

L¹-CONVERGENCE OF COSINE SERIES WITH HYPER SEMI-CONVEX COEFFICIENTS

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Abstract. In this paper we obtain a necessary and sufficient condition for L^1 -convergence of the Fourier cosine series with hyper semi-convex coefficients. Results of Bala R. and Ram B. [1] have been obtained as a special case.

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1. Introduction. Consider

$$(1.1) \quad g(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

to be the cosine series with partial sums defined by

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^n a_k \cos kx$$

and let $g(x) = \lim_{n \rightarrow \infty} S_n(x)$

Concerning the L^1 -convergence of cosine series (1.1) Kolmogorov [3] proved his well known theorem:

Theorem A. If $\{a_n\}$ is a quasi-convex null sequence, then for the L^1 -convergence of the cosine series (1.1) it is necessary and sufficient that $\lim_{n \rightarrow \infty} a_n \log n = 0$.

Definition. A sequence $\{a_n\}$ is said to be semi-convex if $\{a_n\} \rightarrow 0$ as $n \rightarrow \infty$, and

$$\sum_{n=1}^{\infty} n |\Delta^2 a_{n-1} + \Delta^2 a_n| < \infty, \quad (a_0 = 0)$$

where

$$\Delta^2 a_n = \Delta a_n - \Delta a_{n+1}$$

It may be remarked here that every quasi-convex null sequence is semi-convex.

Bala R. and Ram B. [1] have proved that Theorem A holds true for cosine series with semi-convex null coefficients in the following form:

Theorem B. If $\{a_k\}$ is a semi-convex null sequence, then for the convergence of the cosine series in the metric space L , it is necessary and sufficient that $a_{k-1} \log k = o(1)$.

We define $\{a_n\}$ to be hyper semi-convex of order α , in the following way:

Definition. A sequence $\{a_n\}$ is said to be hyper semi-convex, if

$$\begin{aligned} \{a_n\} &\rightarrow 0 \text{ as } n \rightarrow \infty, \\ \sum_{n=1}^{\infty} n^{\alpha+1} |(\Delta^{\alpha+2} a_{n-1} + \Delta^{\alpha+2} a_n)| &< \infty, \quad \text{for } \alpha = 0, 1, 2, \dots, \\ &(a_0 = 0). \end{aligned}$$

By definition, hyper semi-convexity of order zero is same as semi-convexity.

The purpose of this paper is to generalize the Theorem B for the cosine series with hyper semi-convex null coefficients.

2. Notation and Formulae. In what follows, we use the following notation [4]:

Given a sequence S_0, S_1, S_2, \dots , we define for every $\alpha = 0, 1, 2, \dots$,

the sequence $S_0^\alpha, S_1^\alpha, S_2^\alpha, \dots$, by the conditions

$$\begin{aligned} S_n^0 &= S_n, \\ S_n^\alpha &= S_0^{\alpha-1} + S_1^{\alpha-1} + S_2^{\alpha-1} + \dots + S_n^{\alpha-1} \quad (\alpha = 1, 2, \dots, n = 0, 1, 2, \dots). \end{aligned}$$

Similarly for $\alpha = 0, 1, 2, \dots$, we define the sequence of numbers

$$\begin{aligned} A_0^\alpha, A_1^\alpha, A_2^\alpha, \dots &\text{ by the conditions} \\ A_n^0 &= 1, \\ A_n^\alpha &= A_0^{\alpha-1} + A_1^{\alpha-1} + A_2^{\alpha-1} + \dots + A_n^{\alpha-1} \quad (\alpha = 1, 2, \dots, n = 0, 1, 2, \dots). \end{aligned}$$

where A_p^α denotes the binomial coefficients and are given by the following relations.

$$\sum_{p=0}^{\infty} A_p^\alpha x^p = (1-x)^{-\alpha-1}$$

and \tilde{S}_n^α are given by

$$\sum_{p=0}^{\infty} S_p^\alpha x^p = (1-x)^{-\alpha} \sum_{p=0}^{\infty} S_p x^p$$

