

Precision entropy of spinning black holes

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ABSTRACT: We construct spinning black hole solutions in five dimensions that take into account the mixed gauge-gravitational Chern-Simons term and its supersymmetric completion. The resulting entropy formula is discussed from several points of view. We include a Taub-NUT base space in order to test recent conjectures relating 5D black holes to 4D black holes and the topological string. Our explicit results show that certain charge shifts have to be taken into account for these relations to hold. We also compute corrections to the entropy of black rings in terms of near horizon data.

KEYWORDS: Black Holes, Black Holes in String Theory.

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1. Introduction

The importance of black holes for quantum gravity and string theory has motivated a sustained effort to achieve a computational control of black hole entropy that goes beyond the leading Bekenstein-Hawking area law [1–8]. While much has been achieved, there are still many unanswered questions (for recent reviews summarizing the current state of the subject, see [9–12]). On the gravity side, the leading corrections to the entropy come from higher derivative terms in the spacetime effective action, and we would certainly like to know how these affect the standard black hole solutions of string theory. However, the results in this direction have so far been limited to 4D black holes, which is surprising

given that the simplest supersymmetric black holes in string theory arise in 5D. We have recently begun to fill this gap [13, 14] (see also [15]). In this paper we continue this program by constructing asymptotically flat spinning black holes with higher derivative corrections taken into account. Our solutions are generalizations of the BMPV solution [16]. They are simple enough that we can be quite explicit, yet intricate enough that we can shed light on a number of important conceptual issues.

The setting for our analysis is 5D supergravity corrected by the mixed gauge-gravitational Chern-Simons term

$$\frac{c_{2I}}{24 \cdot 16\pi^2} \int A^I \wedge \text{Tr} R^2, \tag{1.1}$$

and terms related to this by supersymmetry. We use the off-shell formalism which has supersymmetry transformations that do not depend on the explicit Lagrangian. The supersymmetric completion of (1.1) was constructed in this formalism in [17]. Taking advantage of the universal supersymmetry variations, and also using the complete action, we find the solution for the spinning black hole.

The next step is to determine the Bekenstein-Hawking-Wald [18] entropy of the black hole. The near horizon geometry consists of a circle fibered over $\text{AdS}_2 \times S^2$. After KK reduction on the circle, Wald's entropy formula is equivalent to entropy extremization [8, 19–23]. A well known subtlety in this procedure arises from the presence of Chern-Simons terms, since these are not gauge invariant [19]. After this is taken into account, we find the entropy of a spinning black hole with higher derivative corrections.

Our result for the entropy is simplest when expressed in terms of the near horizon moduli. In this form we can also demonstrate precise agreement with results inferred from 4D black holes, the topological string, and the 4D/5D connection [9, 3, 12, 24]. However, the more physically relevant result is the entropy expressed in terms of the conserved charges, and in these variables the relation with the 4D results exhibits some new features.

The 5D electric charges are defined unambiguously in terms of flux integrals over a S^3 at infinity surrounding the black hole. The comparison to 4D black holes is made by placing the 5D black hole at the tip of a Taub-NUT space. Taub-NUT is asymptotically $\mathbb{R}^3 \times S^1$, and a 4D black hole is obtained via KK reduction on the S^1 . The 4D electric charges are thereby defined via flux integrals over an asymptotic S^2 . We can think of recovering the 5D black hole by sending the radius of the circle (which is a modulus) to infinity [25–29].

At lowest (i.e. two-derivative) order, the 4D and 5D electric charges are equal, and in the literature it seems to be assumed that this holds in general. However, we show explicitly that the charges are different in the presence of higher derivatives. In particular, the electric charges differ by $\Delta q_I = \frac{1}{24} c_{2I}$. The reason is simple: the operations of computing the flux integrals and decompactifying the Taub-NUT circle do not commute. This in turn follows from the fact that the Taub-NUT space itself carries a delocalized electric charge proportional to its Euler number, as implied by the Chern-Simons term (1.1).

Angular momentum adds further structure, and we find that another higher derivative shift is required to relate J to the corresponding 4D electric charge q_0 . Our conclusion is

that all these shifts need to be taken into account in order to use the 4D/5D connection to reproduce the correct 5D entropy formulas derived here.

While the main topic of this paper is 5D black holes, our entropy analysis can be easily extended to the case of black rings. We thereby find the corrected black ring entropy formula, albeit expressed in terms of near horizon data. Giving an expression in terms of the charges of the ring requires knowledge of the full asymptotically flat solution, which is not yet available (alternatively, one might try to employ the techniques developed in [30, 31]).

This paper is organized as follows. In section 2 we outline the derivation of the spinning black hole solution and discuss some of its properties. Some further details are provided in the appendix. In section 3 we derive the black hole entropy using entropy extremization. As an aside, we also find the entropy of the black ring with higher derivatives. In section 4 we discuss interpretational issues with emphasis on aspects related to the definition of charge. We explain why results motivated by 4D topological string theory fail to capture the full story. Finally, we construct the spinning black hole on a Taub-NUT base space and use this to carry out the 4D-5D reduction explicitly.

2. 5D spinning black hole solutions

We want to find the rotating supersymmetric black holes in five dimensions with higher derivatives taken into account. The procedure for deriving the solution is the same as in the spherically symmetric case [14] so we shall focus on results rather than methodology. Some details of our derivation are given in appendix A.

2.1 The supersymmetry conditions

The starting point is an *ansatz* for the solution. Since the supersymmetry variations in the off-shell formalism are unaffected by the presence of higher derivatives terms in the action, the form of the solution is the same as in the two-derivative context [32]. In particular, supersymmetry implies the existence of a timelike Killing vector, which we build in by writing

$$ds^2 = e^{4U(x)}(dt + \omega)^2 - e^{-2U(x)}h_{mn}dx^m dx^n, \tag{2.1}$$

where $\omega = \omega_i(x)dx^i$ is a one-form on the 4D base manifold equipped with metric $h_{mn}dx^m dx^n$. The base space is generally Hyper-Kähler; for the present it is just taken to be flat space (we discuss the case of Taub-NUT later). We will use the obvious local frame

$$e^{\hat{0}} = e^{2U}(dt + \omega), \quad e^{\hat{i}} = e^{-U}dx^i. \tag{2.2}$$

The matter in the theory consists of n_V vector multiplets of $\mathcal{N} = 2$ supersymmetry. Supersymmetry relates the gauge field strength in each multiplet to the corresponding scalar field through the attractor flow

$$F^I = d(M^I e^{\hat{0}}). \tag{2.3}$$

Generally, supersymmetry also permits the addition to F^I of an anti-self-dual form on the base space. Such a contribution is needed for black ring solutions, but vanishes for

the black hole solutions considered here. With this restriction, we also have that $d\omega$ is self-dual¹

$$\star_4 d\omega = d\omega . \tag{2.4}$$

Supersymmetry further determines the auxiliary fields completely in terms of the geometry (2.1). The auxiliary two-form is fixed to be

$$v = -\frac{3}{4}de^{\hat{0}} = -\frac{3}{4}\left(2\partial_i U e^U e^{\hat{i}} e^{\hat{0}} + \frac{1}{2}e^{2U} d\omega_{ij} e^{\hat{i}} e^{\hat{j}}\right), \tag{2.5}$$

and the auxiliary scalar is determined as

$$D = 3e^{2U} (\nabla^2 U - 6(\nabla U)^2) + \frac{3}{2}e^{8U} (d\omega)^2 . \tag{2.6}$$

2.2 Equations of motion

At this point the constraints of supersymmetry have been exhausted and we must use the explicit action [17]. First of all, we need the equations of motion for the gauge field, namely the Maxwell equation

$$2\nabla_\mu \left(\frac{\partial \mathcal{L}}{\partial F^I_{\mu\nu}} \right) = \frac{\partial \mathcal{L}}{\partial A^I_\nu} . \tag{2.7}$$

It is straightforward in principle (although tedious in practice) to insert a solution of the general form (2.1)–(2.6) into the Maxwell equation (2.7). After reorganization, we find that the spatial components of the equation are satisfied automatically. We also find that the temporal component can be cast in the simple form

$$\nabla^2 \left[e^{-2U} M_I - \frac{c_{2I}}{8} \left((\nabla U)^2 - \frac{1}{12} e^{6U} (d\omega)^2 \right) \right] = 0, \tag{2.8}$$

with

$$M_I = \frac{1}{2} c_{IJK} M^J M^K . \tag{2.9}$$

All indices in (2.8) are contracted with the base space metric h_{mn} , e.g.,

$$(d\omega)^2 = h^{mn} h^{pq} d\omega_{mp} d\omega_{nq} . \tag{2.10}$$

(2.8) is the generalized Gauss' law, and is simply a harmonic equation on the flat base space.² We will later discuss how conserved charges can be read off from this equation, with nontrivial shifts due to higher derivatives encoded in the term proportional to c_{2I} . At this point we just note that the one-form ω enters Gauss' law when higher derivatives are taken into account. The decoupling between angular momentum and radial evolution found in the leading order theory is therefore not preserved in general.

In order to fully specify the solution we also need the equation of motion for the auxiliary field D . It is

$$\mathcal{N} = 1 - \frac{c_{2I}}{72} (F^I_{\mu\nu} v^{\mu\nu} + M^I D) . \tag{2.11}$$

¹ \star_4 denotes the dual taken with respect to the metric $h_{mn} dx^m dx^n$.

