

function K are given by (11) and (19), respectively, the optimal control is

$$u = \begin{bmatrix} -2b_1x_1/q_1[x_1^2 + 2x_2^2] \\ 2b_2x_2^2/q_2[x_2(x_1^2 + 2x_2^2)] \end{bmatrix} \quad (20)$$

Proof: Evaluating the mathematical expectation in (8) we find that

$$\exp[-F(x, t)/2] = x_2/[x_1^2 + 2x_2^2]^{1/2}, \quad (21)$$

i.e.,

$$F(x, t) = \ln[2 + (x_1/x_2)^2] \quad (22)$$

and formula (20) then follows from (10). ■

The termination set D considered in this section depended only on $x_2(T)$, so that we had a one-barrier problem. In the next section we will obtain the optimal control in the first quadrant; that is, we will solve a two-barrier problem.

III. OPTIMAL CONTROL IN THE FIRST QUADRANT

In this section we assume that the process $x(t)$ starts in the region

$$C = \{(x_1(t), x_2(t)) : x_1(t) > 0, x_2(t) > 0\} \quad (23)$$

and we want to find the control that will allow us to leave C at minimal cost. We consider the terminal loss function

$$K[x(T), T] = K[(y_1, y_2), s] = \begin{cases} y_1^2/s & \text{if } y_2 = 0 \\ y_2^2/s & \text{if } y_1 = 0. \end{cases} \quad (24)$$

That is, we want the process $x(t)$ to leave the continuation region C through the origin.

When the continuation region is the first quadrant, the first passage from $((x_1, x_2), t)$ to $((y_1, y_2), T)$ for the uncontrolled process

$$dx_i/dt = \epsilon_i \quad (i = 1, 2) \quad (25)$$

has the probability density

$$f(x_1, x_2, t; y_1, y_2, T) = \begin{cases} (x_2/2\pi s^2)H(x_1, y_1, x_2) & \text{if } y_2 = 0 \\ (x_1/2\pi s^2)H(x_2, y_2, x_1) & \text{if } y_1 = 0, \end{cases} \quad (26)$$

where $s = T - t$ and

$$H(x, y, z) = \exp\{-[(x-y)^2 + z^2]/2s\} - \exp\{-[(x+y)^2 + z^2]/2s\}.$$

We may check that

$$\int_0^\infty \int_0^\infty f dy_1 ds + \int_0^\infty \int_0^\infty f dy_2 ds = 1, \quad (27)$$

so that the optimal control can be obtained from (8) and (10).

If the proportionality constant c in formula (7) is equal to 2, we easily find that

$$\exp[-F(x, t)/2] = U(x_1, x_2) + U(x_2, x_1) \quad (28)$$

where

$$U(x, y) = [2x/\pi(y^2 + 2x^2)^{1/2}] \arctan[y/(y^2 + 2x^2)^{1/2}].$$

Proposition 3: When the continuation region C is the first quadrant and the terminal loss function is given by (24) then, if $c = 2$, the optimal control is

$$u = \begin{bmatrix} (2b_1/q_1) G_{x1}/G \\ (2b_2/q_2) G_{x2}/G \end{bmatrix}, \quad (29)$$

where $G = \exp[-F(x, t)/2]$ is given by (28) and

$$G_{x1} = 2x_2^2/\pi M^{1/2} \arctan(x_2/M^{1/2}) - x_1x_2/\pi N^{3/2} \arctan(x_1/N^{1/2}) + 2x_2(x_2^2 - x_1^2)/\pi NM \quad (30)$$

with

$$M = x_2^2 + 2x_1^2 \text{ and } N = x_1^2 + 2x_2^2$$

(and G_{x2} is the same as G_{x1} but with x_1 and x_2 interchanged).

Proof: Equation (29) follows from (10) since

$$F_x = -2G_x/G. \quad \blacksquare$$

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Optimal Output Feedback for Nonzero Set Point Regulation

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Abstract—Motivated by the results of Artstein and Leizarowitz [2] on steady-state periodic tracking, a continuous-time nonzero set point regulation problem is considered which involves 1) noisy and nonnoisy measurements, 2) weighted and unweighted controls, 3) correlated plant/measurement noise and cross weighting, 4) nonzero-mean disturbances, and 5) state-, control-, and measurement-dependent white noise. It is shown that in the absence of multiplicative disturbances the closed-loop control can be designed independently of the open-loop control. Unlike [2], the results are obtained without using the overtaking criterion.

