Technical Notes and Correspondence

Some Explicit Formulas for the Matrix Exponential

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Abstract—**The matrix exponential plays a central role in linear systems and control theory. In this note, we give explicit formulas for computing the exponential of some special matrices.**

I. Introduction

The linear vector differential equation $\dot{x}(t) = Ax(t)$, where $x(t)$ is an n-vector and A is an $n \times n$ matrix, plays a fundamental role in the study of dynamical systems [3] and linear control systems [5]. As is well known, the solution to this equation is given by $x(t) = e^{At}x(0)$, where $e^{At} = \sum_{k=0}^{\infty}(k!)^{-1}(At)^k$ denotes the exponential of the matrix At. The theoretical and computational properties of the matrix exponential function have been widely studied in [6] and the numerous references given therein. Nevertheless, despite its classical nature, the matrix exponential possesses numerous interesting properties that are still being explored [2], [8], [9].

Because of the ubiquitous presence of the matrix exponential in the study of linear dynamical systems, our objective is to provide explicit formulas to facilitate its exposition and usage. This note appears to be the first attempt to collect together as many such formulas as possible in one place. In addition to their usefulness in linear system theory, these formulas should be helpful in future research concerning the matrix exponential.

In Section II we begin by deriving formulas for the exponential of an arbitrary 2×2 matrix in terms of either its eigenvalues or entries. These results are then applied to the second-order mechanical vibration equation with weak or strong damping. In Section III some formulas for the exponential of $n \times n$ matrices are given for matrices that satisfy an arbitrary quadratic polynomial. Besides the 2×2 matrices considered in Section II these results encompass involutory, rank 1, and idempotent matrices. In Section IV, we consider $n \times n$ matrices that satisfy a special cubic polynomial. These results are applied to the case of a 3×3 skew symmetric matrix whose exponential represents the constant rotation of a rigid body about a fixed axis. For further discussion of the role of the matrix exponential in rotational kinematics, see [7]. An extensive treatment of this topic including the relationship between rotations and quaternions is discussed in [1].

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II. The Exponential of a General 2 × 2 Matrix

In this section, we derive formulas for the exponential of a general 2×2 complex matrix A. Formulas are given in terms of either the eigenvalues of A or the entries of A. The results are specialized to the case in which A is a real matrix. Let $\boldsymbol{R}$ and $\boldsymbol{C}$ denote the real and complex numbers respectively.

Lemma 2.1: Let $A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \boldsymbol{C}^{2\times 2}$.

i) If $a = d$ then $e^A = e^a\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$.

ii) If $a \neq d$ then $e^A = \begin{pmatrix} e^a & b(e^a - e^d)/(a-d) \\ 0 & e^d \end{pmatrix}$.

The following result characterizes e^A in terms of the eigenvalues of A.

Theorem 2.2: Let λ and μ denote the eigenvalues of $A \in \boldsymbol{C}^{2\times 2}$.

i) If $\mu = \lambda$ then

$$e^A = e^{\lambda}[(1-\lambda)I + A].$$

ii) If $\mu \neq \lambda$ then

$$e^A = \frac{\mu e^{\lambda} - \lambda e^{\mu}}{\mu - \lambda}I + \frac{e^{\mu} - e^{\lambda}}{\mu - \lambda}A.$$

Proof:

i) Since $\mu = \lambda$, there exists an invertible matrix X such that $A = X\begin{pmatrix} \lambda & x \\ 0 & \lambda \end{pmatrix}X^{-1}$ for some x. Hence $e^A = e^{\lambda}X\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}X^{-1} = e^{\lambda}[(1-\lambda)I + A]$.

ii) Since $\mu \neq \lambda$, there exists an invertible matrix X such that $A = X\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}X^{-1}$. Hence $e^A = X\begin{pmatrix} e^{\lambda} & 0 \\ 0 & e^{\mu} \end{pmatrix}X^{-1}$. Then, noting that $\begin{pmatrix} e^{\lambda} & 0 \\ 0 & e^{\mu} \end{pmatrix} = (\mu e^{\lambda} - \lambda e^{\mu})/(\mu - \lambda)I + (e^{\mu} - e^{\lambda})/(\mu - \lambda)\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ yields the expression in ii). □

Next we characterize e^A in terms of the entries of A.