²Later we will find that a curved base metric induces a source on the right hand side of this equation.

where $\mathcal{N} = \frac{1}{6}c_{IJK}M^I M^J M^K$. Inserting (2.1)–(2.6) for the spinning black hole we find

$$\frac{1}{6}c_{IJK}M^I M^J M^K = 1 - \frac{c_{2I}}{24} \left[e^{2U} M^I \left(\nabla^2 U - 4(\nabla U)^2 + \frac{1}{4}e^{6U} (d\omega)^2 \right) + e^{2U} \nabla^i M^I \nabla_i U \right]. \quad (2.12)$$

In the two-derivative theory the scalar fields are constrained by the special geometry condition $\mathcal{N} = 1$. In the corrected theory we must instead impose the much more complicated condition (2.12).

2.3 Assembling the solution

We have now determined all the necessary equations and it only remains to solve them. This is simplified by writing the flat base space in the Gibbons-Hawking coordinates³

$$h_{mn} dx^m dx^n = \rho(dx^5 + \cos \theta d\phi)^2 + \frac{1}{\rho} (d\rho^2 + \rho^2(d\theta^2 + \sin^2 \theta d\phi^2)), \quad (2.13)$$

with $x^5 \cong x^5 + 4\pi$.

Let us recall the BMPV solution to the two derivative theory [16]. When written in Gibbons-Hawking coordinates the one-form on the base space takes the form⁴

$$\omega = \frac{J}{8\rho} (dx^5 + \cos \theta d\phi), \quad (2.14)$$

so that

$$d\omega = -\frac{J}{8\rho^2} (e^{\hat{\rho}} e^{\hat{5}} + e^{\hat{\theta}} e^{\hat{\phi}}), \quad (2.15)$$

in the obvious orthonormal frame on the base space. In this form the self-duality condition $d\omega = \star_4 d\omega$ is manifest. In fact, the ρ -dependence of $d\omega$ is completely determined by the Bianchi identity and the self-duality condition. Therefore (2.14)–(2.15) will be maintained when higher derivatives are taken into account.

Let us next turn to the generalized Gauss' law (2.8). As already noted, this is just a harmonic equation. Writing out the Laplacian in Gibbons-Hawking coordinates we are lead to introduce the harmonic function

$$H_I = M_I^\infty + \frac{q_I}{4\rho} = e^{-2U} M_I - \frac{c_{2I}}{8} \left((\nabla U)^2 - \frac{1}{12} e^{6U} (d\omega)^2 \right), \quad (2.16)$$

where the constants of integration M_I^∞ are identified with the asymptotic moduli.

The solution we seek is specified by the conserved charges (J, q_I) and the asymptotic moduli. With these inputs, the one-form ω was given in (2.14), the gauge field strengths were found in (2.3), and (2.16) determines the scalar fields as

$$M_I(\rho) = e^{2U} \left[M_I^\infty + \frac{q_I}{4\rho} + \frac{c_{2I}}{8} \left((\nabla U)^2 - \frac{1}{12} e^{6U} (d\omega)^2 \right) \right]. \quad (2.17)$$

³The transformation $\rho = \frac{r^2}{4}$, $x^5 = \tilde{\phi} + \tilde{\psi}$, $\phi = \tilde{\phi} - \tilde{\psi}$, $\theta = 2\tilde{\theta}$ brings the line element to the form $ds^2 = dr^2 + r^2(d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\psi}^2 + \cos^2 \tilde{\theta} d\tilde{\phi}^2)$.

⁴We use units $G_5 = \frac{\pi}{4}$. In these units the angular momentum and the charges are quantized as integers.

Up to this point the solution has been given not only in terms of the conserved charges, but also in terms of the metric function $U(\rho)$, which has not yet been computed. This function is determined by the constraint (2.12). In order to make this additional equation completely explicit we should first invert the equation (2.9) that determines M^I in terms of M_I . The result should be inserted in (2.12), which then becomes an second order ordinary differential equation that can be easily integrated numerically to find $U(\rho)$. In [14] we carried out this procedure for some examples with spherical symmetry. The rotating solution is qualitatively similar, but not identical. In particular, we mention again that the radial profile depends on the angular momentum when higher derivative corrections are taken into account.

2.4 Near horizon geometry

We are especially interested in the near horizon region, and here we can make the geometry more explicit. In order to do that we consider a radial function of the form

$$e^{2U} = \frac{\rho}{\ell^2} . \tag{2.18}$$

The parameter ℓ sets the physical scale of the solution. We will see later that it can be identified with the radii of a near horizon $AdS_2 \times S^2$.⁵ With this radial function the scalar fields (2.17) reduce to the constants

$$M_I = \frac{1}{4\ell^2} \left(q_I + \frac{c_{2I}}{8} \left(1 - \frac{1}{48\ell^6} J^2 \right) \right) . \tag{2.19}$$

These are the attractor values for the moduli in the geometry modified by higher derivatives. In particular, the attractor values depend on the conserved charges alone, and not the asymptotic moduli.

The constraint equation (2.12) also becomes an algebraic relation

$$\frac{1}{6} c_{IJK} M^I M^J M^K = 1 + \frac{c_{2I} M^I}{48\ell^2} \left(1 - \frac{J^2}{32\ell^6} \right) . \tag{2.20}$$

Taken together with the relations (2.9) we have a set of algebraic equations that determine the near horizon geometry completely. In order to solve these equations it is convenient to introduce the scaled variables

$$\hat{M}^I = 2\ell M^I, \hat{J} = \frac{1}{8\ell^3} J . \tag{2.21}$$

We then have the following procedure: given asymptotic charges (J, q_I) we find the rescaled variables (\hat{J}, \hat{M}^I) by solving the equations (2.19)–(2.20) written in the form

$$\begin{aligned} J &= \left(\frac{1}{3!} c_{IJK} \hat{M}^I \hat{M}^J \hat{M}^K - \frac{c_{2I} \hat{M}^I}{12} (1 - 2\hat{J}^2) \right) \hat{J}, \\ q_I &= \frac{1}{2} c_{IJK} \hat{M}^J \hat{M}^K - \frac{c_{2I}}{8} \left(1 - \frac{4}{3} \hat{J}^2 \right) . \end{aligned} \tag{2.22}$$

⁵In the nonrotating case there is a near horizon $AdS_2 \times S^3$ with radii $\ell_A = \frac{1}{2}\ell_S = \ell$. This was the notation used in [14].

With the solution in hand we compute

$$\begin{aligned} \ell^3 &= \frac{1}{8} \left(\frac{1}{3!} c_{IJK} \hat{M}^I \hat{M}^J \hat{M}^K - \frac{c_{2I} \hat{M}^I}{12} (1 - 2\hat{J}^2) \right), \\ M^I &= \frac{1}{2\ell} \hat{M}^I, \end{aligned} \tag{2.23}$$

to find the values for the physical scale of the solution ℓ and the physical moduli M^I , written as functions of (J, q_I) .

In general it is of course rather difficult to invert (2.22) explicitly. This is the situation also before higher derivative corrections have been taken into account and/or if angular momentum is neglected.

The formulae can be made more explicit for large charges. Let us define the dual charges q^I through

$$q_I = \frac{1}{2} c_{IJK} q^J q^K. \tag{2.24}$$

We also define

$$Q^{3/2} = \frac{1}{3!} c_{IJK} q^I q^J q^K, \tag{2.25}$$

and

$$C_{IJ} = c_{IJK} q^K. \tag{2.26}$$

Each of these quantities depend on charges and Calabi-Yau data but not on moduli.

With the definitions (2.24)–(2.26) we can invert (2.22) for large charges (i.e. expand to first order in c_{2I}) and find

$$\begin{aligned} \hat{M}^I &= q^I + \frac{1}{8} \left(1 - \frac{4}{3} \frac{J^2}{Q^3} \right) C^{IJ} c_{2J} + \dots, \\ \hat{J} &= \frac{J}{Q^{3/2}} \left(1 + \frac{c_2 \cdot q}{48 Q^{3/2}} \left[1 - 4 \frac{J^2}{Q^3} \right] \right) + \dots \end{aligned} \tag{2.27}$$

Then (2.23) gives the physical scale of the geometry and the physical moduli as

$$\begin{aligned} \ell &= \frac{1}{2} Q^{1/2} \left(1 - \frac{c_2 \cdot q}{144 Q^{3/2}} \left[1 - 4 \frac{J^2}{Q^3} \right] \right) + \dots, \\ M^I &= \frac{q^I}{Q^{1/2}} \left(1 + \frac{c_2 \cdot q}{144 Q^{3/2}} \left[1 - 4 \frac{J^2}{Q^3} \right] \right) + \frac{1}{8 Q^{1/2}} \left(1 - \frac{4}{3} \frac{J^2}{Q^3} \right) C^{IJ} c_{2J} + \dots \end{aligned} \tag{2.28}$$

2.5 The 4D-5D connection

One of the advantages in introducing the Gibbons-Hawking coordinates (2.13) is that they facilitate the comparison between 5D and 4D points of view.

