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Partition functions and elliptic genera from supergravity

Per Kraus[a] **and Finn Larsen**[b]

[a] *Department of Physics and Astronomy, UCLA*
Los Angeles, CA 90095-1547, U.S.A.
[b] *Michigan Center for Theoretical Physics, Department of Physics*
University of Michigan
Ann Arbor, MI 48109-1120, U.S.A.
E-mail: pkraus@physics.ucla.edu, larsenf@umich.edu

Abstract: We develop the spacetime aspects of the computation of partition functions for string/M-theory on $\mathrm{AdS}_3 \times M$. Subleading corrections to the semi-classical result are included systematically, laying the groundwork for comparison with CFT partition functions via the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence. This leads to a better understanding of the "Farey tail" expansion of Dijkgraaf et. al. from the point of view of bulk physics. Besides clarifying various issues, we also extend the analysis to the $\mathcal{N} = 2$ setting with higher derivative effects included.

Keywords: Chern-Simons Theories, Black Holes in String Theory, AdS-CFT Correspondence.

Contents

1. Introduction

In situations where string theory accounts for black hole thermodynamics in quantitative detail the microscopic theory is a conformal field theory describing the bound states of various branes. Schematically, we can write an equivalence between black hole and CFT partition functions:

$$Z_{\mathrm{BH}} = Z_{\mathrm{CFT}} \ . \tag{1.1}$$

The early successes in establishing (1.1) involved matching the large charge asymptotics of the two sides. In the last few years there has been much work (including [1−5]) on refining this identification to include the sub-leading asymptotics as well. On the black hole side this requires the inclusion of higher order spacetime effects, such as higher derivative corrections to the action. In favorable cases one has sufficient control to compute both sides of (1.1) beyond leading order and verify that the equality is upheld [1−5].

The identification (1.1) is most naturally viewed as an example of the AdS/CFT correspondence. This interpretation is possible because the near horizon geometry of the black holes considered is locally AdS_3 [6]. The AdS/CFT point of view explains, for example, why the partition functions of the black hole and the CFT agree in their leading exponential dependence. Our working assumption is that the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence is responsible also for the more detailed agreements seen when additional higher order spacetime effects are taken into account. Although there are ways that this assumption could fail (for example, by AdS_2 playing an essential role) it seems rather conservative to us.

The leading exponential behavior of the partition function at high (left or right moving) temperature is determined by the central charge. In [5] we showed that central charges on the two sides must agree due to cancellation of local anomalies. The argument is an adaptation to the AdS/CFT correspondence of the anomaly inflow mechanism explained in [7] (see also [8]). Since it is the exact central charges that agree, higher derivative corrections are taken into account. In particular small black holes (with vanishing classical area, corresponding to vanishing central charge at the leading order) can be considered; and the agreement also extends to non-supersymmetric and near extremal black holes. The significance of the anomaly approach is that it is robust, since anomalies are captured completely by a single term at one loop order. In this way we can bypass the need to determine all the explicit higher derivative terms in the spacetime action.

In this paper we develop spacetime aspects of the relation (1.1) in more detail and show how to compute the left hand side to an accuracy that extends beyond knowing the central charge. A key point is that Chern-Simons terms dominate the theory close to the boundary of AdS_3. This motivates us to develop Chern-Simons theory carefully in the spirit of the AdS/CFT correspondence and holographic renormalization. In particular, we systematically derive the boundary stress tensor and currents by studying the bulk theory in the presence of appropriate sources. We also consider the spacetime implementation of modular invariance and spectral flow in detail, and we make precise statements about the lattice of currents corresponding to particular string theory realizations.

A major precursor to the present work is the "Farey tail" [9]. These authors showed that the elliptic genus of the D1-D5 system admits an expansion that is highly suggestive of a supergravity interpretation in terms of a sum over geometries. One of our main objectives is to give a more first principles derivation of the gravitational side of this story. Another primary goal is to extend all this to the $\mathcal{N} = 2$ context, which is more sensitive to higher derivative terms and other higher order spacetime effects, and has been the subject of much recent discussion. We set up our formalism so that we can consider the $\mathcal{N} = 4$ and $\mathcal{N} = 2$ cases in parallel. The end result is that we can give a coherent formalism for computing partition functions from the spacetime point of view.

An important stimulus for the recent interest in this subject was the OSV conjecture [2] that the black hole CFT is related to the topological string through

$$Z_{\mathrm{CFT}}(p^I, q_I) = \int \frac{d\phi}{2\pi}\, |\psi_{\mathrm{top}}(X^I = p^I + \frac{i}{\pi}\phi^I)|^2 e^{-\pi\phi^I \cdot q_I}\ . \tag{1.2}$$

Much of the work in analyzing (1.2) has taken the approach of keeping only higher derivative F-terms in a near horizon $\mathrm{AdS}_2 \times S^2$. There is clearly a lot that is right about these conjectures but many details remain confusing. For example, the measure of the integral in (1.2) is unspecified and the right hand side is defined only perturbatively. Also, since OSV interpret each of the expressions (1.1)-(1.2) as an index, it is not clear why Wald's entropy formula applies. We will not resolve all these issues here but we will give further evidence that the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence is a useful setting for addressing them.

We have recently learned that several related works on the OSV conjecture and the Farey tail are in progress [10, 11]. Although there is some overlap, our work is complementary in that we emphasize the spacetime issues brought up by the AdS/CFT correspondence, and the general aspects of the problem. We also pay particular attention to situations with enhanced supersymmetry, such as M-theory on $K3 \times T^2$.

This paper is organized as follows. We begin in section 2 with a brief review of the elliptic genus from a CFT point of view. In section 3 we introduce the ingredients we need in our supergravity approach, with particular emphasis on gauge fields. Section 4 contains several simple but important examples that illustrate the approach. In particular we reconsider the computation of the black hole entropy in the saddle point approximation. In section 5 we turn to our main interest, laying out the strategy for computing the full elliptic genus from supergravity. In carrying out the computation we consider first, in section 6, the supergravity contributions, and next, in section 7, the contributions from wrapped branes. Finally, in section 8, we combine the various pieces and discuss the result.

2. Review: partition functions in CFT

We begin by reviewing the definitions and properties of the CFT partition functions that we will be trying to reproduce from supergravity/string theory. Our discussion here parallels that in [9]; also useful are [12, 13]. The focus will be on theories with either $(4,4)$ or $(0,4)$ supersymmetry, though in fact replacing any of the 4's by 2's makes almost no difference. We make reference to the two chiralities of the CFT with the convention (holomorphic, anti-holomorphic) $\sim$ (left,right).

Besides the Virasoro algebras, we play close attention to $U(1)$ and R-symmetry current algebras. We write out the relevant formulas for the holomorphc currents; the anti-holomorphic currents are included by making the obvious substitutions. We write $U(1)$ current algebra OPEs as

$$j^I(z)j^J(0) \sim \frac{k^{IJ}}{2z^2} \ , \tag{2.1}$$

and the $SU(2)_R$ current algebra OPEs as

$$J^i(z)J^j(0) \sim \frac{k}{2z^2}\delta^{ij} + \frac{i\epsilon^{ijk}}{z}J^k(0) \ . \tag{2.2}$$

The leftmoving central charge of the theory is

$$c = 6k \ . \tag{2.3}$$

In terms of the usual expansion of the currents in modes j^I_n and J^i_n, the commutation relations with the Virasoro generators L_n are

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n} \ , \\ [L_m, j^I_n] &= -nj^I_{m+n} \ , \\ [j^I_m, j^J_n] &= \tfrac{1}{2}mk^{IJ}\delta_{m+n} \ , \\ [L_m, J^i_n] &= -nJ^i_{m+n} \ , \\ [J^i_m, J^j_n] &= \tfrac{1}{2}mk\delta_{m+n}\delta^{ij} + i\epsilon_{ijk}j^k_{m+n} \ . \end{aligned} \tag{2.4}$$

When we refer to "charges" we will mean

$$\begin{aligned} J_0 &= 2J^3_0 \ , \quad J_0 \in \mathbb{Z} \ , \\ q^I &= 2j^I_0 \ , \end{aligned} \tag{2.5}$$

in a basis that diagonalizes the operators.

The combined Virasoro/current algebras admit the following spectral flow automorphisms:

$$\begin{aligned} L_0 &\rightarrow L_0 + \eta J_0 + k\eta^2 \ , \\ J_0 &\rightarrow J_0 + 2k\eta \ , \end{aligned} \tag{2.6}$$

in the $SU(2)$ case, and

$$\begin{aligned} L_0 &\rightarrow L_0 + \eta_I q^I + k^{IJ}\eta_I\eta_J \ , \\ q^I &\rightarrow q^I + 2k^{IJ}\eta_J \ , \end{aligned} \tag{2.7}$$

in the $U(1)$ case. Integer η preserves fermion periodicities, while half-integer η interchanges the NS and R sectors. There is an analogous statement for the η^I, depending on the relevant charge lattice, $q^I \in \Gamma$.

2.1 Elliptic genus

To define the elliptic genus we introduce potentials for the charges J_0 and q^I. Since these charges appear symmetrically, it is natural to relabel the R-charge as

$$q^0 \equiv J_0 \ , \tag{2.8}$$

so that $I = 0, 1, 2, \ldots$. We also extend the definition of k^{IJ} such that $k^{00} = k$, $k^{0,I>0} = k^{I>0,0} = 0$.

The elliptic genus is now defined as[1]

$$\chi(\tau, z_I) = \mathrm{Tr}_{RR}\left[e^{2\pi i\tau(L_0 - c/24)} e^{-2\pi i\overline{\tau}(\tilde{L}_0 - \tilde{c}/24)} e^{2\pi i z_I q^I} (-1)^F\right] \ . \tag{2.9}$$

Only rightmoving ground states, with $\tilde{L}_0 - \tilde{c}/24 = 0$, contribute, so the elliptic genus does not depend explicitly on $\overline{\tau}$. On the other hand, all leftmoving states can contribute. The elliptic genus is invariant under smooth deformations of the CFT. This follow from the quantization of the charges and of $L_0 - \tilde{L}_0$, together with the fact that only rightmoving ground states contribute.

In certain cases relevant for AdS/CFT the index defined in (2.9) vanishes, and one needs to consider a modified index by inserting a factor of $\tilde{F}^2$ [14]. An example is for the case of the $(0,4)$ CFT of wrapped M5-branes, where the vanishing comes from the contribution of the center of mass multiplet. As we'll discuss later, the center of mass degrees of freedom are absent in the AdS description, and so from the bulk point of view we can compute the index as defined in (2.9) and obtain a nonvanishing result. What we will not discuss here is the precise relation of this bulk index to the modified CFT index, although this clearly deserves further study.

We now state the main general properties of the elliptic genus.

