as aids in rapid digestion of gross effects. Provision should be made for such displays to the executive.

(VI) Summary

As a result of this investigation, it appears more evident to the writer that computer simulation should play a bigger role in future management decision processes. It is suggested that the effectiveness of this approach to top level management will depend on how active a role the executive plays in the model construction, which in turn will depend greatly on how much effort is demanded in such a task. It would appear that a general simulation program would demand too much time for direct use in many situations, while a specific program might not be flexible enough to handle the variety of problems facing the executive.

Ar attempt to consider elements of a compromise language resulted in only four building blocks, but the flow diagram model was still quite unwieldy. It is evident that the task of writing a flexible yet easily manipulated language is not a small one, but it would seem necessary if the benefits in allowing top management to play active roles in simulation model construction are to be realized.

FIGURE 3



CONSTANTS: F(t), P_{ij}, COSTS, DECISION RULE FOR HYPOTHESIS

REFERENCES

(1) G. Gordon, General Purpose Systems Simulator, IBM Publication.

(2) R. Ackoff (ed.) Progress in Operations Research, John Wiley and Sons, 1961.

Bibliography on

Integer Linear Programming

Arnoldo Hax

1.
Balinsky, H. L., "Notes on Integer Linear Programming". Foundations and Tools for Operations Research and the Management Sciences. The University of Michigan. Summer 1962.

This paper represents, as far as our knowledge is concerned, the only attempt made up to now to provide a general discussion of all the methods available in the field of integer linear programming. Gomory's work is analyzed in detail; a list of uses of integer programming is given, and the necessary background and notation concerning linear programming and the simplex method is presented.

We strongly recommend this paper for anybody interested in gaining a rapid and general insight in the field.

2.
Balinsky, H. L., "Fixed Cost Transportation Problem". Naval Research Logistic Quarterly. Vol. 8 (1961), pp. 41-54

This paper formulates a fixed-cost transportation problem as an integer program, describes some of its special properties and suggests an approximate method of solution. Examples are given to demonstrate the approximation technique.

3.
Berge, C., "The Theory of Graphs". John Wiley and Sons, 1962.

This book gives a general survey of the theory of graphs and its applications. It provides an analysis of p-coloring map problem in terms of integer linear programming. (pp. 27-34)

4.
Charnes, A. and W. W. Cooper, "Management Models and Industrial Applications of Linear Programming" (2 volumes). John Wiley and Sons. 1961

These volumes illustrate all aspects of underlying theory of linear programming with concrete numerical examples accompanied by explanations.

In volume2, pp. 695-712, a discussion of Gomory's algorithms for Integer and Mixed Integer programming problems is given. The methods are illustrated by examples, nevertheless the presentation is far from complete.

5.
Charnes, A. and C. E. Lenke, "Optimization of Non-Linear Separables Convex Functionals". Naval Research Quarterly, Vol. 1, N°4, (1954), pp. 301-312.

Programming problems may arise in which the variables

are subjected to linear equations and inequalities, but the objective function may not be a linear function of the variable. This paper shows how the methods of linear programming may be extended to cover any objective function which is the sum of convex functions of each of the variables. The technique consists of constructing a large linear program whose solution yields the solution to a polygonal approximation of the convex programming problem. An efficient computational algorithm for solving the auxiliary linear program is also presented.

6.
Dantzig, G. B., "Discrete-Variable Extremum Problems". J.O.R.S.A. Vol. 5 (1957), pp. 266-277.

This paper presents an outline of the use of linear programming methods for the solution of discrete variable extremum problems. Three types of these problems are discussed, a) the assignment problem, b) the problem of the shortest route in a network, c) the knapsack problem.

Certain techniques on solving these problems are provided by the author, although no foolproof technique is offered, and no guarantee is given that they will work in all cases. Nevertheless, the paper is interesting primarily because it brings a good discussion of several applications of integer linear programming to certain classes of problems that are combinatorial in nature and easy to formulate.

7.
Dantzig, G. B. "Note on Solving Linear Programs in Integers". Naval Research Logistic Quarterly. Vol. 6 (1959), pp. 75-76.

The paper considers the method presented by Gomory (ref. 13) for solving linear program in integers. Gomory showed how to add linear inequality constraints to a linear programming problem automatically in such a way that the extreme points of the resulting convex contain only integral solutions in the neighborhood of the minimum. In this paper an alternative method is given for generating these additional contraints.

