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ANTI-WINDUP COMPENSATOR SYNTHESIS FOR SYSTEMS WITH SATURATION ACTUATORS

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SUMMARY

In this paper we synthesize linear and nonlinear output feedback dynamic compensators for plants with saturating actuators. Our approach is direct in the sense that it accounts for the saturation nonlinearity throughout the design procedure as distinct from traditional design techniques that first obtain a linear controller for the 'unsaturated' plant and then employ controller modification. We utilize fixed-structure techniques for output feedback compensation while specifying the structure and order of the controller. In the full-order case the controller gains are given by LQG-type Riccati equations that account for the saturation nonlinearity.

KEY WORDS saturation; small gain; domain of attraction; dynamic compensation

1. INTRODUCTION

Virtually all control actuation devices are subject to amplitude saturation. Whether or not these saturation effects need to be accounted for in the control-system design process depends on the required closed-loop performance in relation to the capacity of the actuators. In many applications, particularly in the field of aerospace engineering, actuator saturation is often the principal impediment to achieving significant closed-loop performance.[1,2] In fact, the effects of actuator saturation often constitute a greater source of performance limitation than even modelling uncertainty.

Techniques for addressing actuator saturation have been studied since the advent of modern control theory,[3–9] while recent activity in this area has been steadily increasing; see for example, References 10–16. Performance optimization under saturation constraints is addressed in References 5, 6, 11, 13 and 17, while global stabilization of plants with closed left-half plane poles is discussed in References 4, 14 and 15. A variety of approaches to the classical problem of integrator windup due to saturation are developed in References 8, 10, 12, 18 and 20. These references are merely representative of the extensive research activity in this area.

In the present paper we consider the problem of synthesizing nonlinear output feedback dynamic compensators for plants with saturating actuators. Our approach is direct in the sense that it accounts for the saturation nonlinearity throughout the design procedure and provides an explicit expression for a guaranteed domain of attraction. This approach is thus distinct from the more common two-step design strategy that first designs a linear controller for the 'unsaturated' plant and then accounts for the saturation by means of suitable controller modification. The two-step approach includes the classical problem of designing anti-windup circuitry for controllers with integrators.[8,10,12]

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In this paper we consider both linear and nonlinear controllers. In both cases closed-loop stability is enforced by a bounded real condition. The linear controller given in Section 3 is thus related to the LQG controller with an H_∞ bound given in Reference 21. The proof of stability given herein, however, is considerably more complicated than the results of Reference 21 since we invoke no *a priori* assumption on the magnitude of the control signal as in References 16 and 19.

A primary goal of our work is to design realistic controllers that have access only to the available measurements. Since full-state feedback control is often unrealistic in practice, we utilize fixed-structure techniques.[21–23] In fixed-structure controller synthesis the structure of the controller, including details such as order, degree of decentralization, and availability of measurements, is specified prior to optimization. Finite-dimensional optimization techniques are then applied to the free controller gains within the given controller structure. In the present paper the controller gains are chosen to minimize an LQG-type cost to provide a measure of performance beyond closed-loop stability.

The contents of the paper are as follows. In Section 2 we state and prove a sufficient condition (Theorem 2.1) for stability of a closed-loop system with a saturation nonlinearity. This result involves a small gain condition along with a guaranteed domain of attraction. In Sections 3 and 4 we apply Theorem 2.1 to the problem of controller synthesis to obtain linear and nonlinear dynamic compensators, respectively. The nonlinear controller, which is developed in Section 4, has an observer structure with a nonlinear input to account for the input saturation. Similar anti-windup controller structures were considered in References 12, 20. Illustrative numerical results are given in Section 5.

1.1. Notation

I_r	$r \times r$ identity matrix
$\mathbf{S}^n$, $\mathbf{N}^n$, $\mathbf{P}^n$	$n \times n$ symmetric, nonnegative-definite, positive-definite matrices
$\lambda_{\max}(F)$	maximum eigenvalue of matrix F having real eigenvalues
$\sigma_{\max}(G)$	maximum singular value of matrix G
$\|x\|$	Euclidian norm of x, that is, $\|x\| = \sqrt{x^{\mathrm{T}}x}$
$\|G(j\omega)\|_\infty$	$\sup_{\omega \in \mathbb{R}} \sigma_{\max}[G(j\omega)]$

2. ANALYSIS OF SYSTEMS WITH SATURATION NONLINEARITIES

Consider the closed-loop system

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}(\sigma(u(t)) - u(t)),\ \tilde{x}(0) = \tilde{x}_0 \tag{1}$$

$$u(t) = \tilde{C}\tilde{x}(t) \tag{2}$$

where $\tilde{x} \in \mathbb{R}^{\tilde{n}}$, $u \in \mathbb{R}^m$, $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ are real matrices of compatible dimension, and $\sigma: \mathbb{R}^m \to \mathbb{R}^m$ is a multivariable saturation nonlinearity. We assume that $\sigma(\cdot)$ is a *radial* ellipsoidal saturation function, that is, $\sigma(u)$ has the same direction as u and is confined to an ellipsoidal region in $\mathbb{R}^n$. Letting R denote an $m \times m$ positive-definite matrix, $\sigma(u)$ is defined by

$$\sigma(u) = u, \qquad u^{\mathrm{T}}Ru \leq 1 \tag{3}$$

$$= (u^{\mathrm{T}}Ru)^{-1/2}u, \qquad u^{\mathrm{T}}Ru > 1 \tag{4}$$

Alternatively, $\sigma(u)$ can be written as

$$\sigma(u) = \beta(u)u \tag{5}$$

where the function β: $\mathbb{R}^m \to (0, 1]$ is defined by

$$\beta(u) = 1, \qquad u^{\mathrm{T}}Ru \leq 1 \tag{6}$$

$$= \frac{1}{\sqrt{u^{\mathrm{T}}Ru}}, \quad u^{\mathrm{T}}Ru > 1 \tag{7}$$

The closed-loop system (1), (2) can be represented by the block diagram shown in Figure 1. Note that in the SISO case $m = 1$, the function $u - \sigma(u)$ shown in Figure 1 is a deadzone nonlinearity.

For the statement of Theorem 2.1, define the function β_0: $(0, \infty] \to [0, 1]$ by

$$\beta_0(\gamma) = 0, \qquad 0 < \gamma \leq 1$$

$$= 1 - \frac{1}{\gamma}, \quad \gamma > 1$$

$$= 1, \qquad \gamma = \infty$$

The following result provides the foundation for our synthesis approach.

Theorem 2.1

Let $\tilde{R}_1 \in \mathbf{N}^{\tilde{n}}$, $R_2 \in \mathbf{P}^m$, and $\gamma \in (0, \infty]$, and assume that $(\tilde{A}, \tilde{R}_1 + \tilde{C}^T R_2 \tilde{C})$ is observable. Furthermore, suppose there exists $\tilde{P} \in \mathbf{P}^{\tilde{n}}$ satisfying

$$0 = \tilde{A}^{\mathrm{T}}\tilde{P} + \tilde{P}\tilde{A} + \tilde{R}_1 + \tilde{C}^{\mathrm{T}}R_2\tilde{C} + \gamma^{-2}\tilde{P}\tilde{B}R_2^{-1}\tilde{B}^{\mathrm{T}}\tilde{P} \tag{8}$$

Then the closed-loop system (1) and (2) is asymptotically stable with Lyapunov function $V(\tilde{x}) = \tilde{x}^{\mathrm{T}}\tilde{P}\tilde{x}$, and the set

$$\tilde{\mathfrak{D}} \triangleq \left\{\tilde{x}_0 \in \mathbb{R}^{\tilde{n}}: \tilde{x}_0^{\mathrm{T}}\tilde{P}\tilde{x}_0 < \frac{1}{\beta_0^2(\gamma)\lambda_{\max}(\tilde{C}^{\mathrm{T}}R\tilde{C}\tilde{P}^{-1})}\right\} \tag{9}$$

is a subset of the domain of attraction of the closed-loop system. Finally, the cost functional

$$J(\tilde{x}_0) \triangleq \int_0^\infty [\tilde{x}^{\mathrm{T}}(t)(\tilde{R}_1 + \gamma^{-2}\tilde{P}\tilde{B}R_2^{-1}\tilde{B}^{\mathrm{T}}\tilde{P})\tilde{x}(t) + u^{\mathrm{T}}(t)R_2u(t) + 2\tilde{x}^{\mathrm{T}}(t)\tilde{P}\tilde{B}(u(t) - \sigma(u(t)))]\,\mathrm{d}t \tag{10}$$

is given by $J(\tilde{x}_0) = \tilde{x}_0^{\mathrm{T}}\tilde{P}\tilde{x}_0$.