Also

$$\begin{aligned} A_n^\alpha &= \sum_{p=0}^n A_p^{\alpha-1}, \quad A_n^\alpha - A_{n-1}^\alpha = A_n^{\alpha-1} \\ A_n^\alpha &= \binom{n+\alpha}{n} \simeq \frac{n^\alpha}{\Gamma\alpha+1} \quad (\alpha \neq -1, -2, -3, \dots) \end{aligned}$$

Also for $0 < x \leq \pi$, let

$$\begin{aligned} \tilde{S}_n^0 &= \tilde{S}_n = \sin x + \sin 2x + \dots + \sin nx \\ \tilde{S}_n^1 &= \tilde{S}_1 + \tilde{S}_2 + \dots + \tilde{S}_n \\ \tilde{S}_n^2 &= \tilde{S}_1^1 + \tilde{S}_2^1 + \dots + \tilde{S}_n^1 \\ &\vdots \\ &\vdots \end{aligned}$$

$$\tilde{S}_n^k = \tilde{S}_1^{k-1} + \tilde{S}_2^{k-1} \dots \dots \dots + \tilde{S}_n^{k-1}$$

The conjugate Cesàro means \tilde{T}_n^α of order α of $\sum a_n$ will be defined by

$$\tilde{T}_n^\alpha = \frac{\tilde{S}_n^\alpha}{A_n^\alpha}.$$

2. Lemma. The following Lemma will be used for the proof of our result.

Lemma [2]. If $\alpha \geq 0, p \geq 0,$

- (i) $\epsilon_n = o(n^{-p}),$
- (ii) $\sum_{n=0}^\infty A_n^{\alpha+p} |\Delta^{\alpha+1} \epsilon_n| < \infty,$ then
- (iii) $\sum_{n=0}^\infty A_n^{\lambda+p} |\Delta^{\lambda+1} \epsilon_n| < \infty,$ for $-1 \leq \lambda \leq \alpha$ and
- (iv) $A_n^{\lambda+p} \Delta^\lambda \epsilon_n$ is of bounded variation for $0 \leq \lambda \leq \alpha$ and tends to zero as $n \rightarrow \infty.$

3. Main Result. We prove the following theorem:

Theorem 3.1. Suppose $\{a_n\}$ is a hyper semi-convex null sequence. Then the cosine series (1.1) converges in the metric space L if and only if $|a_{n-1}| \log n \rightarrow 0$ as $n \rightarrow \infty.$

If we take $\alpha = 0,$ then this theorem reduces to the Theorem B of Bala R. and Ram B. [2].

Proof. We have

$$\begin{aligned} S_n(x) &= \sum_{k=1}^n a_k \cos kx \\ &= \frac{1}{2 \sin x} \sum_{k=1}^n a_k \cos kx 2 \sin x \\ &= \frac{1}{2 \sin x} \sum_{k=1}^{n-1} (a_{k-1} - a_{k+1}) \sin kx + a_{n-1} \frac{\sin nx}{2 \sin x} \\ &\quad + a_n \frac{\sin(n+1)x}{2 \sin x} \\ &= \frac{1}{2 \sin x} \sum_{k=1}^{n-1} (\Delta a_k + \Delta a_{k-1}) \sin kx + a_{n-1} \frac{\sin nx}{2 \sin x} + a_n \frac{\sin(n+1)x}{2 \sin x} \end{aligned}$$

Applying Abel's transformation, we have

$$\begin{aligned} g_n(x) &= \frac{1}{2 \sin x} \sum_{k=1}^{n-2} (\Delta^2 a_k + \Delta^2 a_{k-1}) \sum_{v=1}^k \sin vx + (\Delta a_{n-1} + \Delta a_{n-2}) \sum_{v=1}^{n-1} \sin vx \\ &\quad + a_{n-1} \frac{\sin nx}{2 \sin x} + a_n \frac{\sin(n+1)x}{2 \sin x} \end{aligned}$$

$$= \frac{1}{2 \sin x} \left[\sum_{k=1}^{n-2} (\Delta^2 a_{k-1} + \Delta^2 a_k) \tilde{S}_k^0(x) + (\Delta a_{n-1} + \Delta a_{n-2}) \tilde{S}_{n-1}^0(x) \right. \\ \left. + a_{n-1} \frac{\sin nx}{2 \sin x} + a_n \frac{\sin(n+1)x}{2 \sin x} \right]$$