Dantzig's idea is based on the following theorem: If a linear programming problem in variables $x_1, x_2, \ldots, x_n$ has a basic feasible solution for basic variables $x_1, x_2, \ldots, x_m$ say, which is inadmissible for any reason, then the partial sum condition $x_{m+1} + x_{m+2} + \ldots x_n \geq 1$ is satisfied for all admissible solutions with integral values and not the basic solution.

In general suppose the convex of solutions in the n-dimensional space of $x_1, x_2, \ldots x_n$ is defined by K linear inequalities.

$$\pi_i(x) = \sum_{j=1}^{j=n} n_{ij}\, x_j \geq n_i \qquad i=1,2,\ldots K$$

where n_{ij} and n_i are positive and negative integers. An extreme point of the convex would be defined by the intersection of some n of the hyperplanes $\pi_i(x) = n_j$ for $i=i_1, i_2, \ldots i_n$.

If the extreme point is inadmissible for any reason, then at least one of the conditions $\pi_i(x)=n_i$ must be violated for an admissible integral solution, hence for at least one $i=i_1, i_2, \ldots, i_n$ $\pi_i(x) \geq n_i + 1$ because n_{ij} and n_i are integers and therefore

$$\sum_i \pi_i(x) \geq 1 + \sum_i n_i \qquad i = i_1, i_2, \ldots, i_n$$

is a linear inequality not satisfied by the extreme point but satisfied by all admissible integral solutions.

Gomory and Hoffman (ref. 19) and Balinsky (ref. 1) have shown that this is a deficient algorithm which cannot obtain the optimal integer solution under certain circumstances. The reason for us to present Dantzig's idea is due to the fact that it represents a new approach to the problem, and it is considered in almost all the literature in the field.

8. Dantzig, G. B., "On the Signifiance of Solving Linear Programming Problems with some Integer Variables". Econometrica Vol. 28 (1960), pp. 30-44.

Keeping in mind Gomory's methods for solving linear programs involving integer-valued variables, Dantzig reviews and classifies problems that can be reduced to this class and thereby solved. After considering the general principles of the cutting plane method, the author analyzes problems involving multiple dichotomies and k-fold alternatives which include problems with discrete variables, nonlinear seperable minimizing functions, conditional constraints, global minimum of general concave functions and combinatorial problems such as the fixed charge problem, traveling salesman problem, orthogonal latin square problem, and map coloring problem.

The paper gives one of the most complete discussions on uses of integer programming.

9. Dantzig, G. B., "Solution of a Large-Scale Traveling-Salesman Problem". (et al. Fulkerson R. and S. Johnson) J.O.R.S.A. Vol. 2 (1954) pp. 393-410.

The paper gives a complete analysis of the traveling salesman problem, a classic one in the field. The authors present some examples and the way in which it is possible to make a mathematical statement of the problem, putting it into a linear programming form. An estimation procedure for solving the problem is considered. It is shown that a certain tour of 49 cities, one in each of the 48 states and Washington, D. C., has the shortest road distance. In connection with integer programming, the paper is important because the cutting plane approach was proposed in this paper for the first time.

10.
Dorn, W. S., "Non-Linear Programming. A Survey" RC 707. IBM Research Center. June 12, 1962.

Some of the more recent theoretical and computational developments in non-linear programming are surveyed. The notion of Lagrange multipliers and duality are discussed together with applications of these ideas to scientific and business problems. Moreover, several algorithms for solving quadrate programming problems are reviewed. Explicit rules are given for two of these algorithms, and a simple example is solved by both methods. A large step gradient method for the solution of convex programs is given and the all-integer integer programming Gomory's algorithm is described. Simple examples are solved using both of these techniques. Linear fractional programming is also discussed briefly.

11.
Eisemann, K., Management Sciences. Vol. 3 (1957), pp. 279-284 "The Trim Problem"

A problem of primary significance to a variety of industries is the suppression of trim losses in cutting rolls of paper, textiles, cellophane, metallic foil, or other material, for the execution of business orders. The trim problem then consists in fitting orders to rolls and machines in such a way as to reduce trim losses to an absolute minimum.

