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Arbitrary-speed quantum gates within large ion crystals through minimum control of laser beams

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Abstract. – We propose a scheme to implement arbitrary-speed quantum entangling gates on two trapped ions immersed in a large linear crystal of ions, with minimal control of laser beams. For gate speeds slower than the oscillation frequencies in the trap, a single appropriately detuned laser pulse is sufficient for high-fidelity gates. For gate speeds comparable to or faster than the local ion oscillation frequency, we discover a five-pulse protocol that exploits only the local phonon modes. This points to a method for efficiently scaling the ion trap quantum computer without shuttling ions.

Significant advances have been made towards trapped ion quantum computation in the last decade [1]. Many ingredients of quantum computing have been demonstrated experimentally with this system [2–12]; and different versions of quantum gate schemes have been proposed, each offering particular advantages [13–20]. In conventional approaches to trapped ion quantum gates, the interaction between the ions is mediated by a particular phonon mode (PM) in the ion crystal through the sideband addressing with laser beams. In these types of gates, the control of the laser beams is relatively simple, requiring only a continuous-wave beam with an appropriate detuning; but to resolve individual motional sidebands, the gate speed must be much smaller than the ion trap oscillation frequencies. More recently, fast quantum gates have also been proposed, which can operate with a speed comparable with or greater than the trap frequencies [19, 20]. These types of gates involve simultaneous excitation of all PMs [12, 18–20] and require more complicated control of either the pulse shape [19] and/or the timing of a fast pulse sequence [19, 20].

In this paper, we develop a gate scheme that combines the desirable features of the above two types of gates. A conditional phase gate with arbitrary speed is constructed in a large ion array by optimization of few relevant experimental parameters. As a result, first we show that with simple control of the detuning of a continuous-wave laser beam, one can achieve a high-fidelity gate with the gate speed approaching the ion trap frequency. This result is a bit surprising as many PMs are excited during the gate. However, with control of just one experimental parameter (the detuning), each of the modes becomes nearly disentangled with the ion internal states at the end of the gate. Secondly, we show that as the gate speed becomes larger than the local ion oscillation frequency (specified below and see also ref. [20]), only "local" PMs will be primarily excited during the gate. This yields a scaling method for trapped ion quantum computation: a significant scaling obstacle to trapped ion quantum computation is that due to the long-range Coulomb interaction, any collective gate on two ions is necessarily influenced by all the other ions in the architecture, which makes the gate control increasingly difficult with growth of the qubit number. Conventionally, one needs to use the ion shuttling in a complicated trap architecture to avoid this undesirable influence [2,5,10,16]. However, if the gate speed becomes comparable with the local ion oscillation frequency, we have an alternative scaling method without the requirement of ion shuttling: one can perform the gate by exciting only the local PMs, which avoids the complicated influence from the background ions. This result also improves the fast gate scaling method proposed in ref. [20], as here to excite only the local PMs, instead of using hundreds of short pulses, we only need to apply five long pulses with optimized amplitudes chopped from a continuous-wave laser beam.

The system we have in mind is N ions in a linear trap with a global trap frequency ω. To perform arbitrary-speed quantum gates, we need to consider all the PMs [19, 20]. Without laser beams, the ion motional Hamiltonian has the standard form $H_0 = \sum_{k=1}^{N} \hbar\omega_k (a_k^\dagger a_k + 1/2)$ with $a_k, a_k^\dagger$ as the annihilation and creation operators of the k-th PM. The eigen-frequency of the PM $\omega_k \equiv \sqrt{\mu_k}\omega$ is determined by solving the eigen-equations $\sum_n A_{nl} \boldsymbol{b}_n^k = \mu_k \boldsymbol{b}_l^k$, where the matrix elements $A_{nl} = 1 + 2\sum_{p=1,p\neq l}^{N} 1/|u_l - u_p|^3$ for $n = l$, and $A_{nl} = -2/|u_l - u_n|^3$ for $n \neq l$. The parameter $u_n = x_n^0/\sqrt[3]{e^2/4\pi\epsilon_0 M\omega^2}$ with x_n^0 representing the equilibrium position of the n-th ion and M denoting the mass [21]. To perform quantum gates, we need to apply some spin-dependent force on the ions, which can be induced, for instance, through the ac-Stark shift from two propagating laser beams with a relative angle and detuning [7]. As it is the case in experiments [7, 11], we assume that when the ions are in their equilibrium positions, the ac-Stark shifts for the ion qubit states $|0\rangle$ and $|1\rangle$ are equivalent. Then, under the Lamb-Dicke condition and in the interaction picture with respect to H_0, the Hamiltonian for the spin-dependent force is given by