Modular transformation

$$\chi\left(\frac{a\tau + b}{c\tau + d}, \frac{z_I}{c\tau + d}\right) = e^{2\pi i \frac{cz^2}{c\tau + d}} \chi(\tau, z_I) \ , \tag{2.10}$$

where we define

$$z^2 \equiv k^{IJ} z_I z_J \ . \tag{2.11}$$

Note that c in (2.10) is *not* the central charge!

In Appendix A we review the proof of (2.10). It is most easily understood in terms of the relation between the Hamiltonian and path integral expression for the elliptic genus. There is a natural modular invariant path integral expression, but the Hamiltonian corresponding to this action differs from that appearing in the exponential of (2.9). The modular transformation of this extra factor yields the prefactor in (2.10).

Spectral flow

The spectral flow automorphisms imply the relation

$$\chi(\tau, z_I + \ell_I \tau + m_I) = e^{-2\pi i(\ell^2\tau + 2\ell\cdot z)} \chi(\tau, z_I) \ , \tag{2.12}$$

[1] In the $(0,4)$ case RR is replaced by R.

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where m_I obeys $m_I q^I \in \mathbb{Z}$, and we defined $\ell^2 = k^{IJ}\ell_I\ell_J$, $\ell \cdot z = k^{IJ}\ell_I z_J$.

This also implies that if we expand the elliptic genus as

$$\chi(\tau, z_I) = \sum_{n,r^I} c(n, r^I) e^{2\pi i n\tau + 2\pi i z_I r^I} \ , \tag{2.13}$$

then the expansion coefficients are a function of a single spectral flow invariant combination:

$$c(n, r^I) = c(n - \frac{r^2}{4}) \ . \tag{2.14}$$

Here we defined $r^2 = k_{IJ} r^I r^J$, where k_{IJ} denotes the inverse of k^{IJ}.

Factorization of dependence on potentials

We can explicitly write the dependence of the elliptic genus on the potentials z_I. The intuition behind this is that we can always separate the CFT into the currents plus everything else, and the current part can be realized in terms of free bosons. We have:

$$\chi(\tau, z_I) = \sum_{\mu^I} h_\mu(\tau) \Theta_{\mu,k}(\tau, z_I) \ , \tag{2.15}$$

with

$$\Theta_{\mu,k}(\tau, z_I) = \textstyle\sum_{\eta_I} e^{\frac{i\pi\tau}{2}(\mu+2k\eta)^2} e^{2\pi i z_I(\mu^I + 2k^{IJ}\eta_J)} \ . \tag{2.16}$$

We are using the shorthand notation

$$(\mu + 2k\eta)^2 \equiv k_{IJ}(\mu^I + 2k^{IK}\eta_K)(\mu^J + 2k^{JL}\eta_L) \ . \tag{2.17}$$

The combined sum over μ^I and η_I includes the complete spectrum of charges. The sum over η_I corresponds to shifts of the charges by spectral flow, and so the sum on μ_I is over a fundamental domain with respect to these shifts.

NS sector elliptic genus

Using spectral flow, we can alternatively write the elliptic genus in the NS sector. In particular, by performing a half-integer spectral flow on the R-symmetry charges, we obtain

$$\begin{aligned} \chi_{RR}(\tau, z_0, z_I) &= e^{-2\pi i k z_0} \chi_{NS,NS}(\tau, z_0 - \tfrac{1}{2}\tau, z_I) \ , \\ \chi_{NS,NS}(\tau, z_0, z_I) &= \mathrm{Tr}_{NS,NS}\left[e^{2\pi i\tau L_0} e^{-2\pi i\bar{\tau}(\tilde{L}_0 - \frac{1}{2}\tilde{J}_0)} e^{2\pi i(z_0 q^0 + z_I q^I)} (-1)^F \right] \ , \end{aligned} \tag{2.18}$$

where in the above $I = 1, 2, \ldots$. $\chi_{NS,NS}$ receives contributions from chiral primaries, $\tilde{L}_0 - \frac{1}{2}\tilde{J}_0 = 0$. These chiral primaries are, as usual, the spectral flows of R ground states.

Farey tail expansion

The main observation of [9] was that upon applying the "Farey tail transform", the elliptic genus admits an expansion that is suggestive of a supergravity interpretation in terms of a sum over geometries. We will essentially state the result here, referring to [9] for the detailed derivation.

The properties (2.10) and (2.12) are the definitions of a "weak Jacobi form" of weight $w = 0$ and index k. Actually, the definition strictly applies when k is a single number rather than a matrix, but we will still use this langauge.

The Farey tail transformed elliptic genus is

$$\tilde{\chi}(\tau, z_I) = \left(\frac{1}{2\pi i}\partial_\tau - \frac{1}{4}\frac{\partial_z^2}{(2\pi i)^2}\right)^{3/2} \chi(\tau, z_I) \ , \tag{2.19}$$

where $\partial_z^2 = k_{IJ}\partial_{z_I}\partial_{z_J}$. $\tilde{\chi}$ is a weak Jacobi form of weight 3 and index k, and admits the expansion

$$\tilde{\chi}(\tau, z_I) = e^{-\frac{\pi z^2}{\tau_2}} \sum_{\Gamma_\infty \backslash \Gamma} \frac{1}{(c\tau + d)^3} \hat{\chi}\left(\frac{a\tau + b}{c\tau + d}, \frac{z_I}{ct + d}\right) \ , \tag{2.20}$$

with

$$\hat{\chi}(\tau, z_I) = e^{\frac{\pi z^2}{\tau_2}} \hat{\sum}_{\mu, \tilde{\mu}, m, \tilde{m}} \tilde{c}(m, \mu^I) e^{2\pi i (m - \frac{1}{4}\mu^2)\tau} \Theta_{\mu,k}(\tau, z_I) \ , \tag{2.21}$$

and $\Theta_{\mu,k}(\tau, z_I)$ was defined in (2.16). The hatted summation appearing in (2.21) is over states with $m - \frac{1}{4}\mu^2 < 0$. From the gravitational point of view these will be states below the black hole threshold and the sum over $\Gamma_\infty \setminus \Gamma$ then adds the black holes back in. In mathematical terminology (2.21) defines $\hat{\chi}$ as the "polar part" of the elliptic genus. The coefficients $\tilde{c}(m, \mu^I)$ in (2.21) are related to those in (2.13) by

$$\tilde{c}(m, \mu^I) = (m - \frac{\mu^2}{4})^{3/2} c(m - \frac{\mu^2}{4}) \ , \tag{2.22}$$

as follows from (2.19) and from using (2.14). The main point is that the transformed elliptic genus $\tilde{\chi}$ can be reconstructed in terms of its polar part $\hat{\chi}$.

2.2 General partition function

If we include potentials for both left and right moving charges we define a partition function that receives contributions from all states of both chiralities. This object is no longer invariant under deformations of the CFT, and we have little hope of computing it exactly. Nevertheless, we can infer some general properties, and do an approximate computation in the regime of weakly coupled supergravity.

We define

$$Z(\tau, z_I; \overline{\tau}, \tilde{z}_I) = \mathrm{Tr}_{RR}\left[e^{2\pi i \tau (L_0 - c/24)} e^{-2\pi i \overline{\tau}(\tilde{L}_0 - \tilde{c}/24)} e^{2\pi i z_I q^I} e^{-2\pi i \tilde{z}_I \tilde{q}^I}\right] \ . \tag{2.23}$$

The properties of the elliptic genus that we reviewed above all have their obvious analogs for the partition function (2.23).

3. Supergravity analysis: preliminaries

Having reviewed the definitions and properties of the CFT partition functions, we now turn to their study in supergravity. The CFTs in question are dual to string theories on AdS_3 times some compact space. In general, these theories have all manner of complications from higher derivative terms and massive string/brane states. Fortunately, for computing the elliptic genus not all of this information is necessary, and we can get remarkably far by taking advantage of all the symmetries, and by carefully studying the long-distance part

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of the theory. The remaining input which will still be needed to complete the calculation is the spectrum of massive string/brane states, which can be computed exactly in certain cases. We proceed step-by-step, upward in energy scales. We start with the low energy effective action where the relevant *massless* fields on AdS_3 are the metric and a collection of gauge fields. Later we discuss supergravity Kaluza-Klein modes and finally nonperturbative states.

3.1 Gravity action

The action for the metric is

$$S_{grav} = \frac{1}{16\pi G}\int d^3x\sqrt{g}(R - \frac{2}{\ell^2}) + \int d^3x\ \Omega_3(\Gamma) + \frac{1}{8\pi G}\int_{\partial AdS} d^2x\sqrt{g}\left(\mathrm{Tr}\, K - \frac{1}{\ell}\right) + \dots\ . \tag{3.1}$$

We work in Euclidean signature. The second term is a gravitational Chern-Simons term, $\Omega_3(\Gamma) = \beta\,\mathrm{Tr}(\Gamma d\Gamma + \frac{2}{3}\Gamma^3)$. It plays a crucial role in theories with $c \neq \tilde{c}$ (specifically, $c - \tilde{c} = 96\pi\beta$) and its effects were studied in [5]. The next terms are the Gibbons-Hawking and boundary counterterms that are standard for gravity in asymptotically AdS_3 spacetimes. Also indicated by the $\dots$ are the higher derivative terms that are present in string theory constructions. Although these certainly contribute, even without knowing their precise form we can incorporate all their effects provided we carefully implement all symmetries and anomalies.

The statement that the metric is asymptotically AdS_3 means that it takes the Fefferman-Graham form

$$ds^2 = d\eta^2 + e^{2\eta/\ell} g^{(0)}_{\alpha\beta} dx^\alpha dx^\beta + g^{(2)}_{\alpha\beta} dx^\alpha dx^\beta + \dots\ . \tag{3.2}$$

Here $g^{(0)}_{\alpha\beta}$ is the "conformal boundary metric". The boundary stress tensor is defined by computing the on-shell variation of the action with respect to the conformal boundary metric [15]

$$\delta S = \frac{1}{2}\int_{\partial AdS} d^2x\sqrt{g^{(0)}}T^{\alpha\beta}\delta g^{(0)}_{\alpha\beta}\ , \tag{3.3}$$

yielding

$$T^{grav}_{\alpha\beta} = \frac{1}{8\pi G\ell}\left(g^{(2)}_{\alpha\beta} - \mathrm{Tr}(g^{(2)})g^{(0)}_{\alpha\beta}\right) + (\text{higher deriv.})\ . \tag{3.4}$$

We added the *grav* superscript because the stress tensor also receives a contribution from the gauge fields discussed in the following.