The authors illustrate the general treatment of such a problem by a numerical example. The problem, of course, falls into the category of integer linear programming.

12.
Gilmore, P. C. and R. E. Gomory, "A Linear Programming Approach to the Cutting-Stock Problem". J.O.R.S.A. Vol. 9. Dec. 1961, pp. 849-859.

The cutting-stock problem is the problem of filling an order at minimum cost for specified quanitity of material to be cut from given stock lengths of given cost. When expressed as an integer programming problem the large number of variables involved generally makes computation infeasible. This same difficulty persists when only an approximate solution is being sought by linear programming. In this paper, a technique is described for overcoming the difficulty in the linear programming formulation of the problem. The technique enables one to compute with a matrix which has no more columns than it has rows.

13.
Gomory, R. E., "An Algorithm for Integer Solutions to Linear Programs". Princeton-IBM. Mathematics Research Project. Technical Report N°1. Nov. 17, 1958.

This report describes a method, based on G. B. Dantzig's simplex algorithm, for solving linear programming problems in integer. The paper is, perhaps, the most important one written in the field of integer programming.

The paper is divided into ten sections: A general description of the method is given in section 1. In section 2 the main class of inequalities used in the method is derived and shown to form a group. Section 3 gives a geometrical interpretation of the inequalities. In section 4 some properties of the inequality group are derived. Section 5 discusses briefly, ways of choosing particularly effective inequalities. In section 6 a variant of the basic inequalities is discussed. Section 7 contains a description of the lexicographical dual simplex method used in the finiteness proofs. Section 8 gives two versions of the method and shows that they obtain the integer answer in a finite number of steps. Section 9 contains miscellaneous comments including remarks on possible extensions, programming experience, etc. Section 10 contains a summary of the procedure and small worked out problems illustrating some of the results of the preceding sections.

14.
Gomory, R. E., "Outline of an Algorithm for Integer Solutions to Linear Programs". Bulletin of the American Mathematical Society. Vol 64, (1958), pp. 275-278.

This article is a short presentation of ref. 13. Here Gomory only describes the algorithm, without going into deep theoretical considerations.

15.
Gomory, R. E., "Solving Linear Programming Problems in Integers". Proceedings of Symposia in Applied Mathematics. Vol. X. American Mathematical Society. (1960), pp. 211-216.

Same as in ref. 14, this article gives a short presentation of ref. 13. An example is also presented. The paper offers a good analysis of Gomory's algorithm for those who do not like to go into all the mathematical background of the method.

16.
Gomory, R. E., "An Algorithm for the Mixed Integer Problem". RM-2597 Rand Corporation. July 7, 1960.

An algorithm is given for the numerical solution of the "mixed integer" linear programming problem, the problem of maximizing a linear form in finitely many variables constrained both by linear equalities and the requirement that a proper subset of variables assume only integral values. The algorithm is an extension of the cutting plane technique for the solution of the "pure integer" problem, given in ref. 13.

17.
Gomory, R. E. "All-Integer Integer Programming Algorithm". RC 198 IMB Research Center. Jan. 29, 1960.

The purpose of this paper is to describe a new method of integer programming which differs from its predecessors in two main points:

It is an all-integer method, that is, if the coefficients in the original matrix are integers all coefficients remain integer during the whole calculation.

It is a uniform procedure closely resembling the ordinary dual simplex method with the difference that the pivot element is always a-1. The cycle of maximizing by adding an inequality, etc. characteristic of ref. 12 has been eliminated.

18.
Gomory, R. E. and J. W. Banmol, "Integer Programming and Pricing". Econometrica. Vol. 28 (1960), pp. 521-550.

In this article Gomory's method of solution of integer linear programming problems (based on ref. 13 and ref. 14) is described briefly, with an example of the method of solution. The bulk of the paper is devoted to a discussion of the dual prices and their relationship to the marginal yields of scarce indivisible resources and their efficient allocation. The article also gives a geometrical interpretation of the integer programming algorithm, and explains the dual simplex calculations necessary for the applications of Gomory's method.

19.
Gomory, R. E. and A. J. Hoffman, "On the Convergence of an Integer-Programming Process". RC-650 IBM Research Center. Mar. 30, 1962.