Corollary 2.3: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \boldsymbol{C}^{2\times 2}$.

i) If $(a-d)^2 + 4bc = 0$ then

$$e^A = e^{(a+d)/2}\begin{pmatrix} 1 + \dfrac{a-d}{2} & b \\ c & 1 - \dfrac{a-d}{2} \end{pmatrix}.$$

ii) If $(a-d)^2 + 4bc \neq 0$ then

$$e^A = e^{(a+d)/2}\begin{pmatrix} \cosh(\Delta) + \dfrac{a-d}{2}\dfrac{\sinh(\Delta)}{\Delta} & b\dfrac{\sinh(\Delta)}{\Delta} \\ c\dfrac{\sinh(\Delta)}{\Delta} & \cosh(\Delta) - \dfrac{a-d}{2}\dfrac{\sinh(\Delta)}{\Delta} \end{pmatrix}$$

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where

$$\Delta = \frac{1}{2}\sqrt{(a-d)^2 + 4bc}.$$

Proof: The eigenvalues of A are $\lambda = (a + d - \sqrt{(a-d)^2 + 4bc})/2$ and $\mu = (a + d + \sqrt{(a-d)^2 + 4bc})/2$. Hence, $\lambda = \mu$ if and only if $(a-d)^2 + 4bc = 0$. Then the desired results follow by substituting λ and μ into the formulas in Theorem 2.2. □

We now specialize Corollary 2.3 to the case in which A is real.

Corollary 2.4: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \boldsymbol{R}^{2\times 2}$.

i) If $(a-d)^2 + 4bc = 0$ then

$$e^A = e^{(a+d/2)} \begin{pmatrix} 1 + \dfrac{a-d}{2} & b \\ c & 1 - \dfrac{a-d}{2} \end{pmatrix}.$$

ii) If $(a-d)^2 + 4bc > 0$ then

$$e^A = e^{(a+d)/2} \begin{pmatrix} \cosh(\delta) + \dfrac{a-d}{2}\dfrac{\sinh(\delta)}{\delta} & b\dfrac{\sinh(\delta)}{\delta} \\ c\dfrac{\sinh(\delta)}{\delta} & \cosh(\delta) - \dfrac{a-d}{2}\dfrac{\sinh(\delta)}{\delta} \end{pmatrix}$$

where $\delta = (1/2)\sqrt{(a-d)^2 + 4bc}$.

iii) If $(a-d)^2 + 4bc < 0$ then

$$e^A = e^{(a+d)/2} \begin{pmatrix} \cos(\delta) + \dfrac{a-d}{2}\dfrac{\sin(\delta)}{\delta} & b\dfrac{\sin(\delta)}{\delta} \\ c\dfrac{\sin(\delta)}{\delta} & \cos(\delta) - \dfrac{a-d}{2}\dfrac{\sin(\delta)}{\delta} \end{pmatrix}$$

where $\delta = (1/2)\sqrt{|(a-d)^2 + 4bc|}$.

Proof: For i) and ii), they are essentially the same as Corollary 2.3. For iii), we observe that $\Delta = \delta i$. Hence, $\cosh(\Delta) = \cos(\delta)$, and $\sinh(\Delta)/\Delta = \sin(\delta)/\delta$. □

We illustrate the use of these formulas by two examples given in [5, p. 172].

Example 1: If A is the real matrix $\begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}$ then

$$e^{At} = e^{\sigma t}\begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix}.$$

Example 2: If A is the real matrix $\begin{pmatrix} \sigma & \omega \\ \omega & -\sigma \end{pmatrix}$ then

$$e^A = \begin{pmatrix} \cosh(\delta) + \dfrac{\sigma}{\delta}\sinh(\delta) & \dfrac{\omega}{\delta}\sinh(\delta) \\ \dfrac{\omega}{\delta}\sinh(\delta) & \cosh(\delta) - \dfrac{\sigma}{\delta}\sinh(\delta) \end{pmatrix},$$

where $\delta = \sqrt{\omega^2 + \sigma^2}$.

