VI. Conclusion

In this paper, the finite algorithm for strict Hurwitz invariance of a convex combination of polynomials (due to Bialas [5] and Fu and Barmish [6]) is extended to a strict Schur invariance test and to generalized stability tests. These tests have a main advantage over the "Edge Theorem" in [4], i.e., they are computationally efficient and exact. However, the Edge Theorem is more versatile since it applies to arbitrary regions of the complex plane. In contrast, the results presented here are restricted to images of the open complex left-half plane under real linear fractional transformations.

Acknowledgment

The authors would like to thank one of the reviewers for pointing out [5].

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The Optimal Projection Equations with Petersen–Hollot Bounds: Robust Stability and Performance Via Fixed-Order Dynamic Compensation for Systems with Structured Real-Valued Parameter Uncertainty

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Dedicated to the memory of Vicki Ringer Jones
October 26, 1951–December 12, 1987

Abstract—**A feedback control-design problem involving structured real-valued plant parameter uncertainties is considered. A quadratic Lyapunov bound suggested by recent work of Petersen and Hollot is utilized in conjunction with the guaranteed cost approach of Chang and Peng to guarantee robust stability with robust performance bound. Necessary conditions which generalize the optimal projection equations for fixed-order dynamic compensation are used to characterize the controller which minimizes the performance bound. The design equations thus effectively serve as sufficient conditions for synthesizing dynamic output-feedback controllers which provide robust stability and performance.**

Manuscript received December 29, 1986; revised September 17, 1987. Paper recommended by Associate Editor, D. P. Looze. The work of D. S. Bernstein was supported in part by the U.S. Air Force Office of Scientific Research under Contract F49620-86-C-0002.
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IEEE Log Number 8718561.

I. Introduction

As is well known, LQR and LQG controllers lack guaranteed robustness with respect to arbitrary parameter variations [1], [2]. Thus, it is not surprising that there is considerable interest in the analysis and synthesis of feedback controllers which are robust with respect to structured real-valued plant parameter uncertainty. The present paper was motivated in particular by the guaranteed cost control approach of Chang and Peng [3], [4] and the robust stability technique of Petersen and Hollot [5]–[7]. In [3], Chang and Peng consider a modified Riccati equation whose solutions are guaranteed to provide both robust stability and performance over a specified range of parameter variations. On the other hand, Petersen and Hollot in [5]–[7] consider a different modified Riccati equation which utilizes a quadratic Lyapunov bound to provide robust stability over a range of structured plant variations. In the present paper, we combine aspects of both of these approaches to obtain both robust stability and performance.

Our preference for the Petersen–Hollot bound over the bound originally proposed by Chang and Peng is based upon the fact that the former is differentiable with respect to the Riccati solution, while the latter is not. We exploit this smoothness by utilizing the optimal projection approach for fixed-order dynamic compensation [8] in place of full-state feedback considered in [3], [4], [6], [7]. A systematic, in-depth treatment of the Chang–Peng, Petersen–Hollot, and other bounds (such as the right shift/multiplicative white noise bound considered in [9]–[11]) will be the subject of a future paper [12].

As discussed in [8], the optimal projection approach to fixed-order dynamic compensation is based upon a system of two modified algebraic Riccati equations and two modified algebraic Lyapunov equations which directly generalize LQG theory to the case of reduced-order controllers. To ensure robust stability and performance for reduced-order controllers, the present paper utilizes the Petersen–Hollot quadratic Lyapunov technique to bound the performance of controllers of fixed dimension. The performance bound is then interpreted as the cost functional for an *auxiliary minimization problem* whose optimality conditions directly generalize the results of [8]. Specifically, we again obtain a coupled system of algebraic Riccati and Lyapunov equations with additional terms arising from the Petersen–Hollot bound. When uncertainty is absent, these equations specialize immediately to the result of [8] which, in turn, specializes to LQG when the compensator order is equal to the plant dimension.

Although the optimal projection equations are necessary conditions for optimality, it is important to stress that in the present paper they are obtained not for the original cost function, but rather for a *bound* on the cost. The necessary conditions for the auxiliary minimization problem thus effectively serve as sufficient conditions for the original problem. Hence, even if a numerical solution of the extended optimal projection equations fails to produce the globally optimal controller, robust stability and performance are still guaranteed for all local extremals. Our approach thus seeks to rectify one of the main drawbacks of necessity theory by guaranteeing both robust stability and performance. Nevertheless, a numerical algorithm for computing the global optimum is given in [15].

In summary, *the main contribution of the present paper is the generalization of the optimal projection equations by means of the Petersen–Hollot quadratic Lyapunov bound to synthesize robustly stabilizing fixed-order dynamic compensators with guaranteed performance bound.* It is interesting to note that even in the full-order case, our results, which specialize to a coupled system of *three* matrix equations, are distinct from the results of [5] which involve a pair of modified Riccati equations and an auxiliary inequality. Furthermore, the present paper provides a robust performance bound not obtained in [5]–[7]. An additional, conceptual benefit of our approach is a rigorous optimization interpretation for the Petersen–Hollot Riccati equation approach. Finally, as shown in [20] for full-state feedback, the results given herein can be directly applied to the H_∞ design problem. For details, see [21].