The purpose of this paper is to analyze the finiteness of a procedure for integer programming described by G. B. Dantzig in ref. 7 which left the finitness question open. The result given here shows that the process will not be finite or even converge to the optimal integer answer x^o unless certain necessary conditions are satisfied. In particular, the procedure will not be finite unless x^o already lies on at least n-1 of the faces of the polynedron cut out by the inequalities of the linear programming problem.

20.
Land, A. H. and A. G. Doig, "An Automatic Method of Solving Discrete Programming Problems", Econometrica Vol. 28 (1960) pp.497-520.

This paper presents a simple numerical algorithm for the solution of programming problems in which some or all of the variables can take only discrete values. The algorithm requires no special technique beyond those used in ordinary linear programming, and lends itself to automatic computing. Its use is illustrated on two numerical examples.

The algorithm was completed by the authors at the same time that Gomory published his first paper (ref. 14). It had the advantage that the method could work in the mixed case (i.e. in which not all the variables are required to be discrete). Further work made by Gomory (ref. 16), also solved this problem in a more efficient way.

21.
Manne, Alan S., "On the Job-Shop Scheduling Problem". J.O.R.S.A. Vol. 8 (1960), pp. 219-223.

The article is a proposal for the application of discrete linear programming to the typical job-shop scheduling problem, one that involves both sequencing

restriction and also noninterference constraints for individual pieces of equipment. Thus far, no attempt has been made to establish the computational feasibility of the approach in the case of large-scale realistic problems. This formulation seems, however, to involve considerably fewer variables than two other proposals: a) E. H. Bowman "The Schedule-Sequencing Problem", J.O.R.S.A. Vol. 17 (1960) pp. 621-624, b) H. Wagner, "An Integer Linear-Programming Model for Machine Scheduling". Naval Research Logistic Quarterly. June, 1959.

22.
Markowitz, H. M., and A. S. Manne, "On the Solution of Discrete Programming Problems". Econometrica. Vol. 25, (1957), pp. 84 - 110.

This paper considers optimization problems in which some or all variables must take on integral values. The authors do not present an automatic algorithm for solving such problems. Rather they present a general approach susceptible to individual variations, depending upon the problem and the judgment of the user. Two moderate size and interesting examples are presented to illustrate the method.

The paper is often cited as having suggested the general line of attack employed by Gomory.

23.
Miller, C. E., A. W. Tucker and R. A. Zemlin, "Integer Programming Formulation of Traveling Salesman Problems". Journal of the Association for Computing Machinery. Vol. 7 (1960), pp.326-329.

The paper provides yet another example of the versatility of integer programming as a mathematical modeling device by representing a generalization of the well-known "Traveling Salesman Problem" in integer programming terms. After formulating the problem in analytical form, the authors give the results of five machine experiments. The solution procedure used was the All-Integer Algorithm of R. E. Gomory. The answers obtained were sufficiently irregular in their behaviour to cast doubt on the heuristic value of machine experiments with the model. It seems hopeful that more efficient programming procedures now under development will yield a satisfactory algorithmic solution to the traveling salesman problem. In any case, the models served to illustrate how problems of this sort may be succinctly formulated in integer programming terms.

24.
Thrall, R. M., "Linear Algebra with Applications to Linear Programming, Game Theory and other Models". Foundations and Tools for Operations Research and The Management Sciences. The University of Michigan, Summer, 1962.

This set of notes provides a general analysis of linear programming and its applications. The theory underlying the several linear programming techniques is very well covered, and many examples are given.

25.
Vadja, S., "Mathematical Programming". Addison-Wesley Publishing Company, Inc. Chapter 10, "Discrete Linear Programming". pp. 191-205.

The chapter starts with a discussion of certain classical applications of integer linear programming, such as the traveling salesman problem, the allocation problem, the introduction of logical relations, and the fixed charge problem. The problems are formulated in analytical form and some examples are given.

As far as the formulation of algorithms to solve the integer linear programming problem, the author presents two approaches. First, the method proposed by R. E. Gomory (ref. 14), and, second, the method presented by A.H. Land and A. Doig (ref. 20) for the mixed case. Examples are given to illustrate the use of algorithms.

Although the general discussion is far from being complete, and no attempt is made to provide the theoretical background of the methods proposed, we consider that the chapter gives a nice introduction to the topic, for those who want to get a general view of the subject.