To see how this works, start with the rotating black hole solution presented above and then reorganize the metric into a form suitable for KK reduction along x^5 ,

$$ds^2 = -e^{-4\phi} (dx^5 + \cos \theta d\phi + A_t^0 dt)^2 + e^{2\phi} (g dt^2 - g^{-1} (d\rho^2 + \rho^2 d\Omega_2^2)). \tag{2.29}$$

Our *ansatz* gives

$$\begin{aligned}
 e^{-4\phi} &= e^{-2U} \rho \left(1 - \frac{1}{\rho} e^{6U} \omega_5^2 \right) &\Rightarrow & \ell^2 (1 - \hat{J}^2), \\
 g^2 &= \frac{e^{6U} \rho}{1 - \frac{1}{\rho} e^{6U} \omega_5^2} &\Rightarrow & \frac{\rho^4}{\ell^6 (1 - \hat{J}^2)}, \\
 A_t^0 &= -\frac{e^{6U} \omega_5}{\rho \left(1 - \frac{1}{\rho} e^{6U} \omega_5^2 \right)} &\Rightarrow & -\frac{\hat{J}}{1 - \hat{J}^2} \frac{\rho}{\ell^3}.
 \end{aligned} \tag{2.30}$$

The arrows implement the near horizon limit where the metric function takes the form (2.18). Since

$$e^{-2\phi} g = \frac{\rho^2}{\ell^2}, \tag{2.31}$$

we see that the 4D string metric has $AdS_2 \times S^2$ near horizon geometry with the AdS_2 and the S^2 both having radii ℓ . The 4D Einstein metric

$$ds_{4E}^2 = g dt^2 - g^{-1} (d\rho^2 + \rho^2 d\Omega_2^2), \tag{2.32}$$

describes an extremal black hole. The 4D matter fields are the dilaton ϕ , the KK gauge field A^0 , and additional gauge fields A_4^I and scalars a^I coming from the reduction of the 5D gauge field *via* the decomposition

$$\begin{aligned}
 A^I &= e^{2U} M^I (dt + \omega) = e^{2U} M^I (1 - \omega_5 A_t^0) dt + e^{2U} M^I \omega_5 (dx^5 + \cos \theta d\phi + A_t^0 dt) \\
 &= A_4^I + a^I (dx^5 + \cos \theta d\phi + A_t^0 dt).
 \end{aligned} \tag{2.33}$$

The 4D point of view will play a central role in the following.

3. Entropy of 5D spinning black holes (and black rings)

In this section we compute the entropy of our black holes. This is most conveniently done via the entropy function approach [8], which essentially amounts to evaluating the Lagrangian density on the near horizon geometry. The one complication is that the entropy function method assumes a gauge invariant Lagrangian, whereas we have non-gauge invariant Chern-Simons terms in the action. The remedy for this is well known [19]: we should reduce the action to 4D, and then add a total derivative term to the Lagrangian to cancel the non-gauge invariant piece. Applications of the entropy function to rotating black holes can be found in [19–23]. In the last subsection we consider black rings; for previous work on the entropy function for black rings see [33, 23, 34].

3.1 Near horizon geometry and the entropy function

We first review the general procedure for determining the entropy from the near horizon solution, mainly following [23]. The general setup is valid for spinning black holes as well as black rings.

The near horizon geometries of interest take the form of a circle fibered over an $\text{AdS}_2 \times S^2$ base:

$$\begin{aligned}
 ds^2 &= w^{-1} \left[v_1 \left(\rho^2 d\tau^2 - \frac{d\rho^2}{\rho^2} \right) - v_2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] - w^2 \left(dx^5 + e^0 \rho d\tau + p^0 \cos \theta d\phi \right)^2, \\
 A^I &= e^I \rho d\tau + p^I \cos \theta + a^I \left(dx^5 + e^0 \rho d\tau + p^0 \cos \theta d\phi \right), \\
 v &= -\frac{1}{4\mathcal{N}} M_I F^I.
 \end{aligned}
 \tag{3.1}$$

The parameters w , $v_{1,2}$, a^I and all scalar fields are assumed to be constant. KK reduction along x^5 yields a 4D theory on $\text{AdS}_2 \times S^2$. The solution carries the magnetic charges p^I , while e^I denote electric potentials.⁶

Omitting the Chern-Simons terms for the moment, let the action be

$$I = \frac{1}{4\pi^2} \int d^5x \sqrt{g} \mathcal{L} . \tag{3.2}$$

Define

$$f = \frac{1}{4\pi^2} \int d\theta d\phi dx^5 \sqrt{g} \mathcal{L} . \tag{3.3}$$

Then the black hole entropy is

$$S = 2\pi \left(e^0 \frac{\partial f}{\partial e^0} + e^I \frac{\partial f}{\partial e^I} - f \right) . \tag{3.4}$$

Here w , $v_{1,2}$ etc. take their on-shell values. One way to find these values is to extremize f while holding fixed the magnetic charges and electric potentials. The general extremization problem would be quite complicated given the complexity of our four-derivative action. Fortunately, in the cases of interest we already know the values of all fields from the explicit solutions.

The Chern-Simons term is handled by first reducing the action along x^5 and then adding a total derivative to \mathcal{L} to restore gauge invariance.

3.2 Computation of the on-shell action

Starting from our solution written in the form (2.29) we insert the near horizon values given in (2.30) and then change coordinates $t = \tau \ell^3 \sqrt{1 - \hat{J}^2}$ so that the solution takes the form (3.1). We then read off the magnetic charges $p^0 = 1, p^I = 0$ and the electric fields

$$e^0 = -\frac{\hat{J}}{\sqrt{1 - \hat{J}^2}}, \quad e^I = \frac{\hat{M}^I}{2\sqrt{1 - \hat{J}^2}} . \tag{3.5}$$

⁶An important point, discussed at length below, is that e^I are conjugate to 4D electric charges, which differ from the 5D charges.

Expressing the remaining quantities in terms of $e^{0,I}$ and $w = \ell\sqrt{1 - \hat{J}^2}$ we find

$$\begin{aligned}
 v_1 = v_2 \equiv V &= [1 + (e^0)^2]w^3, \\
 a^I &= -\frac{e^0 e^I}{[1 + (e^0)^2]}, \\
 M^I &= \frac{1}{w} \frac{e^I}{[1 + (e^0)^2]}, \\
 v &= \frac{3}{4}wd\tau \wedge d\rho - \frac{3}{4}we^0 \sin\theta d\theta \wedge d\phi, \\
 D &= -\frac{3}{w^2} \frac{[1 - (e^0)^2]}{[1 + (e^0)^2]^2}. \tag{3.6}
 \end{aligned}$$

The Gibbons-Hawking coordinates (2.13) have the periodicity $x^5 \cong x^5 + 4\pi$, so that (3.3) becomes

$$f = \frac{4V^2}{w} \mathcal{L}. \tag{3.7}$$

To proceed we need to evaluate the various terms in \mathcal{L} using (3.6).

Two-derivative gauge invariant contribution. As we have emphasized, the Chern-Simons terms require special considerations because they are not gauge invariant. The remaining terms in the two-derivative action are

$$\mathcal{L}_{GI}^{(2)} = -\frac{1}{2}D - \frac{3}{4}R + v^2 + \mathcal{N} \left(\frac{1}{2}D - \frac{1}{4}R + 3v^2 \right) + 2\mathcal{N}_I v^{ab} F_{ab}^I + \frac{1}{4}\mathcal{N}_{IJ} F_{ab}^I F^{Jab}. \tag{3.8}$$

Inserting the *ansatz* (3.1) with the relations (3.6) we find

$$f_{GI}^{(2)} = 4 \frac{[1 - (e^0)^2]}{[1 + (e^0)^2]^3} \cdot \frac{1}{6} c_{IJK} e^I e^J e^K. \tag{3.9}$$

Two-derivative Chern-Simons term. We next turn to the special treatment needed for the gauge Chern-Simons term

$$I_{CS} = \frac{1}{24\pi^2} \int c_{IJK} A^I \wedge F^J \wedge F^K. \tag{3.10}$$

The reduction to 4D amounts to the decomposition

$$A^I = A_4^I + a^I(dx^5 + A_4^0). \tag{3.11}$$

If we simply insert this into (3.10) the resulting action has the form

$$c_{IJK} A^I \wedge F^J \wedge F^K = 2c_{IJK} A_4^I \wedge (F_4^J + a^J F_4^0) \wedge da^K \wedge dx^5 + \text{gauge invariant}, \tag{3.12}$$

where the first term is not gauge invariant because A_4^I appears by itself rather than as part of the field strength. The remedy for this is to redefine our original action by the addition of a total derivative

$$I_{CS} \Rightarrow I'_{CS} = \frac{1}{24\pi^2} \int c_{IJK} \left(A^I \wedge F^J \wedge F^K + d \left[A_4^I \wedge (2F_4^J + F_4^0 a^J) a^K \wedge dx^5 \right] \right). \tag{3.13}$$

This new action is not meant to replace our original 5D action in general, but it is the correct action to use in the 4D entropy function because it is gauge invariant. It is now straightforward to compute

$$f_{CS}^{(2)} = \frac{4(e^0)^2(3 + (e^0)^2)}{(1 + (e^0)^2)^3} \cdot \frac{1}{6} c_{IJK} e^I e^J e^K. \quad (3.14)$$