As an application of Corollary 2.4, we consider the second-order mechanical vibration equation

$$m\ddot{x} + c\dot{x} + kx = 0,$$

where m, c, and k are real. If $m \neq 0$ then we can write this equation in companion form as the system

$$\dot{z} = Az,$$

where

$$z = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 \\ -k/m & -c/m \end{pmatrix}.$$

Hence, we wish to obtain an expression for e^{At}. The case $k = 0$ is the simplest one since A is reduced to triangular form. In this case, we apply Lemma 2.1 to obtain

Case 1) Rigid Body: If $k = 0$ and $c = 0$ then

$$e^{At} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Case 2) Damped Rigid Body: If $k = 0$ and $c \neq 0$ then

$$e^{At} = \begin{pmatrix} 1 & \dfrac{m}{c}(1 - e^{-ct/m}) \\ 0 & e^{-ct/m} \end{pmatrix}.$$

Next we consider the case $m > 0$, $c \geq 0$, and $k > 0$. In this case we define

$$\omega_n = \sqrt{\frac{k}{m}}, \qquad \zeta = \frac{c}{2\sqrt{mk}},$$

where ω_n denotes the (undamped) natural frequency of vibration and ζ denotes the damping ratio. Now A can be written as

$$A = \begin{pmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{pmatrix}.$$

Note that $\omega_n > 0$ and $\zeta \geq 0$. Now we apply Corollary 2.4 to obtain

Case 3) Undamped: If $k > 0$ and $\zeta = 0$ then

$$e^{At} = \begin{pmatrix} \cos(\omega_n t) & \dfrac{1}{\omega_n}\sin(\omega_n t) \\ -\omega_n \sin(\omega_n t) & \cos(\omega_n t) \end{pmatrix}.$$

Case 4) Underdamped: If $k > 0$ and $0 < \zeta < 1$ then

$$e^{At} = e^{-\zeta\omega_n t}\begin{pmatrix} \cos(\omega_d t) + \dfrac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t) & \dfrac{1}{\omega_d}\sin(\omega_d t) \\ \dfrac{-\omega_d}{1-\zeta^2}\sin(\omega_d t) & \cos(\omega_d t) - \dfrac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t) \end{pmatrix},$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is the damped natural frequency.

Case 5) Critically Damped: If $k > 0$ and $\zeta = 1$ then

$$e^{At} = e^{-\omega_n t}\begin{pmatrix} 1+\omega_n t & t \\ -\omega_n^2 t & 1-\omega_n t \end{pmatrix}.$$

Case 6) Overdamped: If $k > 0$ and $\zeta > 1$ then

$$e^{At} = e^{-\zeta\omega_n t}\begin{pmatrix} \cos(\omega_d t) + \dfrac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t) & \dfrac{1}{\omega_d}\sin(\omega_d t) \\ \dfrac{-\omega_d}{1-\zeta^2}\sin(\omega_d t) & \cos(\omega_d t) - \dfrac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t) \end{pmatrix},$$

where $\omega_d = \omega_n\sqrt{\zeta^2-1}$.

II. The Exponential of $n \times n$ Matrices Satisfying a Quadratic Polynomial

In this section, we derive formulas for $n \times n$ matrices that satisfy a quadratic polynomial. Since, by the Cayley–Hamilton Theorem, this class includes all 2×2 matrices, the results of the previous section are recovered as a special case. In addition, these results apply to certain $n \times n$ matrices such as involutory, rank 1, and idempotent matrices.

Lemma 3.1: Let $A \in \mathbf{C}^{n\times n}$ and suppose that $A^2 = \rho I$, where $\rho \in \mathbf{C}$.

i) If $\rho = 0$ then $e^A = I + A$.

ii) If $\rho \neq 0$ then $e^A = \cosh(\sqrt{\rho})I + (\sinh(\sqrt{\rho})/\sqrt{\rho})A$.