$$H = -\sum_{n,k=1}^{N} F_n(t) g_n^k (a_k^\dagger e^{i\omega_k t} + a_k e^{-i\omega_k t})\sigma_n^z, \tag{1}$$

where $\sigma_n^z \equiv |1\rangle\langle 1| - |0\rangle\langle 0|$ is the Pauli operator, $F_n(t)$ is the force on the n-th ion, and $g_n^k = \sqrt{\hbar/2M\omega_k}\boldsymbol{b}_n^k$ is the coupling constant between the n-th ion and the k-th PM. Using Magnus' formula, the evolution operator corresponding to the Hamiltonian (1) is found as [22]

$$U(\tau) = \exp\left[i\sum_n \phi_n(\tau)\sigma_n^z + i\sum_{l,n} \phi_{ln}(\tau)\sigma_l^z\sigma_n^z\right], \tag{2}$$

where $\phi_n(\tau) = \sum_k [\alpha_n^k(\tau) a_k^\dagger - \alpha_n^{k*}(\tau) a_k]$ with $\alpha_n^k(\tau) = \frac{i}{\hbar}\int_0^\tau F_n(t) g_n^k e^{i\omega_k t}\mathrm{d}t$, and $\phi_{ln}(\tau) = \frac{2}{\hbar^2}\int_0^\tau \int_0^{t_2} \sum_k F_l(t_2) g_l^k g_n^k F_n(t_1) \sin\omega_k(t_2 - t_1)\mathrm{d}t_1\mathrm{d}t_2$.

A conditional phase flip (CPF) gate on two arbitrary ions i and j can be accomplished with identical spin-dependent forces on only these two ions with $F_i(t) = F_j(t) = F(t)$. In this case, the evolution operator $U(\tau)$ in eq. (2) exactly corresponds to a CPF gate $U_{ij} = \exp[i\pi\sigma_i^z\sigma_j^z/4]$ if $\phi_{ij}(\tau) = \pi/4$ and $\alpha_{i(j)}^k(\tau) = 0$ for all the modes k. In principle, it is always possible to satisfy this set of constraints by designing a sufficiently complicated pulse shape for the forces. However, this kind of solution typically requires exquisite control of many parameters that determine the exact shape of $F(t)$, which may be difficult experimentally. In the following, we show that in typical cases it is only necessary to approximately satisfy these constraints, allowing a much simpler class of laser pulse shapes to be used.

To design gate, we optimize the gate fidelity subject to a certain class of laser pulses, with simple control parameters. With an initial state $|\Psi_0\rangle$, the final state would be given by $|\Psi_f\rangle = U_{ij}|\Psi_0\rangle$ after a perfect CPF gate. However, with imperfect control, some of the PMs will not evolve along a closed loop in the phase space corresponding to $\alpha_{i,j}^k(\tau) \neq 0$. In that case, the final internal state of the ions is mixed and described by the density operator $\rho_r = Tr_m[U(\tau)|\Psi_0\rangle\langle\psi_0|U^\dagger(\tau)]$, where the trace is over the motional state of all the ions. The overlap between the ideal state $|\Psi_f\rangle$ and the actual density operator ρ_r defines the fidelity $F_g = \langle\Psi_f|\rho_r|\Psi_f\rangle$. Without loss of generality, we choose here a typical initial state with $|\Psi_0\rangle = (|0\rangle_i + |1\rangle_i) \otimes (|0\rangle_j + |1\rangle_j)/2$ for calculation of the gate fidelity F_g. We assume that the PMs are initially in thermal states with an effective temperature T. Then, with the evolution operator $U(\tau)$ given in eq. (2), the gate fidelity F_g is found to be