3.2 Gauge field action

Leftmoving $SU(2)$ currents

Next, consider the gauge fields. Associated with 4 left or right moving supercharges is an $SU(2)$ current algebra that is realized in the bulk by $SU(2)$ gauge fields. The $SU(2)$ gauge fields have a Chern-Simons term as the leading long-distance part of the action,

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with a coefficient related to the level of the current algebra. See [16, 17] for some relevant earlier work on Chern-Simons theory. For the leftmoving gauge fields we write

$$S_{gauge} = -\frac{ik}{4\pi}\int d^3x\ \mathrm{Tr}(AdA + \frac{2}{3}A^3) + S^{bndy}_{gauge}\ , \tag{3.5}$$

with $A = A^a \frac{i\sigma^a}{2}$. From a higher dimensional point of view, the $SU(2)$ gauge fields can be thought of as being associated with the isometries of a sphere, and the Chern-Simons term (3.5) can be derived by dimensional reductiion [18]. Invariance of the path integral under large gauge transformation fixes k to be an integer, and we'll rederive below the standard fact that k is the level of the boundary current algebra. Supersymmetry relates k to the leftmoving central charge as $c = 6k$. The boundary term indicated in (3.5) and given explicitly below is necessary in order that the currents have only leftmoving components.

The gauge fields admit the expansion

$$A = A^{(0)} + e^{-2\eta/\ell}A^{(2)} + \dots\ , \tag{3.6}$$

and we choose the gauge $A_\eta = 0$. Analysis of the field equations (including the effect of Maxwell type terms) shows that $A^{(0)}$ is a flat connection; that is, the field strength corresponding to (3.6) falls off as $e^{-2\eta/\ell}$. The boundary current is obtained from the on-shell variation of the action with respect to $A^{(0)}$

$$\delta S = \frac{i}{2\pi}\int_{\partial AdS} d^2x\sqrt{g^{(0)}}J^{\alpha a}\delta A^{(0)a}_\alpha\ . \tag{3.7}$$

We expect the boundary current corresponding to $SU(2)_L$ to be purely leftmoving. In a general coordinate system this amounts to the imaginary anti-self dual condition $\star J = -iJ$, with $\star$ defined with respect to $g^{(0)}_{\alpha\beta}$. (The opposite sign holds for the rightmoving case). However, the variation of the action (3.5) does not give a purely leftmoving current unless we take the boundary term as

$$S^{bndy}_{gauge} = -\frac{k}{16\pi}\int_{\partial AdS} d^2x\sqrt{g}g^{\alpha\beta}A^a_\alpha A^a_\beta\ . \tag{3.8}$$

This yields the imaginary anti-self dual current

$$J^a_\alpha = \tfrac{ik}{4}(A^{(0)a}_\alpha - i\epsilon_\alpha{}^\beta A^{(0)a}_\beta)\ . \tag{3.9}$$

In conformal gauge, $g^{(0)}_{\alpha\beta}dx^\alpha dx^\beta = dwd\overline{w}$, we find

$$J^a_w = \frac{ik}{2}A^{(0)a}_w\ , \quad J^a_{\overline{w}} = 0\ . \tag{3.10}$$

This is an exact expression for the current, uncorrected by the higher derivative terms in (3.5), because of the flatness of $A^{(0)}$. This will be important.

An equivalent way to motivate the boundary term (3.8) is to demand that in the variational principle we only fix boundary conditions for $A^{(0)a}_{\overline{w}}$ and not $A^{(0)a}_w$. Fixing both components is too strong in that there will typically not be any smooth solutions in the bulk with the chosen boundary conditions. The condition that the variation of the action takes the form $\delta S \sim \int J_w \delta A^{(0)}_{\overline{w}}$ then leads to the same conclusions as above.

Rightmoving $SU(2)$ currents

The rightmoving gauge fields are described by the action

$$S_{gauge} = \frac{i\tilde{k}}{4\pi} \int d^3x \ \mathrm{Tr}(\tilde{A}d\tilde{A} + \frac{2}{3}\tilde{A}^3) - \frac{\tilde{k}}{16\pi} \int_{\partial AdS} d^2x \sqrt{g} g^{\alpha\beta} \tilde{A}^a_\alpha \tilde{A}^a_\beta \ . \tag{3.11}$$

Note that the Chern-Simons term appears with an opposite sign from (3.5), and the boundary term was fixed by demanding that the current be purely rightmoving:

$$\tilde{J}^a_\alpha = \tfrac{i\tilde{k}}{4}(\tilde{A}^{(0)a}_\alpha + i\epsilon_\alpha{}^\beta \tilde{A}^{(0)a}_\beta) \ , \tag{3.12}$$

or

$$\tilde{J}^a_w = 0 \ , \quad J^a_{\overline{w}} = \frac{i\tilde{k}}{2}\tilde{A}^{(0)a}_{\overline{w}} \ . \tag{3.13}$$

Gauge field contribution to stress tensor

The gauge field boundary terms are metric dependent and hence contribute to the stress tensor as:[2]

$$T^{gauge}_{\alpha\beta} = \frac{k}{8\pi}(A^{(0)a}_\alpha A^{(0)a}_\beta - \tfrac{1}{2}A^{(0)a\gamma}A^{(0)a}_\gamma g^{(0)}_{\alpha\beta}) \ + \ \left[(k, A) \to (\tilde{k}, \tilde{A})\right] \ , \tag{3.14}$$

or

$$\begin{aligned} T^{gauge}_{ww} &= \frac{k}{8\pi}A^{(0)a}_w A^{(0)a}_w + \frac{\tilde{k}}{8\pi}\tilde{A}^{(0)a}_w \tilde{A}^{(0)a}_w \ , \\ T^{gauge}_{\overline{w}\overline{w}} &= \frac{k}{8\pi}A^{(0)a}_{\overline{w}} A^{(0)a}_{\overline{w}} + \frac{\tilde{k}}{8\pi}\tilde{A}^{(0)a}_{\overline{w}} \tilde{A}^{(0)a}_{\overline{w}} \ , \\ T^{gauge}_{w\overline{w}} &= T^{gauge}_{\overline{w}w} = 0 \ . \end{aligned} \tag{3.15}$$

The index (0) on the gauge field reminds us that boundary expressions strictly refer to just the leading term in the Fefferman-Graham expansion (3.6) for the bulk gauge field. In the following we will reduce clutter by dropping this index.

$U(1)$ currents

Besides the $SU(2)$ currents, we will typically have some number of left and right moving $U(1)$ currents. Only those gauge fields that appear in Chern-Simons terms will contribute to boundary currents. The Chern-Simons term has the form $S \sim \int C^{IJ} A_I dA_J$. By a change of basis we can put C^{IJ} in block diagonal form,

$$C^{IJ} = (\ k^{IJ} \quad 00 \quad \tilde{k}^{IJ}\ ) \ , \tag{3.16}$$

where k^{IJ} ($\tilde{k}^{IJ}$) has positive(negative) eigenvalues. We then write the relevant part of the action as

$$S = \frac{i}{8\pi}\int d^3x \, (k^{IJ}A_I dA_J - \tilde{k}^{IJ}\tilde{A}_I d\tilde{A}_J) - \frac{1}{16\pi}\int_{\partial AdS} d^2x \sqrt{g} g^{\alpha\beta}(k^{IJ}A_{I\alpha}A_{J\beta} + \tilde{k}^{IJ}\tilde{A}_{I\alpha}\tilde{A}_{J\beta}) \ . \tag{3.17}$$

[2] The variation of the metric is computed while holding fixed the lower components of the gauge fields.

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We combine this with the $SU(2)$ gauge fields as follows. We will be considering solutions in which in which only the $a = 3$ component of $A^{(0)a}$ and $\tilde{A}^{(0)a}$ is nonvanishing. Then, so far as getting the correct currents is concerned, these can thought of as $U(1)$ currents, and can be incorporated in (3.17) by extending the I indices to include $I = 0$ with

$$A^{(3)} = A_{I=0} \ , \quad \tilde{A}^{(3)} = \tilde{A}_{I=0} \ . \tag{3.18}$$

With this in mind, we can now take (3.17) as our general gauge field action. We will write it in condensed form as

$$S = \frac{i}{8\pi} \int d^3x \, (AdA - \tilde{A}d\tilde{A}) - \frac{1}{16\pi} \int_{\partial AdS} d^2x \sqrt{g} g^{\alpha\beta} (A_\alpha A_\beta + \tilde{A}_\alpha \tilde{A}_\beta) \ , \tag{3.19}$$

where the appropriate index contractions with k^{IJ} and $\tilde{k}^{IJ}$ are implicit.

In conformal gauge, the gauge fields contribute to the currents and stress tensor as,

$$\begin{aligned}
T^{gauge}_{ww} &= \frac{1}{8\pi} A_w^2 + \frac{1}{8\pi} \tilde{A}_w^2 \ , \\
T^{gauge}_{\overline{w}\overline{w}} &= \frac{1}{8\pi} A_{\overline{w}}^2 + \frac{1}{8\pi} \tilde{A}_{\overline{w}}^2 \ , \\
T^{gauge}_{w\overline{w}} &= T^{gauge}_{\overline{w}w} = 0 \ , \\
J^I_w &= \frac{i}{2} k^{IJ} A_{Jw} \ , \quad J^I_{\overline{w}} = 0 \ , \\
\tilde{J}^I_w &= 0 \ , \quad \tilde{J}^I_{\overline{w}} = \frac{i}{2} \tilde{k}^{IJ} \tilde{A}_{J\overline{w}} \ .
\end{aligned} \tag{3.20}$$

Again, the appropriate contraction of indices is implicit.

3.3 Anomalies

The currents defined in (3.20) satisfy

$$\begin{aligned}
\partial_{\overline{w}} J^I_w &= \frac{i}{2} k^{IJ} \partial_w A_{J\overline{w}} \ , \\
\partial_w \tilde{J}^I_{\overline{w}} &= \frac{i}{2} \tilde{k}^{IJ} \partial_{\overline{w}} \tilde{A}_{Jw} \ ,
\end{aligned} \tag{3.21}$$

where we used flatness of the boundary potentials. By comparing with the chiral anomalies of the boundary CFT, we see that k_{IJ} can be identified with the k_{IJ} matrix appearing in Section 2 (and similarly for $\tilde{k}_{IJ}$). Since they are related to chiral anomalies, for a given string/brane realization of the AdS$_3$ geometry we can compute k_{IJ} and $\tilde{k}_{IJ}$ exactly. By the anomaly inflow mechanism, one further knows that they must agree with their CFT counterparts, otherwise there will be an inconsistency in coupling the string/brane system to bulk fields. We refer to [5] for the detailed story.