Proof:

i) Immediate.

ii) Since $A^2 = \rho I$, $A^{2k} = \rho^k I$ and $A^{2k+1} = \rho^k A$ for $k \geq 0$. Hence,

$$\begin{aligned} e^A &= \sum_{k=0}^{\infty}\frac{A^k}{k!} \\ &= \left[I + \frac{A^2}{2!} + \frac{A^4}{4!} + \cdots\right] + \left[A + \frac{A^3}{3!} + \frac{A^5}{5!} + \cdots\right] \\ &= \left(1 + \frac{\rho}{2!} + \frac{\rho^2}{4!} + \cdots\right)I + \left(I + \frac{\rho}{3!} + \frac{\rho^2}{5!} + \cdots\right)A \\ &= \cosh\left(\sqrt{\rho}\right)I + \frac{\sinh\left(\sqrt{\rho}\right)}{\sqrt{\rho}}A. \qquad \square \end{aligned}$$

Lemma 3.1 applies to nilpotent and involutory matrices.

Corollary 3.2: Let $A \in \mathbf{C}^{n\times n}$.

i) If $A^2 = 0$ then $e^A = I + A$.

ii) If $A^2 = I$ then $e^A = \cosh(1)I + \sinh(1)A$.

Theorem 3.3: Let $A \in \mathbf{C}^{n\times n}$ and suppose that $A^2 + 2\lambda A + \mu I = 0$, *where* $\lambda, \mu \in \mathbf{C}$.

i) If $\lambda^2 = \mu$ then

$$e^A = e^{-\lambda}[(1+\lambda)I + A].$$

ii) If $\lambda^2 \neq \mu$ then

$$e^A = e^{-\lambda}\left\{\left[\cosh\left(\sqrt{\lambda^2-\mu}\right) + \frac{\lambda}{\sqrt{\lambda^2-\mu}}\sinh\left(\sqrt{\lambda^2-\mu}\right)\right]I + \frac{1}{\sqrt{\lambda^2-\mu}}\sinh\left(\sqrt{\lambda^2-\mu}\right)A\right\}.$$

Proof: Note that $B^2 = (\lambda^2 - \mu)I$ where $B = A + \lambda I$. By Lemma 4.1, we have

i)
$$\begin{aligned} e^A &= e^{-\lambda}e^B \\ &= e^{-\lambda}[I + B] \\ &= e^{-\lambda}[(1+\lambda)I + A]. \end{aligned}$$

ii)
$$\begin{aligned} e^A &= e^{-\lambda}e^B \\ &= e^{-\lambda}\left[\cosh\left(\sqrt{\lambda^2-\mu}\right)I + \frac{\sinh\left(\sqrt{\lambda^2-\mu}\right)}{\sqrt{\lambda^2-\mu}}B\right] \\ &= e^{-\lambda}\left\{\left[\cosh\left(\sqrt{\lambda^2-\mu}\right) + \frac{\lambda}{\sqrt{\lambda^2-\mu}}\sinh\left(\sqrt{\lambda^2-\mu}\right)\right]I + \frac{1}{\sqrt{\lambda^2-\mu}}\sinh\left(\sqrt{\lambda^2-\mu}\right)A\right\}. \qquad \square \end{aligned}$$

We now specialize Theorem 3.3 to the case in which A is real.

Corollary 3.4: Let $A \in \mathbf{R}^{n\times n}$ and suppose that $A^2 + 2\lambda A + \mu I = 0$, where $\lambda, \mu \in \mathbf{R}$.

i) If $\lambda^2 = \mu$ then

$$e^A = e^{-\lambda}[(1+\lambda)I + A].$$

ii) If $\lambda^2 > \mu$ then

$$e^A = e^{-\lambda}\left\{\left[\cosh\left(\sqrt{\lambda^2-\mu}\right) + \frac{\lambda}{\sqrt{\lambda^2-\mu}}\sinh\left(\sqrt{\lambda^2-\mu}\right)\right]I + \frac{1}{\sqrt{\lambda^2-\mu}}\sinh\left(\sqrt{\lambda^2-\mu}\right)A\right\}.$$

iii) If $\lambda^2 < \mu$ then

$$e^A = e^{-\lambda}\left\{\left[\cos\left(\sqrt{\mu-\lambda^2}\right) + \frac{\lambda}{\sqrt{\mu-\lambda^2}}\sin\left(\sqrt{\mu-\lambda^2}\right)\right]I + \frac{1}{\sqrt{\mu-\lambda^2}}\sin\left(\sqrt{\mu-\lambda^2}\right)A\right\}.$$