I. INTRODUCTION

The quadratic performance criterion

$$J \triangleq \int_0^{t_1} x^T(t)Qx(t) + u^T(t)Ru(t) dt \quad (1.1)$$

expresses the desire to minimize deviations of the state $x(t)$ of the system

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (1.2)$$

from the regulation point $x = 0$. As is well known [1, pp. 270-276], the nonzero set point criterion

$$J_{\bar{x}} = \int_0^{t_1} [x(t) - \bar{x}]^T Q[x(t) - \bar{x}] + u^T(t)Ru(t) dt \quad (1.3)$$

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presents no additional difficulty as long as $x(t)$ and $u(t)$ are replaced by $x(t) - \bar{x}$ and $u(t) - \bar{u}$, where \bar{u} satisfies

$$0 = A\bar{x} + B\bar{u}. \tag{1.4}$$

Closer inspection, however, reveals that this approach is suboptimal. Specifically, the offset \bar{u} in the control may correspond to an unacceptably high level of control effort when $\bar{u}^T R \bar{u}$ is large. Hence, this approach overlooks design tradeoffs concerning the control effort required for maintaining the nonzero regulation point \bar{x} . Moreover, such an approach is impossible when \bar{u} satisfying (1.4) does not exist.

A significant advance in extending the LQR formulation to steady-state tracking problems (and, hence, to nonzero set point regulation) was given by Artstein and Leizarowitz in [2]. They consider the performance criterion

$$J_\infty \triangleq \int_0^\infty [x(t) - \Gamma(t)]^T Q [x(t) - \Gamma(t)] + u^T(t) R u(t) dt \tag{1.5}$$

where $\Gamma(\cdot)$ is periodic on $[0, \infty)$ and the minimization of J_∞ is performed in the sense of the overtaking criterion. For the nonzero set point problem ($\Gamma(t) \equiv \bar{x}$) with full-state feedback plus constant offset control law

$$u(t) = Kx(t) + \alpha \tag{1.6}$$

it follows from [2, Theorem 2] that K and α are given by

$$K = -R^{-1} B^T P, \tag{1.7}$$

$$\alpha = -R^{-1} B^T (A - \Sigma P)^{-T} Q \bar{x} \tag{1.8}$$

where P satisfies the Riccati equation

$$0 = A^T P + P A + Q - P \Sigma P \tag{1.9}$$

with $\Sigma \triangleq B R^{-1} B^T$.

Two features of the control law (1.6)–(1.8) are noteworthy. First, (1.6) consists of both a closed-loop feedback component $Kx(t)$ and an open-loop component α depending upon the regulation point (Fig. 1). Second (and more important), is the observation that the closed-loop control component is independent of the open-loop component. From a practical point of view this feature is quite useful since it implies that the feedback gain K can be determined without regard to the set point. Hence, a change in the desired set point \bar{x} during on-line operation does not necessitate resolving the Riccati equation in real time; only α requires updating. For a new value of \bar{x} , α can readily be recomputed on-line via the matrix multiplication operation (1.8).

The contribution of the present note is an extension of the result of [2] as applied to the nonzero set point regulation problem without using the overtaking criterion. We extend this result in the following different ways.

1) *Output Feedback with Noisy and Nonnoisy Measurements:* To obtain a more realistic problem setting, we consider the case in which the full state is not available, but rather only measured linear combinations of states. Moreover, we consider the possibility that some of the measurements are corrupted by white noise while others are noise free. Note that the noise-free case was considered in [3] while the fully noisy case is the standard assumption in LQG theory. As in [4]–[6] we express the solution in terms of a projection corresponding to the noise-free measurements.

2) *Singular Control Weighting:* As noted in [6], [7] static continuous-time feedback of noise-corrupted measurements results in unbounded cost unless the corresponding controls are unweighted. Hence, we allow for both weighted and unweighted controls to which the noise-free and noisy measurements are fed, respectively. This setting leads to an additional projection dual to the projection arising from the noise-free measurements [6].

3) *Correlated Plant and Measurement Noise and Cross Weighting:* To allow greater design flexibility we allow the possibility that the plant and measurement noise are correlated. In addition, we consider the dual design feature, namely, cross weighting in the performance criterion.

4) *Nonzero-Mean Disturbances:* In addition to the presence of zero-

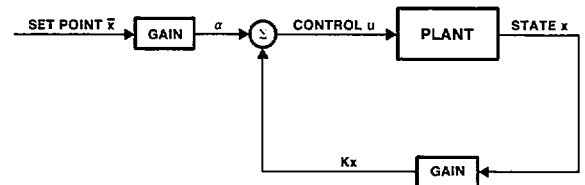


Fig. 1.

mean white plant disturbances we allow for the possibility of a nonzero constant disturbance offset. In contrast to [1, pp. 277–281], our result shows that the presence of a constant disturbance offset leads to an additional offset in the open-loop component of the control.

5) *Multiplicative White Noise:* In addition to the above generalizations we allow for the presence of multiplicative disturbances in the plant. The control law thus generalizes previous results involving state-, control- and measurement-dependent noise [8]–[11]. As shown in [12]–[14], the multiplicative white noise model can be used to guarantee robustness with respect to deterministic plant parameter variations.