3.4 Charges

Charges are defined as contour integrals around the AdS$_3$ boundary cylinder. We work with the complex boundary coordinate $w \cong w + 2\pi$. The $U(1)$ charges are then

$$\begin{aligned}
q^I &= 2 \oint \frac{dw}{2\pi i} J^I_w = i \oint \frac{dw}{2\pi i} k^{IJ} A_{Jw} \ , \\
\tilde{q}^I &= -2 \oint \frac{d\overline{w}}{2\pi i} \tilde{J}^I_{\overline{w}} = -i \oint \frac{d\overline{w}}{2\pi i} \tilde{k}^{IJ} \tilde{A}_{J\overline{w}} \ ,
\end{aligned} \tag{3.22}$$

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where the factors of 2 were included to agree with our convention in (2.5). Similarly, the Virasoro zero mode generators are[3]

$$
\begin{aligned}
L_0^{gauge} &= \oint dw \; T_{ww}^{gauge} = \frac{1}{8\pi} \oint dw \; A_w^2 \; , \\
\tilde{L}_0^{gauge} &= \oint d\overline{w} \; T_{\overline{w}\overline{w}}^{gauge} = \frac{1}{8\pi} \oint d\overline{w} \; \tilde{A}_w^2 \; .
\end{aligned} \tag{3.23}
$$

More generally, we can define all the modes of the currents, J_n^I, and stress tensor, L_n, by inserting factors of w^n into the above integrals. These modes then satisfy the commutation relations in (2.4).

3.5 Spectral flow

Spectral flow corresponds to a constant shift in the gauge potentials. This is equivalent to shifting the periodicities of charged fields. In the presence of a nonzero potential, the holonomy associated with a charged particle taken around the AdS$_3$ boundary cylinder is

$$
e^{\frac{1}{2} i q^I \oint dw \, A_{Iw}} \; , \tag{3.24}
$$

so that the shift,

$$
A_{Iw} \to A_{Iw} + 2\eta_I \; , \tag{3.25}
$$

introduces the phase factor $e^{2\pi i q^I \eta_I}$. The factor of $\frac{1}{2}$ in the exponent of (3.24) came from our definition of charge (3.22).

Under the shift (3.25) we have,

$$
\begin{aligned}
L_0 &\to L_0 + \eta_I q^I + k^{IJ}\eta_I\eta_J \; , \\
q^I &\to q^I + 2k^{IJ}\eta_J \; ,
\end{aligned} \tag{3.26}
$$

in agreement with (2.7). We also have the analogous formulas for a rightmoving spectral flow.

While we can perform a spectral flow with respect to any of the $U(1)$ currents, the terminology is often reserved for the R-symmetry. Recalling that the R-symmetry charge is q^0, such a spectral flow correspond to η_0. We pass back and forth between the NS and R sectors by flipping the periodicity of the supercurrent; this corresponds to taking $\eta_0 = m + \frac{1}{2}$, with $m \in \mathbb{Z}$.

4. Supergravity analysis: explicit examples

Before considering the computation of partition functions we illustrate the above results with some simple examples.

[3] Note that our normalization of the stress tensor differs by a factor -2π from the standard CFT definition in, e.g., [19].

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4.1 NS-NS vacuum

The NS-NS (or simply NS in the case of $(0,4)$ susy) vacuum is invariant under $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$. In other words, it is invariant under the full group of AdS$_3$ isometries, which means that it is precisely global AdS$_3$,

$$ds^2 = (1 + r^2/\ell^2)\ell^2 dt^2 + \frac{dr^2}{1 + r^2/\ell^2} + r^2 d\phi^2 \ . \tag{4.1}$$

The contractibility of the ϕ circle forces the fermions to be anti-periodic in ϕ. Invariance under the isometry group means that this geometry has

$$L_0 = \tilde{L}_0 = 0 \ . \tag{4.2}$$

4.2 Spectral flow to the R sector

As described at the end of section 3.5, to get R sector geometries we take the geometry (4.1) with

$$A_{0w} = 1 \ , \tag{4.3}$$

and fermions to be periodic in ϕ. The gauge field contribution (3.15) increases the stress tensor from (4.2) to

$$L_0 = \frac{k}{4} = \frac{c}{24} \ . \tag{4.4}$$

Since the charge (3.22) is

$$q^0 = k = \frac{c}{6} \ , \tag{4.5}$$

this is the maximally charged R vacuum state. To get the maximally negatively charged R vacuum we flip the sign in (4.3). The rightmoving side is treated analogously.

4.3 Conical defects

A more general class of RR vacua are the conical defect geometries [20, 21]. For these we take

$$\begin{aligned} ds^2 &= (\frac{1}{N^2} + \frac{r^2}{\ell^2})\ell^2 dt^2 + \frac{dr^2}{(\frac{1}{N^2} + \frac{r^2}{\ell^2})} + r^2 d\phi^2 \ , \\ A_{0w} &= \tilde{A}_{0\overline{w}} = \frac{1}{N} \ , \end{aligned} \tag{4.6}$$

with $N \in \mathbb{Z}$. The angular coordinate ϕ has the standard 2π periodicity, and fermions are taken to be periodic in ϕ.

To read off the Virasoro charges we just note that by rescaling coordinates all these geometries are locally equivalent to the $N = 1$ case discussed above in section 4.2. In the $N = 1$ case the stress tensor vanishes, and it will clearly continue to vanish after rescaling coordinates. Thus (4.4) still applies and so $L_0 = \frac{k}{4}$ and $\tilde{L}_0 = \frac{\tilde{k}}{4}$ as before. The R-charge is read off from (3.22)

$$q^0 = \frac{k}{N} \ , \quad \tilde{q}^0 = \frac{\tilde{k}}{N} \ . \tag{4.7}$$

Upper and lower bounds on N are given by the quantization of R-charge. For example, in the D1-D5 case, the condition that $q^0, \tilde{q}^0$ are integral gives $|N| \leq N_1 N_5$ since $k = \tilde{k} = N_1 N_5$.

These conical defect geometries are singular at the origin unless the holonomy is ± 1, which corresponds to $N = \pm 1$. In the context of the D1-D5 system, the singular geometries are known to be physical in that the singularity corresponds to the presence of N coincident Kaluza-Klein monopoles. Another way of viewing this is that these singular geometries are special limits of the much larger class of smooth RR vacua geometries that have been heavily studied in recent years [22, 23].

We also note that any of the RR-vacua in (4.6) can be spectral flowed to the NSNS sector to give chiral primary geometries.

4.4 Black holes

We now consider black hole geometries, and give a simple derivation of the entropy of charged black holes that incorporates higher derivative corrections. This is a slight refinement of our earlier work [5]. We use the well known trick of relating black holes to thermal AdS by a modular transformation; the main novelty here is the inclusion of charge and higher derivative corrections.

The starting point is global AdS_3, as in (4.1). The complex boundary coordinate is $w = \phi + it/\ell$, and we identify $w \cong w + 2\pi \cong w + 2\pi\tau$. To add charge we also want to turn on flat potentials for the gauge fields. Now, the ϕ circle is contractible in the bulk, so to avoid a singularity at the origin we need to set to zero the ϕ component of all potentials. We therefore allow nonzero $A_{Iw} = -A_{I\overline{w}}$, and $\tilde{A}_{I\overline{w}} = -\tilde{A}_{Iw}$.

What is the action associated with this solution? From the discussion in section 3 we know the exact expressions for the stress tensor and currents

$$\begin{aligned} T_{ww} &= -\frac{k}{8\pi} + \frac{1}{8\pi} A_w^2 + \frac{1}{8\pi} \tilde{A}_w^2 \ , \\ T_{\overline{w}\overline{w}} &= -\frac{\tilde{k}}{8\pi} + \frac{1}{8\pi} A_{\overline{w}}^2 + \frac{1}{8\pi} \tilde{A}_{\overline{w}}^2 \ , \\ J_w^I &= \frac{i}{2} k^{IJ} A_{Jw} \ , \\ \tilde{J}_{\overline{w}}^I &= \frac{i}{2} \tilde{k}^{IJ} \tilde{A}_{J\overline{w}} \ . \end{aligned} \tag{4.8}$$

To obtain the exact action from these formulae we need to integrate the equation

$$\delta S = \int_{\partial AdS} d^2x \sqrt{g^{(0)}} \left(\frac{1}{2} T^{\alpha\beta} \delta g^{(0)}_{\alpha\beta} + \frac{i}{2\pi} J^{I\alpha} \delta A_{I\alpha} \right) \ . \tag{4.9}$$

In doing so, let us first note that the conformal gauge used hitherto fixed the conformal boundary metric as $dw d\overline{w}$ and encoded the conformal structure in the periodicities of the coordinates. To exploit (4.9) it is advantageous to define a new coordinate z

$$z = \frac{i - \overline{\tau}}{\tau - \overline{\tau}} w - \frac{i - \tau}{\tau - \overline{\tau}} \overline{w} \ , \tag{4.10}$$

with fixed periodicity $z \cong z + 2\pi \cong z + 2\pi i$. In terms of this coordinate the conformal structure is encoded in the metric

$$ds^2 = dwd\bar{w} = \left| \frac{1 - i\tau}{2} dz + \frac{1 + i\tau}{2} d\bar{z} \right|^2 . \tag{4.11}$$

Using the new coordinates to compute the variations $\delta g^{(0)}_{\alpha\beta}$ with respect to τ and $\bar{\tau}$, and also transforming the z and $\bar{z}$ components of $T^{\alpha\beta}$ back to the original coordinates, we can rewrite (4.9) as

$$\delta S = (2\pi)^2 i \left[-T_{ww}\delta\tau + T_{\bar{w}\bar{w}}\delta\bar{\tau} + \frac{\tau_2}{\pi} J^I_w \delta A_{I\overline{w}} + \frac{\tau_2}{\pi} \tilde{J}^I_{\overline{w}} \delta \tilde{A}_{Iw} \right]_{\text{const}} . \tag{4.12}$$

The *const* subscript indicates that we keep just the zero mode part. Inserting (4.8) into this equation we can now integrate and find our desired action as

$$S = \frac{i\pi k}{2}\tau - \frac{i\pi \tilde{k}}{2}\overline{\tau} + \pi\tau_2 (A^2_{\overline{w}} + \tilde{A}^2_w) . \tag{4.13}$$

A simpler derivation of this result is to just compute (4.8) by directly evaluating the action on the solution. The gauge field contribution just comes from the boundary term (3.8). The reason we proceeded in terms of (4.9) was to emphasize that the result (4.8) is exact for an arbitrary higher derivative action, and also because we will generalize this computation later.