Proof. First consider the case $\gamma = \infty$, that is, $\beta_0(\gamma) = 1$. Letting $\tilde{x}_0 \in \tilde{\mathfrak{D}}$ it follows that

$$\begin{aligned} u^{\mathrm{T}}(0)Ru(0) &= \tilde{x}_0^{\mathrm{T}}\tilde{C}^{\mathrm{T}}R\tilde{C}\tilde{x}_0 \\ &= \tilde{x}_0^{\mathrm{T}}\tilde{P}^{1/2}\tilde{P}^{-1/2}\tilde{C}^{\mathrm{T}}R\tilde{C}\tilde{P}^{-1/2}\tilde{P}^{1/2}\tilde{x}_0 \\ &\leq \tilde{x}_0^{\mathrm{T}}\tilde{P}^{1/2}\lambda_{\max}(\tilde{P}^{-1/2}\tilde{C}^{\mathrm{T}}R\tilde{C}\tilde{P}^{-1/2})\tilde{P}^{1/2}\tilde{x}_0 \\ &= \tilde{x}_0^{\mathrm{T}}\tilde{P}\tilde{x}_0\lambda_{\max}(\tilde{C}^{\mathrm{T}}R\tilde{C}\tilde{P}^{-1}) \\ &< 1 \end{aligned}$$

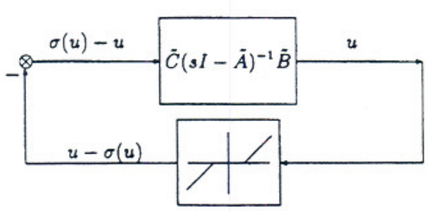

Figure 1. Closed-loop system with a deadzone nonlinearity in negative feedback

so that $\beta(u(0)) = 1$. Letting $\tilde{x}(t)$ satisfy (1) and (2) and using (8) it follows that

$$\begin{aligned}\dot{V}(\tilde{x}(t)) &= \tilde{x}^{\mathrm{T}}(t)(\tilde{A}^{\mathrm{T}}\tilde{P} + \tilde{P}\tilde{A})\tilde{x}(t) + (\sigma(u(t)) - u(t))^{\mathrm{T}}\tilde{B}^{\mathrm{T}}\tilde{P}\tilde{x}(t) + \tilde{x}^{\mathrm{T}}(t)\tilde{P}\tilde{B}(\sigma(u(t)) - u(t)) \\ &= -\tilde{x}^{\mathrm{T}}(t)[\tilde{R}_1 + \tilde{C}^{\mathrm{T}}R_2\tilde{C} + (1 - \beta(u(t)))(\tilde{C}^{\mathrm{T}}\tilde{B}^{\mathrm{T}}\tilde{P} + \tilde{P}\tilde{B}\tilde{C}) + \gamma^{-2}\tilde{P}\tilde{B}R_2^{-1}\tilde{B}^{\mathrm{T}}\tilde{P}]\tilde{x}(t) \qquad (11)\end{aligned}$$

and thus $\dot{V}(\tilde{x}(0)) \leq 0$. Two cases, that is, $\dot{V}(\tilde{x}(0)) < 0$ and $\dot{V}(\tilde{x}(0)) = 0$, will be treated separately.

First consider the case $\dot{V}(\tilde{x}(0)) < 0$. Suppose there exist $T_1 > T > 0$ such that $\dot{V}(\tilde{x}(t)) < 0$ for all $t \in [0, T)$, $\dot{V}(\tilde{x}(T)) = 0$, and $\dot{V}(\tilde{x}(t)) > 0$, $t \in (T, T_1]$. Since $\dot{V}(\tilde{x}(t)) < 0$, $t \in [0, T)$, there exists T_2 satisfying $T < T_2 \leq T_1$ and sufficiently close to T such that $\tilde{x}^{\mathrm{T}}(t)\tilde{P}\tilde{x}(t) = V(\tilde{x}(t)) < V(\tilde{x}_0) = \tilde{x}_0^{\mathrm{T}}\tilde{P}\tilde{x}_0$, $t \in (0, T_2]$, and thus

$$\begin{aligned}u^{\mathrm{T}}(t)Ru(t) &= \tilde{x}^{\mathrm{T}}(t)\tilde{C}^{\mathrm{T}}R\tilde{C}\tilde{x}(t) \\ &\leq \tilde{x}^{\mathrm{T}}(t)\tilde{P}\tilde{x}(t)\lambda_{\max}(\tilde{C}^{\mathrm{T}}R\tilde{C}\tilde{P}^{-1}) \\ &< \tilde{x}_0^{\mathrm{T}}\tilde{P}\tilde{x}_0\lambda_{\max}(\tilde{C}^{\mathrm{T}}R\tilde{C}\tilde{P}^{-1}) \\ &< 1\end{aligned}$$

$t \in [0, T_2]$. Therefore, $\beta(u(\mathrm{t})) = 1$, $t \in [0, T_2]$. Since $\dot{V}(\tilde{x}(t)) > 0$, $t \in (T, T_1]$, it follows from (11) that $\beta(u(t)) < 1$, $t \in (T, T_1]$. Therefore, $\beta(u(T_2)) < 1$, which is a contradiction. Hence, $\beta(u(t)) = 1$ for all $t \geq 0$ and thus $\sigma(u(t)) = u(t)$ and $\tilde{x}(t) = \exp(\tilde{A}t)\tilde{x}_0$ for all $t \geq 0$, which implies that

$$\dot{V}(\tilde{x}(t)) = -\tilde{x}_0^{\mathrm{T}}\exp(\tilde{A}^{\mathrm{T}}t)\,[\tilde{R}_1 + \tilde{C}^{\mathrm{T}}R_2\tilde{C} + \gamma^{-2}\tilde{P}\tilde{B}R_2^{-1}\tilde{B}^{\mathrm{T}}\tilde{P}]\exp(\tilde{A}t)\tilde{x}_0 \leq 0$$

for all $t \geq 0$. Since $(\tilde{A}, \tilde{R}_1 + \tilde{C}^{\mathrm{T}}R_2\tilde{C})$ is observable, it follows that the invariant set consists of $\tilde{x} = 0$. Hence $V(\tilde{x}(t))$ is nonincreasing and approaches zero as $t \to \infty$.[25]

Next, consider the case $\dot{V}(\tilde{x}(0)) = 0$. Since $\beta(u(0)) = 1$, it follows that $\dot{V}(\tilde{x}(0)) = 0$ implies that $u(0) = 0$, that is, $u^{\mathrm{T}}(0)Ru(0) = 0$.Since, for $t > 0$, $\dot{V}(\tilde{x}(t)) > 0$ implies that $\beta(u(t)) < 1$, that is, $u^{\mathrm{T}}(t)Ru(t) > 1$, it follows that there exists $T_0 > 0$ sufficiently close to 0 such that $\dot{V}(\tilde{x}(t)) \leq 0$ for all $t \in (0, {}_0]$. Using similar arguments as in the case $\dot{V}(\tilde{x}(0)) < 0$, it can be shown that $\dot{V}(\tilde{x}(t)) \neq 0$ for all $t \in (0, T_0]$. Therefore, $\dot{V}(\tilde{x}(t)) < 0$ for all $t \in (0, T_0]$. In particular, $\dot{V}(\tilde{x}(T_0)) < 0$. Hence we can proceed as in the previous case where $\dot{V}(\tilde{x}(0)) < 0$ with the time 0 replaced by T_0. It thus follows that $\dot{V}(\tilde{x}(t)) \to 0$ as $t \to \infty$ and the closed-loop system (1), (2) is asymptotically stable.