$$F_g = \frac{1}{8}[2 + 2(\Gamma_i + \Gamma_j) + \Gamma_+ + \Gamma_-], \tag{3}$$

where $\Gamma_{i(j)} = \exp[-\sum_k |\alpha_{i(j)}^k(\tau)|^2\bar{\beta}_k/2]$, and $\Gamma_\pm = \exp[-\sum_k |\alpha_i^k(\tau) \pm \alpha_j^k(\tau)|^2\bar{\beta}_k/2]$. The parameter $\bar{\beta}_k$ is given by $\bar{\beta}_k = \coth(\hbar\omega_k/k_BT) = \coth[\frac{\sqrt{\mu_k}}{2}\ln(1 + 1/\bar{n}_c)]$, with k_B denoting the Boltzman constant and $\bar{n}_c = (e^{\hbar\omega/k_BT} - 1)^{-1}$ representing the mean phonon number of the center-of-mass mode [23].

To maximize the gate fidelity, we choose our control parameters to be simply the detuning and the amplitude of the laser beams that introduce the spin-dependent force. With the ac Stark shift from the Raman laser beams [7, 11], the force function $F(t)$ has the form of $\Omega\sin(\mu t)$, where μ is determined by the detuning between the Raman laser beams and Ω is the two-photon Rabi frequency. To introduce more control parameters, we can chop the continuous-wave laser beam into m equal-time segments with the Rabi frequency for the p-th ($p=1, 2, \cdots, m$) segment given by a controllable value Ω_p. The force $F(t)$ then takes the form $F(t) = \Omega_p \sin(\mu t)$ for the interval $(p-1)\tau/m \leq t < p\tau/m$. This kind of amplitude control for the Raman beams can be done, for example, with simple acoustic- or electro-optical modulators.

With a sufficient number of control parameters Ω_p, it is always possible to make the gate infidelity $F_{in} \equiv 1 - F_g = 0$. In this case, the conditions $\alpha_{i,j}^k(\tau) = 0$ require $\sum_{p=1}^m \Omega_p \times \int_{(p-1)\tau/m}^{p\tau/m} \sin(\mu t)\exp[i\omega_k t]\mathrm{d}t = 0$ for any k mode, which are a set of linear constraints for the ratios $f_p \equiv \Omega_p/\Omega_1$. For the case of N PMs, it is possible to satisfy these N complex constraints with $2N$ real parameters f_i ($i = 2, 3, \cdots, 2N + 1$), so the required number of segments is $m = 2N + 1$. In the following, we will show that we can actually use a much smaller number of segments (control parameters) to reduce the gate infidelity to almost zero.

First we consider the case of the minimal control of the laser beams: a single amplitude and detuning of the laser beam, or a single segment ($m = 1$). This situation corresponds exactly to current experimental configurations [7, 11]. Without shape control of the laser beams, all the known gate schemes require the gate speed to be much smaller than the ion trap frequency for sideband addressing of a particular PM. Here, by taking into account all the PMs, we show that one can still get a high-fidelity gate even if the gate speed approaches the ion trap frequency, which is well beyond the limit set by the sideband addressing.

In our calculation, we first consider the gate acting on the two central ions in a 20-ion array. In fig. 1a, the gate fidelity calculated from eq. (3) is shown as a function of the laser detuning for various gate speeds. When the gate time τ is significantly larger than τ_0 with $\tau_0 \equiv 2\pi/\omega$, the gate fidelity has local maximum at the detuning $\mu = \omega_k + 2\pi l/\tau$ with an integer l. This corresponds to the well-known condition in the phase (the Milburn-Sorensen-Molmer) gate [7, 14, 15] . When τ approaches τ_0, it is better to choose a detuning with either $\mu < \omega$ or $\mu > \max\{\omega_k\}$ to have a higher gate fidelity. The optimal detunings shift a bit downwards in