Four-derivative gauge invariant contribution. We next turn to the higher derivative terms in the action. Again, the Chern-Simons term requires special consideration. Putting that term aside we have the action

$$\begin{aligned} \mathcal{L}_{GI}^{(4)} = & \frac{c_{2I}}{24} \left(\frac{1}{8} M^I C^{abcd} C_{abcd} + \frac{1}{12} M^I D^2 + \frac{1}{6} F^{Iab} v_{ab} D \right. \\ & + \frac{1}{3} M^I C_{abcd} v^{ab} v^{cd} + \frac{1}{2} F^{Iab} C_{abcd} v^{cd} + \frac{8}{3} M^I v_{ab} \hat{D}^b \hat{D}_c v^{ac} \\ & + \frac{4}{3} M^I \mathcal{D}^a v^{bc} \mathcal{D}_a v_{bc} + \frac{4}{3} M^I \mathcal{D}^a v^{bc} \mathcal{D}_b v_{ca} - \frac{2}{3} M^I \epsilon_{abcde} v^{ab} v^{cd} \mathcal{D}_f v^{ef} \\ & + \frac{2}{3} F^{Iab} \epsilon_{abcde} v^{cf} \mathcal{D}_f v^{de} + F^{Iab} \epsilon_{abcde} v^c{}_f \mathcal{D}^d v^{ef} \\ & \left. - \frac{4}{3} F^{Iab} v_{ac} v^{cd} v_{db} - \frac{1}{3} F^{Iab} v_{ab} v^2 + 4M^I v_{ab} v^{bc} v_{cd} v^{da} - M^I (v^2)^2 \right). \quad (3.15) \end{aligned}$$

with

$$v_{ab} \hat{D}^b \hat{D}_c v^{ac} = v_{ab} \mathcal{D}^b \mathcal{D}_c v^{ac} - \frac{2}{3} v^{ac} v_{cb} R_a{}^b - \frac{1}{12} v^2 R. \quad (3.16)$$

Inserting the *ansatz* (3.1) with the relations (3.6) we find

$$f_{GI}^{(4)} = -\frac{1}{8} \frac{[1 + (e^0)^2 + (e^0)^4]}{[1 + (e^0)^2]^3}, \quad (3.17)$$

after algebra using MAPLE. It is worth noting that every term in the action contributes to this result.

Four-derivative Chern-Simons term. Finally we must consider the mixed gauge-gravitational Chern-Simons term:

$$I_{CS} = \frac{1}{4\pi^2} \frac{c_{2I}}{24 \cdot 16} \int d^5x \sqrt{g} \epsilon_{abcde} A^{Ia} R^{bcfg} R^{de}{}_{fg}. \quad (3.18)$$

Again we reduce to 4D variables by inserting the decomposition (3.11). Since there will be a term with A_4^I appearing by itself and not in a field strength, the result will not be gauge invariant in 4D. After implementing the 4+1 split on the curvature tensor and writing $\epsilon_{abcde} R^{bcfg} R^{de}{}_{fg}$ as a total derivative the relevant term becomes

$$\begin{aligned} I_{CS} = & -\frac{1}{4\pi^2} \frac{c_{2I}}{24 \cdot 16} w^2 \int dx^5 \int d^4x \sqrt{-g_4} \epsilon_{ijkl} A_4^I \nabla^l \left(2F_{4mn}^0 R^{jkmn} \right. \\ & \left. + \frac{1}{2} w^2 F_4^{0jk} F_{4mn}^0 F_4^{0mn} + w^2 F_{4mn}^0 F_4^{0jm} F_4^{0kn} \right) + \text{gauge invariant}, \quad (3.19) \end{aligned}$$

where indices are raised and lowered by the $AdS_2 \times S^2$ metric

$$ds_4^2 = \frac{V}{w} \left[\left(\rho^2 d\tau^2 - \frac{d\rho^2}{\rho^2} \right) - (d\theta^2 + \sin^2 \theta d\phi^2) \right]. \quad (3.20)$$

Also, $\sqrt{-g_4}$ and ϵ_{ijkl} are defined with respect to this metric.

We then cancel off the non-gauge invariant part by modifying (3.18) as $I_{CS} \Rightarrow I'_{CS} = I_{CS} + \Delta I_{CS}$, with

$$\begin{aligned} \Delta I_{CS} &= \frac{1}{4\pi^2} \frac{c_{2I}}{24 \cdot 16} w^2 \int dx^5 \int d^4x \sqrt{-g_4} \epsilon_{ijkl} \nabla^l \\ &\times \left[A_4^{Ii} \left(2F_{4mn}^0 R^{jkmn} + \frac{1}{2} w^2 F_4^{0jk} F_{4mn}^0 F_4^{0mn} + w^2 F_{4mn}^0 F_4^{0jm} F_4^{0kn} \right) \right]. \end{aligned} \quad (3.21)$$

We now compute

$$\begin{aligned} f_{CS}^{(4)} &= -\frac{1}{16} \frac{(e^0)^2 [1 - (e^0)^2]}{[1 + (e^0)^2]^3} c_{2I} e^I - \frac{1}{48} \frac{[2 + 5(e^0)^2]}{[1 + (e^0)^2]^2} c_{2I} e^I, \\ &= -\frac{1}{24} \frac{[1 + 5(e^0)^2 + (e^0)^4]}{[1 + (e^0)^2]^3} c_{2I} e^I, \end{aligned} \quad (3.22)$$

where in the top line we showed the separate contribution of I_{CS} and ΔI_{CS} . Note that ΔI_{CS} is nonvanishing even in the nonrotating case $e^0 = 0$.

3.3 Computation of entropy

Our final result for the on-shell action f is found by adding the contributions determined in the previous subsection

$$f = f_{GI}^{(2)} + f_{CS}^{(2)} + f_{GI}^{(4)} + f_{CS}^{(4)} = \frac{4}{[1 + (e^0)^2]} \left(\frac{1}{6} c_{IJK} e^I e^J e^K - \frac{1}{24} c_{2I} e^I \right). \quad (3.23)$$

The entropy (3.4) is

$$S = 2\pi \left(e^0 \frac{\partial f}{\partial e^0} + e^I \frac{\partial f}{\partial e^I} - f \right) = \frac{16\pi}{[1 + (e^0)^2]^2} \left(\frac{1}{6} c_{IJK} e^I e^J e^K + \frac{1}{24} (e^0)^2 c_{2I} e^I \right). \quad (3.24)$$

We can rewrite this in terms of rescaled moduli using (3.5):

$$S = 2\pi \sqrt{1 - \hat{J}^2} \left(\frac{1}{6} c_{IJK} \hat{M}^I \hat{M}^J \hat{M}^K + \frac{1}{6} \hat{J}^2 c_{2I} \hat{M}^I \right). \quad (3.25)$$

This is our final result for the entropy of the spinning black hole, expressed in terms of the near-horizon moduli.

We can also express the entropy in terms of the conserved charges. We first use (2.23) to find an expression in terms of geometrical variables

$$S = 2\pi \sqrt{(2\ell)^6 - J^2} \left(1 + \frac{c_{2I} M^I}{48\ell^2} \right), \quad (3.26)$$

and then expand to first order in c_{2I} using (2.28) to find

$$S = 2\pi \sqrt{Q^3 - J^2} \left(1 + \frac{c_2 \cdot q}{16} \frac{Q^{3/2}}{(Q^3 - J^2)} + \dots \right). \quad (3.27)$$

This is our expression for the black hole entropy as a function of charges.

The microscopic understanding of these black holes is quite limited. However, our formulae do agree with the microscopic corrections to the entropy where such results are available [35, 36]. Note that these special cases do not involve rotation, and amount to reproducing the $\frac{c_{2I}}{8}$ term in (2.22).

3.4 Black ring entropy

The entropy computation we have presented for the spinning black hole is readily modified to the black ring. So although black rings are not the focus of the present work we make a detour to present the relevant entropy formula. Since we just use the entropy function computed from the near horizon geometry we will only be able to give a formula for the entropy in terms of the electric potentials. To express the entropy in terms of charges requires more details of the full black ring solution than are presently available.

For the black ring the near horizon solution is

$$\begin{aligned}
 ds^2 &= w^{-1}V \left[\left(\rho^2 d\tau^2 - \frac{d\rho^2}{\rho^2} \right) - d\Omega^2 \right] - w^2 \left(dx^5 + e^0 \rho d\tau \right)^2, \\
 A^I &= -\frac{1}{2}p^I \cos \theta d\phi - \frac{e^I}{e^0} dx^5.
 \end{aligned}
 \tag{3.28}$$

Further details of the solution follow from the fact that the near horizon geometry is a magnetic attractor, as studied in [13]. The near horizon geometry is a product of a BTZ black hole and an S^2 , and there is enhanced supersymmetry. These conditions⁷ imply

$$\begin{aligned}
 M^I &= \frac{p^I}{2we^0}, \\
 V &= w^3(e^0)^2, \\
 D &= \frac{3}{w^2(e^0)^2}, \\
 v &= -\frac{3}{4}we^0 \sin \theta d\theta \wedge d\phi,
 \end{aligned}
 \tag{3.29}$$

as can be read off from [13].

The computation of the f function now proceeds just as for the rotating black hole. The result is

$$f = f_{GI}^{(2)} + f_{CS}^{(2)} + f_{GI}^{(4)} + f_{CS}^{(4)} = -\frac{1}{2e^0} \left(\frac{1}{6} c_{IJK} p^I p^J p^K + \frac{1}{6} c_{2I} p^I \right) + 2 \frac{c_{IJK} e^I e^J p^K}{e^0}, \tag{3.30}$$

and the entropy is

$$S = 2\pi \left(e^0 \frac{\partial f}{\partial e^0} + e^I \frac{\partial f}{\partial e^I} - f \right) = \frac{2\pi}{e^0} \left(\frac{1}{6} c_{IJK} p^I p^J p^K + \frac{1}{6} c_{2I} p^I \right). \tag{3.31}$$

The entropy can also be expressed as

$$S = (2 - \mathcal{N}) \frac{A}{\pi} = (2 - \mathcal{N}) \frac{A}{4G_5}, \tag{3.32}$$

where A is the area of the event horizon. In the two-derivative limit we have $\mathcal{N} = 1$ and we recover the Bekenstein-Hawking entropy.