Due to space constraints, the contents of the paper will not be reviewed here. We note only that the proof of Theorem 8.1, which has been omitted for this reason, can be found in [13], [14]. Finally, although numerical algorithms are outside the scope of this note, related results can be found in [15].

II. Notation and Definitions

Note: All matrices have real entries.

$\mathbb{R}$, $\mathbb{R}^{r\times s}$, $\mathbb{R}^r$, $\mathbb{E}$	Real numbers, $r \times s$ real matrices, $\mathbb{R}^{r\times 1}$, expected value.
I_r, $(\,)^T$	$r \times r$ identity matrix, transpose.
$\mathbb{S}^r$, $\mathbb{N}^r$, $\mathbb{P}^r$	$r \times r$ symmetric, nonnegative-definite, positive-definite matrices.
$Z_1 \le Z_2$, $Z_1 < Z_2$	$Z_2 - Z_1 \in \mathbb{N}^r$, $Z_2 - Z_1 \in \mathbb{P}^r$, $Z_1, Z_2 \in \mathbb{S}^r$.
$n, m, l, n_c; \tilde{n}$	Positive integers; $n + n_c$.
$x, u, y, x_c, \tilde{x}$	$n, m, l, n_c, \tilde{n}$-dimensional vectors.
$A, \Delta A; B, \Delta B; C, \Delta C$	$n \times n$ matrices; $n \times m$ matrices; $l \times n$ matrices.
A_c, B_c, C_c	$n_c \times n_c$; $n_c \times l$; $m \times n_c$ matrices.
$\tilde{A}, \Delta\tilde{A}$	$\begin{bmatrix} A & BC_c \\ B_cC & A_c \end{bmatrix}, \begin{bmatrix} \Delta A & \Delta BC_c \\ B_c\Delta C & 0 \end{bmatrix}.$
R_1, R_2	$n \times n$, $m \times m$ state, control weighting matrices; $R_1 \ge 0$, $R_2 > 0$.
R_{12}	$n \times m$ cross weighting matrix; $R_1 - R_{12}R_2^{-1}R_{12}^T \ge 0$.
$w_1(\cdot), w_2(\cdot)$	n, l-dimensional white noise.
V_1, V_2	Intensity of $w_1(\cdot)$, $w_2(\cdot)$; $V_1 \ge 0$, $V_2 > 0$.
V_{12}	$n \times l$ cross intensity of $w_1(\cdot)$, $w_2(\cdot)$.
$\tilde{w}(\cdot), \tilde{V}$	$\begin{bmatrix} w_1(\cdot) \\ B_cw_2(\cdot) \end{bmatrix}, \begin{bmatrix} V_1 & V_{12}B_c^T \\ B_cV_{12}^T & B_cV_2B_c^T \end{bmatrix}$
$\tilde{R}$	$\begin{bmatrix} R_1 & R_{12}C_c \\ C_c^TR_{12}^T & C_c^TR_2C_c \end{bmatrix}.$

III. Robust Stability and Robust Performance Problems

Let $\mathfrak{U} \subset \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{l\times n}$ denote the set of uncertain perturbations $(\Delta A, \Delta B, \Delta C)$ of the nominal plant matrices A, B, and C.

Robust Stability Problem: For fixed $n_c \le n$, determine (A_c, B_c, C_c) such that the closed-loop system consisting of the nth-order controlled plant

$$\dot{x}(t) = (A+\Delta A)x(t) + (B+\Delta B)u(t), \qquad t \in [0, \infty), \tag{3.1}$$

measurements

$$y(t) = (C+\Delta C)x(t), \tag{3.2}$$

and n_cth-order dynamic compensator

$$\dot{x}_c(t) = A_cx_c(t) + B_cy(t), \tag{3.3}$$

$$u(t) = C_cx_c(t) \tag{3.4}$$

is asymptotically stable for all $(\Delta A, \Delta B, \Delta C) \in \mathfrak{U}$.

Robust Performance Problem: For fixed $n_c \le n$, determine (A_c, B_c, C_c) such that, for the closed-loop system consisting of the nth-order disturbed plant

$$\dot{x}(t) = (A+\Delta A)x(t) + (B+\Delta B)u(t) + w_1(t), \qquad t \in [0, \infty), \tag{3.5}$$

noisy measurements

$$y(t) = (C+\Delta C)x(t) + w_2(t), \tag{3.6}$$

and n_cth-order dynamic compensator (3.3), (3.4), the performance criterion

$$J(A_c, B_c, C_c) \triangleq \sup_{(\Delta A, \Delta B, \Delta C)\in\mathfrak{U}} \limsup_{t\to\infty} \mathbb{E}[x^T(t)R_1x(t) + 2x^T(t)R_{12}u(t) + u^T(t)R_2u(t)] \tag{3.7}$$

is minimized.

Remark 3.1: Note that (3.7) is precisely the LQG criterion except for the supremum over $\mathfrak{U}$ for worst-case performance.