The result (4.8) is the action for the AdS_3 ground state with a flat connection turned on. Next, we perform the modular transformation $\tau \to -1/\tau$ in order to reinterpret the solution as a Euclidean black hole. This is implemented by

$$w \to -w/\tau, \quad A_{I\overline{w}} \to -\bar{\tau} A_{I\overline{w}} , \quad \tilde{A}_{Iw} \to -\tau \tilde{A}_{Iw} . \tag{4.14}$$

The action is of course invariant since we are just rewriting it in new variables. Using $\overline{\tau}/\tau = 1 - 2i\tau_2/\tau$ and introducing the potentials $z = -i\tau_2 A_{\overline{w}}$ and $\tilde{z} = i\tau_2 \tilde{A}_w$ (defined in equation A.7) we can present the result as

$$\begin{aligned} S &= -\frac{i\pi k}{2\tau} + \frac{i\pi\tilde{k}}{2\overline{\tau}} - \frac{2\pi i \tau_2^2 A^2_{\overline{w}}}{\tau} + \frac{2\pi i \tau_2^2 \tilde{A}^2_{\overline{w}}}{\overline{\tau}} + \pi\tau_2 (A^2_{\overline{w}} + \tilde{A}^2_w) \\ &= -\frac{i\pi k}{2\tau} + \frac{i\pi\tilde{k}}{2\overline{\tau}} + \frac{2\pi i z^2}{\tau} - \frac{2\pi i \tilde{z}^2}{\overline{\tau}} - \frac{\pi}{\tau_2}(z^2 + \tilde{z}^2) . \end{aligned} \tag{4.15}$$

This is the Euclidean action of a black hole with modular parameter τ and potentials specified by z and $\tilde{z}$.

Our result (4.15) is the leading saddle point contribution to the path integral. As we discuss in appendix A, the standard canonical form of the partition function, defined as a trace, is related to the path integral as

$$Z = e^{-\frac{\pi}{\tau_2}(z^2 + \tilde{z}^2)} Z_{PI} = e^{-\frac{\pi}{\tau_2}(z^2 + \tilde{z}^2)} \sum e^{-S} . \tag{4.16}$$

The exponential prefactor cancels the last term in (4.15) so that

$$\ln Z = \frac{i\pi k}{2\tau} - \frac{i\pi\tilde{k}}{2\overline{\tau}} - \frac{2\pi i z^2}{\tau} + \frac{2\pi i \tilde{z}^2}{\overline{\tau}} , \tag{4.17}$$

on the saddle point. We define the entropy s by writing the partition function as

$$Z = e^s e^{2\pi i\tau(L_0-c/24)} e^{-2\pi i\bar{\tau}(\tilde{L}_0-\tilde{c}/24)} e^{2\pi i z_I q^I} e^{-2\pi i\tilde{z}_I q^I} \ , \tag{4.18}$$

where we assume that Z is dominated by a single a single charge configuration with, e.g., $q^I = \frac{1}{2\pi i}\frac{\partial}{\partial z_I}\ln Z$.

Putting everything together we read off the black hole entropy as

$$s = 2\pi\sqrt{\frac{c}{6}(L_0 - \frac{c}{24} - \frac{1}{4}q^2)} + 2\pi\sqrt{\frac{\tilde{c}}{6}(\tilde{L}_0 - \frac{\tilde{c}}{24} - \frac{1}{4}\tilde{q}^2)} \ . \tag{4.19}$$

The expression (4.19) gives the entropy for a general nonextremal, rotating, charged, black hole in AdS_3, including the effect of higher derivative corrections as incorporated in the central charges. Since we used the saddle point approximation the formula is only valid to leading order in $\tilde{L}_0 - \frac{\tilde{c}}{24} - \frac{1}{4}\tilde{q}^2$; including the subleading contribution is the topic of the next section. It is striking that we have control over higher derivative corrections to the entropy even for nonsupersymmetric black holes. In earlier work [5] we explained this in terms of anomalies, and showed that (4.19) is in precise agreement with the microscopic entropy counting coming from brane constructions.

5. Computation of partition functions in supergravity

Let's now look at the supergravity computation of the elliptic genus

$$\chi(\tau, z_I) = \mathrm{Tr}_{RR}\left[e^{2\pi i\tau(L_0-c/24)} e^{-2\pi i\bar{\tau}(\tilde{L}_0-\tilde{c}/24)} e^{2\pi i z_I q^I}(-1)^F\right] \ . \tag{5.1}$$

Once we understand this it is straightforward to incorporate potentials for the rightmoving charges $\tilde{q}^I$, if desired. We'll consider both the Hamiltonian and path integral approaches, which, as explained in Appendix A, are useful for making manifest the behavior under spectral flow and modular transformation, respectively. In keeping with the Farey tail philosophy [9], we will explicitly compute the contribution to the elliptic genus from states below the black hole threshold. With this in hand, black holes are included by the construction (2.20).

5.1 Hamiltonian approach

In the Hamiltonian approach we need to enumerate the allowed set of bulk solutions and their charge assignments. For the elliptic genus we consider states of the form (anything, R-ground state), which have $\tilde{L}_0 = \frac{\tilde{k}}{4}$. There are three classes of such states: smooth solutions in the effective three dimensional theory; states coming from Kaluza-Klein reduction of the higher dimensional supergravity theory; and non-supergravity string/brane states. Some members of the first class were discussed above, and we consider the other types of states later.

Just as was done in the CFT approach (2.15), it is useful to factorize the dependence on the potentials. In the gravitational context it is manifest that the stress tensor consists of a metric part plus a gauge field part. Suppose we are given a state carrying leftmoving charges

$$(L_0 - \frac{c}{24}, q^I) = (m, \mu^I) \; . \tag{5.2}$$

We can apply spectral flow to generate the family of states with charges

$$\begin{aligned} L_0 - \frac{c}{24} &= m + \eta_I q^I + k^{IJ}\eta_I\eta_J = m - \frac{1}{4}\mu^2 + \frac{1}{4}(\mu + 2k\eta)^2 \\ q^I &= \mu^I + 2k^{IJ}\eta_J \; , \end{aligned} \tag{5.3}$$

where we are using the same shorthand notation as in (2.17). This class of states will then contribute to the elliptic genus as

$$\chi(\tau, z_I) = (-1)^F e^{2\pi i \tau (m - \frac{1}{4}\mu^2)} \Theta_{\mu,k}(\tau, z_I) \; , \tag{5.4}$$

in terms of the Θ-function (2.16). Each such spectral flow orbit has a certain degeneracy from the number distinct states with these charges. We call this degeneracy $c(m - \frac{1}{4}\mu^2)$, where the functional dependence is fixed by the spectral flow invariance, and we also include $(-1)^F$ in the definition. We can now write down the "polar" part of the elliptic genus, that is, the contribution below the black hole threshold: $m - \frac{1}{4}\mu^2 < 0$. We then have

$$\chi'(\tau, z_I) = \sum\nolimits'_{m,\mu} c(m - \frac{1}{4}\mu^2)\Theta_{\mu,k}(\tau, z_I) e^{2\pi i (m - \frac{1}{4}\mu^2)\tau} \; . \tag{5.5}$$

In the Hamiltonian approach it is easy to write down the polar part of the elliptic genus in terms of the degeneracies $c(m - \frac{1}{4}\mu^2)$. But the full elliptic genus also has a contribution from black holes, and these are not easily incorporated since black holes do not correspond to individual states of the theory. To incorporate black holes we need to turn to a Euclidean path integral, as we do now.

5.2 Path integral approach

In the path integral approach we sum over bulk solutions with fixed boundary conditions

$$\chi_{PI}(\tau, z_I) = \sum e^{-S} \; . \tag{5.6}$$

The action appearing in (5.6) is the full string/M-theory effective action reduced to AdS_3, though we fortunately do not require its explicit form to compute the elliptic genus. In particular, in (5.6) we only sum over stationary points of S since the fluctuations have already been incorporated through higher derivative corrections to the action.

The boundary conditions on the metric are that the boundary geometry is a torus of modular parameter τ. z_I fix the boundary conditions for the gauge potentials. As explained in Appendix A, the relation is, in conformal gauge,

$$A_{I\overline{w}} = \frac{i z_I}{\tau_2} \; . \tag{5.7}$$

A_{Iw} is not fixed as a boundary condition. Since the potential $\tilde{z}_I$ is set to zero in the elliptic genus, we also have the boundary condition

$$\tilde{A}_{Iw} = 0 \; . \tag{5.8}$$

Now we turn to the allowed values of A_{Iw} and $\tilde{A}_{I\overline{w}}$. The allowed boundary values of A_{Iw} are determined from the holonomies around the contractible cycle of the AdS$_3$ geometry. Recall that when we write $w = \sigma_1 + i\sigma_2$ we are taking σ_1 to be the 2π periodic spatial angular coordinate. The corresponding cycle on the boundary torus is contractible in the bulk, and so any nonzero holonomy must match onto an appropriate source in order to be physical. The holonomy of a charge q^I particle is

$$e^{\frac{1}{2}iq^I \int d\sigma_1 A_{I\sigma_1}} = e^{\frac{1}{2}iq^I \int d\sigma_1 (A_{Iw}+A_{I\overline{w}})} \, . \tag{5.9}$$

Choosing a gauge with constant A_{Iw}, we write the allowed values as

$$A_{Iw} = k_{IJ}\mu^I + 2\eta_I - \frac{iz_I}{\tau_2} \ , \quad q^I\eta_I \in \mathbb{Z} \ , \tag{5.10}$$

where we have written the charge of the source as μ^I.

In the same way we can determine the allowed values of $\tilde{A}_{I\overline{w}}$. In this case we know that only geometries with $\tilde{L}_0 - \frac{\tilde{c}}{24} = 0$ contribute to the elliptic genus, and so we do not include the spectral flowed geometries as we did above. Instead, we just have

$$\tilde{A}_{I\overline{w}} = \tilde{k}_{IJ}\tilde{\mu}^I \ . \tag{5.11}$$

Given the gauge fields, we know the exact stress tensor (3.23) and also the exact currents (3.10) and (3.13). We can therefore find the action by integrating

$$\begin{aligned} \delta S &= \int_{\partial AdS} d^2x \sqrt{g^{(0)}} \left(\frac{1}{2} T^{\alpha\beta} \delta g^{(0)}_{\alpha\beta} + \frac{i}{2\pi} J^{I\alpha} \delta A_{I\alpha} \right) \\ &= (2\pi)^2 i \left[-T_{ww}\delta\tau + T_{\overline{w}\overline{w}}\delta\overline{\tau} + \frac{\tau_2}{\pi} J^I_w \delta A_{I\overline{w}} + \frac{\tau_2}{\pi} \tilde{J}^I_{\overline{w}} \delta \tilde{A}_{Iw} \right]_{\text{const}} \ , \end{aligned} \tag{5.12}$$

as in section 4.4. The result is

$$\begin{aligned} S &= -2\pi i\tau (L_0^{grav} - \frac{c}{24}) + 2\pi i\overline{\tau}(\tilde{L}_0^{grav} - \frac{\tilde{c}}{24}) \\ &\quad -\frac{i\pi}{2}\left[\tau A_w^2 + \overline{\tau}A_{\overline{w}}^2 + 2\overline{\tau}A_w A_{\overline{w}}\right] + \frac{i\pi}{2}\left[\tau\tilde{A}_w^2 + \overline{\tau}\tilde{A}_{\overline{w}}^2 + 2\tau\tilde{A}_w\tilde{A}_{\overline{w}}\right] \ . \end{aligned} \tag{5.13}$$

In verifying that (5.13) satisfies (5.12) one has to take care to consider only variations consistent with the equations of motion and the assumed boundary conditions. We maintain fixed holonomies by taking $\delta A_{Iw} = -\delta A_{I\overline{w}}$ and $\delta\tilde{A}_{Iw} = -\delta\tilde{A}_{I\overline{w}}$. Also, the variation of the complex structure must be taken with the gauge field fixed in the z-coordinates introduced in (4.10).