As mentioned above, the final step is to trade e^0 for the charges of the black ring, but for this one needs knowledge of more than just the near horizon geometry.

⁷which can also be verified by extremizing the full entropy function.

4. Comparison with topological strings, the 4D-5D connection, and all that

In this section we discuss various interpretational aspects and the relation to previous work.

4.1 Comparison with 4D black hole entropy from the topological string

The OSV conjecture relates the free energy of the topological string to the Legendre transform of the 4D black hole entropy [3]. It has further been proposed that the OSV conjecture lifts to five dimensions [24]. It is instructive to compare this 5D version of the OSV conjecture with our explicit computations. Our analysis has been at the level of the 1-loop correction to the free energy, and at this level the OSV conjecture for the entropy *by design* reproduces the known 1-loop correction for the 4D black hole. So from a logical standpoint, our comparison below really refers to the relation between 4D and 5D black hole entropy. We nevertheless find it useful to cast the discussion in the language of the OSV conjecture, although this is not strictly necessary.

The one-loop free energy from the topological string is

$$\mathcal{F} = \frac{i}{\pi\mu} \left(\frac{1}{6} c_{IJK} \phi^I \phi^J \phi^K - \frac{\pi^2}{6} c_{2I} \phi^I \right) + \text{c.c.} = -\frac{1}{\pi^2} \frac{\frac{1}{6} c_{IJK} \phi^I \phi^J \phi^K - \frac{\pi^2}{6} c_{2I} \phi^I}{\left(\frac{\text{Re}\mu}{2\pi}\right)^2 + 1}, \quad (4.1)$$

where $\mu = \text{Re}\mu - 2\pi i$. The relation to our notation is

$$\begin{aligned} \text{Re}\mu &= 2\pi e^0 = -\frac{2\pi\hat{J}}{\sqrt{1-\hat{J}^2}}, \\ \phi^I &= 2\pi e^I = \frac{\pi\hat{M}^I}{\sqrt{1-\hat{J}^2}}, \\ \mathcal{F} &= -2\pi f. \end{aligned} \quad (4.2)$$

With these identifications we see that the free energy from the topological string (4.1) agrees precisely with our f function (3.23). The 5D OSV conjecture gives the entropy

$$S = \mathcal{F} - \phi^I \frac{\partial \mathcal{F}}{\partial \phi^I} - \text{Re}\mu \frac{\partial \mathcal{F}}{\partial \text{Re}\mu} = \frac{2}{\pi^2 \left(\left(\frac{\text{Re}\mu}{2\pi}\right)^2 + 1\right)^2} \left(\frac{1}{6} c_{IJK} \phi^I \phi^J \phi^K + (\text{Re}\mu)^2 \frac{c_{2I}}{24} \phi^I \right). \quad (4.3)$$

This agrees precisely with our result (3.25) for the entropy. Of course this second agreement is not independent from the first, since we Legendre transform the same expression on the two sides.

So far we expressed the free energy and the entropy as functions of the potentials. However, we are usually more interested in these quantities written in terms of the conserved charges (J, q_I) . According to our explicit construction of the solution the charges are related to rescaled potentials through (2.22). Rewriting in terms of the electric fields (3.5)

and then using the dictionary (4.2) to the topological string we have

$$\begin{aligned}
 q_I &= \frac{\frac{1}{2}c_{IJK}\phi^J\phi^K - \frac{\pi^2}{6}c_{2I}}{\pi^2\left(1 + \left(\frac{\text{Re}\mu}{2\pi}\right)^2\right)} + \frac{1}{24}c_{2I}, \\
 J &= -\frac{\frac{1}{3!}c_{IJK}\phi^I\phi^J\phi^K - \frac{\pi^2}{6}c_{2I}\phi^I}{\pi^3\left(1 + \left(\frac{\text{Re}\mu}{2\pi}\right)^2\right)^2} \frac{\text{Re}\mu}{2\pi} - \frac{1}{12\pi}c_{2I}\phi^I \frac{1}{1 + \left(\frac{\text{Re}\mu}{2\pi}\right)^2} \frac{\text{Re}\mu}{2\pi}.
 \end{aligned}
 \tag{4.4}$$

The 5D OSV conjecture [24] instead defines the charges as

$$\begin{aligned}
 \bar{q}_I &= -\frac{\partial\mathcal{F}}{\partial\phi^I} = \frac{\frac{1}{2}c_{IJK}\phi^J\phi^K - \frac{\pi^2}{6}c_{2I}}{\pi^2\left(1 + \left(\frac{\text{Re}\mu}{2\pi}\right)^2\right)}, \\
 \bar{J} &= -\frac{\partial\mathcal{F}}{\partial\text{Re}\mu} = -\frac{\frac{1}{3!}c_{IJK}\phi^I\phi^J\phi^K - \frac{\pi^2}{6}c_{2I}\phi^I}{\pi^3\left(1 + \left(\frac{\text{Re}\mu}{2\pi}\right)^2\right)^2} \frac{\text{Re}\mu}{2\pi},
 \end{aligned}
 \tag{4.5}$$

and these do *not* agree with our expressions (4.4).

A consequence of this discrepancy is that our expression for the entropy disagrees with that conjectured in [24] when both are written in terms of conserved charges. In the notation used here the topological string gives⁸

$$S = 2\pi\sqrt{\bar{Q}^3 - \bar{J}^2} \left(1 + \frac{c_2 \cdot \bar{q}}{12\bar{Q}^{3/2}} + \dots\right).
 \tag{4.6}$$

This does not take the same form as our expression (3.27).

The discrepancy arises because the 4D-5D charge map used in [24] misidentifies the 5D charges. The charges we have been using, (J, q_I) , are the 5D conserved charges as measured by surface integrals at infinity. In contrast, the charges from the topological string, (\bar{J}, \bar{q}_I) , are defined via the 4D effective theory. The black hole with the prescribed near horizon geometry, and which asymptotes to 4D asymptotically flat spacetime (times a circle), has a Taub-NUT base space. As we show explicitly in the next two subsections, the Taub-NUT itself has a delocalized contribution to the 4D charges. This contribution is absent for the 5D black hole.

4.2 Spinning black hole on a Taub-NUT base space: the solution

In order to carry out the 4D-5D reduction explicitly we now construct the spinning black hole on a Taub-NUT base space. To do so we need to generalize some previous results to the case of a curved base space. Most of the analysis goes through essentially unchanged, so we can be brief.

The analysis of the Killing spinor equations is unchanged except that derivatives on the base space now become covariant. As a result (2.1)–(2.6) remain valid on the Taub-NUT base space. Supersymmetry also demands that $d\omega$ is a self-dual two-form on the

⁸Notation: $q_{\text{here}}^I = y_{\text{there}}^I$. Also note that [24] introduce moduli Y^I which satisfy the tree-level special geometry condition even when higher derivative corrections are taken into account so these moduli are shifted relative to M^I used here.

base space. Finally, supersymmetry requires a Killing spinor which is covariantly constant on the base-space. This in turn implies that the base-space is hyper-Kähler and so also Ricci-flat, with anti-self-dual Riemann tensor. Using this information it is straightforward to generalize Gauss' law (2.8),

$$\nabla^2 \left[e^{-2U} M_I - \frac{c_{2I}}{24} \left(3(\nabla U)^2 - \frac{1}{4} e^{6U} (d\omega)^2 \right) \right] = \frac{c_{2I}}{24 \cdot 8} R^{ijkl} R_{ijkl} . \quad (4.7)$$

Indices are contracted with the four-dimensional base space metric, and the Riemann tensor and derivatives are that of the base space. We see that the only change is the new contribution on the right hand side. This in turn comes from the $A \wedge \text{Tr} R^2$ term in the action, which represents a curvature induced charge density.

We first consider the case of a charge $p^0 = 1$ Taub-NUT space, and then generalize to the case of general charge. We write Taub-NUT in Gibbons-Hawking form

$$ds_4^2 = \frac{1}{H^0(\rho)} (dx^5 + \cos \theta d\phi)^2 + H^0(\rho) (d\rho^2 + \rho^2 (d\theta^2 + \sin^2 \theta d\phi^2)) , \quad (4.8)$$

with $x^5 \cong x^5 + 4\pi$ and orientation $\epsilon_{\rho\theta\phi x_5} = 1$. The harmonic function H^0 is

$$H^0(\rho) = 1 + \frac{1}{\rho} . \quad (4.9)$$

As in (2.15), the anti-self-duality and closure conditions determine $d\omega$ completely, *viz.*

$$d\omega = -\frac{J}{8\rho^2} (e^\rho e^5 + e^\theta e^\phi) , \quad (4.10)$$

where the e^i are the obvious vielbeins of Taub-NUT.

For Taub-NUT the source on the right hand side of (4.7) can be expressed as

$$R^{ijkl} R_{ijkl} = \nabla^2 \left(\frac{2}{\rho(\rho+1)^3} - \frac{2}{\rho} \right) . \quad (4.11)$$

Using this we can easily solve (4.7) as

$$H_I = M_I^\infty + \frac{q_I}{4\rho} = e^{-2U} M_I - \frac{c_{2I}}{24} \left(3(\nabla U)^2 - e^{6U} \frac{J^2}{64\rho^4} + \frac{1}{4} \frac{1}{\rho(\rho+1)^3} - \frac{1}{4\rho} \right) , \quad (4.12)$$

where we have substituted in (4.10) for $d\omega$. The radial function $U(\rho)$ is determined again by the D equation of motion which remains of the form (2.12).