If we use Abel's transformation $\alpha + 1$ times, we have

$$S_n(x) = \frac{1}{2 \sin x} \left[\sum_{k=1}^{n-(\alpha+2)} (\Delta^{\alpha+2} a_{k-1} + \Delta^{\alpha+2} a_k) \tilde{S}_k^\alpha(x) + \sum_{k=0}^{\alpha} (\Delta^{k+1} a_{n-k-1}) \tilde{S}_{n-k-1}^k(x) \right] \\ + \frac{1}{2 \sin x} \left[\sum_{k=0}^{\alpha} (\Delta^{k+1} a_{n-k-2}) \tilde{S}_{n-k-1}^k(x) \right] \\ + a_{n-1} \frac{\sin nx}{2 \sin x} + a_n \frac{\sin(n+1)x}{2 \sin x} \\ = \frac{1}{2 \sin x} \left[\sum_{k=1}^{n-(\alpha+2)} (\Delta^{\alpha+2} a_{k-1} + \Delta^{\alpha+2} a_k) \tilde{S}_k^\alpha(x) + \sum_{k=0}^{\alpha} (\Delta^{k+1} a_{n-k-1}) A_{n-k-1}^k \tilde{T}_{n-k-1}^k(x) \right] \\ + \frac{1}{2 \sin x} \left[\sum_{k=0}^{\alpha} (\Delta^{k+1} a_{n-k-2}) A_{n-k-1}^k \tilde{T}_{n-k-1}^k(x) \right] + a_{n-1} \frac{\sin nx}{2 \sin x} + a_n \frac{\sin(n+1)x}{2 \sin x}$$

Since \tilde{S}_n and \tilde{T}_n are uniformly bounded on every segment $[\varepsilon, \pi - \varepsilon]$, $\varepsilon > 0$.

$$f(x) = \lim_{n \rightarrow \infty} S_n(x) \\ = \frac{1}{2 \sin x} \left[\sum_{k=1}^{\infty} (\Delta^{\alpha+2} a_{k-1} + \Delta^{\alpha+2} a_k) \tilde{S}_k^\alpha(x) \right]$$

Thus

$$f(x) - S_n(x) = \frac{1}{2 \sin x} \left[\sum_{k=n-(\alpha+1)}^{\infty} (\Delta^{\alpha+2} a_{k-1} + \Delta^{\alpha+2} a_k) \tilde{S}_k^\alpha(x) \right] \\ - \frac{1}{2 \sin x} \left[\sum_{k=0}^{\alpha} (\Delta^{k+1} a_{n-k-1}) A_{n-k-1}^k \tilde{T}_{n-k-1}^k(x) \right] \\ - \frac{1}{2 \sin x} \left[\sum_{k=0}^{\alpha} (\Delta^{k+1} a_{n-k-2}) A_{n-k-1}^k \tilde{T}_{n-k-1}^k(x) \right] \\ - \left(a_{n-1} \frac{\sin nx}{2 \sin x} + a_n \frac{\sin(n+1)x}{2 \sin x} \right)$$