For each controller (A_c, B_c, C_c) and plant variation $(\Delta A, \Delta B, \Delta C) \in \mathfrak{U}$, the undisturbed closed-loop system (3.1)–(3.4) is given by

$$\dot{\tilde{x}}(t) = (\tilde{A} + \Delta\tilde{A})\tilde{x}(t), \qquad t \in [0, \infty), \tag{3.8}$$

while the disturbed closed-loop system (3.3)–(3.6) is

$$\dot{\tilde{x}}(t) = (\tilde{A} + \Delta\tilde{A})\tilde{x}(t) + \tilde{w}(t), \qquad t \in [0, \infty), \tag{3.9}$$

where $\tilde{x}(t) \triangleq [x^T(t), x_c^T(t)]^T$ and $\tilde{w}(\cdot)$ is white noise with intensity $\tilde{V} \in \mathbb{N}^{\tilde{n}}$.

IV. Sufficient Conditions for Robust Stability and Performance

In practice, steady-state performance is only of interest when the closed-loop system (3.8) is stable over $\mathfrak{U}$. The following result expresses the performance in terms of the steady-state closed-loop second-moment matrix.

Lemma 4.1: Suppose (3.8) is stable for all $(\Delta A, \Delta B, \Delta C) \in \mathfrak{U}$. Then

$$J(A_c, B_c, C_c) = \sup_{(\Delta A, \Delta B, \Delta C)\in\mathfrak{U}} \operatorname{tr} \tilde{Q}_{\Delta\tilde{A}}\tilde{R}, \tag{4.1}$$

where $\tilde{Q}_{\Delta\tilde{A}} \triangleq \lim_{t\to\infty} \mathbb{E}[\tilde{x}(t)\tilde{x}^T(t)] \in \mathbb{N}^{\tilde{n}}$ is the unique solution to

$$0 = (\tilde{A} + \Delta\tilde{A})\tilde{Q}_{\Delta\tilde{A}} + \tilde{Q}_{\Delta\tilde{A}}(\tilde{A} + \Delta\tilde{A})^T + \tilde{V}. \tag{4.2}$$

We now seek upper bounds for $J(A_c, B_c, C_c)$.

Theorem 4.1: Let Ω: $\mathbb{N}^{\tilde{n}} \times \mathbb{R}^{n_c\times l} \times \mathbb{R}^{m\times n_c} \to \mathbb{S}^{\tilde{n}}$ be such that

$$\Delta\tilde{A}\mathcal{Q} + \mathcal{Q}\Delta\tilde{A}^T \le \Omega(\mathcal{Q}, B_c, C_c),$$
$$(\Delta A, \Delta B, \Delta C) \in \mathfrak{U}, \; (\mathcal{Q}, B_c, C_c) \in \mathbb{N}^{\tilde{n}} \times \mathbb{R}^{n_c\times l} \times \mathbb{R}^{m\times n_c}, \tag{4.3}$$

and, for given (A_c, B_c, C_c), suppose there exists $\mathcal{Q} \in \mathbb{N}^{\tilde{n}}$ satisfying

$$0 = \tilde{A}\mathcal{Q} + \mathcal{Q}\tilde{A}^T + \Omega(\mathcal{Q}, B_c, C_c) + \tilde{V}, \tag{4.4}$$

and suppose the pair $(\tilde{V}^{1/2}, \tilde{A} + \Delta\tilde{A})$ is stabilizable for all $(\Delta A, \Delta B, \Delta C) \in \mathfrak{U}$. Then $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $(\Delta A, \Delta B, \Delta C) \in \mathfrak{U}$,

$$\tilde{Q}_{\Delta\tilde{A}} \le \mathcal{Q}, \qquad (\Delta A, \Delta B, \Delta C) \in \mathfrak{U}, \tag{4.5}$$

where $\tilde{Q}_{\Delta\tilde{A}}$ satisfies (4.2), and

$$J(A_c, B_c, C_c) \le \operatorname{tr} \mathcal{Q}\tilde{R}. \tag{4.6}$$

Proof: For all $(\Delta A, \Delta B, \Delta C) \in \mathfrak{U}$, (4.4) is equivalent to

$$0 = (\tilde{A} + \Delta\tilde{A})\mathcal{Q} + \mathcal{Q}(\tilde{A} + \Delta\tilde{A})^T + \Psi(\mathcal{Q}, B_c, C_c, \Delta\tilde{A}) + \tilde{V}, \tag{4.7}$$

where

$$\Psi(\mathcal{Q}, B_c, C_c, \Delta\tilde{A}) \triangleq \Omega(\mathcal{Q}, B_c, C_c) - (\Delta\tilde{A}\mathcal{Q} + \mathcal{Q}\Delta\tilde{A}^T).$$

Note that by (4.3), $\Psi(\mathcal{Q}, B_c, C_c, \Delta\tilde{A}) \ge 0$ for all $(\Delta A, \Delta B, \Delta C) \in \mathfrak{U}$. Since $(\tilde{V}^{1/2}, \tilde{A} + \Delta\tilde{A})$ is stabilizable for all $(\Delta A, \Delta B, \Delta C) \in \mathfrak{U}$, it follows from [16, Theorem 3.6] that $((\tilde{V} + \Psi(\mathcal{Q}, B_c, C_c, \Delta\tilde{A}))^{1/2}, \tilde{A} + \Delta\tilde{A})$ is stabilizable for all $(\Delta A, \Delta B, \Delta C) \in \mathfrak{U}$. Hence, [16, Lemma 12.2] implies $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $(\Delta A, \Delta B, \Delta C) \in$