4.3 Relation between 4D and 5D charges

The above construction incorporates both 4D and 5D black holes. Specifically, if we drop the 1 in the harmonic function H^0 then the base space is simply \mathbb{R}^4 and we recover the 5D black hole. Now, we have been using the symbol q_I , but we need to check its relation to the physical electric charge of the 4D and 5D black holes. From the gauge field dependent terms in the action the conserved electric charge Q_I is

$$Q_I = -\frac{1}{4\pi^2} \int_{\Sigma} \left(\frac{1}{2} \mathcal{N}_{IJ} \star_5 F^J + 2M_I \star_5 v \right) , \quad (4.13)$$

where Σ denotes the $S^2 \times S^1$ at infinity spanned by (θ, ϕ, x^5) . Note that only the two-derivative terms in the action contribute to (4.13) since the four-derivative contributions to the surface integral die off too quickly at infinity.

Using the explicit solution we find

$$Q_I = -4 \left[\rho^2 \partial_\rho (M_I e^{-2U}) \right]_{\rho=\infty} . \quad (4.14)$$

In the case of the 5D black hole we have $M_I e^{-2U} = H_I + \dots$, where \dots denote terms falling off faster than $\frac{1}{\rho}$, and hence we find

$$Q_I^{(5D)} = q_I . \quad (4.15)$$

For the 4D black hole we should instead use (4.12), and we see that the final term in parenthesis contributes an extra $\frac{1}{\rho}$ piece. Hence, for the 4D black hole we have $M_I e^{-2U} = H_I - \frac{c_{2I}}{4 \cdot 24 \rho} + \dots$, which gives

$$Q_I^{(4D)} = q_I - \frac{c_{2I}}{24} = \bar{q}_I . \quad (4.16)$$

A similar story holds for the relation between the 5D angular momentum J and the 4D charge q_0 . So we see that the 4D and 5D charges are different. This has important implications for the 4D-5D connection: it is *not* true that $S_{5D}(J, q_I) = S_{4D}(q_0 = J, q_I)$. Rather, one should first convert from barred to unbarred charges in the 4D entropy formula before writing the result for the 5D entropy. In general, if we write $(\bar{J}, \bar{q}_I) = (J + \Delta J, q_I + \Delta q_I)$, then we should instead use $S_{5D}(J, q_I) = S_{4D}(J + \Delta J, q_I + \Delta q_I)$.

The physical reason for this is simple: due to higher derivative effects the Taub-NUT space itself carries a delocalized charge. The 4D black hole sees the charge as measured at infinity, while the 5D black hole effectively sees the charge as measured near the tip of Taub-NUT (since the 5D black hole is obtained by dropping the 1 in H^0). To see how these two notions of charge are related, we define a ρ dependent ‘‘charge’’ via the left-hand side of (4.7),

$$Q_I(\Sigma_\rho) = -\frac{1}{4\pi^2} \int_{\Sigma_\rho} \sqrt{h} n^\mu \nabla_\mu \left\{ e^{-2U} M_I - \frac{c_{2I}}{24} \left(3(\nabla U)^2 - \frac{1}{4} e^{6U} (d\omega)^2 \right) \right\} \quad (4.17)$$

where Σ_ρ is surface of constant ρ with unit normal, n^μ , and h is the induced metric on Σ_ρ . Because of the curvature term in (4.7), this quantity is dependent on ρ . The difference between the charges at the center (5D) and at infinity (4D) is given by integrating the right-hand side of (4.7),

$$\begin{aligned} Q_I(\Sigma_\infty) - Q_I(\Sigma_0) &= -\frac{1}{4\pi^2} \int_{\Sigma_\infty - \Sigma_0} \sqrt{h} n^\mu \nabla_\mu \left\{ e^{-2U} M_I - \frac{c_{2I}}{24} \left(3(\nabla U)^2 - \frac{1}{4} e^{6U} (d\omega)^2 \right) \right\} , \\ &= -\frac{1}{4\pi^2} \int_{\mathcal{M}} \sqrt{g} \nabla^2 \left\{ e^{-2U} M_I - \frac{c_{2I}}{24} \left(3(\nabla U)^2 - \frac{1}{4} e^{6U} (d\omega)^2 \right) \right\} , \\ &= -\frac{1}{4\pi^2} \frac{c_{2I}}{24 \cdot 8} \int_{\mathcal{M}} R^{ijkl} R_{ijkl} . \end{aligned} \quad (4.18)$$

For a 4D Ricci-flat manifold, the Euler number is given by

$$\chi(\mathcal{M}) = \frac{1}{32\pi^2} \int_{\mathcal{M}} R_{abcd} R^{abcd}, \quad (4.19)$$

which for Taub-NUT gives $\chi = 1$. Thus

$$Q_I(\Sigma_\infty) - Q_I(\Sigma_0) = -\frac{c_{2I}}{24}, \quad (4.20)$$

which accounts for the relation between \bar{q}_I and q_I .

We emphasize again that charges are completely unambiguous in 5D. Also, in 5D the asymptotic charge $Q_I(\Sigma_\infty)$ agrees with the near horizon charge $Q_I(\Sigma_0)$ because the base is flat. The nontrivial relation is between the 4D and 5D charges in the presence of higher derivatives.

4.4 Generalization to charge p^0

We can easily generalize the above to Taub-NUT with arbitrary charge p^0 . This is defined by taking a Z_{p^0} orbifold of the charge 1 solution. We identify $x^5 \cong x^5 + \frac{4\pi}{p^0}$. To keep the asymptotic size of the Taub-NUT circle fixed we take $H^0 = \frac{1}{(p^0)^2} + \frac{1}{\rho}$, which is a choice of integration constant. Finally, to put the solution back in standard form we define $(\tilde{x}^5 = p^0 x^5, \tilde{\rho} = \frac{1}{p^0} \rho)$. The general charge p^0 solution then has (dropping the tildes)

$$\begin{aligned} ds_4^2 &= \frac{1}{H^0(\rho)} (dx^5 + p^0 \cos \theta d\phi)^2 + H^0(\rho) (d\rho^2 + \rho^2 (d\theta^2 + \sin^2 \theta d\phi^2)), \\ H^0(\rho) &= 1 + \frac{p^0}{\rho}, \\ H_I &= M_I^\infty + \frac{q_I}{4\rho}. \end{aligned} \quad (4.21)$$

Again, q_I is the 5D electric charge. The 4D electric charge is now

$$\bar{q}_I = q_I - \frac{c_{2I}}{24p^0}. \quad (4.22)$$

4.5 Example: $K3 \times T^2$

We conclude the paper by making our formulae completely explicit in the special case of $K3 \times T^2$. In this case $c_{1ij} = c_{ij}$, $i, j = 2, \dots, 23$ are the only nontrivial intersection numbers and $c_{2i} = 0$, $c_{2,1} = 24$ are the 2nd Chern-classes.

Our procedure instructs us to first find the hatted variables in terms of conserved charges by inverting (2.22). In the present case we find

$$\begin{aligned} \hat{M}^1 &= \sqrt{\frac{\frac{1}{2}c^{ij}q_iq_j + \frac{4J^2}{(q_1+1)^2}}{q_1+3}}, \\ \hat{M}^i &= \sqrt{\frac{q_1+3}{\frac{1}{2}c^{ij}q_iq_j + \frac{4J^2}{(q_1+1)^2}}} c^{ij} q_j, \\ \hat{J} &= \sqrt{\frac{q_1+3}{\frac{1}{2}c^{ij}q_iq_j + \frac{4J^2}{(q_1+1)^2}}} \frac{J}{q_1+1}. \end{aligned} \quad (4.23)$$

All quantities of interest are given in terms of these variables. For example, the relation between 4D charges (4.5) and 5D charges (4.2) is

$$\begin{aligned}\bar{\mathcal{J}} &= \frac{q_1 + 3}{q_1 + 1} J, \\ \bar{q}_1 &= q_1 - 1, \\ \bar{q}_i &= q_i,\end{aligned}\tag{4.24}$$

and the entropy as function of the conserved charges becomes

$$\begin{aligned}S &= 2\pi \sqrt{\frac{1}{2} c^{ij} q_i q_j (q_1 + 3) - \frac{(q_1 - 1)(q_1 + 3)}{(q_1 + 1)^2} J^2} \\ &= 2\pi \sqrt{\frac{1}{2} c^{ij} \bar{q}_i \bar{q}_j (\bar{q}_1 + 4) - \frac{\bar{q}_1}{\bar{q}_1 + 4} \bar{\mathcal{J}}^2}.\end{aligned}\tag{4.25}$$

In the special case of $K3 \times T^2$ the charge corresponding to D2-branes wrapping T^2 is special, and it is apparently that charge which undergoes corrections due to higher order derivatives. The precise form of the corrections is reminiscent of the shifts in level that are characteristic of σ -models.⁹

Our formulae for general Calabi-Yau black holes are democratic between the various charges.

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A. Derivation of the spinning black hole

In this appendix we show how to obtain rotating black hole solutions by imposing the Killing spinor equations and Maxwell equations, including higher derivatives corrections. Our conventions follow [14].