Next consider the case $1 < \gamma < \infty$, that is, $\beta_0(\gamma) = 1 - \gamma^{-1}$. Letting $\tilde{x}_0 \in \tilde{\mathfrak{D}}$, we have

$$u^{\mathrm{T}}(0)Ru(0) = \tilde{x}_0^{\mathrm{T}}\tilde{C}^{\mathrm{T}}R\tilde{C}\tilde{x}_0 \leq \tilde{x}_0^{\mathrm{T}}\tilde{P}\tilde{x}_0\lambda_{\max}(\tilde{C}^{\mathrm{T}}R\tilde{C}\tilde{P}^{-1}) < \frac{1}{\beta_0^2(\gamma)}$$

which, since $\beta_0(\gamma) < 1$, implies that $\beta(u(0)) > \beta_0(\gamma)$. Next, note that $\dot{V}(\tilde{x}(t))$ can be written as

$$\begin{aligned}\dot{V}(\tilde{x}(t)) = &-\frac{\beta(u(t)) - \beta_0(\gamma)}{1 - \beta_0(\gamma)}\,\tilde{x}^{\mathrm{T}}(t)(\tilde{R}_1 + \tilde{C}^{\mathrm{T}}R_2\tilde{C} + \gamma^{-2}\tilde{P}\tilde{B}R_2^{-1}\tilde{B}^{\mathrm{T}}\tilde{P})\tilde{x}(t) \\ &-\frac{1 - \beta(u(t))}{1 - \beta_0(\gamma)}\,\tilde{x}^{\mathrm{T}}(t)[\tilde{R}_1 + \tilde{C}^{\mathrm{T}}R_2\tilde{C} + \gamma^{-2}\tilde{P}\tilde{B}R_2^{-1}\tilde{B}^{\mathrm{T}}\tilde{P} + (1 - \beta_0(\gamma))(\tilde{C}^{\mathrm{T}}\tilde{B}^{\mathrm{T}}\tilde{P} + \tilde{P}\tilde{B}\tilde{C})]\tilde{x}(t) \\ = &-\frac{\beta(u(t)) - \beta_0(\gamma)}{1 - \beta_0(\gamma)}\,\tilde{x}^{\mathrm{T}}(t)(\tilde{R}_1 + \tilde{C}^{\mathrm{T}}R_2\tilde{C} + \gamma^{-2}\tilde{P}\tilde{B}R_2^{-1}\tilde{B}^{\mathrm{T}}\tilde{P})\tilde{x}(t) \\ &-\frac{1 - \beta(u(t))}{1 - \beta_0(\gamma)}\,\tilde{x}^{\mathrm{T}}(t)[\tilde{R}_1 + (\tilde{C} + \gamma^{-1}R_2^{-1}\tilde{B}^{\mathrm{T}}\tilde{P})^{\mathrm{T}}R_2(\tilde{C} + \gamma^{-1}R_2^{-1}\tilde{B}^{\mathrm{T}}\tilde{P})]\tilde{x}(t) \qquad (12)\end{aligned}$$

Hence, $\beta(u(0)) > \beta_0(\gamma)$ implies that $\dot{V}(\tilde{x}(0)) \leqslant 0$. For the case $\dot{V}(\tilde{x}(0)) = 0$ the procedure used to prove asymptotic stability is similar to the case $\gamma = \infty$. Here, we prove only the case $\dot{V}(\tilde{x}(0)) < 0$.

Assuming $\dot{V}(\tilde{x}(0)) < 0$, there exist $T_1 > T > 0$ such that $\dot{V}(\tilde{x}(t)) < 0$ for all $t \in [0, T)$, $\dot{V}(\tilde{x}(T)) = 0$, and $\dot{V}(\tilde{x}(t)) > 0$, $t \in (T, T_1]$. Since $\dot{V}(\tilde{x}(t)) < 0$, $t \in [0, T)$, there exists T_2 satisfying $T < T_2 \leqslant T_1$ and sufficiently close to T such that $\tilde{x}^{\mathrm{T}}(t)\tilde{P}\tilde{x}(t) = V(\tilde{x}(t)) < V(\tilde{x}_0) = \tilde{x}_0^{\mathrm{T}}\tilde{P}\tilde{x}_0$, $t \in (0, T_2]$. Thus, if $t \in [0, T_2]$ is such that $\beta(u(t)) < 1$, then

$$\begin{aligned}
\beta(u(t)) &= \frac{1}{\sqrt{u^{\mathrm{T}}(t)Ru(t)}} \\
&= \frac{1}{\sqrt{\tilde{x}^{\mathrm{T}}(t)\tilde{C}^{\mathrm{T}}R\tilde{C}\tilde{x}(t)}} \\
&= \frac{1}{\sqrt{\tilde{x}^{\mathrm{T}}(t)\tilde{P}^{1/2}\tilde{P}^{-1/2}\tilde{C}^{\mathrm{T}}R\tilde{C}\tilde{P}^{-1/2}\tilde{P}^{1/2}\tilde{x}(t)}} \\
&\geqslant \frac{1}{\sqrt{\lambda_{\max}(\tilde{P}^{-1/2}\tilde{C}^{\mathrm{T}}R\tilde{C}\tilde{P}^{-1/2})\tilde{x}^{\mathrm{T}}(t)\tilde{P}\tilde{x}(t)}} \\
&= \frac{1}{\sqrt{\lambda_{\max}(\tilde{C}^{\mathrm{T}}R\tilde{C}\tilde{P}^{-1})(\tilde{x}_0^{\mathrm{T}}\tilde{P}\tilde{x}_0 - \delta(t))}} \\
&\geqslant \frac{1}{\sqrt{\lambda_{\max}(\tilde{C}^{\mathrm{T}}R\tilde{C}\tilde{P}^{-1})\tilde{x}_0^{\mathrm{T}}\tilde{P}\tilde{x}_0}} \\
&> \beta_0(\gamma)
\end{aligned}$$

where $\delta(t) \triangleq \int_0^t -\dot{V}(\tilde{x}(t))\,dt = V(\tilde{x}_0) - V(\tilde{x}(t)) \geqslant 0$, $t \in [0, T_2]$. Therefore, $\beta_0(\gamma) < \beta(u(t))$. On the other hand, if $t \in [0, T_2]$ is such that $\beta(u(t)) = 1$, then $\beta_0(\gamma) < \beta(u(t))$. Hence, $\beta_0(\gamma) < \beta(u(t))$ for all $t \in [0, T_2]$. In particular, $\beta(u(T_2)) > \beta_0(\gamma)$. Since $\dot{V}(\tilde{x}(t)) > 0$, $t \in (T, T_1]$, it follows from (12) that $\beta(u(t)) < \beta_0(\gamma)$, $t \in (T, T_1]$. Therefore, $\beta(u(T_2)) < \beta_0(\gamma)$, which is a contradiction. Therefore, $\dot{V}(\tilde{x}(t)) \leqslant 0$ for all $t \geqslant 0$ and thus $\beta(u(t)) > \beta_0(\gamma)$ for all $t \geqslant 0$.

If $\dot{V}(\tilde{x}(t)) = 0$, it follows from (12) that $u(t) = \tilde{C}\tilde{x}(t) = 0$, which gives $\tilde{x}(t) = \exp(\tilde{A}t)\tilde{x}_0$. Since $(\tilde{\mathrm{A}}, \tilde{\mathrm{R}}_1 + \tilde{C}^{\mathrm{T}}R_2\tilde{C})$ is observable it follows that $(\tilde{A}, \tilde{R}_1 + \tilde{C}^{\mathrm{T}}R_2\tilde{C} + \gamma^{-2}\tilde{P}\tilde{B}R_2^{-1}\tilde{B}^{\mathrm{T}}\tilde{P})$ is observable. Therefore, the invariant set consists of $\tilde{x} = 0$. It thus follows that $V(\tilde{x}(t)) \to 0$ as $t \to \infty$ and closed-loop system (1), (2) is asymptotically stable.

For the case $0 < \gamma \leqslant 1$, that is, $\beta_0(\gamma) = 0$, we have

$$\begin{aligned}
\dot{V}(\tilde{x}(t)) = &-\beta(u(t))\tilde{x}^{\mathrm{T}}(t)[\tilde{\mathrm{R}}_1 + \tilde{C}^{\mathrm{T}}R_2\tilde{C} + \gamma^{-2}\tilde{P}\tilde{B}R_2^{-1}\tilde{B}^{\mathrm{T}}\tilde{P}]\tilde{x}(t) \\
&- (1 - \beta(u(t)))\tilde{x}^{\mathrm{T}}(t)[\tilde{R}_1 + (1 - \gamma^2)\tilde{C}^{\mathrm{T}} R_2\tilde{C}]\tilde{x}(t) \\
&- (1 - \beta(u(t)))\tilde{x}^{\mathrm{T}}(t)[\gamma\tilde{C} + \gamma^{-1}R_2^{-1}\tilde{B}^{\mathrm{T}}\tilde{P}]^{\mathrm{t}}R_2[\gamma\tilde{C} + \gamma^{-1}R_2^{-1}\tilde{B}^{\mathrm{T}}\tilde{P}]\tilde{x}(t)
\end{aligned}$$

The remaining steps are similiar to those in the case $1 < \gamma < \infty$.