$\mathfrak{U}$. Next, subtracting (4.2) from (4.7) yields

$$0=(\tilde{A}+\Delta\tilde{A})(\mathfrak{Q}-\tilde{Q}_{\Delta\tilde{A}})+(\mathfrak{Q}-\tilde{Q}_{\Delta\tilde{A}})(\tilde{A}+\Delta\tilde{A})^T+\Psi(\mathfrak{Q}, B_c, C_c, \Delta\tilde{A})$$

or, equivalently (since $\tilde{A}+\Delta\tilde{A}$ is asymptotically stable),

$$\mathfrak{Q}-\tilde{Q}_{\Delta\tilde{A}}=\int_0^\infty e^{(\tilde{A}+\Delta\tilde{A})t}\Psi(\mathfrak{Q}, B_c, C_c, \Delta\tilde{A})e^{(\tilde{A}+\Delta\tilde{A})^Tt}\,dt\geq 0,$$

which implies (4.5). Finally, (4.5) and (4.1) yield (4.6). □

V. Uncertainty Structure

To obtain explicit expressions for (A_c, B_c, C_c), we require that $\Delta B=0$, $(\Delta A, \Delta B, \Delta C)\in\mathfrak{U}$. Hence, for simplicity, we write $(\Delta A, \Delta C)\in\mathfrak{U}$. The dual case $\Delta B\neq 0$ and $\Delta C=0$ is treated in Section X. Thus, $\mathfrak{U}$ is assumed to be of the form

$$\mathfrak{U}=\left\{(\Delta A, \Delta C)\in\mathbb{R}^{n\times n}\times\mathbb{R}^{l\times n}: \Delta A=\sum_{i=1}^p D_iM_iN_iE_i,\right.$$
$$\left.\Delta C=\sum_{i=1}^p F_iM_iN_iE_i, \qquad M_iM_i^T\leq\bar{M}_i,\ N_i^TN_i\leq\bar{N}_i,\ i=1,\cdots,p\right\}, \tag{5.1}$$

where, for $i=1,\cdots,p$: $D_i\in\mathbb{R}^{n\times r_i}$, $E_i\in\mathbb{R}^{t_i\times n}$, and $F_i\in\mathbb{R}^{l\times r_i}$ are fixed matrices denoting the structure of the uncertainty; $\bar{M}_i\in\mathbb{N}^{r_i}$ and $\bar{N}_i\in\mathbb{N}^{t_i}$ are given uncertainty bounds; and $M_i\in\mathbb{R}^{r_i\times s_i}$ and $N_i\in\mathbb{R}^{s_i\times t_i}$ are uncertain matrices. The closed-loop system thus has structured uncertainty of the form

$$\Delta\tilde{A}=\sum_{i=1}^p\tilde{D}_iM_iN_i\tilde{E}_i,$$

where

$$\tilde{D}_i\triangleq\begin{bmatrix}D_i\\ B_cF_i\end{bmatrix},\quad \tilde{E}_i\triangleq[E_i\quad 0].$$

The special case $\bar{M}_i=\mu_i^2I_{r_i}$, $\bar{N}_i=\nu_i^2I_{t_i}$ is worth noting.

Proposition 5.1: Let $\mu_i, \nu_i\geq 0$, $i=1,\cdots,p$. Then $M_iM_i^T\leq\mu_i^2I_{r_i}$ and $N_i^TN_i\leq\nu_i^2I_{t_i}$ if and only if $\sigma_{\max}(M_i)\leq\mu_i$ and $\sigma_{\max}(N_i)\leq\nu_i$.

Remark 5.1: The form of $\mathfrak{U}$ given by (5.1) is directly related to the structured stability radius introduced by Hinrichsen and Pritchard [17], [18]. Specifically, let $p=1$, $\bar{M}_1=\mu_1I_{r_1}$, $r_1=s_1$, and $N_1\equiv\bar{N}_1=I_{t_1}$.

VI. The Petersen–Hollot Bound

Given $\mathfrak{U}$, we now specify the bound Ω satisfying (4.3). Note that because of $\Delta B=0$, Ω is independent of C_c. Hence, we write $\Omega(\mathfrak{Q}, B_c)$ for $\Omega(\mathfrak{Q}, B_c, C_c)$.

Proposition 6.1: The function

$$\Omega(\mathfrak{Q}, B_c)\triangleq\sum_{i=1}^p\tilde{D}_i\bar{M}_i\tilde{D}_i^T+\mathfrak{Q}\tilde{E}_i^T\bar{N}_i\tilde{E}_i\mathfrak{Q} \tag{6.1}$$

satisfies (4.3) with $\mathfrak{U}$ given by (5.1).