II. NOTATION AND DEFINITIONS

$\mathbb{R}, \mathbb{R}^{r \times s}, \mathbb{R}^r$	real numbers, $r \times s$ real matrices, $\mathbb{R}^{r \times 1}$
\mathbb{E}	expectation
$I_r, (\)^T$	$r \times r$ identity, transpose
\oplus, \otimes	Kronecker sum, Kronecker product
asymptotically stable matrix	matrix with eigenvalues in open left-half plane
$n, m_1, m_2, l_1, l_2, p, r$	positive integers
x, u_1, u_2, y_1, y_2	n, m_1, m_2, l_1, l_2 -dimensional vectors
$A, A_i; B_1, B_{1i}; C_1, C_{1i}$	$n \times n$ matrices; $n \times m_1$ matrices; $l_1 \times n$ matrices, $i = 1, \dots, p$
B_2, C_2, K_1, K_2, L	$n \times m_2, l_2 \times n, m_1 \times l_2, m_2 \times l_1, r \times n$ matrices
$\alpha_1, \alpha_2, \gamma, \delta$	m_1, m_2, n, r -dimensional vectors
$w_i(t)$	unit variance white noise, $i = 1, \dots, p$
$w_0(t), w_1(t)$	n -dimensional, l_1 -dimensional white noise
V_0, V_1	intensities of w_0, w_1 ; $V_0 \geq 0, V_1 > 0$
V_{01}	$n \times l_1$ cross intensity of w_0, w_1
R_0, R_1	$r \times r$ and $m_1 \times m_1$ state and control weightings; $R_0 \geq 0, R_1 > 0$
R_{01}	$r \times m_1$ cross weighting; $R_0 - R_{01} R_1^{-1} R_{01}^T \geq 0$
\bar{A}, \bar{A}_i	$A + B_1 K_1 C_2 + B_2 K_2 C_1, A_i + B_{1i} K_1 C_2 + B_{2i} K_2 C_{1i}, i = 1, \dots, p$
\bar{B}, \bar{B}_i	$B_1 \alpha_1 + B_2 \alpha_2 + \gamma, B_{1i} \alpha_i, i = 1, \dots, p$
$\bar{w}(t)$	$w_0(t) + B_2 K_2 w_1(t)$
\bar{V}	$V_0 + V_{01} K_2^T B_2^T + B_2 K_2 V_{01}^T + B_2 K_2 V_1 K_2^T B_2^T$
\bar{R}	$L^T R_0 L + L^T R_{01} K_1 C_2 + C_2^T K_1^T R_{01}^T L + C_2^T K_1^T R_1 K_1 C_2$

For arbitrary $n \times n$ Q, P such that the indicated inverses exist, define:

$$\tau_1 \triangleq Q C_2^T (C_2 Q C_2^T)^{-1} C_2, \quad \tau_2 \triangleq B_2 (B_2^T P B_2)^{-1} B_2^T P,$$

$$\tau_{1\perp} \triangleq I_n - \tau_1, \quad \tau_{2\perp} \triangleq I_n - \tau_2,$$

$$V_{1s} \triangleq V_1 + \sum_{i=1}^p C_{1i} (Q + \hat{m}_s \hat{m}_s^T) C_{1i}^T, \quad R_{1s} \triangleq R_1 + \sum_{i=1}^p B_{1i}^T P B_{1i},$$

$$Q_c \triangleq Q C_1^T + V_{01} + \sum_{i=1}^p A_i (Q + \hat{m}_s \hat{m}_s^T) C_{1i}^T,$$

$$\Phi_s \triangleq B_1^T P + R_{01}^T L + \sum_{i=1}^p B_{1i}^T P A_{1i},$$

$$\hat{A}_s \triangleq A - B_1 R_{1s}^{-1} \Phi_s \tau_1 - \tau_2 Q_s V_{1s}^{-1} C_1, \quad \hat{m}_s \triangleq -\hat{A}_s^{-1} \bar{B},$$

$$\hat{A}_{1s} \triangleq A_1 - B_{11} R_{1s}^{-1} \Phi_s \tau_1 - \tau_2 Q_s V_{1s}^{-1} C_{1s},$$

$$\Phi_s \triangleq R_{1s} - (\Phi_s \tau_{1\perp} \hat{A}_s^{-1} B_1 + B_1^T \hat{A}_s^{-T} \tau_{1\perp}^T \Phi_s^T),$$

$$\Delta_{1s} \triangleq \Phi_s \tau_{1\perp} \hat{A}_s^{-1} + B_1^T \hat{A}_s^{-T} P, \quad \Delta_{2s} \triangleq B_2^T (P \hat{A}_s^{-1} + \hat{A}_s^{-T} P),$$

$$\Omega_{1s} \triangleq \Delta_{1s} B_2, \quad \Omega_{2s} \triangleq \Delta_{2s} B_2,$$