Finally, since $\tilde{x}(t) \to 0$ as $t \to \infty$, the cost cost (10) is given by

$$\begin{aligned}
J(\tilde{x}(t)) &= \int_0^\infty [\tilde{x}^{\mathrm{T}}(t)(\tilde{R}_1 + \gamma^{-2}\tilde{P}\tilde{B}R_2^{-1}\tilde{B}^{\mathrm{T}}\tilde{P})\tilde{x}(t) + u^{\mathrm{T}}(t)R_2u(t) + 2\tilde{x}^{\mathrm{T}}(t)\tilde{P}\tilde{B}(\sigma(u(t)) - u(t))]\,\mathrm{d}t \\
&= \int_0^\infty \tilde{x}^{\mathrm{T}}(t)[\tilde{R}_1 + \tilde{C}^{\mathrm{T}}R_2\tilde{C} + (\beta(u(t)) - 1)(\tilde{P}\tilde{B}\tilde{C} + \tilde{C}^{\mathrm{T}}\tilde{B}^{\mathrm{T}}\tilde{P}) + \gamma^{-2}\tilde{P}\tilde{B}R_2^{-1}\tilde{B}^{\mathrm{T}}\tilde{P}]\tilde{x}(t)\,\mathrm{d}t \\
&= \int_0^\infty -\dot{V}(\tilde{x}(t))\,\mathrm{d}t \\
&= V(\tilde{x}_0)
\end{aligned}$$

□

Remark 2.1

Theorem 2.1 can be viewed as an application of the small gain theorem to a deadzone nonlinearity. To see this, note that since (8) has a positive-definite solution $\tilde{P}$ it follows that $\|\tilde{G}(j\omega)\|_\infty \leq \gamma$, where $\tilde{G}(s) \triangleq \tilde{C}(sI-\tilde{A})^{-1}\tilde{B}$. Furthermore, since $\beta(u(t)) \geq \beta_0(\gamma)$ for all $t \geq 0$, the gain of the deadzone nonlinearity lies in $[0, 1-\beta_0(\gamma)]$. Since $1-\beta_0(\gamma)=1/\gamma$ for $\gamma \geq 1$ and $1-\beta_0(\gamma)<1/\gamma$ for $0<\gamma<1$, we have $\gamma(1-\beta_0(\gamma)) \leq 1$ for all $\gamma>0$. If, furthermore, $\gamma(1-\beta_0(\gamma))<1$, then it follows from the small gain theorem that the closed-loop system is asymptotically stable. This approach to stability with a saturation nonlinearity is closely related to References 9, 16 and 19. However, the novel feature of Theorem 2.1 is the proof that $\beta(u(t)) \geq \beta_0(\gamma)$, $t \in [0,\infty)$, for initial conditions that lie in the subset $\tilde{\mathfrak{D}}$ of the domain of attraction. In contrast, the results of References 9, 16, 19 require an explicit *a priori* assumption on the magnitude of the control input.

Remark 2.2

If $0<\gamma<1$ then $\beta(\gamma)=0$ and the system (1), (2) is globally asymptotically stable.

Remark 2.3

The cost $J(\tilde{x}_0)$ defined by (10) is similar to the H_2 cost of LQG theory with additional terms. The quadratic terms involving $\tilde{x}$ and u can be used to adjust the control authority. Although the additional terms are indefinite, Theorem 2.1 shows that the integrand of $J(\tilde{x}_0)$ is nonnegative. The closed-form expression $J(\tilde{x}_0)=\tilde{x}_0^{\mathrm{T}}\tilde{P}\tilde{x}_0$ will be used within an optimization procedure to determine stabilizing feedback gains. This procedure is carried out in the following section.

3. LINEAR CONTROLLER SYNTHESIS

Consider the plant

$$\dot{x}(t)=Ax(t)+B\sigma(u(t)), \qquad x(0)=x_0 \tag{13}$$

$$y(t)=Cx(t) \tag{14}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^l$, (A,B) is controllable, (A,C) is observable, and let the linear controller have the form

$$\dot{x}_c(t)=A_c x_c(t)+B_c y(t), \qquad x_c(0)=x_{c0} \tag{15}$$

$$u(t)=C_c x_c(t) \tag{16}$$

where $x_c \in \mathbb{R}^{n_c}$ and $n_c \leq n$. Then the closed-loop system can be written in the form of (1), (2) with

$$\tilde{x} \triangleq \begin{bmatrix} x \\ x_c \end{bmatrix}, \quad \tilde{x}_0 \triangleq \begin{bmatrix} x_0 \\ x_{c0} \end{bmatrix}, \quad \tilde{A} \triangleq \begin{bmatrix} A & BC_c \\ B_cC & A_c \end{bmatrix}, \quad \tilde{B} \triangleq \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \tilde{C} \triangleq [0 \quad C_c]$$

Our goal is to optimize the closed-loop cost $J(\tilde{x}(0))=\tilde{x}_0^{\mathrm{T}}\tilde{P}\tilde{x}_0$ given by Theorem 2.1 with respect to the controller matrices A_c, B_c, C_c. To do this note that $J(\tilde{x}(0))=\operatorname{tr}\tilde{P}\tilde{x}_0\tilde{x}_0^{\mathrm{T}}$, which has the same form as the H_2 cost in LQG theory, which has the form

$$\hat{J}(A_c,B_c,C_c)=\operatorname{tr}\tilde{P}\tilde{V}, \quad \tilde{V}=\begin{bmatrix} V_1 & 0 \\ 0 & B_cV_2B_c^{\mathrm{T}} \end{bmatrix} \tag{17}$$

where $V_1 \in \mathbf{N}^n$ and $V_2 \in \mathbf{P}^l$ denote plant disturbance and measurement noise intensity matrices, respectively. It is therefore convenient to replace $\tilde{x}_0\tilde{x}_0^{\mathrm{T}}$ by $\tilde{V}$ and proceed by determining controller matrices that minimize this LQG-type cost. Furthermore, let

$$\tilde{R}_1 = \begin{bmatrix} R_1 & 0 \\ 0 & 0 \end{bmatrix}$$

where $R_1 \in \mathbb{N}^n$.

We first consider the full-order controller case, that is, $n_c = n$. The following result as well as Proposition 3.2 and later results is obtained by minimizing $\tilde{J}(A_c, B_c, C_c)$ with respect to A_c, B_c, C_c. These necessary conditions then provide sufficient conditions for closed-loop stability by applying Theorem 2.1. For convenience define $\Sigma \triangleq BR_2^{-1}B^{\mathrm{T}}$ and $\bar{\Sigma}C^{\mathrm{T}}V_2^{-1}C$.

Proposition 3.1

Let $n_c = n$, $\gamma \in (0, \infty]$, suppose there exist $n \times n$ nonnegative-definite matrices P, Q, $\hat{P}$ satisfying

$$0 = A^{\mathrm{T}}P + PA + R_1 - (1 - \gamma^{-2})P\Sigma P \tag{18}$$

$$0 = (A - Q\bar{\Sigma} + \gamma^{-2}\Sigma P)^{\mathrm{T}}\hat{P} + \hat{P}(A - Q\bar{\Sigma} + \gamma^{-2}\Sigma P) + P\Sigma P + \gamma^{-2}\hat{P}\Sigma\hat{P} \tag{19}$$

$$0 = (A + \gamma^{-2}\Sigma(P + \hat{P}))Q + Q(A + \gamma^{-2}\Sigma(P + \hat{P}))^{\mathrm{T}} + V_1 - Q\bar{\Sigma}Q \tag{20}$$

and let A_c, B_c, C_c be given by

$$A_c = A + (1 - \gamma^{-2})BC_c - B_cC \tag{21}$$

$$B_c = QC^{\mathrm{T}}V_2^{-1} \tag{22}$$

$$C_c = -R_2^{-1}B^{\mathrm{T}}P \tag{23}$$

Furthermore, suppose that $(\tilde{A}, \tilde{R}_1 + \tilde{C}^{\mathrm{T}}R_2\tilde{C})$ is observable. Then

$$\tilde{P} = \begin{bmatrix} P + \hat{P} & -\hat{P} \\ -\hat{P} & \hat{P} \end{bmatrix}$$

satisfies (8), and (A_c, B_c, C_c) is an extremal of $\hat{J}(A_c, B_c, C_c)$. Furthermore, the closed-loop system (1), (2) is asymptotically stable, and $\tilde{\mathfrak{D}}$ defined by (9) is a subset of the domain of attraction of the closed-loop system.