$$\|f(x) - S_n(x)\| \leq \int_0^\pi \left| \frac{1}{2 \sin x} \left[\sum_{k=n-(\alpha+1)}^{\infty} (\Delta^{\alpha+2} a_{k-1} + \Delta^{\alpha+2} a_k) \tilde{S}_k^\alpha(x) \right] \right| dx \\ + \int_0^\pi \left| \frac{1}{2 \sin x} \sum_{k=0}^{\alpha} (\Delta^{k+1} a_{n-k-1}) A_{n-k-1}^k \tilde{T}_{n-k-1}^k(x) \right| dx \\ + \int_0^\pi \left| \frac{1}{2 \sin x} \left[\sum_{k=0}^{\alpha} (\Delta^{k+1} a_{n-k-2}) A_{n-k-1}^k \tilde{T}_{n-k-1}^k(x) \right] \right| dx \\ + \int_0^\pi \left| \left(a_{n-1} \frac{\sin nx}{2 \sin x} + a_n \frac{\sin(n+1)x}{2 \sin x} \right) \right| dx$$

$$\begin{aligned}
\|f(x) - S_n(x)\| &\leq C \left[\sum_{k=n-(\alpha+1)}^{\infty} (\Delta^{\alpha+2} a_{k-1} + \Delta^{\alpha+2} a_k) \int_0^{\pi} |\tilde{S}_k^{\alpha}(x)| dx \right] \\
&+ C \left[\sum_{k=0}^{\alpha} A_{n-k-1}^k |\Delta^{k+1} a_{n-k-1}| \int_0^{\pi} |\tilde{T}_{n-k-1}^k(x)| dx \right] \\
&+ C \left[\sum_{k=0}^{\alpha} A_{n-k-1}^k |\Delta^{k+1} a_{n-k-2}| \int_0^{\pi} |\tilde{T}_{n-k-1}^k(x)| dx \right] \\
&+ \int_0^{\pi} \left| \left(a_{n-1} \frac{\sin nx}{2 \sin x} + a_n \frac{\sin(n+1)x}{2 \sin x} \right) \right| dx \\
&\leq C \left[\sum_{k=n-(\alpha+1)}^{\infty} A_k^{\alpha} |\Delta^{\alpha+2} a_{k-1} + \Delta^{\alpha+2} a_k| \int_0^{\pi} |\tilde{T}_k^{\alpha}(x)| dx \right] \\
&+ C \left[\sum_{k=0}^{\alpha} A_{n-k}^k |\Delta^{k+1} a_{n-k-1}| \int_0^{\pi} |\tilde{T}_{n-k-1}^k(x)| dx \right] \\
&+ C \left[\sum_{k=0}^{\alpha} A_{n-k}^{\alpha} |\Delta^k a_{n-k-2}| \int_0^{\pi} |\tilde{T}_{n-k-1}^k(x)| dx \right] \\
&+ \int_0^{\pi} \left| \left(a_{n-1} \frac{\sin nx}{2 \sin x} + a_n \frac{\sin(n+1)x}{2 \sin x} \right) \right| dx \\
&\leq C_1 \sum_{k=n-(\alpha+1)}^{\infty} A_k^{\alpha+1} |\Delta^{\alpha+2} a_{k-1} + \Delta^{\alpha+2} a_k| \\
&+ C_1 \sum_{k=0}^{\alpha} A_{n-k+1}^k |\Delta^{k+1} a_{n-k-1}| \\
&+ C_1 \sum_{k=0}^{\alpha} A_{n-k+1}^{\alpha} |\Delta^k a_{n-k-2}| \\
&+ O(a_{n-1} \log n) \quad (C_1 \text{ is an absolute constant})
\end{aligned} \tag{3.1}$$

The first three terms of the above inequality are of $o(1)$ by the Lemma and the hypothesis of theorem.

Because,

$$\begin{aligned}
&\int_0^{\pi} \left| \left(a_{n-1} \frac{\sin nx}{2 \sin x} + a_n \frac{\sin(n+1)x}{2 \sin x} \right) \right| dx \\
&\leq |a_{n-1}| \int_0^{\pi} |D_n(x)| dx \\
&= O(a_{n-1} \log n) \quad \text{as } \int_0^{\pi} |D_n(x)| dx \sim \log n.
\end{aligned}$$

$\int_0^{\pi} |f(x) - S_n(x)| dx \rightarrow 0$ if and only if $|a_{n-1}| \log n \rightarrow 0$ as $n \rightarrow \infty$.

This completes the proof of the theorem.

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