$$\Lambda_s \triangleq \hat{A}_s^{-T} (L^T R_0 - \tau_1^T \Phi_s^T R_{1s}^{-1} R_{01}^T),$$

$$\Lambda_{1s} \triangleq B_1^T \Lambda_s - R_{01}^T, \quad \Lambda_{2s} \triangleq B_2^T \Lambda_s,$$

$$Q \triangleq Q C_1^T + V_{01}, \quad \Phi \triangleq B_1^T P + R_{01}^T,$$

$$\hat{A} \triangleq A - B_1 R_1^{-1} \Phi \tau_1 - \tau_2 Q V_1^{-1} C_1, \quad \hat{m} \triangleq -\hat{A}^{-1} \bar{B},$$

$$\Phi \triangleq R_1 - (\Phi \tau_{1\perp} \hat{A}^{-1} B_1 + B_1^T \hat{A}^{-T} \tau_{1\perp}^T \Phi^T),$$

$$\Delta_1 \triangleq \Phi \tau_{1\perp} \hat{A}^{-1} + B_1^T \hat{A}^{-T} P, \quad \Delta_2 \triangleq B_2^T (P \hat{A}^{-1} + \hat{A}^{-T} P),$$

$$\Omega_1 \triangleq \Delta_1 B_2, \quad \Omega_2 \triangleq \Delta_2 B_2,$$

$$\Lambda \triangleq \hat{A}^{-T} (L^T R_0 - \tau_1 \Phi^T R_1^{-1} R_{01}^T),$$

$$\Lambda_1 \triangleq B_1^T \Lambda - R_{01}^T, \quad \Lambda_2 \triangleq B_2^T \Lambda,$$

$$\hat{A}_0 = A - B_1 R_1^{-1} \Phi \tau_1, \quad \Phi_0 \triangleq R_1 - (\Phi \tau_{1\perp} \hat{A}_0^{-1} B_1 + B_1^T \hat{A}_0^{-T} \tau_{1\perp}^T \Phi^T),$$

$$\Delta_{10} \triangleq \Phi \tau_{1\perp} \hat{A}_0^{-1} + B_1^T \hat{A}_0^{-T} P,$$

$$\Lambda_{10} \triangleq B_1^T \hat{A}_0^{-T} (L^T R_0 - \tau_1 \Phi^T R_1^{-1} R_{01}^T) - R_{01}^T,$$

$$\hat{A}_1 = A - B_1 R_1^{-1} \Phi, \quad \Sigma \triangleq B_1 R_1^{-1} B_1^T, \quad A_p \triangleq A - \Sigma P.$$

III. NONZERO SET POINT REGULATION

Nonzero Set Point Problem: Given the controlled system

$$\begin{aligned} \dot{x}(t) = & \left(A + \sum_{i=1}^p v_i(t) A_i \right) x(t) \\ & + \left(B_1 + \sum_{i=1}^p v_i(t) B_{1i} \right) u_1(t) + B_2 u_2(t) + w_0(t) + \gamma \end{aligned} \quad (3.1)$$

with measurements

$$y_1(t) = \left(C_1 + \sum_{i=1}^p v_i(t) C_{1i} \right) x(t) + w_1(t), \quad (3.2)$$

$$y_2(t) = C_2 x(t) \quad (3.3)$$

where $t \in [0, \infty)$, determine K_1, K_2, α_1 , and α_2 such that the static output feedback law

$$u_1(t) = K_1 y_2(t) + \alpha_1, \quad (3.4)$$

$$u_2(t) = K_2 y_1(t) + \alpha_2 \quad (3.5)$$

minimizes the performance criterion

$$\begin{aligned} J(K_1, K_2, \alpha_1, \alpha_2) = & \lim_{t \rightarrow \infty} \mathbb{E}[(Lx(t) - \delta)^T R_0 (Lx(t) - \delta) \\ & + 2(Lx(t) - \delta)^T R_{01} u_1(t) + u_1^T(t) R_1 u_1(t)]. \end{aligned} \quad (3.6)$$

The closed-loop system (3.1)–(3.5) can be written as

$$\dot{x}(t) = \left(\bar{A} + \sum_{i=1}^p v_i(t) \bar{A}_i \right) x(t) + \bar{B} + \sum_{i=1}^p v_i(t) \bar{B}_i + \bar{w}(t). \quad (3.7)$$