Proof. The proof is similar to the proof of Proposition 3.2 below with $n_c = n$ and $\Gamma = G^{\mathrm{T}} = \tau = I$.

Remark 3.1

Proposition 3.1 can be viewed as a direct extension of the standard LQG result. Specifically, by setting $\gamma = \infty$, equations (18) and (20) specialize to the usual regulator and estimator Riccati equations, while equation (19) plays no role.

Next we consider the case $n_c \leq n$. The following lemma is required.

Lemma 3.1

Let $\hat{P}$, $\hat{Q}$ be $n \times n$ nonnegative-definite matrices and suppose that rank $\hat{Q}\hat{P} = n_c$. Then there exist $n_c \times n$ matrices G, Γ and an $n_c \times n_c$ invertible matrix M, unique except for a change of basis in $\mathbb{R}^{n_c}$, such that

$$\operatorname{rank} \hat{Q} = \operatorname{rank} \hat{P} = \operatorname{rank} \hat{Q}\hat{P} = n_c \tag{24}$$

$$\hat{Q}\hat{P} = G^{T}M\Gamma, \qquad \Gamma G^{T} = I_{n_c} \tag{25}$$

Furthermore, the $n \times n$ matrices

$$\tau \triangleq G^{T}\Gamma, \qquad \tau_{\perp} \triangleq I_n - \tau \tag{26}$$

are idempotent and have rank n_c and $n - n_c$, respectively.

Proof. See Reference 21 for details. □

Proposition 3.2

Let $n_c \leqslant n$, $\gamma \in (0, \infty]$, suppose there exist $n \times n$ nonnegative-definite matrices P, Q, $\hat{P}$, $\hat{Q}$ satisfying

$$0 = A^{T}P + PA + R_1 - (1 - \gamma^{-2})P\Sigma P + \tau_{\perp}^{T}P\Sigma P\tau_{\perp} \tag{27}$$

$$0 = (A - Q\bar{\Sigma} + \gamma^{-2}\Sigma P)\hat{P} + \hat{P}(A - Q\bar{\Sigma} + \gamma^{-2}\Sigma P) + P\Sigma P + \gamma^{-2}\hat{P}\Sigma\hat{P} - \tau_{\perp}^{T}P\Sigma P\tau_{\perp} \tag{28}$$

$$0 = (A + \gamma^{-2}\Sigma(P + \hat{P}))Q + Q(A + \gamma^{-2}\Sigma(P + \hat{P}))^{T} + V_1 - Q\Sigma Q + \tau_{\perp}Q\bar{\Sigma}Q\tau_{\perp}^{T} \tag{29}$$

$$0 = (A - (1 - \gamma^{-2})\Sigma P)\hat{Q} + \hat{Q}(A - (1 - \gamma^{-2})\Sigma P)^{T} + Q\bar{\Sigma}Q - \tau_{\perp}Q\bar{\Sigma}Q\tau_{\perp}^{T} \tag{30}$$

and let A_c, B_c, C_c be given by

$$A_c = \Gamma AG^{T} + (1 - \gamma^{-2})\Gamma BC_c - B_cCG^{T} \tag{31}$$

$$B_c = \Gamma QC^{T}V_2^{-1} \tag{32}$$

$$C_c = -R_2^{-1}B^{T}PG^{T} \tag{33}$$

Furthermore, suppose that $(\tilde{A}, \tilde{R}_1 + \tilde{C}^{T}R_2\tilde{C})$ is observable. Then

$$\tilde{P} = \begin{bmatrix} P + \hat{P} & -\hat{P}G^{T} \\ -G\hat{P} & G\hat{P}G^{T} \end{bmatrix}$$

satisfies (8), and (A_c, B_c, C_c) is an extremal of $\hat{J}(A_c, B_c, C_c)$. Furthermore, the closed-loop system (1), (2) is asymptotically stable, and $\tilde{\mathfrak{D}}$ defined by (9) is a subset of the domain of attraction of the closed-loop system.

Proof. The result is obtained by applying the Lagrange multiplier technique to performance subject to (8) and by partitioning $\tilde{P}$, $\tilde{Q}$ as

$$\tilde{P} = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^{T} & P_2 \end{bmatrix}, \quad \tilde{Q} = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{21}^{T} & Q_2 \end{bmatrix}$$

Here, we show only the key steps. First, define the Lagrangian

$$\mathcal{L} = \operatorname{tr} \tilde{P}\tilde{V} + \operatorname{tr} \tilde{Q}(\tilde{A}^{T}\tilde{P} + \tilde{P}\tilde{A} + \tilde{R}_1 + \tilde{C}^{T}R_2\tilde{C} + \gamma^{-2}\tilde{P}\tilde{B}R_2^{-1}\tilde{B}^{T}\tilde{P})$$

Taking derivatives with respect to A_c, B_c, C_c and $\tilde{P}$, and setting them to zero yields

$$0 = \frac{\partial \mathcal{L}}{\partial A_c} = 2(P_{12}^{\mathrm{T}} Q_{12} + P_2 Q_2) \tag{34}$$

$$0 = \frac{\partial \mathcal{L}}{\partial B_c} = 2P_2 B_c V_2 + 2(P_{12}^{\mathrm{T}} Q_1 + P_2 Q_{12}^{\mathrm{T}})C^{\mathrm{T}} \tag{35}$$

$$0 = \frac{\partial \mathcal{L}}{\partial C_c} = 2R_2 C_c Q_2 + 2B^{\mathrm{T}}(P_1 Q_{12} + P_{12} Q_2) \tag{36}$$

$$0 = \frac{\partial \mathcal{L}}{\partial \tilde{P}} = (\tilde{A} + \gamma^{-2}\tilde{B}R_2^{-1}\tilde{B}^{\mathrm{T}}\tilde{P})\tilde{Q} + \tilde{Q}(\tilde{A} + \gamma^{-2}\tilde{B}R_2^{-1}\tilde{B}^{\mathrm{T}}\tilde{P})^{\mathrm{T}} + \tilde{V} \tag{37}$$

Next, define $P, Q, \hat{P}, \hat{Q}, \Gamma, G, M$ by

$$P \triangleq P_1 - \hat{P},\ \hat{P} \triangleq P_{12}P_2^{-1}P_{12}^{\mathrm{T}},\ Q \triangleq Q_1 - \hat{Q},\ \hat{Q} \triangleq Q_{12}Q_2^{-1}Q_{12}^{\mathrm{T}},$$
$$G^{\mathrm{T}} \triangleq Q_{12}Q_2^{-1},\ M \triangleq Q_2 P_2,\ \Gamma \triangleq -P_2^{-1}P_{12}^{\mathrm{T}}$$

Algebraic manipulation yields B_c and C_c, given by (32) and (33). The expression (31) for A_c is obtained by combining the (1,2) and (2,2) blocks of equation (8) or (37) using (34). Equations (27) and (28) are obtained by combining the (1,1) and (1,2) blocks of equation (8). Similarly, (29) and (30) are obtained by combining the (1,1) and (1,2) blocks of equation (37). See Reference 21 for details of the algebraic manipulation. □

Remark 3.2

Suppose that $0 < \gamma < 1$ and there exists $\tilde{P} \in \mathbf{P}^{\tilde{n}}$ satisfying (8). Then Theorem 2.1 implies that the open-loop system and the compensator are both asymptotically stable. To see this note that

$$\begin{aligned}
&(\tilde{A} - \tilde{B}\tilde{C})^{\mathrm{T}}\tilde{P} + \tilde{P}(\tilde{A} - \tilde{B}\tilde{C}) \\
&\quad = -(\tilde{C}^{\mathrm{T}}\tilde{B}^{\mathrm{T}}\tilde{P} + \tilde{P}\tilde{B}\tilde{C} + \tilde{R}_1 + \tilde{C}^{\mathrm{T}}R_2\tilde{C} + \gamma^{-2}\tilde{P}\tilde{B}R_2^{-1}\tilde{B}^{\mathrm{T}}\tilde{P}) \\
&\quad = -\tilde{R}_1 - (1-\gamma^2)\tilde{C}^{\mathrm{T}}R_2\tilde{C} - (\gamma\tilde{C}^{\mathrm{T}}R_2 + \gamma^{-1}\tilde{P}\tilde{B})R_2^{-1}(\gamma R_2\tilde{C} + \gamma^{-1}\tilde{B}^{\mathrm{T}}\tilde{P}) \\
&\quad \leqslant 0,
\end{aligned}$$

where

$$\tilde{A} - \tilde{B}\tilde{C} = \begin{bmatrix} A & 0 \\ B_c C & A_c \end{bmatrix}$$

Since $(\tilde{A}, \tilde{R}_1 + \tilde{C}^{\mathrm{T}}R_2\tilde{C})$ is observable it follows that $(\tilde{A} - \tilde{B}\tilde{C}, \tilde{R}_1 + \tilde{C}^{\mathrm{T}}R_2\tilde{C})$ is observable or, equivalently, $(\tilde{A} - \tilde{B}\tilde{C}, \tilde{R}_1 + (1-\gamma^2)\tilde{C}^{\mathrm{T}}R_2\tilde{C})$ is observable, which implies that $(\tilde{A} - \tilde{B}\tilde{C}, \tilde{R}_1 + (1-\gamma^2)\tilde{C}^{\mathrm{T}}R_2\tilde{C} + (\gamma\tilde{C}^{\mathrm{T}} R_2 + \gamma^{-1}\tilde{P}\tilde{B})R_2^{-1}(\gamma R_2\tilde{C} + \gamma^{-1}\tilde{B}^{\mathrm{t}}\tilde{P})$ is observable. Lyapunov's lemma now implies that $\tilde{A} - \tilde{B}\tilde{C}$ is asymptotically stable.

