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Regular reflection in potential flow and the sonic criterion

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We study the classical problem of self-similar reflection of shocks at a ramp, modeled by potential flow with γ-law pressure. Depending on corner angle θ and upstream Mach number M_I, either regular (RR) or Mach reflections occur. There are several conflicting transition criteria predicting the corner angle at which the type of reflection changes. We show that in some cases, in particular $M_I = 1$ and $\gamma = 5/3$, an exact RR solution exists for all θ specified by the sonic criterion. Thus all weaker criteria are false.

Reflection of oblique shock waves at a solid wall has been one of the most studied problems of compressible flow, originated by Mach [1] and drawing the attention of von Neumann [2]. In a common variant, a straight vertical *incident* shock wave (inflow Mach number M_I, downstream velocity zero) moves along a horizontal solid wall; at $t = 0$ it reaches a wall corner with deflection angle θ (see Figure 1 left). For time $t > 0$ a "reflected shock" is thrown back from the corner whereas the incident shock continues to move along the slope. The solution is *self-similar*: density, velocity and temperature are constant in $\vec{\xi} = (\xi, \eta) = (x/t, y/t)$.

In a *regular reflection* (*RR*), incident and reflected shock meet in a single point on the slope (see Figure 1 middle). But in many cases more complicated *Mach reflections* (MR) patterns are observed, such as the *single Mach reflection* (*SMR*) shown on Figure 1 right, where reflected and incident shock meet in a *triple point* with a contact discontinuity and a third shock (*Mach stem*) connecting to the slope. Other types are *double* and *complex Mach reflection* (*DMR, CMR*); some more have been conjectured (see [3] for a classification).

It is not clear for which values of M_I and θ we should expect RR or MR. In fact for some of them RR is theoretically impossible since there is no reflected shock that passes through the reflection point at all times *and* has a velocity tangential to the slope below it (slip condition). The *detachment criterion* is the strongest possible: it predicts RR whenever a reflected shock exists. However, usually *two* shocks exists, a strong and a weak one (this puzzle has been studied by the author and Liu [4] in a different context). The *sonic criterion* is more restrictive: it predicts RR exactly if the flow on both sides of the reflected shock is supersonic, in the reference frame of an observer moving in the reflection point. Several other criteria have been proposed, most importantly the *von Neumann criterion* (see [3]).

Numerics and experiments have failed to decide which of the many criteria is correct. The sonic and detachment criterion predict transition at angles θ_s resp. θ_d which are less than a degree apart (see Figure 2 left). Even careful experiments [5] have not been able to reach this accuracy, due to various error sources. For example numerical or viscous boundary layers can cause *spurious Mach stems* that make RR look like MR [6, Figure 7c].

Drawing on techniques from [4] we can show:

Theorem 1 ([7], Theorem 1) *For $\theta > \theta_s$ and the (γ, M_I) pairs enclosed by the solid and dashed curve in Figure 2 right, there is a weak solution of compressible potential flow with γ-law pressure, defined for all $t > 0$, that attains the initial data shown Figure 1 left. It is of RR type.*

In earlier work, Chen and Feldman [8] have already obtained a similar result for the case $\theta \approx 90°$.

The values covered include in particular $M_I = 1$ and $\gamma = 5/3$, but currently not $\gamma = 7/5$ or $\gamma = 4/3$. (However, in some cases of non-vertical incident shocks, RR can also be constructed for all $\theta \downarrow \theta_s$ in these cases.)

Self-similar potential flow is a mixed-type equation:

$$(c^2 - (\psi_\xi - \xi)^2)\psi_{\xi\xi} - 2(\psi_\xi - \xi)(\psi_\eta - \eta)\psi_{\xi\eta} + (c^2 - (\psi_\eta - \eta)^2)\psi_{\eta\eta} = 0$$

where $\nabla\psi$ is velocity, $c^2 = c_0^2 + (1-\gamma)(-\psi + \vec{\xi} \cdot \nabla\psi - \frac{1}{2}|\nabla\psi|^2)$. The type is elliptic for $L := |\nabla\psi - \vec{\xi}|/c < 1$, parabolic for $L = 1$, hyperbolic otherwise. RR is elliptic in the region below reflected shock and dashed arc in Figure 1; it is parabolic on the arc; all other regions are hyperbolic and trivial (constant $\vec{v}, \rho$). The curved part of the reflected shock is a free boundary, which is compensated by having two boundary conditions. The elliptic region degenerates at the arc which is particularly difficult.