To analyze (3.7) define the second-moment and covariance matrices

$$\bar{Q}(t) \triangleq \mathbb{E}[x(t)x^T(t)], \quad Q(t) \triangleq \bar{Q}(t) - m(t)m^T(t)$$

where $m(t) \triangleq \mathbb{E}[x(t)]$. It follows from [15, p. 142], that $\bar{Q}(t), Q(t)$, and $m(t)$ satisfy

$$\begin{aligned} \dot{\bar{Q}}(t) = & \bar{A} \bar{Q}(t) + \bar{Q}(t) \bar{A}^T + \bar{V} + \bar{B} m^T(t) + m(t) \bar{B}^T \\ & + \sum_{i=1}^p [\bar{A}_i \bar{Q}(t) \bar{A}_i^T + \bar{A}_i m(t) \bar{B}_i^T + \bar{B}_i m^T(t) \bar{A}_i^T + \bar{B}_i \bar{B}_i^T], \end{aligned} \quad (3.8)$$

$$\begin{aligned} \dot{Q}(t) = & \bar{A} Q(t) + Q(t) \bar{A}^T + \bar{V} + \sum_{i=1}^p [\bar{A}_i Q(t) \bar{A}_i^T + \bar{A}_i m(t) m^T(t) \bar{A}_i^T \\ & + \bar{A}_i m(t) \bar{B}_i^T + \bar{B}_i m^T(t) \bar{A}_i^T + \bar{B}_i \bar{B}_i^T], \end{aligned} \quad (3.9)$$

$$\dot{m}(t) = \bar{A} m(t) + \bar{B}. \quad (3.10)$$

To consider the steady state, we restrict our consideration to the set of second-moment stabilizing gains

$$S_s \triangleq \{(K_1, K_2): \bar{A} \oplus \bar{A} + \sum_{i=1}^p \bar{A}_i \otimes \bar{A}_i \text{ is asymptotically stable}\}.$$

It follows from fundamental properties of Lyapunov equations that if $(K_1, K_2) \in S_s$, then \bar{A} is also asymptotically stable. Hence, for $(K_1, K_2) \in S_s$, $\bar{Q} \triangleq \lim_{t \rightarrow \infty} \bar{Q}(t)$, $Q \triangleq \lim_{t \rightarrow \infty} Q(t)$ and $m \triangleq \lim_{t \rightarrow \infty} m(t)$ exist and satisfy

$$\begin{aligned} 0 = & \bar{A} \bar{Q} + \bar{Q} \bar{A}^T + \bar{B} m + m \bar{B}^T + \bar{V} + \sum_{i=1}^p [\bar{A}_i \bar{Q} \bar{A}_i^T \\ & + \bar{A}_i m \bar{B}_i^T + \bar{B}_i m^T \bar{A}_i^T + \bar{B}_i \bar{B}_i^T], \end{aligned} \quad (3.11)$$

$$\begin{aligned} 0 = & \bar{A} Q + Q \bar{A}^T + \bar{V} + \sum_{i=1}^p [\bar{A}_i Q \bar{A}_i^T + \bar{A}_i m m^T \bar{A}_i^T \\ & + \bar{A}_i m \bar{B}_i^T + \bar{B}_i m^T \bar{A}_i^T + \bar{B}_i \bar{B}_i^T], \end{aligned} \quad (3.12)$$

$$0 = \bar{A} m + \bar{B}. \quad (3.13)$$

Now $J(K_1, K_2, \alpha_1, \alpha_2)$ is given by

$$\begin{aligned} J(K_1, K_2, \alpha_1, \alpha_2) = & \text{tr} [(Q + m m^T) \bar{R}] - 2m^T L^T R_0 \delta \\ & + \delta^T R_0 \delta + 2m^T L^T R_{01} \alpha_1 \\ & - 2\delta^T R_{01} K_1 C_2 m - 2\delta^T R_{01} \alpha_1 \\ & + 2m^T C_2^T K_1^T R_1 \alpha_1 + \alpha_1^T R_1 \alpha_1. \end{aligned} \quad (3.14)$$

Associated with Q is its dual $P \geq 0$ which is the unique solution of

$$0 = \bar{A}^T P + P \bar{A} + \bar{R} + \sum_{i=1}^p \bar{A}_i^T P \bar{A}_i. \quad (3.15)$$

To obtain closed-form expressions for the feedback gains we further restrict consideration to the set

$$S_s^+ \triangleq \{(K_1, K_2) \in S_s: C_2 Q C_2^T, B_2^T P B_2, \Phi, \text{ and } \Omega_{1s}^T \Phi_s^{-1} \Omega_{1s} + \Omega_{2s} \text{ are invertible}\},$$

and assume

$$[B_{1i} \neq 0 \Rightarrow C_{1i} = 0], \quad i = 1, \dots, p. \quad (3.16)$$

Optimizing (3.14) subject to (3.12) and (3.13) yields the following result illustrated in Fig. 2.