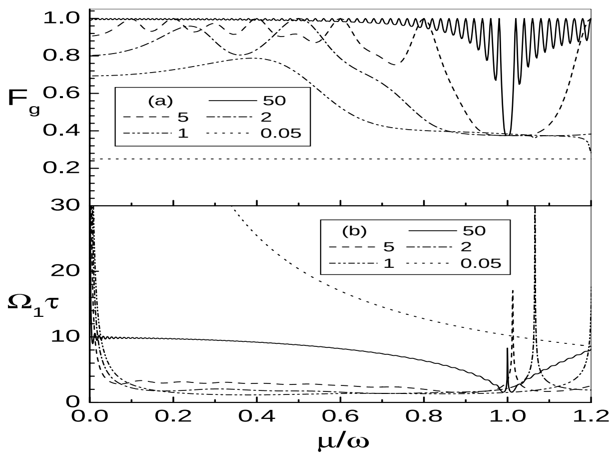

Fig. 1 – For the two center ions in a 20-ion array, the gate fidelity (a) and the required Rabi frequency (b) shown as a function of the detuning μ with $\tau = 50\tau_0, 5\tau_0, 2\tau_0, \tau_0, 0.05\tau_0$, respectively. The other parameters: $\bar{n}_c = 3$ and $m = 1$.

the region $\mu < \omega$ and upwards in the region $\mu > \max\{\omega_k\}$ compared with the values given by $\mu = \omega_k + 2\pi l/\tau$. A distinct result from this calculation is that the gate fidelity can be still very high even if the gate speed goes well beyond the sideband addressing limit. For instance, the optimal fidelity $F_g \simeq 99.97\%$ ($F_g \simeq 99\%$) for the gate time $\tau = 2\tau_0$ ($\tau = 1.5\tau_0$), respectively, with the corresponding detuning μ pretty close to $\omega - 2\pi/\tau$. If we further increase the gate speed, the fidelity quickly goes down. For instance, the optimal fidelity is only 80% for $\tau = \tau_0$ and reduces to the minimum of 25% (corresponding to a completely mixed state after the gate) when $\tau \leq 0.05\tau_0$.

As the gate time τ approaches τ_0, many PMs are involved during the gate, and they ultimately get nearly disentangled with the ion internal state. To see this, we checked the contribution to the conditional phase ϕ_{ij} from all the other (non center-of-mass) PMs for the case of the optimal detuning very close to $\omega - 2\pi/\tau$. The relative contributions from the "spectator" PMs is about 1.3%, 10%, and 18.1% for the gate time $\tau = 50\tau_0$, $5\tau_0$, and $2\tau_0$, respectively. It is a bit surprising that, for instance at $\tau = 2\tau_0$, the "spectator" PMs contribute 18.1% of the conditional phase but induce an infidelity of only 0.03%.

We have also calculated the required laser power (proportional to Ω_p for the Raman configuration) for achieving the high-speed gates, and the result is shown in fig. 1b. Note that the optimal detuning μ not only maximizes the gate fidelity, but also requires the least amount of laser power. We can see from this figure that with increase of the gate speed, the required laser power grows slower than a linear increase in the region $\tau \geq \tau_0$. In current experiments, typically $\tau \sim 100\tau_0$ [7], so with moderate increase of the laser power, one can expect a significant increase of the gate speed even without any laser shape control. Similar calculations are also done for gates on different pairs of ions in the array. The results are qualitatively similar, although the gate fidelity is somewhat lower for the pair of ions with a larger distance. For instance, with $\tau = 2\tau_0$ and an optimal μ, the gate fidelity F_g is about 99% for the 1st and 2nd ions (at the edge of the trap), and is 95% for the 1st and 20th ions (the worst case).