The result (5.13) for the action agrees with (4.8) when the geometry is in the ground state where $A_{Iw} = -A_{I\overline{w}}$ and $\tilde{A}_{Iw} = -\tilde{A}_{I\overline{w}}$, but it is valid also more generally in the presence of charged sources. In fact, it is equivalent to the Hamiltonian result discussed in section 5.1. To see this we consider again the charge assignments (5.2). Writing $L_0 = L_0^{grav} + L_0^{gauge} = L_0^{grav} + \frac{1}{4}\mu^2$ (and analogously for $\tilde{L}_0$) we insert into (5.13) and find

$$S = -2\pi i\tau(m - \frac{1}{4}\mu^2) - \frac{i\pi\tau}{2}(\mu + 2k\eta)^2 - 2\pi i z_I(\mu^I + 2k^{IJ}\eta_J) - \frac{\pi z^2}{\tau_2} \ . \tag{5.14}$$

Summing over the geometries below the black hole threshold we find

$$\begin{aligned}\chi'_{PI}(\tau, z_I) &= \sum\nolimits'_{m,\mu} c(m - \frac{1}{4}\mu^2) e^{-S} \\ &= e^{\frac{\pi z^2}{\tau_2}} \sum\nolimits'_{m,\mu} c(m - \frac{1}{4}\mu^2) \Theta_{\mu,k}(\tau, z_I) e^{2\pi i (m - \frac{1}{4}\mu^2)\tau} \\ &= e^{\frac{\pi z^2}{\tau_2}} \chi'(\tau, z_I) \ ,\end{aligned} \tag{5.15}$$

where χ' is the Hamiltonian result (5.5). As discussed in Appendix A, the overall exponential factor is precisely the one we expect.

5.3 Including black holes

Black holes are readily included in the path integral approach since they are just rewritten versions of solutions below the black hole threshold. Taking a solution below the black threshold and performing the coordinate transformation $w \to \frac{aw+b}{cw+d}$ generates a black hole. Using the manifest invariance of the action under such coordinate transformations, the contribution of such a black is then

$$\chi_{PI}(\tau, z_I) = \chi'_{PI}\left(\frac{a\tau + b}{c\tau + d}, \frac{z_I}{c\tau + d}\right) \ . \tag{5.16}$$

On the other hand, from the relation (5.15) between χ'_{PI} and χ' we have

$$\chi'_{PI}\left(\frac{a\tau + b}{c\tau + d}, \frac{z_I}{c\tau + d}\right) = e^{-2\pi i \frac{cz^2}{c\tau+d}} e^{\frac{\pi z^2}{\tau_2}} \chi'\left(\frac{a\tau + b}{c\tau + d}, \frac{z_I}{c\tau + d}\right) \ . \tag{5.17}$$

Thus the black hole contribution to χ is

$$\chi(\tau, z_I) = e^{-\frac{\pi z^2}{\tau_2}} \chi_{PI}(\tau, z_I) = e^{-2\pi i \frac{cz^2}{c\tau+d}} \chi'\left(\frac{a\tau + b}{c\tau + d}, \frac{z_I}{c\tau + d}\right) \ . \tag{5.18}$$

The next step is to sum over all inequivalent black holes to get the complete elliptic genus. This means summing over the subgroup of $\Gamma = SL(2, \mathbb{Z})$ corresponding to inequivalent black holes or, more precisely, distinct ways of labelling the contractible cycle in terms of time and space coordinates. As explained in [9] the inequivalent cycles are parameterized by $\Gamma_\infty \setminus \Gamma$; so it seems natural to write

$$\chi(\tau, z_I) = \sum_{\Gamma_\infty \setminus \Gamma} e^{-2\pi i \frac{cz^2}{c\tau+d}} \chi'\left(\frac{a\tau + b}{c\tau + d}, \frac{z_I}{c\tau + d}\right) \ . \tag{5.19}$$

However, as emphasized in [9], this cannot be correct since the sum is not convergent. Instead we should compute not the elliptic genus but instead its Farey transform, introduced in section 2. This amounts to first replacing χ' by

$$\hat{\chi}'(\tau, z_I) = \sum\nolimits'_{m,\mu} \tilde{c}(m - \frac{1}{4}\mu^2) \Theta_{\mu,k}(\tau, z_I) e^{2\pi i (m - \frac{1}{4}\mu^2)\tau} \tag{5.20}$$

with $\tilde{c}$ defined as in (2.22). We interpret this as the polar part of a weak Jacobi form of weight 3 and index k. Instead of (5.19) we therefore write

$$\hat{\chi}(\tau, z_I) = \sum_{\Gamma_\infty \setminus \Gamma} (c\tau + d)^{-3} e^{-2\pi i \frac{cz^2}{c\tau+d}} \hat{\chi}'\left(\frac{a\tau + b}{c\tau + d}, \frac{z_I}{c\tau + d}\right) \ . \tag{5.21}$$

In the D1-D5 system this agrees with the Farey transform of the CFT elliptic genus.

5.4 High temperature behavior

The high temperature ($\tau_2 \to 0$) behavior of (5.21) is governed by the free energy of a BPS black hole. The leading exponential behavior can be read off from the term

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \quad m = 0 , \quad \eta_I = 0, \quad \mu^I = k\delta^{I0} , \tag{5.22}$$

which gives

$$\hat{\chi}(\tau, z_I) \approx e^{-\frac{2\pi i z^2}{\tau} + \frac{2\pi i k z_0}{\tau}} . \tag{5.23}$$

We can compare with (4.17) by performing the spectral flow (2.18) $z_0 \to z_0 + \frac{1}{2}$. This yields

$$\ln \hat{\chi}(\tau, z_I) \approx \frac{i\pi k}{2\tau} - \frac{2\pi i z^2}{\tau} . \tag{5.24}$$

Noting that this agrees with the holomorphic part of (4.17), we find that the corresponding entropy is is indeed that of a BPS black hole,

$$s = 2\pi\sqrt{\frac{c}{6}(L_0 - \frac{c}{24} - \frac{1}{4}q^2)} . \tag{5.25}$$

This is just the leading part of the entropy, and is insensitive to the distinction between the elliptic genus and its Farey-tail transformed version.

6. Supergravity fluctuations

To compute the elliptic genus we need to know the spectrum of BPS states, as described by the coefficients $c(m - \frac{1}{4}\mu^2)$. In this section we consider the contribution from supergravity states, which are obtained from the fluctuation spectrum of 10 or 11 dimensional supergravity compactified on AdS_3 times some compact space. Our computation generalizes one given in [24].

We will focus on the $(0, 4)$ case, corresponding to M-theory on $AdS_3 \times S^2 \times M_6$, as the $(4, 4)$ case is quite well known. The $(0, 4)$ CFT on the AdS_3 boundary describes M5-branes wrapped on 4-cycles in M_6 [8] (the same CFT also describes black rings [25].) Up to a spectral flow, supergravity states can carry vanishing charges, $q^I = 0$. These charges are instead carried by wrapped branes. So the contribution to the polar part of the elliptic genus from such supergravity states is

$$\chi'_{sugra}(\tau, z_I) = \sum\nolimits'_m c_{sugra}(m) e^{2\pi i m\tau} . \tag{6.1}$$

We will now compute $\chi'_{sugra}(\tau, z_I)$ in order to extract the coefficients $c_{sugra}(m)$.

6.1 Spectrum

It is conventional to compute the elliptic genus in the NS sector, related to the R sector by the spectral flow (2.18). In the NS sector the elliptic genus receives contributions from rightmoving chiral primaries obeying $\overline{h} = \frac{1}{2}\tilde{q}^0$, where $\overline{h}$ is the eigenvalue of $\tilde{L}_0$. There will be two types supergravity modes: dynamical modes and "singletons". The latter are pure gauge modes that are nonetheless physical. We first focus on the dynamical modes.

M_6	n_S	n_V	n_H
CY_3	0	$h^{1,1}-1$	$2h^{1,2}+2$
$K3\times T^2$	2	22	42
T^6	6	14	14

Table 1: 5-dimensional supergravity spectra.

$s=h-\tilde{h}$	degeneracy	range of $\tilde{h}=\frac{1}{2}\tilde{q}^0$
$1/2$	n_H	$1/2, 3/2, \ldots$
0	n_V	$1, 2, \ldots$
1	n_V	$1, 2, \ldots$
$-1/2$	n_S	$3/2, 5/2, \ldots$
$1/2$	n_S	$3/2, 5/2, \ldots$
$3/2$	n_S	$1/2, 3/2, \ldots$
-1	1	$2, 3, \ldots$
0	1	$2, 3, \ldots$
1	1	$1, 2, \ldots$
2	1	$1, 2, \ldots$

Table 2: Spectrum of (non-singleton) chiral primaries for $AdS_3 \times S^2 \times M_6$ [26].[5]

We work with a 5-dimensional supergravity obtained by compactifying M-theory on M_6, where M_6 can be CY_3, $K3\times T^2$, or T^6 corresponding to having $\mathcal{N}=2$, 4, or 8 susy. The 5-dimensional spectrum is written in the $\mathcal{N}=2$ language in terms of the number of vectormultiplets n_V, hypermultiplets n_H, and gravitino multiplets n_S, in addition to the graviton multiplet.

The chiral primaries form multiplets under the leftmoving $SL(2,\mathbb{R})$ symmetry. In table 2 we list the spectrum of single particle chiral primaries that are also primary under the leftmoving $SL(2,\mathbb{R})$; i.e. are annihilated by L_1.

Each chiral primary above lies at the bottom of an $SL(2,\mathbb{R})$ multiplet obtained by acting an arbitrary number of times with L_{-1}.

6.2 NS sector elliptic genus

The contribution from supergravity states to the NS sector elliptic genus can be written

$$\chi^{sugra}_{NS}(\tau) = \mathrm{Tr}_{cp}\left[(-1)^{\tilde{q}^0} q^{L_0}\right] , \tag{6.2}$$

where the trace is over chiral primaries, and $q=e^{2\pi i\tau}$. As we have said, the complete spectrum of single particle primaries corresponds to table 2 and their $SL(2,\mathbb{R})$ descendants. Multiparticle chiral primaries are obtained by taking arbitrary tensor products of single particle chiral primaries.