Proof: For $i=1,\cdots,p$,

$$\begin{aligned}0&\leq[\tilde{D}_iM_i-\mathfrak{Q}\tilde{E}_i^TN_i^T][\tilde{D}_iM_i-\mathfrak{Q}\tilde{E}_i^TN_i^T]^T\\&=\tilde{D}_iM_iM_i^T\tilde{D}_i^T+\mathfrak{Q}\tilde{E}_i^TN_i^TN_i\tilde{E}_i\mathfrak{Q}-(\tilde{D}_iM_iN_i\tilde{E}_i\mathfrak{Q}+\mathfrak{Q}\tilde{E}_i^TN_i^TM_i^T\tilde{D}_i^T)\\&\leq\tilde{D}_i\bar{M}_i\tilde{D}_i^T+\mathfrak{Q}\tilde{E}_i^T\bar{N}_i\tilde{E}_i\mathfrak{Q}-(\tilde{D}_iM_iN_i\tilde{E}_i\mathfrak{Q}+\mathfrak{Q}\tilde{E}_i^TN_i^TM_i^T\tilde{D}_i^T).\end{aligned}$$

Summing over i yields (4.3). □

Remark 6.1: The bound (6.1) was originally proposed by Petersen in [5] for unit-rank perturbations with scalar uncertain parameters. A more general treatment appears in [7]. Note that we absorb the epsilon used in [7] into D_i and E_i.

VII. The Auxiliary Minimization Problem

To optimize robust performance while guaranteeing robust stability, we consider the following problem.

Auxiliary Minimization Problem: Determine $(\mathfrak{Q}, A_c, B_c, C_c)$ which minimizes

$$\mathfrak{J}(\mathfrak{Q}, A_c, B_c, C_c)\triangleq\operatorname{tr}\mathfrak{Q}\tilde{R} \tag{7.1}$$

subject to

$$\mathfrak{Q}\in\mathbb{N}^{\tilde{n}}, \tag{7.2}$$

$$0=\tilde{A}\mathfrak{Q}+\mathfrak{Q}\tilde{A}^T+\sum_{i=1}^p[\tilde{D}_i\bar{M}_i\tilde{D}_i^T+\mathfrak{Q}\tilde{E}_i^T\bar{N}_i\tilde{E}_i\mathfrak{Q}]+\tilde{V}, \tag{7.3}$$

$$(\tilde{V}^{1/2}, \tilde{A}+\Delta\tilde{A})\text{ is stabilizable, }(\Delta A, \Delta C)\in\mathfrak{U}. \tag{7.4}$$

Proposition 7.1: If $(\mathfrak{Q}, A_c, B_c, C_c)$ satisfies (7.2)–(7.4), then $\tilde{A}+\Delta\tilde{A}$ is asymptotically stable for all $(\Delta A, \Delta C)\in\mathfrak{U}$ and

$$J(A_c, B_c, C_c)\leq\mathfrak{J}(\mathfrak{Q}, A_c, B_c, C_c). \tag{7.5}$$

Proof: With Ω given by (6.1), the hypotheses of Theorem 4.1 are satisfied so that robust stability is guaranteed with performance bound (4.6). □

VIII. Necessary Conditions for the Auxiliary Minimization Problem

Rigorous derivation of the necessary conditions for the Auxiliary Minimization Problem requires additional technical assumptions. Specifically, in addition to (7.2), we restrict $(\mathfrak{Q}, A_c, B_c, C_c)$ to the open set

$$\mathcal{S}\triangleq\{(\mathfrak{Q}, A_c, B_c, C_c): \mathfrak{Q}\in\mathcal{P}^{\tilde{n}},\ \mathfrak{A}\text{ is asymptotically stable,}$$
$$\text{and }(A_c, B_c, C_c)\text{ is controllable and observable}\},$$

where (see [19] for the definition of the Kronecker sum)

$$\mathfrak{A}\triangleq\left(\tilde{A}+\sum_{i=1}^p\mathfrak{Q}\tilde{E}_i^T\bar{N}_i\tilde{E}_i\right)\oplus\left(\tilde{A}+\sum_{i=1}^p\mathfrak{Q}\tilde{E}_i^T\bar{N}_i\tilde{E}_i\right).$$

Furthermore, the constraint (7.4) will not be accounted for explicitly since it can be shown that the compactness of $\mathfrak{U}$ implies that the set of (A_c, B_c, C_c) satisfying (7.4) is open.

Remark 8.1: The constraint $(\mathfrak{Q}, A_c, B_c, C_c)\in\mathcal{S}$ is not required for either robust stability or robust performance since Proposition 7.1 shows that only (7.2)–(7.4) are needed. Rather, the set $\mathcal{S}$ constitutes sufficient conditions under which the Lagrange multiplier technique is applicable to the Auxiliary Minimization Problem. Specifically, the condition $\mathfrak{Q}>0$ replaces (7.2) by an open set constraint, the stability of $\mathfrak{A}$ serves as a normality condition, and (A_c, B_c, C_c) minimal is a nondegeneracy condition.