If A is the 2×2 complex matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then, by the Cayley–Hamilton Theorem, $A^2 - (a+d)A + (ad - bc)I = 0$. By applying Theorem 3.3 with $\lambda = -(a+d)/2$ and $\mu = ad - bc$, we will recover Corollary 2.3.

Corollary 3.5: Let $A \in \mathbf{C}^{n \times n}$ and suppose that rank $A = 1$.

i) If tr $A = 0$ then $e^A = I + A$.
ii) If tr $A \neq 0$ then $e^A = I + ((e^{\mathrm{tr}\,A} - 1)/\mathrm{tr}\,A)A$.

Proof: Since rank $A = 1$, $A^2 = (\mathrm{tr}\,A)A$. Hence, we can apply Theorem 3.3 with $\lambda = (\mathrm{tr}\,A)/2$ and $\mu = 0$ to obtain the desired results. □

Next, we consider the case in which A is idempotent.

Corollary 3.6: Let $A \in \mathbf{C}^{n \times n}$ and suppose that $A^2 = A$. Then $e^A = I + (e - 1)A$.

Proof: Apply Theorem 3.3 with $\lambda = -(1/2)$ and $\mu = 0$. □

IV. The Exponential of $n \times n$ Matrices Satisfying a Special Cubic Polynomial

Theorem 4.1: Let $A \in \mathbf{C}^{n \times n}$ and suppose that $A^3 = \rho A$, where $\rho \in \mathbf{C}$.

i) If $\rho = 0$ then $e^A = I + A + A^2/2$.
ii) If $\rho \neq 0$ then $e^A = I + (\sinh(\sqrt{\rho})/\sqrt{\rho})A + ((\cosh(\sqrt{\rho}) - 1)/\rho)A^2$.

Proof: The first case is easy. For the second case, since $A^3 = \rho A$, $A^{2k+2} = \rho^k A^2$ and $A^{2k+1} = \rho^k A$ for $k \geq 0$. Hence,

$$\begin{aligned} e^A &= \sum_{k=0}^{\infty} \frac{A^k}{k!} \\ &= I + \left[A + \frac{A^3}{3!} + \frac{A^5}{5!} + \cdots\right] + \left[\frac{A^2}{2!} + \frac{A^4}{4!} + \cdots\right] \\ &= I + \left(1 + \frac{\rho}{3!} + \frac{\rho^2}{5!} + \cdots\right)A + \left(\frac{1}{2!} + \frac{\rho}{4!} + \cdots\right)A^2 \\ &= I + \frac{\sinh(\sqrt{\rho})}{\sqrt{\rho}}A + \frac{\cosh(\sqrt{\rho}) - 1}{\rho}A^2. \end{aligned}$$ □

Theorem 4.1 yields the exponential of tripotent matrices.

Corollary 4.2: Let $A \in \mathbf{C}^{n \times n}$ and suppose that $A^3 = A$. Then

$$e^A = I + \sinh(1)A + (\cosh(1) - 1)A^2.$$

When Theorem 4.1 is specialized to real matrices, we have the following result.

Corollary 4.3: Let $A \in \mathbf{R}^{n \times n}$ and suppose that $A^3 = \rho A$, where $\rho \in \mathbf{R}$. Then

i) If $\rho = 0$ then $e^A = I + A + A^2/2$.
ii) If $\rho > 0$ then $e^A = I + (\sinh(\sqrt{\rho})/\sqrt{\rho})A + ((\cosh(\sqrt{\rho}) - 1)/\rho)A^2$.
iii) If $\rho < 0$ then $e^A = I + (\sin(\sqrt{-\rho})/\sqrt{-\rho})A + ((\cos(\sqrt{-\rho}) - 1)/\rho)A^2$.