The single particle spectrum starts at $h_{min} = \tilde{h}_{min} + s$. The contribution of a bosonic tower ($\tilde{q}^0$ even) is then

$$\chi^{bos}_{NS}(\tau) = \prod_{\ell=0}^{\infty}\prod_{p=0}^{\infty}\sum_{m=0}^{\infty} q^{m(h_{min}+\ell+p)} = \prod_{\ell=0}^{\infty}\prod_{p=0}^{\infty} \frac{1}{1-q^{(\tilde{h}_{min}+s+\ell+p)}} \ . \tag{6.3}$$

In the above, m stands for the number of particles; p for acting with $(L_{-1})^p$; and ℓ for $\tilde{h} = \tilde{h}_{min} + \ell$. Now define $n = \ell + p + 1$, so that there are n distinct terms with the same power of q. Then we can write

$$\chi^{bos}_{NS}(\tau) = \prod_{n=1}^{\infty}\left[\frac{1}{1-q^{h_{min}-1+n}}\right]^n \ . \tag{6.4}$$

The computation for fermions is analogous, and gives

$$\chi^{fer}_{NS}(\tau) = \prod_{n=1}^{\infty}\left[1-q^{h_{min}-1+n}\right]^n \ . \tag{6.5}$$

We can simplify the infinite products using

$$\begin{aligned}\prod_{n=1}^{\infty}(1-q^{1+n})^n &= M(q)\prod_{n=1}^{\infty}\frac{1}{(1-q^n)} \ , \\ \prod_{n=1}^{\infty}(1-q^{2+n})^n &= M(q)\prod_{n=1}^{\infty}\frac{1}{(1-q^n)}\frac{1}{(1-q^{n+1})} \ ,\end{aligned} \tag{6.6}$$

where the McMahon function is defined as

$$M(q) = \prod_{n=1}^{\infty}(1-q^n)^n \ . \tag{6.7}$$

The overall power of $M(q)$ will be equal to $n_F - n_B$, the number of fermonic towers minus the number of bosonic towers. From the degeneracies in table 2 and the entries of table 1 we have

$$n_F - n_B = n_H + 3n_S - 2n_V - 4 = \{ 0 \quad K3\times T^2 \text{ or } T^6 2(h^{1,2}-h^{1,1}) = -\text{Euler} \quad CY_3 \tag{6.8}$$

where "Euler" denotes the Euler number.

From table 2 we read off the spectrum of h_{min}. For bosons we have: $n_V + 1$ towers with $h_{min} = 1$; $(n_V + 2)$ towers with $h_{min} = 2$; and 1 tower with $h_{min} = 3$. For fermions we have: $(n_S + n_H)$ towers with $h_{min} = 1$; and $2n_S$ towers with $h_{min} = 2$. We then find the supergravity elliptic genus to be

$$\chi^{sugra}_{NS}(\tau) = M(q)^{-\text{Euler}}\prod_{n=1}^{\infty}(1-q^n)^{n_v+3-2n_s}(1-q^{n+1}) \ . \tag{6.9}$$

6.3 Spectrum of gauge fields and their Chern-Simons couplings

We now discuss the three dimensional spectrum of gauge fields relevant to compactifying M-theory on $\mathrm{AdS}_3 \times S^2 \times M_6$, with M_6 being T^6, $K3 \times T^2$ or CY_3. Besides being an important general property of these theories, the precise spectrum will needed in the next subsection when we work out the singleton contribution to the elliptic genus.

In five dimensions there are a total of $n_V + 2n_S + 1$ gauge fields, the $+1$ coming from the gauge field in the graviton multiplet. The action for these gauge fields includes the Chern-Simons coupling $\int C^{IJK} A_I \wedge F_J \wedge F_K$, where C^{IJK} is the intersection form on M_6. To pass to the three dimensional description we reduce on S^2 in the presence of magnetic flux, $p_I = \frac{1}{2\pi}\int_{S^2} F_I$. The three dimensional Chern-Simons term is therefore $\int C^{IJK} p_K A_I \wedge F_J$. In addition to these $U(1)$ Chern-Simons terms we also have the $SU(2)$ Chern-Simons term for the S^2 Kaluza-Klein gauge fields.

We now discuss each of the choices of M_6 in turn.

$\mathbf{T^6}$

M-theory on T^6 has 27 gauge fields transforming in the **27** of the $E_{6(6)}$ duality group. The Chern-Simons coupling C^{IJK} is the E_6 cubic invariant. We consider the case where we turn on a single magnetic charge. It is then convenient to decompose the **27** under an $SO(5,5)$ subgroup as $\mathbf{27} \to \mathbf{16} + \mathbf{10} + \mathbf{1}$, and to turn on the singlet. The cubic invariant decomposes as $\mathbf{27^3} \to \mathbf{1} \cdot \mathbf{10} \cdot \mathbf{10} + \mathbf{16} \cdot \mathbf{16} \cdot \mathbf{10}$. The three dimensional Chern-Simons term is then $\int g^{IJ} A_I \wedge F_J$ where $g^{IJ} = \mathrm{diag}((+1)^5, (-1)^5)$ is the $SO(5,5)$ invariant quadratic form. So we find 5 leftmoving and 5 rightmoving $U(1)$ currents, in addition to the right-moving $SU(2)$ R-currents.

This result has a simple interpretation if we take, by U-duality, the charge to correspond to a wrapped fundamental string in IIA compactified on T^5. The $(5,5)$ currents correspond to momentum/winding charges on the 5 transverse compact dimensions. In addition, on the worldsheet there are three additional left and right moving currents corresponding to translations in the noncompact directions.

In comparing the AdS and worldsheet descriptions we note two basic facts. First, although this charge configuration is expected on general grounds to yield a near horizon $\mathrm{AdS}_3 \times S^2 \times T^6$ geometry, this is yet to be shown. Just keeping the two derivative terms in the spacetime action leads to a naked singularity. Furthermore, in this case there are no R^2 corrections in the action that might stabilize the geometry in analogy with other examples. Establishing the existence of a stabilized geometry requires a better understanding of R^4 type terms than is currently available. Second, while the $(5,5)$ currents are found to match in the AdS and worldsheet descriptions, we note that the translation currents are absent on the AdS side. On the worldsheet there are 3 left and 3 rightmoving currents corresponding to motion in the transverse noncompact directions. These center of mass degrees of freedom have been "factored out" in passing to the near horizon geometry.

$\mathbf{K3 \times T^2}$

M-theory on $K3 \times T^2$ has duality group $SO(21,5) \times SO(1,1)$, as is readily understood by dualizing to the heterotic string on T^5. There are 27 gauge fields transforming as

$\mathbf{26+1}$ under $SO(21,5)$. By direct computation on the heterotic side one finds that the Chern-Simons coupling in five dimensions is of the form $\int g^{IJ} A_I \wedge F_J \wedge F_{27}$, where g^{IJ} is the $SO(21,5)$ invariant quadratic form, and F_{27} refers to the $SO(21,5)$ singlet gauge field. Magnetic charge with respect to the singlet corresponds to an M5-brane wrapped on $K3$, dual to a fundamental string on the heterotic side. We consider turning on only this magnetic charge, Again, in the two derivative approximation this yields a singular spacetime solution, but in this case it has been shown explicitly [1] that R^2 corrections stabilize the geometry into a smooth $\mathrm{AdS}_3 \times S^2 \times K3 \times T^2$. Reducing the five dimensional Chern-Simons term to three dimensions yields $\int g^{IJ} A_I \wedge F_J$. We thus find that the spectrum of currents matches up with the worldsheet structure of the heterotic string. Again, the translational currents are absent in the AdS picture; adding them in yields the $(24,8)$ spectrum of currents corresponding to the transverse modes of the heterotic string.

$\mathbf{CY_3}$

There are $n_V + 1 = h^{1,1}$ gauge fields with five dimensional Chern-Simons coupling $\int C^{IJK} A_I \wedge F_J \wedge F_K$ where C^{IJK} is the intersection form of the CY_3. Reduction to three dimensions give a signature $(n_V, 1)$ Chern-Simons term, as follows from the Hodge index theorem (see [8]).

6.4 Including singletons

Singleton modes are pure gauge configurations that are nonetheless physical in the presence of the AdS_3 boundary. To see why, consider the case of a $U(1)$ gauge field with Chern-Simons term. The configuration $A_w = \partial_w \Lambda(w)$ is formally pure gauge, but from (3.15) it carries the nonzero energy $T_{ww} = \frac{k}{8\pi}(\partial_w \Lambda)^2$, and hence is physical. This is possible because the true gauge transformations must vanish at the boundary and it is only those that leave the stress tensor invariant. The singleton states are described in the CFT as $J_{-1}|0\rangle$, where J is the current corresponding to A. We also have the $SL(2,\mathbb{R})$ descendants of these states.

A similar story holds for singletons associated with diffeomorphisms that are nonvanishing at the boundary. These correspond to the states $L_{-2}|0\rangle$ and $SL(2,\mathbb{R})$ descendants thereof. The explicit form of the diffeomorphisms is given in [28].

We can now work out the contribution of the singletons to the elliptic genus of the $(0,4)$ theory. The computation is the same as in (6.3) except without the product over harmonics ℓ. The leftmoving currents have $h_{min} = 1$, and the stress tensor has $h_{min} = 2$. If there are n_L leftmoving currents then the contribution of singletons is

$$\chi_{NS}^{sing} = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{n_L}} \frac{1}{(1-q^{n+1})} \tag{6.10}$$

Now, we found in the previous subsection that

$$n_L = \{\, 5 \quad T^6 21 \quad K3 \times T^2 n_V \quad CY_3 \quad . \tag{6.11}$$

We find the full results by multiplying (6.9) and (6.10):

$$\chi_{NS} = \chi_{NS}^{sugra} \chi_{NS}^{sing} = \{\, 1 \quad T^6 \quad 1 \quad K3 \times T^2 \quad M(q)^{-\chi_E} \textstyle\prod_{n=1}^{\infty}(1-q^n)^3 \quad CY_3 \tag{6.12}$$

We find that in the T^6 and $K3 \times T^2$ cases the singletons precisely cancel the dynamical contribution (6.9). For the CY_3 the dependence on n_V cancelled. Note that these conclusion are a result of cancellations between propagating states from table 2 and the singletons.

6.5 R sector

As far as the supergravity fluctuations are concerned, the NS and R sector elliptic genera are identical in the $(0,4)$ case. In general we have the relation (2.18), but in the $(0,4)$ case there is no z_0 since there is no leftmoving R-symmetry. So in (6.12) we can trivially replace NS by R.

7. Contribution from wrapped branes

The final ingredient in the computation of the elliptic genus is the contribution from wrapped branes. In [24] it was shown that this computation is equivalent to the Gopakumar-Vafa derivation [29] of the topological string partition function from M-theory. The same argument applies here, and so we can be brief.