For arbitrary $Q, P\in\mathbb{R}^{n\times n}$ define the following notation:

$$D\triangleq\sum_{i=1}^p D_i\bar{M}_iD_i^T,\quad E\triangleq\sum_{i=1}^p E_i^T\bar{N}_iE_i,$$

$$P_a\triangleq B^TP+R_{12}^T,\quad Q_a\triangleq QC^T+V_{12}+\sum_{i=1}^p D_i\bar{M}_iF_i^T,$$

$$A_P\triangleq A-BR_2^{-1}P_a,\quad A_Q\triangleq A-Q_aV_{2a}^{-1}C,\quad V_{2a}\triangleq V_2+\sum_{i=1}^p F_i\bar{M}_iF_i^T.$$

The following factorization lemma is needed. For details, see [8].

Lemma 8.1: If $\hat{Q}, \hat{P}\in\mathbb{N}^n$ and rank $\hat{Q}\hat{P}=n_c$, then there exist $n_c\times n$ G, Γ, and $n_c\times n_c$ invertible M such that

$$\hat{Q}\hat{P}=G^TM\Gamma, \tag{8.1}$$

$$\Gamma G^T=I_{n_c}. \tag{8.2}$$

Furthermore, G, M, and Γ are unique except for a change of basis in $\mathbb{R}^{n_c}$.

As shown in [8], the matrix τ defined by

$$\tau \triangleq \hat{Q}\hat{P}(\hat{Q}\hat{P})^{\#} = G^T\Gamma \tag{8.3}$$

is an oblique projection where $(\ )^{\#}$ denotes group generalized inverse [8]. For convenience, define the complementary projection $\tau_\perp \triangleq I_n - \tau$.

Theorem 8.1: If $(\mathcal{Q}, A_c, B_c, C_c) \in \mathcal{S}$ solves the Auxiliary Minimization Problem with $\mathfrak{U}$ given by (5.1), then there exist $Q, P, \hat{Q}, \hat{P} \in \mathbb{N}^n$ such that

$$\mathcal{Q} = \begin{bmatrix} Q+\hat{Q} & \hat{Q}\Gamma^T \\ \Gamma\hat{Q} & \Gamma\hat{Q}\Gamma^T \end{bmatrix}, \tag{8.4}$$

$$A_c = \Gamma(A - BR_2^{-1}P_a - Q_aV_{2a}^{-1}C + QE)G^T, \tag{8.5}$$

$$B_c = \Gamma Q_a V_{2a}^{-1}, \tag{8.6}$$

$$C_c = -R_2^{-1}P_aG^T, \tag{8.7}$$

and such that $Q, P, \hat{Q}, \hat{P}$ satisfy

$$0 = AQ + QA^T + V_1 + D + QEQ - Q_aV_{2a}^{-1}Q_a^T + \tau_\perp Q_aV_{2a}^{-1}Q_a^T\tau_\perp^T, \tag{8.8}$$

$$0 = [A + (Q+\hat{Q})E]^TP + P[A + (Q+\hat{Q})E] + R_1 - P_a^TR_2^{-1}P_a + \tau_\perp^TP_a^TR_2^{-1}P_a\tau_\perp, \tag{8.9}$$

$$0 = (A_P + QE)\hat{Q} + \hat{Q}(A_P + QE)^T + \hat{Q}E\hat{Q} + Q_aV_{2a}^{-1}Q_a^T - \tau_\perp Q_aV_{2a}^{-1}Q_a^T\tau_\perp^T, \tag{8.10}$$

$$0 = (A_Q + QE)^T\hat{P} + \hat{P}(A_Q + QE) + P_a^TR_2^{-1}P_a - \tau_\perp^TP_a^TR_2^{-1}P_a\tau_\perp, \tag{8.11}$$

$$\operatorname{rank}\hat{Q} = \operatorname{rank}\hat{P} = \operatorname{rank}\hat{Q}\hat{P} = n_c. \tag{8.12}$$

Furthermore, the auxiliary cost is given by

$$\mathcal{J}(\mathcal{Q}, A_c, B_c, C_c) = \operatorname{tr}[(Q+\hat{Q})R_1 + \hat{Q}PBR_2^{-1}P_a - R_{12}R_2^{-1}P_a\hat{Q}]. \tag{8.13}$$

Conversely, if there exist $Q, P, \hat{Q}, \hat{P} \in \mathbb{N}^n$ satisfying (8.8)–(8.12), then $(\mathcal{Q}, A_c, B_c, C_c)$ given by (8.4)–(8.7) satisfy (7.2) and (7.3) with cost (8.13).

Proof: See [13], [14]. □

Remark 8.2: Theorem 8.1 presents necessary conditions for the Auxiliary Minimization Problem which explicitly characterize extremal quadruples $(\mathcal{Q}, A_c, B_c, C_c)$. These necessary conditions consist of a system of two modified Riccati equations and two modified Lyapunov equations coupled by both the optimal projection τ and uncertainty terms. Several special cases can immediately be discerned. For example, in the full-order case $n_c = n$, set $\tau = I_n$ so that $\tau_\perp = 0$. Now the last term in each of (8.8)–(8.11) can be deleted and G and Γ in (8.5)–(8.7) can be taken to be the identity. Furthermore, $\hat{P}$ plays no role so that (8.11) is superfluous. Note that in this case, (8.8) is independent of P and $\hat{Q}$. Setting further D_i, E_i, and F_i to zero, it can be seen that (8.10) and (8.11) drop out, while (8.8) and (8.9) reduce to the standard separated Riccati equations of LQG theory. If, alternatively, the reduced-order constraint is retained, but the uncertainty terms are deleted, then the results of [8] are recovered.