Remark 3.3

For initial conditions of the form $\tilde{x}_0 = [x_0^{\mathrm{T}}\ 0]^{\mathrm{T}}$, the set $\mathfrak{D} \times \{0\}$ is a subset of $\tilde{\mathfrak{D}}$, where

$$\mathfrak{D} \triangleq \left\{x_0 \in \mathbb{R}^n : x_0^{\mathrm{T}}(P + \hat{P})x_0 < \frac{1}{\beta_0^2(\gamma)\lambda_{\max}(\tilde{C}^{\mathrm{T}}R\tilde{C}\tilde{P}^{-1})}\right\} \tag{38}$$

Remark 3.4

By setting $\gamma = \infty$, Proposition 3.2 specializes to the reduced-order LQG result.[21]

4. NONLINEAR CONTROLLER SYNTHESIS

In this section we consider the nonlinear controller

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t) + E_c(\sigma(u(t)) - u(t)), \qquad x_c(0) = x_{c0} \tag{39}$$

$$u(t) = C_c x_c(t) \tag{40}$$

where $x_c \in \mathbb{R}^{n_c}$ and $n_c \leqslant n$. Note that the compensator now includes a nonlinear term $E_c(\sigma(u(t)) - u(t))$, and the structure shown in Figure 2 is similar to the observer-based anti-windup setup studied in Reference 20. The closed-loop system can be written in the form of (1), (2) with

$$\tilde{x} \triangleq \begin{bmatrix} x \\ x_c \end{bmatrix}, \quad \tilde{x}_0 \triangleq \begin{bmatrix} x_0 \\ x_{c0} \end{bmatrix}, \quad \tilde{A} \triangleq \begin{bmatrix} A & BC_c \\ B_c C & A_c \end{bmatrix}, \quad \tilde{B} \triangleq \begin{bmatrix} B \\ E_c \end{bmatrix}, \quad \tilde{C} \triangleq [0 \quad C_c]$$

Proposition 4.1

Let $n_c = n$, $\gamma \in (0, \infty]$, suppose there exist $n \times n$ nonnegative-definite matrices $P, Q, \hat{P}$ satisfying

$$0 = A^{\mathrm{T}}P + PA + R_1 - (1 - \gamma^{-2})P\Sigma P \tag{41}$$

$$0 = (A - Q\bar{\Sigma})^{\mathrm{T}}\hat{P} + \hat{P}(A - Q\bar{\Sigma}) + P\Sigma P \tag{42}$$

$$0 = AQ + QA^{\mathrm{T}} + V_1 - Q\bar{\Sigma}Q \tag{43}$$

and let A_c, B_c, C_c, E_c, be given by

$$A_c = A + BC_c - B_c C \tag{44}$$

$$B_c = QC^{\mathrm{T}}V_2^{-1} \tag{45}$$

$$C_c = -R_2^{-1}B^{\mathrm{T}}P \tag{46}$$

$$E_c = B \tag{47}$$

Furthermore, suppose that $(\tilde{A}, \tilde{R}_1 + \tilde{C}^{\mathrm{T}}R_2\tilde{C})$ is observable. Then

$$\tilde{P} = \begin{bmatrix} P + \hat{P} & -\hat{P} \\ -\hat{P} & \hat{P} \end{bmatrix}$$

satisfies (8). Furthermore, the closed-loop system (1), (2) is asymptotically stable, and $\tilde{\mathfrak{D}}$ defined by (9) is a subset of the domain of attraction of the closed-loop system.

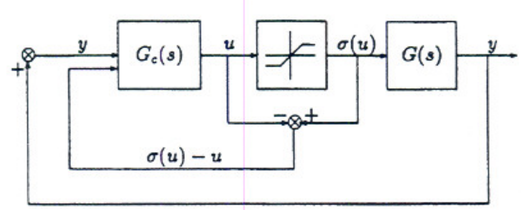

Figure 2. Closed-loop system with nonlinear anti-windup controller

Proof. The result is a special case of Proposition 4.2 with $n_c = n$. □

Proposition 4.2

Let $n_c \leq n$, $\gamma \in (0, \infty]$, suppose there exist $n \times n$ nonnegative-definite matrices $P, Q, \hat{P}, \hat{Q}$ satisfying

$$0 = A^T P + PA + R_1 - (1-\gamma^{-2})P\Sigma P + \tau_\perp^T P \Sigma P \tau_\perp \tag{48}$$

$$0 = (A - Q\bar{\Sigma})^T \hat{P} + \hat{P}(A - Q\bar{\Sigma}) + P\Sigma P - \tau_\perp^T P\Sigma P \tau_\perp \tag{49}$$

$$0 = AQ + QA^T + V_1 - Q\bar{\Sigma}Q + \tau_\perp Q\bar{\Sigma}Q\tau_\perp^T \tag{50}$$

$$0 = (A - \Sigma P)\hat{Q} + \hat{Q}(A - \Sigma P)^T + Q\bar{\Sigma}Q - \tau_\perp Q\bar{\Sigma}Q\tau_\perp^T \tag{51}$$

and let A_c, B_c, C_c, E_c be given by

$$A_c = \Gamma A G^T + \Gamma B C_c - B_c C G \tag{52}$$

$$B_c = \Gamma Q C^T V_2^{-1} \tag{53}$$

$$C_c = -R_2^{-1} B^T P G^T \tag{54}$$

$$E_c = \Gamma B \tag{55}$$

Furthermore, suppose that $(\tilde{A}, \tilde{R}_1 + \tilde{C}^T R_2 \tilde{C})$ is observable. Then

$$\tilde{P} = \begin{bmatrix} P + \hat{P} & -\hat{P} \\ -\hat{P} & \hat{P} \end{bmatrix}$$

satisfies (8). Furthermore, the closed-loop system (1), (2) is asymptotically stable, and $\tilde{\mathfrak{D}}$ defined by (9) is a subset of the domain of attraction of the closed-loop system.

Proof. Letting $V(\tilde{x}) = \tilde{x}^T \tilde{P} \tilde{x}$, it is easy to show that (8) is satisfied, and

$$\begin{aligned}\dot{V}(\tilde{x}(t)) = &-x^T[R_1 + (\gamma^{-2} - (1-\beta(u))^2)P\Sigma P]x \\ &-[C_c x_c + (1-\beta(u))R_2^{-1}BPx]^T R_2 [C_c x_c + (1-\beta(u))R_2^{-1}BPx]\end{aligned}$$

Next let $E_c = \Gamma B$ and require that $\tilde{Q}$ satisfy

$$0 = \tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T + \tilde{V}$$

The remaining steps are similiar to the proof of Proposition 3.2. □

5. NUMERICAL EXAMPLES

Example 5.1

To illustrate Proposition 3.1, consider the asymptotically stable system

$$\dot{x} = \begin{bmatrix} -0{\cdot}03 & 1 & 0 \\ 0 & -0.03 & 1 \\ 0 & 0 & -0{\cdot}03 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \sigma(u(t))$$

$$\dot{y} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x,$$

with the saturation nonlinearity $\sigma(u)$ given by

$$\sigma(u) = u, \ |u| < 4$$
$$= \text{sgn}(u)\, 4, \ |u| \geq 4$$

Choosing $R_1 = I_3$, $R_2 = 100$, $V_1 = I_3$, $V_2 = 1$, and $\gamma = 1{\cdot}05$ yields the linear controller (15), (16) with gains (21)–(23) given by

$$A_c = 10^4 \times \begin{bmatrix} -0{\cdot}0069 & 0{\cdot}0001 & 0 \\ -0{\cdot}2392 & 0{\cdot}0000 & 0{\cdot}0001 \\ -4{\cdot}6611 & 0{\cdot}0000 & 0{\cdot}0001 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0{\cdot}0069\text{e}4 \\ 0{\cdot}2392\text{e}4 \\ 4{\cdot}6611\text{e}4 \end{bmatrix}$$

$$C_c = \begin{bmatrix} -0{\cdot}3686 & -2{\cdot}6772 & -10{\cdot}0025 \end{bmatrix}$$