Theorem 3.1: Suppose $K_1, K_2, \alpha_1, \alpha_2$ solve the nonzero set point

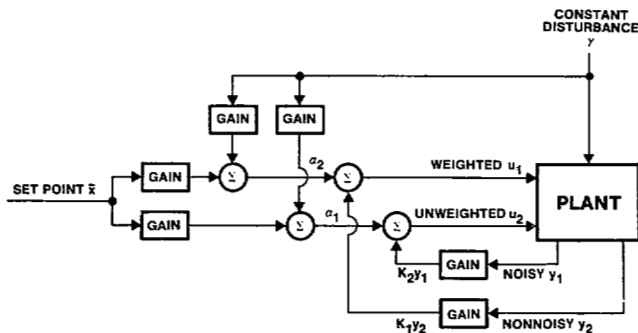


Fig. 2.

problem with $(K_1, K_2) \in \mathcal{S}_s^+$. Then there exist $n \times n$ $Q, P \geq 0$ such that

$$K_1 = -R_{1s}^{-1} \Phi_s Q C_2^T (C_2 Q C_2^T)^{-1}, \quad (3.17)$$

$$K_2 = -(B_2^T P B_2)^{-1} B_2^T P Q_s V_{1s}^{-1}, \quad (3.18)$$

$$\alpha_1 = \Phi_s^{-1} (\Omega_{1s} \alpha_2 + \Delta_{1s} \gamma - \Lambda_{1s} \delta), \quad (3.19)$$

$$\alpha_2 = (\Omega_{1s}^T \Phi_s^{-1} \Omega_{1s} + \Omega_{2s})^{-1} [(\Omega_{1s}^T \Phi_s^{-1} \Delta_{1s} + \Delta_{2s}) \gamma - (\Omega_{1s}^T \Phi_s^{-1} \Lambda_{1s} + \Lambda_{2s}) \delta], \quad (3.20)$$

and such that Q and P satisfy

$$\begin{aligned} 0 &= (A - B_1 R_{1s}^{-1} \Phi_s \tau_1) Q + Q (A - B_1 R_{1s}^{-1} \Phi_s \tau_1)^T + V_0 \\ &+ \sum_{i=1}^p [(A_i - B_{1i} R_{1s}^{-1} \Phi_s \tau_1) Q (A_i - B_{1i} R_{1s}^{-1} \Phi_s \tau_1)^T \\ &+ \hat{A}_{is} \hat{m}_s \hat{m}_s^T \hat{A}_{is}^T + \hat{A}_{is} \hat{m}_s \hat{B}_i^T + \hat{B}_i \hat{m}_s^T \hat{A}_{is}^T + \hat{B}_i \hat{B}_i^T] \\ &- Q_s V_{1s}^{-1} Q_s^T + \tau_{2\perp} Q_s V_{1s}^{-1} Q_s^T \tau_{2\perp}^T, \quad (3.21) \\ 0 &= (A - \tau_2 Q_s V_{1s}^{-1} C_1)^T P + P (A - \tau_2 Q_s V_{1s}^{-1} C_1) + R_0 \\ &+ \sum_{i=1}^p (A_i - \tau_2 Q_s V_{1s}^{-1} C_{1i})^T P (A_i - \tau_2 Q_s V_{1s}^{-1} C_{1i}) \\ &- \Phi_s^T R_{1s}^{-1} \Phi_s + \tau_{1\perp}^T \Phi_s^T R_{1s}^{-1} \Phi_s \tau_{1\perp}. \quad (3.22) \end{aligned}$$

Outline of Proof: As in [16] the result is obtained by forming the Lagrangian while accounting for (3.12) and (3.13). Define

$$\begin{aligned} \mathcal{L}(K_1, K_2, \alpha_1, \alpha_2, Q, m) &= \text{tr} [\lambda_0 J(K_1, K_2, \alpha_1, \alpha_2) \\ &+ P \text{ [RHS of (3.12)]} + \lambda^T (\hat{A} m + \hat{B})] \end{aligned}$$

where $\lambda_0 \geq 0$ and $\lambda \in \mathbb{R}^n$. Setting $\partial \mathcal{L} / \partial Q = 0$ and using the second-moment stability assumption it follows that $\lambda_0 = 1$ without loss of generality. The derivation now follows by setting the partial derivatives of \mathcal{L} with respect to $K_1, K_2, \alpha_1, \alpha_2$, and m to zero and solving for the gains. To assist the reader in carrying out the details we note that λ is given by

$$\begin{aligned} \lambda &= -2 \hat{A}^{-T} [-P \hat{B} - \sum_{i=1}^p \hat{A}_i^T P \hat{B}_i + L^T R_0 \delta \\ &- L^T R_{01} \alpha_1 + C_2^T K_1 R_{01}^T \delta - C_2^T K_1^T R_{01} \alpha_1]. \quad \square \end{aligned}$$

Remark 3.1: Because of the presence of δ in (3.21) via α_1 and α_2 in \hat{B} (see the definition of \hat{m}_s) and α_1 in \hat{B}_i , the closed-loop component of the control law (3.17)–(3.20) cannot be designed independently of the open-loop component. As now shown, independence is recovered when the multiplicative noise terms are absent.