Remark 8.3: When solving (8.8)–(8.12) numerically, the uncertainty terms can be adjusted to examine tradeoffs between performance and robustness. Specifically, the bounds $\bar{M}_i$ and $\bar{N}_i$ and structure matrices D_i, E_i, and F_i appearing in Q_a, D, E, and V_{2a} can be varied systematically to determine the region of solvability of (8.8)–(8.12).

IX. Sufficient Conditions for Robust Stability and Performance

Theorem 9.1: Suppose there exist $Q, P, \hat{Q}, \hat{P} \in \mathbb{N}^n$ satisfying (8.8)–(8.12), and assume that $(\tilde{V}^{1/2}, \tilde{A} + \Delta\tilde{A})$ is stabilizable for all $(\Delta A, \Delta C) \in \mathfrak{U}$ with A_c, B_c, C_c given by (8.5)–(8.7) and $\mathfrak{U}$ given by (5.1). Then $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $(\Delta A, \Delta C) \in \mathfrak{U}$ and the closed-loop performance is bounded by (8.13).

Proof: Theorem 8.1 implies that $\mathcal{Q}$ given by (8.4) satisfies (7.2) and (7.3). With the stabilizability assumption, the result follows from Proposition 7.1. □

X. The Dual Case

In place of (5.1), assume now that $\Delta C = 0$, $(\Delta A, \Delta B, \Delta C) \in \mathfrak{U}$, and define

$$\mathfrak{U} = \left\{(\Delta A, \Delta B) \in \mathbb{R}^{n\times n}\times\mathbb{R}^{n\times m} : \Delta A = \sum_{i=1}^{p} D_iM_iN_iE_i,\right.$$
$$\left.\Delta B = \sum_{i=1}^{p} D_iM_iN_iG_i, \qquad M_iM_i^T \le \bar{M}_i,\ N_i^TN_i \le \bar{N}_i,\ i = 1, \cdots, p\right\}, \tag{10.1}$$

where, for $i = 1, \cdots, p$: $D_i \in \mathbb{R}^{n\times r_i}$, $E_i \in \mathbb{R}^{t_i\times n}$, and $G_i \in \mathbb{R}^{t_i\times m}$ are fixed matrices denoting the structure of the uncertainty; and $\bar{M}_i$, $\bar{N}_i$, M_i, and N_i are as before. For arbitrary $Q, P \in \mathbb{R}^{n\times n}$ define the following notation:

$$\hat{P}_a \triangleq B^TP + R_{12}^T + \sum_{i=1}^{p} G_i^T\bar{N}_iE_i, \quad \hat{Q}_a \triangleq QC^T + V_{12},$$

$$\hat{A}_P \triangleq A - BR_{2a}^{-1}\hat{P}_a, \quad \hat{A}_Q \triangleq A - \hat{Q}_aV_2^{-1}C, \quad R_{2a} \triangleq R_2 + \sum_{i=1}^{p} G_i^T\bar{N}_iG_i.$$

The main result guaranteeing robust stability and performance for the dual problem can now be stated. For details, see [13], [14].

Theorem 10.1: Suppose there exist $P, Q, \hat{P}, \hat{Q} \in \mathbb{N}^n$ satisfying (8.12) and

$$0 = A^TP + PA + R_1 + E + PDP - \hat{P}_a^TR_{2a}^{-1}\hat{P}_a + \tau_\perp^T\hat{P}_a^TR_{2a}^{-1}\hat{P}_a\tau_\perp, \tag{10.2}$$

$$0 = [A + D(P+\hat{P})]Q + Q[A + D(P+\hat{P})]^T + V_1 - \hat{Q}_aV_2^{-1}\hat{Q}_a^T + \tau_\perp\hat{Q}_aV_2^{-1}\hat{Q}_a^T\tau_\perp^T, \tag{10.3}$$

$$0 = (\hat{A}_Q + DP)^T\hat{P} + \hat{P}(\hat{A}_Q + DP) + \hat{P}D\hat{P} + \hat{P}_a^TR_{2a}^{-1}\hat{P}_a - \tau_\perp^T\hat{P}_a^TR_{2a}^{-1}\hat{P}_a\tau_\perp, \tag{10.4}$$

$$0 = (\hat{A}_P + DP)\hat{Q} + \hat{Q}(\hat{A}_P + DP)^T + \hat{Q}_aV_2^{-1}\hat{Q}_a^T - \tau_\perp\hat{Q}_aV_2^{-1}\hat{Q}_a^T\tau_\perp^T, \tag{10.5}$$

and assume that $(\tilde{R}^{1/2}, \tilde{A} + \Delta\tilde{A})$ is detectable for all $(\Delta A, \Delta B) \in \mathfrak{U}$ with A_c, B_c, C_c given by