The above calculation shows that without chopping of the laser beams, the gate fidelity quickly decreases in the region $\tau \leq 1.5\tau_0$. To improve the gate fidelity, we need to introduce

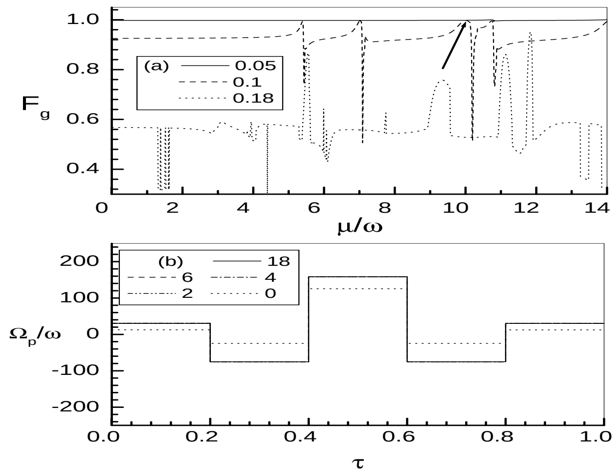

Fig. 2 – (a) The gate fidelity as a function of the detuning μ with $m = 5$ and $\tau = 0.18\tau_0, 0.1\tau_0, 0.05\tau_0$, respectively. (b) The optimal sequence of the force (Ω_p) for the gate with $\tau = 0.1\tau_0$ and $\mu = 10\omega$ (denoted by an arrow in (a)). The numbers n above the curve denote how many neighboring ions' motion is taken into account for calculating the force sequence. With $n = 18$, all the PMs are included. The force sequences are basically indistinguishable for $n = 2, 4, 6, 18$.

more control parameters by dividing the laser beams into m segments. One might expect that to attain a certain gate fidelity, the number of segments m (*i.e.*, the number of control parameters) should continuously increase with $1/\tau$, as more and more normal PMs will be substantially excited during such a fast gate. However, this is actually not the case as we will see here. The key point is that as the gate speed becomes faster than the ion motional response time, only the local PMs (which are superpositions of many normal PMs) of the two ions involved in the gate will be substantially excited, greatly simplifying the control. To make this point more precise, we characterize the ion response time by its local oscillation frequency ω_{Li} [20]. The ω_{Li} for the i-th ion is defined as the eigen-oscillation frequency of this ion if we fix all the other ions in the trap at their equilibrium positions. If the gate speed becomes faster than ω_{Li}, we expect that the gate in a large ion array could be reduced to an effective two-ion problem, so with $m = 2N + 1 = 5$ segments of laser pulses, we should expect good gate fidelity. In the following, we test this idea by calculating the gate fidelity with 5 laser pulses under various gate speeds.

We still take a 20-ion array, and for the center ions the local ion oscillation frequency $\omega_{Li} \simeq 9.2\omega$. We calculate the gate fidelity with $m = 5$ and the optimized parameters $\Omega_1, \Omega_2, \cdots, \Omega_5$, and the result is shown in fig. 2a for $\tau = 0.18\tau_0$, $0.1\tau_0$, and $0.05\tau_0$, respectively. The fidelity F_g is above 99.99% for all μ for $\tau \leq 0.05\tau_0 \sim 2\pi/(2\omega_{Li})$, clearly demonstrating the above idea. Even for $\tau = 0.1\tau_0 \sim 2\pi/\omega_{Li}$, we find that the gate fidelity is above 99% at the optimal values of the detuning with $\mu = 5.4\omega$, 7.0ω, 10.0ω, or 10.7ω. The corresponding force sequence Ω_p for $\mu = 10.0\omega$ (corresponding to a fidelity $F_g = 99.76\%$) is shown in fig. 2(b). To see that only the local PMs are substantially involved during the gate, we also calculate the optimal Ω_p subject to the constraint that only a few neighboring ions around the target ions are allowed to oscillate during the gate (all the other ions are assumed fixed at their equilibrium positions). Including the motion of n ($n = 0, 2, 4, 6$) neighboring ions, the corresponding optimal force sequences are shown in fig. 2(b). The force sequences become indistinguishable as soon as

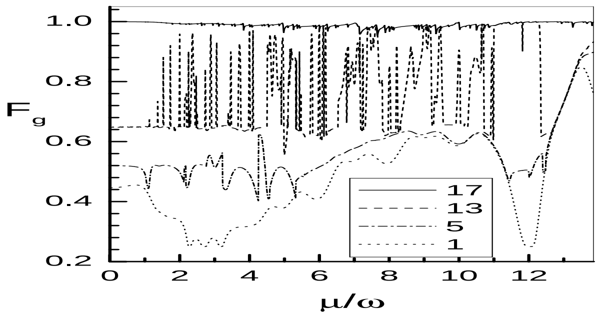

Fig. 3 – The gate fidelity as a function of the detuning μ with $\tau = 0.5\tau_0$ and $m = 1, 5, 13, 17$, respectively. The other parameters are the same as in fig. 1.