IV. SPECIALIZATIONS OF THEOREM 3.1

To draw connections with the previous literature, a series of specializations of Theorem 3.1 is now given. We begin by deleting all multiplicative white noise terms, i.e.,

$$A_i, B_{1i}, C_{1i} = 0, \quad i = 1, \dots, p. \quad (4.1)$$

In this case the stabilizing set \mathcal{S}_s can be characterized by

$$\mathcal{S} = \{(K_1, K_2): \bar{A} \text{ is asymptotically stable}\},$$

and, furthermore, \mathcal{S}_s^+ becomes

$$\begin{aligned} \mathcal{S}^+ &\triangleq \{(K_1, K_2) \in \mathcal{S}: C_2 Q C_2^T, B_2^T P B_2, \\ &\Phi \text{ and } \Omega_1^T \Phi^{-1} \Omega_1 + \Omega_2 \text{ are invertible}\}. \end{aligned}$$

Corollary 4.1: Assume (4.1) is satisfied and suppose $K_1, K_2, \alpha_1, \alpha_2$ solve the nonzero set point problem with $(K_1, K_2) \in \mathcal{S}^+$. Then there exist $n \times n$ $Q, P \geq 0$ such that

$$K_1 = -R_1^{-1} \Phi Q C_2^T (C_2 Q C_2^T)^{-1}, \quad (4.2)$$

$$K_2 = -(B_2^T P B_2)^{-1} B_2^T P Q V_1^{-1}, \quad (4.3)$$

$$\alpha_1 = \Phi^{-1} (\Omega_1 \alpha_2 + \Delta_1 \gamma - \Lambda_1 \delta), \quad (4.4)$$

$$\alpha_2 = (\Omega_1^T \Phi^{-1} \Omega_1 + \Omega_2)^{-1} [(\Omega_1^T \Phi^{-1} \Delta_1 + \Delta_2) \gamma - (\Omega_1^T \Phi^{-1} \Lambda_1 + \Lambda_2) \delta], \quad (4.5)$$

and such that Q and P satisfy

$$\begin{aligned} 0 &= (A - B_1 R_1^{-1} \Phi \tau_1) Q + Q (A - B_1 R_1^{-1} \Phi \tau_1)^T + V_0 \\ &- Q V_1^{-1} Q^T + \tau_{2\perp} Q V_1^{-1} Q^T \tau_{2\perp}^T, \quad (4.6) \end{aligned}$$

$$\begin{aligned} 0 &= (A - \tau_2 Q V_1^{-1} C_1)^T P + P (A - \tau_2 Q V_1^{-1} C_1) + R_0 \\ &- \Phi^T R_1^{-1} \Phi + \tau_{1\perp}^T \Phi^T R_1^{-1} \Phi \tau_{1\perp}. \quad (4.7) \end{aligned}$$

In addition to (4.1) we assume further that all controls are weighted and all measurements are noise free, i.e.,

$$B_2 = 0, \quad C_1 = 0. \quad (4.8)$$

This corresponds to the setting considered in [3]. It follows from the proof of Theorem 3.1 that the assumption $B_2 = 0$ leads to $\tau_2 = 0$, and $C_1 = 0$ corresponds to deleting (3.5). Hence \mathcal{S} and \mathcal{S}^+ are now given by

$$\mathcal{S}_0 \triangleq \{K_1: A + B_1 K_1 C_2 \text{ is asymptotically stable}\},$$

$$\mathcal{S}_0^+ \triangleq \{K_1 \in \mathcal{S}_0: C_2 Q C_2^T \text{ and } \Phi_0 \text{ are invertible}\}.$$

Corollary 4.2: Assume (4.1) and (4.8) are satisfied and suppose K_1 and α_1 solve the nonzero set point problem with $K_1 \in \mathcal{S}_0^+$. Then there exist $n \times n$ $Q, P \geq 0$ such that

$$K_1 = -R_1^{-1} \Phi Q C_2^T (C_2 Q C_2^T)^{-1}, \quad (4.9)$$

$$\alpha_1 = \Phi_0^{-1} (\Delta_{10} \gamma - \Lambda_{10} \delta), \quad (4.10)$$

and such that Q and P satisfy

$$0 = (A - B_1 R_1^{-1} \Phi \tau_1) Q + Q (A - B_1 R_1^{-1} \Phi \tau_1)^T + V_0, \quad (4.11)$$

$$0 = A^T P + P A + R_0 - \Phi^T R_1^{-1} \Phi + \tau_{1\perp}^T \Phi^T R_1^{-1} \Phi \tau_{1\perp}. \quad (4.12)$$