$$A_c = \Gamma(A - \hat{Q}_aV_2^{-1}C - BR_{2a}^{-1}\hat{P}_a + DP)G^T, \tag{10.6}$$

$$B_c = \Gamma\hat{Q}_aV_2^{-1}, \tag{10.7}$$

$$C_c = -R_{2a}^{-1}\hat{P}_aG^T, \tag{10.8}$$

and $\mathfrak{U}$ given by (10.1). Then, with (10.6)–(10.8), $\tilde{A} + \Delta\tilde{A}$ is asymptotically stable for all $(\Delta A, \Delta B) \in \mathfrak{U}$ and the performance of the closed-loop system satisfies

$$J(A_c, B_c, C_c) \le \operatorname{tr}[(P+\hat{P})V_1 + \hat{Q}_aV_2^{-1}CQ\hat{P} - \hat{P}\hat{Q}_aV_2^{-1}V_{12}^T]. \tag{10.9}$$

Remark 10.1: Even in the case $\Delta B = 0$, $\Delta C = 0$, the performance bounds (8.13) and (10.9) are generally different.

Remark 10.2: The case in which ΔB and ΔC are simultaneously nonzero also appears to be tractable and leads to additional terms in the design equations. The bound considered in [11] also permits this case.

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A Frequency Response-Based Model Order Selection Criterion

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Abstract—**The use of weighting sequence models to describe the dynamics of physical systems provides an effective means of translating the uncertainty associated with the model parameter estimates derived from noisy input/output data into corresponding frequency response uncertainty information. However, an appropriate truncation level must be established to accomplish this task. This paper addresses the truncation problem from a frequency response perspective and proposes a new criterion based on frequency response considerations to select the proper truncation.**

Manuscript received January 6, 1987; revised May 18, 1987. This work was supported by the U.S. Air Force and the U. K. Science and Engineering Research Council (SERC).
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IEEE Log Number 8718562.

I. Introduction

Recent work on robustness analysis methods for multivariable systems has led to the development of structured singular value techniques (e.g., [1], [2]) which provide a means of assessing the impact of system uncertainty on closed-loop stability and performance. A key missing element in these analysis methods is the ability to describe the frequency response uncertainty associated with any given system. When system identification techniques are used to derive the system description, however, it becomes possible to quantify system uncertainty statistically, and recent efforts have demonstrated that uncertainty information on the estimated parameters of the system model can be transformed into corresponding information on the frequency response uncertainty of the system [3], [4]. For difference equation models, the transformation from the parameter space to the frequency domain is nonlinear, a result which necessitates the use of linear approximations and produces a statistical description of uncertainty that fails to account for the interfrequency dependence of the frequency response estimates. On the other hand, Cloud and Kouvaritakis [4] have shown that these problems can be avoided by the use of weighting sequence models to describe system dynamics.

The uncertainty description developed in [4] assumes that the system can be accurately described by a finite weighting sequence model, an assumption that is valid for all stable systems. This assumption, in turn, implies that the "correct" model order (i.e., truncation level) is known. As a result, the identification process must not only be able to generate appropriate parameter estimates, it must also be able to identify the "correct" level of truncation. In effect, this second requirement is a restatement of the standard model order selection problem, a problem which has been widely investigated in the literature (e.g., [5]). But for the frequency response applications of interest here, appropriate solutions must focus on generating accurate frequency response information. When this perspective is taken, it becomes clear that the standard order selection criteria are not well suited to the task because they focus on generating accurate input/output descriptions rather than accurate frequency response descriptions.

In this paper, a new criterion is derived to identify the "correct" truncation level based on frequency response considerations. The development begins by highlighting a geometric interpretation of the standard "input/output" order selection problem. These geometric results are then transformed into the frequency domain to produce the new "frequency response-based" criterion for truncation selection, and simulation results are presented to demonstrate its use. Armed with the "correct" truncation level generated by this criterion, it is now possible to implement the techniques described in [4] to produce a valid description of frequency response uncertainty for any given system.

II. Model Order Selection: A Geometric Perspective

Consider the discrete-time system with weighting sequence elements $\{\theta_1, \theta_2, \theta_3, \cdots\}$ whose true response at sample k to the set of inputs $\{u(k-1), u(k-2), \cdots, u(0)\}$ is given by

$$y^0(k)=\sum_{i=1}^{k}\theta_i u(k-i)=d_k^t\theta^0 \tag{1}$$

where $\theta^0=[\theta_1 \cdots \theta_k]^t$ and $d_k^t=[u(k-1) \cdots u(0)]$. The measured output at sample k is then given by

$$y^m(k)=y^0(k)+\epsilon(k) \tag{2}$$

where $\epsilon(k)$ is assumed to be an element of a white noise sequence with variance σ_ϵ^2. For a set of N measurements, we may stack the scalars $y^m(k)$ and $\epsilon(k)$ as elements of the vectors y^m and ϵ, respectively, and may then rewrite (2) as

$$y^m=y^0+\epsilon=D^0\theta^0+\epsilon \tag{3}$$

where the rows of D^0 are given by the vectors d_k^t for $k=1, \cdots, N$.