$n \geq 2$, which means that the motion of the ions beyond the nearest neighbors has no influence on the gate with a speed faster than ω_{Li}. This result has important implications for using the fast gates as a method to scale up ion trap quantum computation [20]. For a large-scale computation with any ion trap architecture, as soon as the gate speed becomes larger than the local ion oscillation frequency, we need only consider the influence of neighboring ions on the target ions. The other ions, near their equilibrium positions, only provide an effective static potential, and the design of the gate can always be well approximated by considering only a few ions. So the control complexity of each gate does not increase with the number of ions in the computation, which provides an effective scaling method.

It turns out that it is most difficult to perform a gate with the gate speed between the trap frequency ω and the local ion oscillation frequency ω_{Li}. In that region, one needs to introduce more control parameters by dividing the laser beams into more segments. But even in this worst case, it is still possible to get a high-fidelity gate with the number of segments m much smaller than $2N + 1$. For instance, with 20 ions, the worst case occurs with a gate time $\tau \sim 0.5\tau_0$, which requires the largest number of control parameters. For this worst case, we plot the gate fidelity in fig. 3 as a function of μ with $m = 1, 5, 13, 17$, respectively. The fidelity F_g has been above 99% at some optimal values of the detuning μ with $m = 13$, and a fidelity larger than 98.5% can be reached at almost any μ with $m \geq 17$. Note that this value is still significantly smaller than $2N + 1 = 41$. We have also done calculations with different number of ions in the array and for gates on different pairs of ions. The results are qualitatively very similar to what we have described. For instance, with $N = 40$ ions, a gate fidelity higher than 98.8% can be reached for the two center ions with the number of segments $m = 1$ if the gate time $\tau \geq 1.7\tau_0$; and a fidelity large than 99.6% is achievable with $m = 5$ segments of laser beams if the gate time $\tau \leq 2\pi/\omega_{Li}$ (in this case $\omega_{Li} = 16.7\omega$ for the center ions).

Before ending the manuscript, we briefly discuss some experimental imperfections. One of them is the intensity and phase fluctuations of the laser beams for the Raman operations (which influences the effective Rabi frequency). In this case, the fixed Rabi frequency is replaced by the one with a small random fluctuation term ϵ, *i.e.*, $\Omega[1+\epsilon]$. The additional infidelity due to this fluctuation can be estimated by $\pi^2\epsilon^2/2$. This term has the same dependence on fluctuation ϵ as the conventional trapped ion quantum gates, and they typically contribute to a small fraction of the infidelity in current experimenters [7]. Similarly, we can also discuss the imperfection in the trap potential, such as its anharmonicity. This dependence is actually the same as some other types of fast gates (see, *e.g.*, the discussion in ref. [19]). With typical experimental configurations, the additional gate infidelity due to the trap anharmonicity is significantly smaller than other contributions (such as those discussed in this manuscript).

In summary, we have described a scheme to achieve arbitrary-speed quantum gates on ions immersed in a large ion array, through minimum control of the amplitude of a continuous-wave laser beam. With the same control complexity as the conventional gates, we have shown how to push the gate speed towards the ion trap frequency. We have also shown a version of fast gates with five laser pulses which can operate in any large ion crystal and thus provide an efficient scaling method for ion trap quantum computation.

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[23] Equation (3) is still valid when different phonon modes have different temperature. In this case $\bar{\beta}_k$ should be replaced by $\coth[\frac{1}{2}\ln(1+1/\bar{n}_k)]$ with $\bar{n}_k$ denoting the mean phonon number of the k-th mode.