We consider M-theory compactified on a Calabi-Yau threefold with intersection numbers, c_{IJK} , and second Chern class coefficients, c_{2I} . The bosonic part of the action up to four-derivative terms is given by

$$S = \frac{1}{4\pi^2} \int d^5x \sqrt{g} (\mathcal{L}_0 + \mathcal{L}_1),\tag{A.1}$$

where the two-derivative Lagrangian is

$$\begin{aligned}\mathcal{L}_0 &= -\frac{1}{2} D - \frac{3}{4} R + v^2 + \mathcal{N} \left(\frac{1}{2} D - \frac{1}{4} R + 3v^2 \right) + 2\mathcal{N}_I v^{ab} F_{ab}^I \\ &\quad + \mathcal{N}_{IJ} \left(\frac{1}{4} F_{ab}^I F^{Jab} + \frac{1}{2} \partial_a M^I \partial^a M^J \right) + \frac{1}{24} c_{IJK} A_a^I F_{bc}^J F_{de}^K \epsilon^{abcde},\end{aligned}\tag{A.2}$$

⁹For example, the σ -model of heterotic string theory on the near horizon geometry $AdS_3 \times S^3/Z_N$ with \bar{q}_1 units of B-flux has spacetime central charge $c = 6(\bar{q}_1 + 4)$ [37] in apparent agreement with (4.25).

and the four-derivative Lagrangian is

$$\begin{aligned}
 \mathcal{L}_1 = & \frac{c_{2I}}{24} \left(\frac{1}{16} \epsilon_{abcde} A^{Ia} C^{bcfg} C^{de}_{fg} + \frac{1}{8} M^I C^{abcd} C_{abcd} + \frac{1}{12} M^I D^2 + \frac{1}{6} F^{Iab} v_{ab} D \right. \\
 & + \frac{1}{3} M^I C_{abcd} v^{ab} v^{cd} + \frac{1}{2} F^{Iab} C_{abcd} v^{cd} + \frac{8}{3} M^I v_{ab} \hat{D}^b \hat{D}_c v^{ac} \\
 & + \frac{4}{3} M^I \hat{D}^a v^{bc} \hat{D}_a v_{bc} + \frac{4}{3} M^I \hat{D}^a v^{bc} \hat{D}_b v_{ca} - \frac{2}{3} M^I \epsilon_{abcde} v^{ab} v^{cd} \hat{D}_f v^{ef} \\
 & + \frac{2}{3} F^{Iab} \epsilon_{abcde} v^{cf} \hat{D}_f v^{de} + F^{Iab} \epsilon_{abcde} v^c{}_f \hat{D}^d v^{ef} \\
 & \left. - \frac{4}{3} F^{Iab} v_{ac} v^{cd} v_{db} - \frac{1}{3} F^{Iab} v_{ab} v^2 + 4M^I v_{ab} v^{bc} v_{cd} v^{da} - M^I (v^2)^2 \right). \quad (\text{A.3})
 \end{aligned}$$

The double superconformal derivative of the auxiliary field has curvature contributions

$$v_{ab} \hat{D}^b \hat{D}_c v^{ac} = v_{ab} \mathcal{D}^b \mathcal{D}_c v^{ac} - \frac{2}{3} v^{ac} v_{cb} R_a{}^b - \frac{1}{12} v_{ab} v^{ab} R. \quad (\text{A.4})$$

The functions defining the scalar manifold are

$$\mathcal{N} = \frac{1}{6} c_{IJK} M^I M^J M^K, \quad \mathcal{N}_I = \partial_I \mathcal{N} = \frac{1}{2} c_{IJK} M^J M^K, \quad \mathcal{N}_{IJ} = c_{IJK} M^K, \quad (\text{A.5})$$

where $I, J, K = 1, \dots, n_V$.

We study supersymmetric configurations so we seek solutions in which both the fermion fields and their first variations under supersymmetry vanish. The supersymmetry variations of the fermions are

$$\begin{aligned}
 \delta\psi_\mu &= \left(\mathcal{D}_\mu + \frac{1}{2} v^{ab} \gamma_{\mu ab} - \frac{1}{3} \gamma_\mu \gamma \cdot v \right) \epsilon = 0, \\
 \delta\Omega^I &= \left(-\frac{1}{4} \gamma \cdot F^I - \frac{1}{2} \gamma^a \partial_a M^I - \frac{1}{3} M^I \gamma \cdot v \right) \epsilon = 0, \\
 \delta\chi &= \left(D - 2\gamma^c \gamma^{ab} \mathcal{D}_a v_{bc} - 2\gamma^a \epsilon_{abcde} v^{bc} v^{de} + \frac{4}{3} (\gamma \cdot v)^2 \right) \epsilon = 0. \quad (\text{A.6})
 \end{aligned}$$

We now examine the consequences of setting these variations to zero.

A.1 The stationary background

We begin by writing our metric ansatz

$$ds^2 = e^{4U_1(x)} (dt + \omega)^2 - e^{-2U_2(x)} dx^i dx^i, \quad (\text{A.7})$$

where $\omega = \omega_i(x) dx^i$ and $i = 1 \dots 4$. The vielbeins are

$$e^{\hat{0}} = e^{2U_1} (dt + \omega), \quad e^{\hat{i}} = e^{-U_2} dx^i, \quad (\text{A.8})$$

which give the following spin connections

$$\begin{aligned}
 \omega^{\hat{i}}{}_{\hat{j}} &= e^{-U_2} (\partial_j U_2 e^{\hat{i}} - \partial_i U_2 e^{\hat{j}}) + \frac{1}{2} e^{2U_1 + 2U_2} d\omega_{ij} e^{\hat{0}}, \\
 \omega^{\hat{0}}{}_{\hat{i}} &= 2e^{U_2} \partial_i U_1 e^{\hat{0}} + \frac{1}{2} e^{2U_1 + 2U_2} d\omega_{ij} e^{\hat{j}}, \quad (\text{A.9})
 \end{aligned}$$

with

$$d\omega = \partial_{[i}\omega_{j]}dx^i \wedge dx^j . \quad (\text{A.10})$$

The Hodge dual on the base space is defined as

$$\star_4 \alpha_{\hat{i}\hat{j}} = \frac{1}{2} \epsilon_{\hat{i}\hat{j}\hat{k}\hat{l}} \alpha^{\hat{k}\hat{l}} , \quad (\text{A.11})$$

with $\epsilon_{\hat{1}\hat{2}\hat{3}\hat{4}} = 1$. A 2-form on the base space can be decomposed into self-dual and anti-self-dual forms,

$$\alpha = \alpha^+ + \alpha^- , \quad (\text{A.12})$$

where $\star_4 \alpha^\pm = \pm \alpha^\pm$. We will use this decomposition for the spatial components of $d\omega$ and the auxiliary 2-form v_{ab}

$$\begin{aligned} v_{\hat{i}\hat{j}} &= v_{\hat{i}\hat{j}}^+ + v_{\hat{i}\hat{j}}^- , \\ d\omega_{\hat{i}\hat{j}} &= d\omega_{\hat{i}\hat{j}}^+ + d\omega_{\hat{i}\hat{j}}^- . \end{aligned} \quad (\text{A.13})$$

For stationary solutions, the Killing spinor ϵ satisfies the projection

$$\gamma^{\hat{0}} \epsilon = -\epsilon . \quad (\text{A.14})$$

Using $\gamma_{abcde} = \epsilon_{abcde}$ and (A.14), it is easy to show that anti-self-dual tensors in the base space satisfy

$$\alpha^{-\hat{i}\hat{j}} \gamma_{\hat{i}\hat{j}} \epsilon = 0 . \quad (\text{A.15})$$

A.2 Supersymmetry variations

There are three supersymmetry constraints we need to solve. Following the same procedure as in [14], we first impose a vanishing gravitino variation,

$$\delta\psi_\mu = \left[\mathcal{D}_\mu + \frac{1}{2} v^{ab} \gamma_{\mu ab} - \frac{1}{3} \gamma_\mu \gamma \cdot v \right] \epsilon = 0 . \quad (\text{A.16})$$

Evaluated in our background, the time component of equation (A.16) reads

$$\left[\partial_t - e^{2U_1+U_2} \partial_i U_1 \gamma_{\hat{i}} - \frac{2}{3} e^{2U_1} v^{\hat{0}\hat{i}} \gamma_{\hat{i}} - \frac{1}{4} e^{4U_1} d\omega_{\hat{i}\hat{j}} \gamma^{\hat{i}\hat{j}} - \frac{1}{6} e^{2U_1} v_{\hat{i}\hat{j}} \gamma^{\hat{i}\hat{j}} \right] \epsilon = 0 , \quad (\text{A.17})$$

where we used the projection (A.14). The terms proportional to $\gamma_{\hat{i}}$ and $\gamma_{\hat{i}\hat{j}}$ give the conditions

$$\begin{aligned} v_{\hat{0}\hat{i}} &= \frac{3}{2} e^{U_2} \partial_i U , \\ v^+ &= -\frac{3}{4} e^{2U_1} d\omega^+ . \end{aligned} \quad (\text{A.18})$$

The spatial component of the gravitino variation (A.16) simplifies to

$$\left[\partial_i + \frac{1}{2} \partial_j U_2 \gamma_{\hat{i}\hat{j}} + v^{\hat{0}\hat{k}} e_{\hat{i}}^{\hat{j}} \left(\gamma_{\hat{j}\hat{k}} - \frac{2}{3} \gamma_{\hat{j}} \gamma_{\hat{k}} \right) - e^{\hat{k}}_{\hat{i}} \left(v_{\hat{k}\hat{j}}^- + \frac{1}{4} e^{2U_1} d\omega_{\hat{k}\hat{j}}^- \right) \gamma^{\hat{j}} \right] \epsilon = 0 , \quad (\text{A.19})$$

where we used the results from (A.18). The last term in (A.19) relates the anti-self-dual pieces of v and $d\omega$,

$$v^- = -\frac{1}{4}e^{2U_1}d\omega^- . \tag{A.20}$$

The remaining components of (A.19) impose equality of the two metric functions $U_1 = U_2 \equiv U$ and determine the Killing spinor as

$$\epsilon = e^{U(x)}\epsilon_0 , \tag{A.21}$$

with ϵ_0 a constant spinor.