The elliptic genus receives contribution from M2 branes and antibranes wrapping 2-cycles in M_6 and carrying angular momentum on S^2 [24]. In the general CY_3 case an explicit result requires a determination of the BPS spectrum of M2-branes. In the T^6 and $K3 \times T^2$ examples emphasized here there are drastic simplifications, as discussed in [30]. In the T^6 case there is no contribution at all since the M2-branes are in sufficiently large multiplets to mutually cancel. In the $K3 \times T^2$ case there is a similar cancellation except for M2-branes wrapping T^2. These M2-branes precisely reproduce the known correction to the topological string partition function on $K3 \times T^2$. From the standpoint of section 6.3, these M2-branes are states electrically charged with respect to F_{27}, which we recall was the one special gauge field not appearing in the Chern-Simons term.

8. Discussion and open questions

Let us review what has been achieved. We have shown, following the ideas in [9], how to systematically compute the elliptic genus (or rather, its Farey tail transform) of string/M-theory on AdS_3 using supergravity. What makes this possible is that the long distance theory on AdS_3 is topological, allowing for the exact determination of currents and stress tensors. The currents and stress tensor can be integrated to find the action, and then summing over the space of solutions yields the elliptic genus. What distinguishes different examples from one another is the precise spectrum of currents and the spectrum of BPS states supporting nontrivial holonomies. We also noted that the same approach can be employed in the computation of the full partition function, but here one would need the full spectrum of states and not just its BPS sector, and so explicit results are not possible.

To be more specific, the path integral evaluation of the Farey tail transformed elliptic genus led to the expressions (5.20)-(5.21). The theta function arises from summing over the allowed class of gauge fields, where it's important that we have correctly included all

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boundary terms in the action. The term in the exponential of (5.20) is the contribution of the purely gravitational part of the action. Finally, the coefficients $\tilde{c}(m - \frac{1}{4}\mu^2)$ encode the spectrum of BPS states, or more accurately their orbits under spectral flow. Once these are known we can feed (5.20) into (5.21) to complete the computation.

The $\tilde{c}(m - \frac{1}{4}\mu^2)$ are computed from the spectrum of supergravity fluctuations and wrapped branes. As discussed in section 6, the contribution from supergravity fluctuations is extracted from the Kaluza-Klein spectrum for the compactification of interest. For the contribution from wrapped branes we relied on the observation of [24] that this is equivalent to the Gopakumar-Vafa computation of the topological string free energy.

Of course, one natural question is why we should be computing the Farey tail transform of the elliptic genus, rather than the elliptic genus itself. Recall that the latter cannot quite be extracted from the former, since states with $m - \frac{1}{4}\mu^2 = 0$ are projected out by the transform. In [9] it was shown that this procedure is necessary in order for the CFT result to take a natural form in terms of buk geometries. It would be nice to have a better understanding of this from the bulk point of view

Our considerations have been from the bulk point of view of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence. An exact expression for the CFT elliptic genus in the $(4,4)$ case was the starting point for [9], and corresponding study of the $(0,4)$ case are the subject of recent investigations [10, 11]. This should lead to a more detailed understanding of the AdS/CFT correspondence, and its connection with topological strings.

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A. Modular properties of the charged boson partition function

In this appendix we determine the modular transformation of the partition function for a single charged boson by comparing the canonical and path integral formulations. This illustrates some general features.

Consider a free compact boson of radius $2\pi R$. We use the conventions of [19] and set $\alpha' = 1$. We define the partition function

$$Z(\tau, z, \tilde{z}) = (q\overline{q})^{-1/24}\,\mathrm{Tr}\left[q^{L_0}\overline{q}^{\tilde{L}_0}e^{2\pi i z p_L}e^{2\pi i\tilde{z}p_R}\right] \;, \tag{A.1}$$

with

$$\begin{aligned} L_0 &= \frac{p_L^2}{4} + L_0^{osc} \;, \quad \tilde{L}_0 = \frac{p_R^2}{4} + \tilde{L}_0^{osc} \\ p_L &= \frac{n}{R} + wR \;, \quad p_R = \frac{n}{R} - wR \;. \end{aligned} \tag{A.2}$$

The partition function obeys the modular transformation rule

$$Z\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}, \frac{\tilde{z}}{c\overline{\tau}+d}\right) = e^{\frac{2\pi i c z^2}{c\tau+d}}\,e^{-\frac{2\pi i c\tilde{z}^2}{c\overline{\tau}+d}}\,Z(\tau, z, \tilde{z}) \;, \tag{A.3}$$

as is readily verified by direct computation. (A.3) is to be compared with (2.10).

To explain the origin of the exponential prefactors in (A.3) we pass to a path integral formulation. We consider

$$Z_{PI}(\tau, A) = \int \mathcal{D}X e^{-S} \tag{A.4}$$

with

$$S = \frac{1}{2\pi} \int_{T^2} d^2\sigma \sqrt{g} \left[\frac{1}{2} g^{ij} \partial_i X \partial_j X - A^i \partial_i X \right] \tag{A.5}$$

and A^i = constant. To relate potentials appearing in (A.1) and (A.5), we use the standard expression for the charges

$$p_L = 2 \oint \frac{dw}{2\pi i} i\partial_w X \ , \quad p_R = -2 \oint \frac{dw}{2\pi i} i\partial_{\overline{w}} X \ , \tag{A.6}$$

and then equate the charge dependent phases in the two versions. This yields

$$z = -i\tau_2 A_{\overline{w}} \ , \quad \tilde{z} = i\tau_2 \tilde{A}_w \ . \tag{A.7}$$

We denoted the holomorphic part of the gauge field $\tilde{A}_w$ because, in the body of the paper, this component arises from an independent bulk 1-form $\tilde{A}$.

In the path integral formulation a modular transformation is a coordinate transformation combined with a Weyl transformation, and so it is manifest that

$$Z_{PI}\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}, \frac{\tilde{z}}{c\overline{\tau} + d}\right) = Z_{PI}(\tau, z, \tilde{z}) \ , \tag{A.8}$$

where the transformation of z and $\tilde{z}$ just expresses the coordinate transformation.

What then is the relation between Z_{PI} and Z? To find this we just carry out the usual steps that relate Hamiltonian and path integral expression: $\int \mathcal{D}X e^{-S} = \mathrm{Tr}\, e^{-\beta H}$. The only point to be aware of is that the Hamiltonian corresponding to the action (A.5) is not the factor appearing in the exponential of (A.1), but differs from this by a contribution quadratic in the potentials. In particular, we find

$$Z_{PI}(\tau, z, \tilde{z}) = e^{\frac{\pi(z+\tilde{z})^2}{\tau_2}} Z(\tau, z, \tilde{z}) \ . \tag{A.9}$$

Combining (A.8) and (A.9) we see that the modular transformation law of Z must be such to precisely offset that of $e^{\frac{\pi(z+\tilde{z})^2}{\tau_2}}$. This is what (A.3) does.

Let us summarize the lessons just learned as applied to a more general setting. In the canonical form (A.1) properties such as spectral flow are manifest. In the path integral form (A.5) the modular transformation law is manifest. By relating the two versions we can understand both properties. Furthermore, the analysis we performed is essentially completely general, in that given an arbitrary CFT we can always realize the $U(1)$ current algebra in terms of free bosons. This is the way we derive (2.10) and (2.12), for example.

B. Conventions

In this appendix we summarize our conventions.

We work in Euclidean signature with $\epsilon_{12\ldots} = \sqrt{g}$ and $\epsilon^{12\cdots} = 1/\sqrt{g}$. The rule for integrating differential forms is

$$\alpha = \alpha(x) dx^1 \wedge \cdots \wedge dx^d \quad \rightarrow \quad \int \alpha = \int d^d x \, \alpha(x) \ . \tag{B.1}$$

The star operation in d-dimensions is

$$\star dx^\mu \wedge \ldots \wedge dx^{\mu_p} = \frac{1}{(d-p)!} \epsilon^{\mu_1 \ldots \mu_p}{}_{\mu_{p+1} \ldots \mu_d} dx^{\mu_{p+1}} \wedge \ldots \wedge dx^{\mu_d} \ . \tag{B.2}$$

The components of a p-form are defined as

$$A^{(p)} = \frac{1}{p} A^{(p)}_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} \ , \tag{B.3}$$

so for a 1-form in $d = 2$:

$$\star A^{(1)} = \epsilon^\mu{}_\nu A^{(1)}_\mu dx^\nu, \quad \star A^{(1)}_\mu = -\epsilon_\mu{}^\nu A^{(1)}_\nu, \quad \star\star A^{(1)} = -A^{(1)} \ . \tag{B.4}$$

Useful wedge products are

$$\begin{aligned} A^{(1)} \wedge B^{(1)} &= A^{(1)}_\mu B^{(1)}_\nu dx^\mu \wedge dx^\nu = \sqrt{g} \epsilon^{\mu\nu} A^{(1)}_\mu B^{(1)}_\nu dx^1 \wedge dx^2 \ , \\ A^{(1)} \wedge A^{(2)} &= \tfrac{1}{2} A^{(1)}_\mu A^{(2)}_{\nu\sigma} dx^\mu \wedge dx^\nu \wedge dx^\sigma = \tfrac{1}{2} \sqrt{g} \epsilon^{\mu\nu\sigma} A^{(1)}_\mu A^{(2)}_{\nu\sigma} dx^1 \wedge dx^2 \wedge dx^3 \ , \\ \star A^{(1)} \wedge B^{(1)} &= -\epsilon_\mu{}^\nu A^{(1)}_\nu B^{(1)}_\sigma dx^\mu \wedge dx^\sigma = -\sqrt{g} g^{\mu\nu} A^{(1)}_\mu B^{(1)}_\nu dx^1 \wedge dx^2 \ , \end{aligned} \tag{B.5}$$

where the last formula applies in $d = 2$. From this it follows that

$$\star A^{(1)} \wedge B^{(1)} = \star B^{(1)} \wedge A^{(1)} = -A^{(1)} \wedge \star B^{(1)} \ . \tag{B.6}$$

Complex coordinates on the boundary are defined as $w = \sigma_1 + i\sigma_2$, where σ_2 is Euclidean time. We then have the components of a vector

$$\begin{aligned} v_w &= \frac{v_1 - iv_2}{2} \ , \quad v_{\overline{w}} = \frac{v_1 + iv_2}{2} \ , \\ v_1 &= 7v_w + v_{\overline{w}} \ , \quad v_2 = i(v_w - v_{\overline{w}}) \ . \end{aligned} \tag{B.7}$$

Also useful are Hodge stars

$$\star dw = -idw \ , \quad \star d\overline{w} = id\overline{w} \ . \tag{B.8}$$

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