Next we apply this result to a 3×3 skew symmetric real matrix.

Corollary 4.4: Let $A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \in \mathbf{R}^{3 \times 3}$. Then $A^3 + \gamma^2 A = 0$, where $\gamma = \sqrt{a^2 + b^2 + c^2}$. If $\gamma \neq 0$ then $e^A = I + (\sin\gamma/\gamma)A + ((1 - \cos\gamma)/\gamma^2)A^2$.

Corollary 4.4 can be interpreted in terms of kinematics of rotating bodies. To see this, let $\omega = (\omega_1, \omega_2, \omega_3)^T \in \mathbf{R}^3$ be a fixed nonzero vector and let B denote a rigid body of arbitrary shape. Now assume that B is rotating with respect to ω in the sense that the motion of each point in B traces out a circle whose center lies on the line containing ω and which lies in a plane that is perpendicular to the line containing ω. Furthermore, assume that each point in B rotates about ω at the constant angular rate of $\omega_0 = \sqrt{\omega^T\omega}$ radians per second. Then ω is called the angular velocity vector [4, p. 22].

Now let $x(t) = (x_1(t), x_2(t), x_3(t))^T$ denote the coordinates of an arbitrary point in B. Then $x(t)$ satisfies

$$\dot{x}(t) = \omega \times x(t),$$

where $x(0) = x_0$. The cross product operation can be represented equivalently by

$$\dot{x}(t) = A(\omega)x(t),$$

where $A(\omega) = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$. Hence, $x(t)$ is given by $x(t) = e^{A(\omega)t}x_0$. Since $e^{A(\omega)t}$ is an orthogonal matrix, $x(t)$ has a constant Euclidean length $\sqrt{x(t)^T x(t)} = \sqrt{x_0^T x_0}$ as is intuitively clear. Now Corollary 4.4 implies that

$$e^{A(\omega)t} = I + \frac{\sin\omega_o t}{\omega_o}A(\omega) + \frac{1 - \cos\omega_o t}{\omega_o^2}A^2(\omega),$$

which shows that the motion of each point in B has a rotational period of $2\pi/\omega_o$ seconds. Since $A^2(\omega) = \omega\omega^T - \omega_o^2 I$, the matrix exponential $e^{A(\omega)t}$ is given alternatively by

$$e^{A(\omega)t} = \cos(\omega_o t)I + \frac{\sin\omega_o t}{\omega_o}A(\omega) + \frac{1 - \cos\omega_o t}{\omega_o^2}\omega\omega^T.$$

Finally, suppose that the body B undergoes an arbitrary series of rotations with a single fixed point at the origin. Then the final orientation can be viewed as an orthogonal transformation of the original orientation. The corresponding orthogonal matrix can then be shown to be the exponential of a skew symmetric matrix of the form $A(\theta)$, where $\theta = (\theta_1, \theta_2, \theta_3)^T$. Normalizing θ by $\hat{\theta} = \theta/\theta_0$, where $\theta_0 = \sqrt{\theta^T\theta}$, it follows that

$$e^{A(\theta)} = (\cos\theta_0)I + (\sin\theta_0)A(\hat{\theta}) + (1 - \cos\theta_0)\hat{\theta}\hat{\theta}^T.$$

In accordance with Euler's Theorem [4, p. 10], this orthogonal transformation represents a rotation of θ_0 radians about the axis of rotation given by $\hat{\theta}$.

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Single-Loop Stability Margins for Multirate and Periodic Control Systems

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***Abstract*—We show how to compute the stability margins of periodic systems with respect to compatible periodic perturbations of such systems. We suggest that the nonstandard time-invariant model for periodic systems introduced in [3] leads to a simpler computation, although the standard model gives the same result.**

I. Introduction

The validation of a control system design requires the computation of stability margins with respect to errors in the model of the system to be controlled. In the absence of a practical technique for computing stability margins with respect to simultaneous variations in real plant parameters, it is common for control practitioners to compute single-loop gain and phase margins at points in a feedback system at which model variations have physical significance. Typically, it is desirable to do these computations in the frequency domain, so that well-developed engineering interpretations can be applied.