The gaugino variation is given by

$$\delta\Omega^I = \left[-\frac{1}{4}\gamma \cdot F^I - \frac{1}{2}\gamma^a\partial_a M^I - \frac{1}{3}M^I\gamma \cdot v \right] \epsilon = 0 . \tag{A.22}$$

This constraint will determine the electric and self-dual pieces of F_{ab}^I . Using (A.14) and (A.15) to solve (A.22) we find

$$\begin{aligned} F^{I\hat{0}\hat{i}} &= e^{-U}\partial_i(e^{2U}M^I) , \\ F^{I+} &= -\frac{4}{3}M^I v^+ . \end{aligned} \tag{A.23}$$

Defining the anti-self-dual form

$$\Theta^I = -e^{2U}M^I d\omega^- + F^{I-} , \tag{A.24}$$

then the field strength can be written as

$$F^I = d(M^I e^{\hat{0}}) + \Theta^I . \tag{A.25}$$

We emphasize that Θ^I , or more precisely F^{I-} , is undetermined by supersymmetry. These anti-self-dual components are important for black ring geometries but for rotating black holes we can take $\Theta^I = 0$ and $d\omega^- = 0$.

Finally, the variation of the auxiliary fermion is

$$\delta\chi = \left[D - 2\gamma^c\gamma^{ab}\mathcal{D}_a v_{bc} - 2\gamma^a\epsilon_{abcde}v^{bc}v^{de} + \frac{4}{3}(\gamma \cdot v)^2 \right] \epsilon = 0 . \tag{A.26}$$

For the background given in section A.1 and using equations (A.18) and (A.20), the terms proportional to one or two gamma matrices cancel identically. The terms independent of $\gamma_{\hat{i}}$ give an equation for D , which reads

$$D = 3e^{2U}(\nabla^2 U - 6(\nabla U)^2) + \frac{1}{2}e^{4U}(3d\omega_{\hat{i}\hat{j}}^+ d\omega^{+\hat{i}\hat{j}} + d\omega_{\hat{i}\hat{j}}^- d\omega^{-\hat{i}\hat{j}}) . \tag{A.27}$$

A.3 Maxwell equation

The part of the action containing the gauge fields is

$$S^{(A)} = \frac{1}{4\pi^2} \int d^5x \sqrt{g} \left(\mathcal{L}_0^{(A)} + \mathcal{L}_1^{(A)} \right), \quad (\text{A.28})$$

where the two-derivative terms are

$$\mathcal{L}_0^{(A)} = 2\mathcal{N}_I v^{ab} F_{ab}^I + \frac{1}{4} \mathcal{N}_{IJ} F_{ab}^I F^{Jab} + \frac{1}{24} c_{IJK} A_a^I F_{bc}^J F_{de}^K \epsilon^{abcde}, \quad (\text{A.29})$$

and the four-derivative contributions are

$$\begin{aligned} \mathcal{L}_1^{(A)} = \frac{c_{2I}}{24} & \left(\frac{1}{16} \epsilon^{abcde} A_a^I C_{bc}^{fg} C_{defg} + \frac{2}{3} \epsilon_{abcde} F^{Iab} v^{cf} \mathcal{D}_f v^{de} + \epsilon_{abcde} F^{Iab} v^c{}_f \mathcal{D}^d v^{ef} \right. \\ & \left. + \frac{1}{6} F^{Iab} v_{ab} D + \frac{1}{2} F^{Iab} C_{abcd} v^{cd} - \frac{4}{3} F^{Iab} v_{ac} v^{cd} v_{db} - \frac{1}{3} F^{Iab} v_{ab} v^2 \right). \end{aligned} \quad (\text{A.30})$$

Variation of (A.28) with respect to A_μ^I gives,

$$\nabla_\mu \left(4\mathcal{N}_I v^{\mu\nu} + \mathcal{N}_{IJ} F^{J\mu\nu} + 2 \frac{\delta \mathcal{L}_1}{\delta F_{\mu\nu}^I} \right) = \frac{1}{8} c_{IJK} F_{\alpha\beta}^J F_{\sigma\rho}^K \epsilon^{\nu\alpha\beta\sigma\rho} + \frac{c_{2I}}{24 \cdot 16} \epsilon^{\nu\alpha\beta\sigma\rho} C_{\alpha\beta\mu\gamma} C_{\sigma\rho}{}^{\mu\gamma}, \quad (\text{A.31})$$

with

$$\begin{aligned} \frac{\delta \mathcal{L}_1}{\delta F_{ab}^I} = \frac{c_{2I}}{24} & \left(\frac{1}{3} v_{ab} D - \frac{8}{3} v_{ac} v^{cd} v_{db} - \frac{2}{3} v_{ab} v^2 + C_{abcd} v^{cd} \right. \\ & \left. + \frac{4}{3} \epsilon_{abcde} v^{cf} \mathcal{D}_f v^{de} + 2 \epsilon_{abcde} v^c{}_f \mathcal{D}^d v^{ef} \right), \end{aligned} \quad (\text{A.32})$$

and

$$\frac{\delta \mathcal{L}_1}{\delta F_{\mu\nu}^I} = e_a{}^\mu e_b{}^\nu \frac{\delta \mathcal{L}_1}{\delta F_{ab}^I}. \quad (\text{A.33})$$

The equations of motion are evidently rather involved, so we will now restrict attention to rotating black hole solutions with

$$d\omega = d\omega^+, \quad d\omega^- = 0, \quad \Theta^I = 0. \quad (\text{A.34})$$

Given the form of the solution imposed by supersymmetry it can be shown that the spatial components of the Maxwell equation are satisfied automatically. The time-component of (A.31) give a non-trivial relation between the geometry of the rotating black hole and the conserved charges. We start by writing this equation as

$$\begin{aligned} \nabla_i (e^{-3U} [4\mathcal{N}_I v^{\hat{i}\hat{0}} + \mathcal{N}_{IJ} F^{J\hat{i}\hat{0}}]) + \nabla_i \left(2e^{-3U} \frac{\delta \mathcal{L}_1}{\delta F_{\hat{i}\hat{0}}^I} \right) - 2e^{-2U} d\omega_{\hat{i}\hat{j}} \frac{\delta \mathcal{L}_1}{\delta F_{\hat{i}\hat{j}}^I} \\ = e^{-4U} \frac{1}{8} c_{IJK} F_{ab}^J F_{cd}^K \epsilon^{\hat{0}abcd} + e^{-4U} \frac{c_{2I}}{24 \cdot 16} \epsilon_{\hat{0}abcd} C^{abfg} C_{fg}{}^{cd}. \end{aligned} \quad (\text{A.35})$$

The two-derivative contribution to (A.35) is

$$\nabla_i (e^{-3U} [4\mathcal{N}_I v^{\hat{i}\hat{0}} + \mathcal{N}_{IJ} F^{J\hat{i}\hat{0}}]) - e^{-4U} \frac{1}{8} c_{IJK} F_{ab}^J F_{cd}^K \epsilon^{\hat{0}abcd} = -\nabla^2 (e^{-2U} M_I), \quad (\text{A.36})$$

where we used the results from section A.2 and (A.34). The higher derivatives terms in (A.35) on this background are

$$2 \frac{\delta \mathcal{L}_1}{\delta F^{I\hat{0}i}} = \frac{c_{2I}}{24} e^{3U} \left(3 \nabla_i (\nabla U)^2 - \frac{9}{32} \nabla_i [e^{6U} (d\omega)^2] - \frac{3}{8} e^{6U} \nabla_i U (d\omega)^2 \right) \quad (\text{A.37})$$

$$-e^{-2U} d\omega_{ij} \frac{\delta \mathcal{L}_1}{\delta F_{ij}^I} = \frac{c_{2I}}{24} \frac{3}{16} e^{6U} \left(\nabla_k U \nabla_k (d\omega)^2 + \frac{1}{4} e^{6U} ((d\omega)^2)^2 + 3(d\omega)^2 \nabla^2 U \right) \quad (\text{A.38})$$

$$e^{-4U} \epsilon_{\hat{0}abcd} C^{abfg} C^{cd}_{fg} = -\frac{1}{2} \nabla^2 [e^{6U} (d\omega)^2] + \frac{3}{4} e^{12U} ((d\omega)^2)^2 + 3e^{6U} (\nabla^2 U - 12(\nabla U)^2) (d\omega)^2 - 3e^{6U} \nabla_k U \nabla_k (d\omega)^2 \quad (\text{A.39})$$

where again we used the form of the solution imposed by supersymmetry and also the self-duality condition of $d\omega$. Inserting (A.36)–(A.39) in (A.35) gives

$$\nabla^2 \left[e^{-2U} M_I - \frac{c_{2I}}{8} \left((\nabla U)^2 - \frac{1}{12} e^{6U} (d\omega)^2 \right) \right] = 0. \quad (\text{A.40})$$

This is the generalized Gauss' law given in (2.8).

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