By applying Remark 3.3, the set $\mathfrak{D}$ is given by $\mathfrak{D} = \{x_0 \colon x_0^{\mathrm{T}}(P + \hat{P})x_0 < 1{\cdot}6498 \times 10^3\}$, where

$$P + \hat{P} = \begin{bmatrix} 4{\cdot}6916e8 & -1{\cdot}8665e7 & 2{\cdot}6908e5 \\ -1{\cdot}8665e7 & 1{\cdot}8959e6 & -6{\cdot}2702e4 \\ 2{\cdot}6908e5 & -6{\cdot}2702e4 & 3{\cdot}6318e3 \end{bmatrix}$$

To illustrate the closed-loop behaviour let $x_0 = [-40 \ \ -25 \ \ 30]^{\mathrm{T}}$ and $x_{c0} = [0 \ \ 0 \ \ 0]^{\mathrm{T}}$, respectively. Note that x_0 is not in the set $\mathfrak{D}$. As can be seen in Figure 3, the closed-loop system consisting of the saturation nonlinearity and the LQG controller designed for the 'unsaturated' plant is unstable. However, the controller designed by Proposition 3.1 provides an asymptotically stable closed-loop system. The actual domain of attraction is thus larger than $\mathfrak{D} \times \{0\}$. Figure 4 illustrates the control input $u(t)$ for the LQG controller with and without saturation as well as the output of the saturation nonlinearity $\sigma(u(t))$ for the LQG controller with saturation. Figures 5 and 6 show the control $u(t)$ and saturation input $\sigma(u(t))$ for the controller obtained from Proposition 3.1.

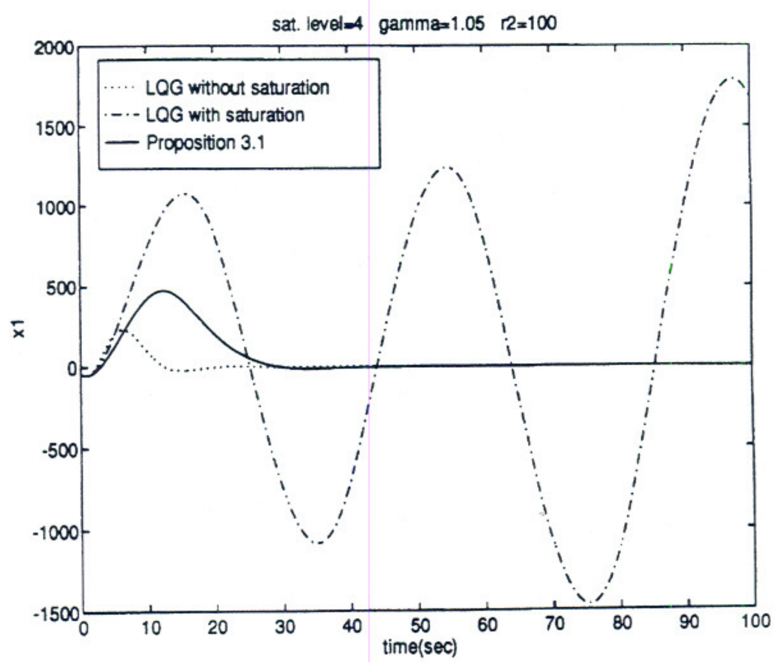

Figure 3. Time response of $x1$

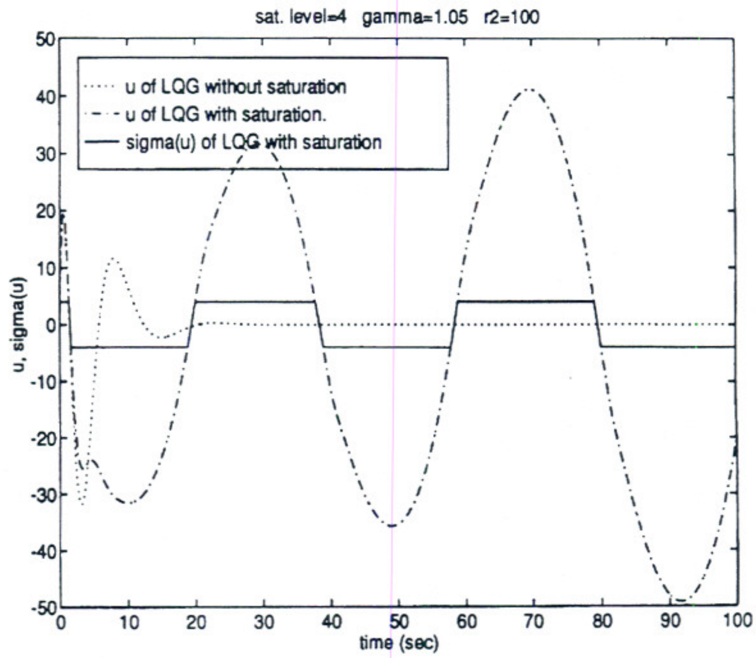

Figure 4. Control effort u and saturated input $\sigma(u)$ of the LQG controller with and without saturation

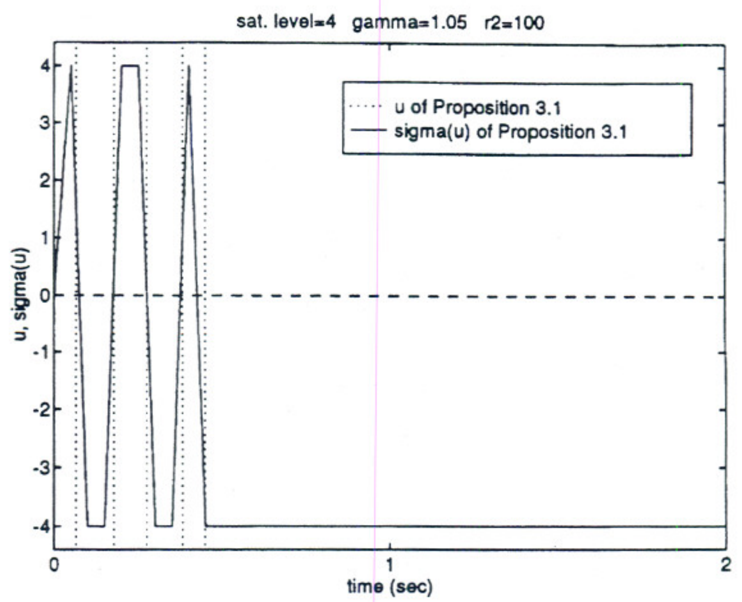

Figure 5. Control effort u and saturated input $\sigma(u)$ using Proposition 3.1 for $0 \leq t \leq 2$

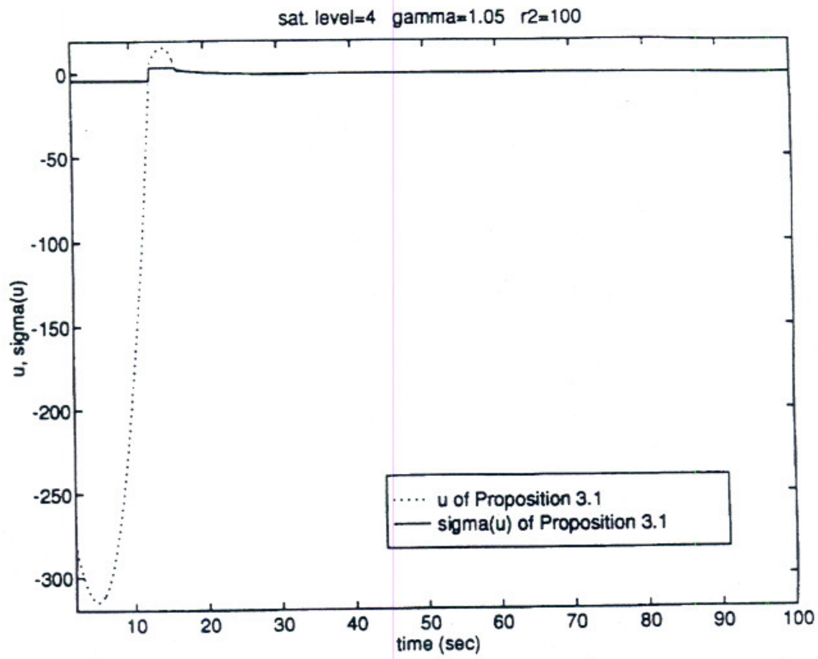

Figure 6. Control effort u and saturated input $\sigma(u)$ using Proposition 3.1 for $2 \leqslant t \leqslant 100$