The proof relies on techniques for nonlinear elliptic equations. An iteration $\mathcal{K} : \mathcal{F} \to C^{2,\alpha}_\beta$ is constructed where $C^{2,\alpha}_\beta$ is a weighted Hölder space and $\mathcal{F}$ a subset defined by many constraints, such as $L < 1$, $\rho > 0$, shock location bounds etc. In each iteration step a nonlinear elliptic *fixed* boundary value problem is solved with one of the two shock conditions, followed by adjusting the free shock to satisfy the other shock condition; a fixed point of the iteration will satisfy both. On the arc,

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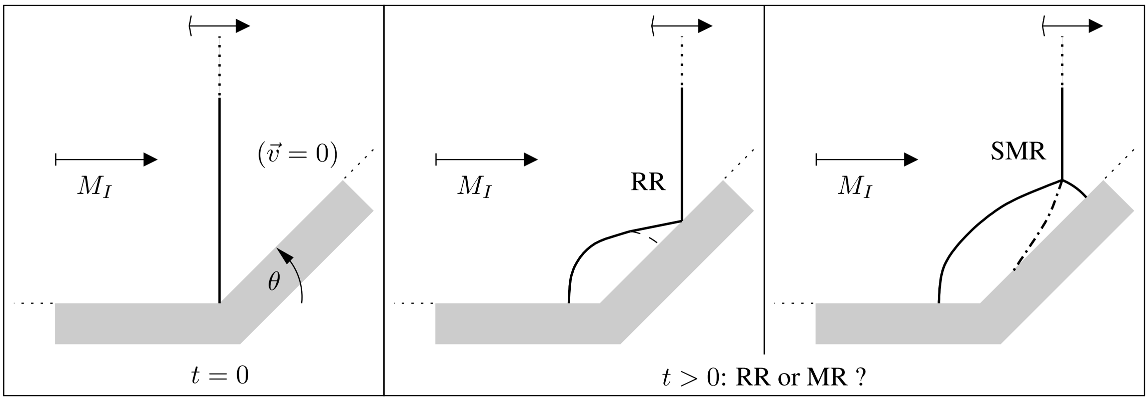

Fig. 1 Left: initial data. Center: RR. Right: SMR (dashed curve = contact, solid = shock)

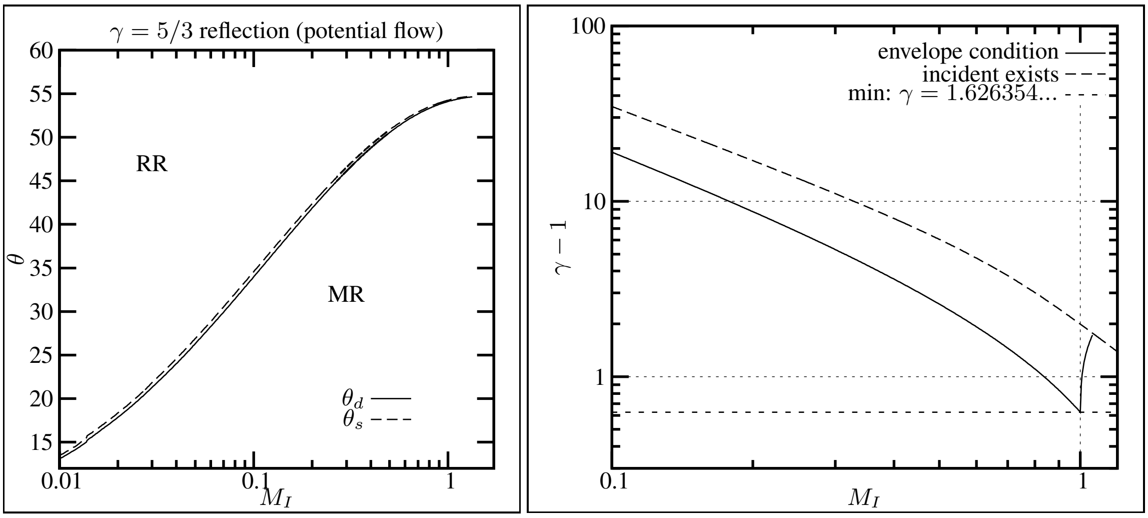

Fig. 2 Left: RR→MR transition criteria for full Euler, $\gamma = 5/3$. Right: for $\theta = \theta_s$, cases between dashed and solid line are covered.

$L^2 = 1 - \epsilon$ with $\epsilon > 0$ is imposed so that the PDE is slightly elliptic there and classical regularity theory can be used; in the end ϵ-independent estimates are used to obtain a limit solution.

Collecting the parameters γ, M_I, θ in a vector λ, we have a continuous family $\lambda \mapsto \mathcal{K}_\lambda$ of compact maps. In some trivial cases ($\theta = 90°$) it can be shown that there is a unique fixed point in $\mathcal{F}$ with Leray-Schauder index $\iota = \pm 1$, so $\mathcal{K}$ has degree ι for those λ. Most of the proof is concerned with showing that $\mathcal{K}$ has no fixed point on $\partial\mathcal{F}$. Therefore K_λ has degree $\iota \neq 0$, hence at least one fixed point, for *all* λ.

The restriction in Figure 2 right is necessary to show uniform strength of the reflected shock. It is expected that new techniques can remove it so that all $\theta > \theta_s$ can be covered regardless of γ, M_I.

The theorem demonstrates that criteria weaker than sonic are invalid, at least for potential flow. There is no qualitative difference between RR for potential flow vs. full Euler, suggesting that the latter admits a similar result. However, MR often requires a contact discontinuity which is not modelled by potential flow, so a treatment in full Euler would be desirable.

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