Many practical control designs are implemented as multirate systems in order to utilize hardware as efficiently as possible when computing power presents a limitation on design. Some techniques for the frequency domain analysis of multirate systems are limited by the fact that the natural models for multirate systems are not time invariant but periodically time varying. It is only when a loop is broken at a point at which the sampling rate is the greatest common divisor of all the sampling rates in the system that the natural input–output model is time invariant.

These limitations were partially addressed in the 1950's by the work of Kranc [8] and Kalman–Bertram [6]. Since it is easy to see, say using the Kalman–Bertram approach, that a multirate system is periodic if the sampling rates are rationally related, we shall phrase our results in the context of periodic systems. The cited work may be summarized by saying that by breaking the loop at any point and accumulating all inputs and outputs over a time interval for which the overall system is periodic into a vector input and a vector output one can obtain a time-invariant model. However, this time-invariant system is not readily amenable to stability margin calculation.

The obvious approach to the control analyst of "modern" inclinations is to compute the frequency dependent singular values of the transfer function matrix for the multiinput/multioutput (MIMO) time-invariant model. The singular value decomposition provides both a lower bound for the gain margin of the time-varying loop and a natural (constant) destabilizing perturbation corresponding to this margin. Unfortunately, this constant perturbation will generally correspond to a noncausal perturbation of the original periodic system.

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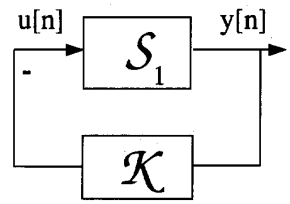

Fig. 1. Periodic feedback system.

In this note (which is a revised version of [4]) we show that the singular value calculation just mentioned gives a stability margin which is nonconservative in the sense that there is a periodic perturbation within the computed stability radius which destabilizes the closed loop.

II. Problem Formulation

We assume that we have a feedback system as in Fig. 1. Here $\mathscr{S}_1$ is a periodic time varying linear system in discrete time, given by the equations

$$x[n+1] = A_n x[n] + B_n u[n] \tag{1}$$

$$y[n] = C_n x[n] + D_n u[n] \tag{2}$$

where $x[i] \in \mathbb{R}^p$, $u[i] \in \mathbb{R}^q$ and $y[i] \in \mathbb{R}^r$, for $i = 0, 1, \cdots$. We assume that this system has period N, so that $A_i = A_{i+N}$, $B_i = B_{i+N}$, etc. The system $\mathscr{K}$ is assumed to be a linear system which may have additional properties as discussed below. $\mathscr{K}$ is a perturbation of the identity system, and we use it to define and analyze stability margins.

Such models cover a large class of "hybrid" (interconnected continuous- and discrete-time) systems. Using the methods presented by Kalman–Bertram [6] one can write down time-invariant models for these systems. Such models have been recently presented in a more elegant mathematical framework in [7] and [5]. This class includes multirate sampled-data systems, as considered, for example, by [1].

For the present purposes, we assume that the feedback loop represented in Fig. 1 is a scalar signal which we have "pulled out" of the overall system in order to do a stability margin analysis. In other words, we assume $q = r = 1$. We shall call this the *periodic single-input/single-output* (PSISO) case.

The problem we consider is the following: Assuming the closed loop system is stable if $\mathscr{K} \equiv 1$, find the largest $\mu > 0$ (the *margin*) such that the closed-loop system is stable for all stable $\mathscr{K}$ (subject to additional assumptions below) satisfying $\|1 - \mathscr{K}\| < \mu$. Here $\|\cdot\|$ is the l^2-induced operator norm.

There are three natural sets of assumptions about $\mathscr{K}$, which are of increasing generality:

1) $\mathscr{K}$ is memoryless. It therefore has a transfer function K which may be a real constant, a unit-magnitude complex number, or a general complex constant. These possibilities result in what may be called the *gain margin*, *phase margin*, and *complex margin* problems, respectively.

2) $\mathscr{K}$ is a time-invariant system with transfer function $K(z)$, which we assume to be stable. This we call the *time-invariant perturbation* stability problem. Here we assume that the sampling

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