Example 5.2

This example illustrates Proposition 4.1 by designing nonlinear controllers with integrators for tracking step commands. Consider the closed-loop system shown in Figure 7, where the plant $G(s) = 1/s^2$ and r is a step command. Let $G(s)$ and $G_c(s)$ have the realizations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sigma(u)$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and

$$\dot{x}_c = A_c x_c + B_c q + E_c(\sigma(u) - u)$$
$$u = C_c x_c$$

respectively. The saturation nonlinearity $\sigma(u)$ is given by

$$\sigma(u) = u, \qquad |u| < 0{\cdot}3$$
$$= \operatorname{sgn}(u)\, 0{\cdot}3, \qquad |u| \geqslant 0{\cdot}3$$

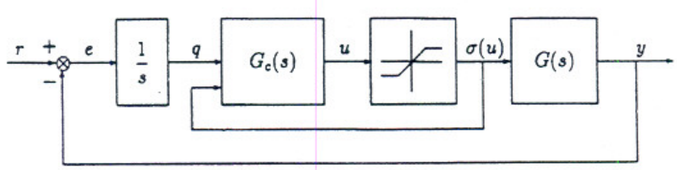

Figure 7. Block diagram for Example 5.2

To apply Theorem 2.1, we combine the plant $G(s)$ with an integrator state q to obtain the augmented plant

$$\begin{bmatrix}\dot{x}_2\\ \dot{e}\\ \dot{q}\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\ -1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_2\\ e\\ q\end{bmatrix} + \begin{bmatrix}1\\ 0\\ 0\end{bmatrix}\sigma(u(t))$$

$$q = \begin{bmatrix}0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_2\\ e\\ q\end{bmatrix}$$

which has the form of (13), (14) with

$$x = \begin{bmatrix}x_2\\ e\\ q\end{bmatrix}$$

Choosing design parameters $R_1 = I_3$, $R_2 = 100$, $V_1 = I_3$, $V_2 = 1$, and $\gamma = 1{\cdot}002$ yields the nonlinear controller (39), (40) with gains (44)–(47) given by

$$A_c = \begin{bmatrix}-165{\cdot}2148 & 27{\cdot}2773 & 3{\cdot}2366\\ -1{\cdot}0000 & 0 & -2{\cdot}4142\\ 0 & 1{\cdot}0000 & -2{\cdot}4142\end{bmatrix}, \quad B_c = \begin{bmatrix}-1\\ 2{\cdot}4142\\ 2{\cdot}4142\end{bmatrix}$$

$$C_c = \begin{bmatrix}-165{\cdot}2148 & 27{\cdot}2773 & 2{\cdot}2366\end{bmatrix}, \quad E_c = \begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$$

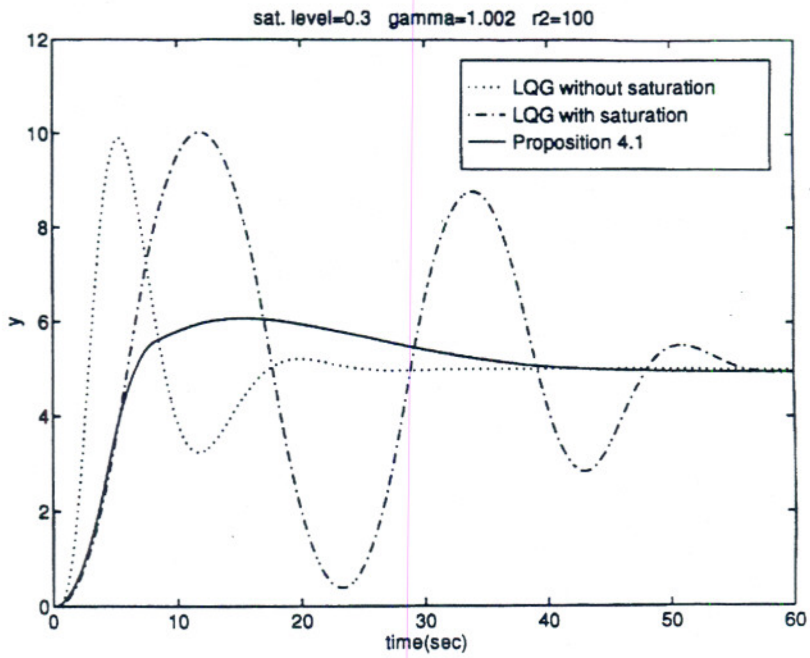

Figure 8. Time response of y with $r = 5$

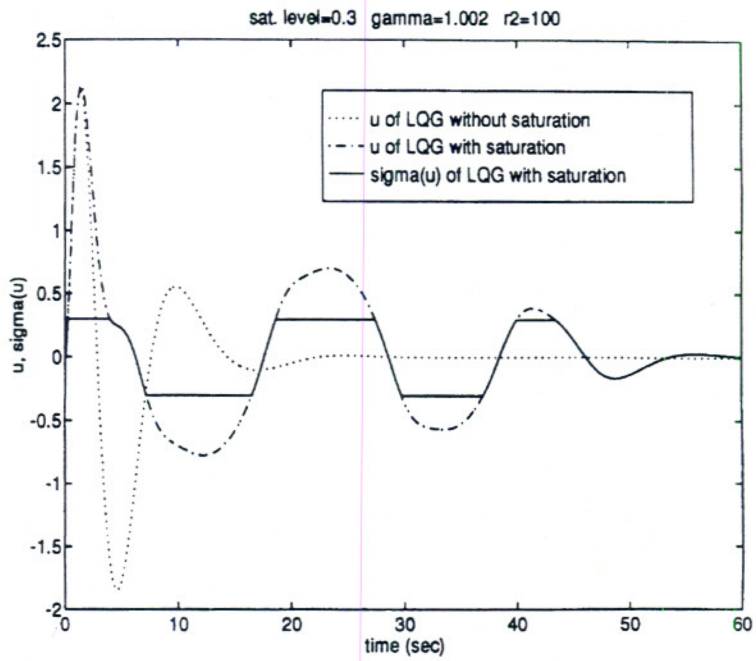

Figure 9. Control effort u and saturated input $\sigma(u)$ of the LQG controller with and without saturation

The set $\mathscr{D}$ is given by $\mathscr{D} = \{x_0\colon x_0^{\mathrm{T}}(P+\hat{P})x_0 < 5{\cdot}8809 \times 10^5\}$, where

$$P + \hat{P} = 10^6 \times \begin{bmatrix} 6{\cdot}4328 & 1{\cdot}3621 & 1{\cdot}2928 \\ 1{\cdot}3621 & 0{\cdot}8430 & -0{\cdot}0371 \\ 1{\cdot}2928 & -0{\cdot}0371 & 0{\cdot}5729 \end{bmatrix}$$

To illustrate the closed-loop behaviour let $r=5$, $x_{20}=q_0=0$, $e_0=r$, and $x_{c0}=[0\ 0\ 0]^{\mathrm{T}}$, respectively. As can be seen from Figure 8, the output y of the closed-loop system with the LQG controller becomes oscillatory and has a large overshoot, while the output of the closed-loop system with the controller given by Proposition 4.1 shows satisfactory response. Figure 9 shows the control input $u(t)$ for the LQG controller with and without saturation as well as the output of the saturation nonlinearity $\sigma(u(t))$ for the LQG controller with saturation.

6. CONCLUSION

In this paper, we developed linear and nonlinear dynamic compensators based upon Theorem 2.1, which accounts for the saturation nonlinearity and provides a guaranteed domain of attraction. Controller gains were characterized by Riccati equations which were obtained by minimizing an LQG-type cost. Propositions 3.1 and 4.1 were demonstrated by two numerical examples using full-order dynamic compensators.

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