We now specialize further to the full-state feedback case, i.e.,

$$C_2 = I_n, \quad (4.13)$$

and hence $\tau_1 = I_n$ and $\tau_{1\perp} = 0$. Now \mathcal{S}_0 and \mathcal{S}_0^+ become

$$\mathcal{S}_1 \triangleq \{K_1: A + B_1 K_1 \text{ is asymptotically stable}\},$$

$$\mathcal{S}_1^+ \triangleq \{K_1 \in \mathcal{S}_1: Q \text{ is invertible}\}.$$

Corollary 4.3: Assume (4.1), (4.8), and (4.13) are satisfied and suppose K_1 and α_1 solve the nonzero set point problem with $K_1 \in \mathcal{S}_1^+$. Then there exist $n \times n$ $Q, P \geq 0$ such that

$$K_1 = -R_1^{-1}\Phi, \tag{4.14}$$

$$\alpha_1 = R_1^{-1}B_1^T\hat{A}_1^{-T}P\gamma - R_1^{-1}[B_1^T\hat{A}_1^{-T}(L^TR_0 - \Phi^TR_1^{-1}R_{01}^T) - R_{01}^T]\delta \tag{4.15}$$

and such that Q and P satisfy

$$0 = (A - B_1R_1^{-1}\Phi)Q + Q(A - B_1R_1^{-1}\Phi)^T + V_0, \tag{4.16}$$

$$0 = A^TP + PA + R_0 - \Phi^TR_1\Phi. \tag{4.17}$$

Finally, setting

$$\gamma = 0, \quad R_{01} = 0, \quad L = I_n \tag{4.18}$$

we obtain the result of [2].

Corollary 4.4: Assume (4.1), (4.8), (4.13), and (4.18) are satisfied and suppose K_1 and α_1 solve the nonzero set point problem with $K_1 \in \mathcal{S}_1^+$. Then there exists $n \times n$ $P \geq 0$ such that

$$K_1 = -R_1^{-1}B_1^TP, \tag{4.19}$$

$$\alpha_1 = -R_1^{-1}B_1^TA_1^{-T}R_0\delta \tag{4.20}$$

and such that P satisfies

$$0 = A^TP + PA + R_0 - P\Sigma P. \tag{4.21}$$

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Analysis of Time-Varying Scaled Systems Via General Orthogonal Polynomials

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Abstract—General orthogonal polynomials are introduced to analyze and approximate the solution of a class of scaled systems. Using the operational matrix of integration, together with the operational matrix of linear transformation, the dynamical equation of a scaled system is reduced to a set of simultaneous linear algebraic equations. The coefficient vectors of the general orthogonal polynomials can be determined recursively by the derived algorithm. An illustrative example is given to demonstrate the validity and applicability of the orthogonal polynomial approximations.

I. INTRODUCTION

An investigation of the dynamics of an overhead current collection mechanism for an electric locomotive by Ockendon and Taylor [12] revealed that under certain conditions, the dynamics of the systems is characterized by a differential equation containing terms with a scaled argument of the form

$$\dot{X}(t) = AX(\lambda t) + BX(t)$$

$$X(0) = X_0$$

where $X(\lambda t)$ and $X(t)$ are n -vectors and A and B are $n \times n$ matrices and the constant $0 < \lambda < 1$. This type of differential equation also plays an important role in several chemical processes [3], [13]. This equation was first studied by Fox *et al.* [11] with the introduction of a finite difference method for $0 < \lambda < 1$. Recently, the solution of such a scaled system has been obtained by several different orthogonal functions, such as block-pulse functions [14], [2], [3], Walsh functions [1], delayed unit step functions [4], Laguerre polynomials [5], Chebyshev polynomials [6], [7], and Legendre polynomials [15]. The common approach of these methods is the use of the operational matrix of integration together with the operational matrix of scaling to reduce the differential equation to a set of linear algebraic equations, which is more suitable for computer programming.

In this note we will employ the operational matrix of integration and product operational matrix of the general orthogonal polynomials, together with the operational matrix of linear transformation, which will be derived later, to obtain the solution of the scaled system. The operational matrix of linear transformation is derived based on the following properties, namely, the pure recurrence relation

$$\phi_{i+1}(z) = (a_i z + b_i)\phi_i(z) - c_i\phi_{i-1}(z) \tag{1}$$

with

$$\phi_0(z) = 1; \phi_1(z) = a_0 z + b_0$$

and the differential recurrence relation

$$\phi_i(z) = A_i \dot{\phi}_{i+1}(z) + B_i \dot{\phi}_i(z) + C_i \dot{\phi}_{i-1}(z) \tag{2}$$

where recurrence coefficients a_i, b_i, c_i and differential recurrence coefficients $A_i, B_i,$ and C_i are specified by the particular orthogonal polynomials under consideration and some are listed in [9]. The aim of this paper is twofold: 1) to derive an operational matrix of linear transformation for general orthogonal polynomials so that the scaled

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