Institute of Mathematical Geography

# SOLSTICE

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VOLUME I, 1990

**Monograph #13** in Institute of Mathematical Geography Monograph Series.



ISBN:
1-877751-44-8

ISSN:
1059-5325

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# Content of "Solstice—I"

## Major papers.

Institute of Mathematical Geography

# SOLSTICE

Journal of the
Institute of Mathematical Geography

Summer, 1990

# SOLSTICE:

## AN ELECTRONIC JOURNAL OF GEOGRAPHY AND MATHEMATICS

SUMMER, 1990

Volume I, Number 1
Institute of Mathematical Geography
Ann Arbor, Michigan

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ISBN: 1-877751-44-8

## SUMMARY OF CONTENT

Note: in this first issue, there is one of each type of article–this need not be the case in the future.

1. REPRINT.

William Kingdon Clifford, *Postulates of the Science of Space.*

This reprint of a portion of Clifford's lectures to the Royal Institution in the 1870's suggests many geographic topics of concern in the last half of the twentieth century. Look for connections to boundary issues, to scale problems, to self- similarity and fractals, and to non–Euclidean geometries (from those based on denial of Euclid's parallel postulate to those based on a sort of mechanical "polishing"). What else did, or might, this classic essay foreshadow?

2. FULL-LENGTH ARTICLE.

Sandra L. Arlinghaus, *Beyond the Fractal.* Figures are transmitted in this e-file only for the half of the article described in the first paragraph below.

An original article. The fractal notion of self- similarity is useful for characterizing change in scale; the reason fractals are effective in the geometry of central place theory is because that geometry is hierarchical in nature. Thus, a natural place to look for other connections of this sort is to other geographical concepts that are also hierarchical. Within this fractal context, this article examines the case of spatial diffusion.

When the idea of diffusion is extended to see "adopters" of an innovation as "attractors" of new adopters, a Julia set is introduced as a possible axis against which to measure one class of geographic phenomena. Beyond the fractal context, fractal concepts, such as "compression" and "space–filling" are considered in a broader graph–theoretic context.

3. SHORT ARTICLE.

William C. Arlinghaus, *Groups, graphs, and God*

An original article based on a talk given before a MIdwest GrapH TheorY (MIGHTY) meeting. The author, an algebraic graph theorist, ties his research interests to a broader philosophical realm, suggesting the breadth of range to which algebraic structure might be applied.

The fact that almost all graphs are rigid (have trivial automorphism groups) is exploited to argue probabilistically for the existence of God. This is presented in the context that applications of mathematics need not be limited to scientific ones.

Note: In this first issue, there is one of each type of article–this need not be the case in the future.

4. REGULAR FEATURES

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ii. **Construction Zone** — a centrally symmetric hexagon is derived from an arbitrary convex hexagon.
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vi. **Solution to puzzle**

## 5. SAMPLE OF HOW TO DOWNLOAD THE ELECTRONIC FILE

This section shows the exact set of commands that work to download this file on The University of Michigan's Xerox 9700. Because different universities will have different installations of TEX, this is only a rough guideline which *might* be of use to the reader or to the reader's computing center.

1. REPRINT

# THE POSTULATES OF THE SCIENCE OF SPACE

William Kingdon Clifford

From a set of lectures given before the Royal Institution, 1873 – "The Philosophy of the Pure Sciences." Reprinted excerpt longer than this one appears in *The World of Mathematics*, edited by James R. Newman, New York: Simon and Schuster, 1956.

In my first lecture I said that, out of the pictures which are all that we can really see, we imagine a world of solid things; and that this world is constructed so as to fulfil a certain code of rules, some called axioms, and some called definitions, and some called postulates, and some assumed in the course of demonstration, but all laid down in one form or another in Euclid's Elements of Geometry. It is this code of rules that we have to consider to-day. I do not, however, propose to take this book that I have mentioned, and to examine one after another the rules as Euclid has laid them down or unconsciously assumed them; notwithstanding that many things might be said in favour of such a course. This book has been for nearly twenty-two centuries the encouragement and guide of that scientific thought which is one thing with the progress of man from a worse to a better state. The encouragement; for it contained a body of knowledge that was really known and could be relied on, and that moreover was growing in extent and application. For even at the time this book was written—shortly after the foundation of the Alexandrian Museum–Mathematic was no longer the merely ideal science of the Platonic school, but had started on her career of conquest over the whole world of Phenomena. The guide; for the aim of every scientific student of every subject was to bring his knowledge of that subject into a form as perfect as that which geometry had attained. Far up on the great mountain of Truth, which all the sciences hope to scale, the foremost of that sacred sisterhood was seen, beckoning to the rest to follow her. And hence she was called, in the dialect of the Pythagoreans, 'the purifier of the reasonable soul.' Being thus in itself at once the inspiration and the aspiration of scientific thought, this Book of Euclid's has had a history as chequered as that of human progress itself. [Deleted text.] The geometer of to-day knows nothing about the nature of actually existing space at an infinite distance; he knows nothing about the properties of this present space in a past or a future eternity. He knows, indeed, that the laws assumed by Euclid are true with an accuracy that no direct experiment can approach, not only in this place where we are, but in places at a distance from us that no astronomer has conceived; but he knows this as of Here and Now; beyond his range is a There and Then of which he knows nothing at present, but may ultimately come to know more. So, you see, there is a real parallel between the work of Copernicus and his successors on the one hand, and the work of Lobatchewsky and his successors on the other. In both of these the knowledge of Immensity and Eternity is replaced by knowledge of Here and Now. And in virtue of these two revolutions the idea of the Universe, the Macrocosm, the All, as subject of human knowledge, and therefore of human interest, has fallen to pieces.

It will now, I think, be clear to you why it will not do to take for our present consideration the postulates of geometry as Euclid has laid them down. While they were all certainly true, there might be substituted for them some other group of equivalent propositions; and the choice of the particular set of statements that should be used as the groundwork of the science was to a certain extent arbitrary, being only guided by convenience of exposition.

But from the moment that the actual truth of these assumptions becomes doubtful, they fall of themselves into a necessary order and classification; for we then begin to see which of them may be true independently of the others. And for the purpose of criticizing the evidence for them, it is essential that this natural order should be taken; for I think you will see presently that any other order would bring hopeless confusion into the discussion.

Space is divided into parts in many ways. If we consider any material thing, space is at once divided into the part where that thing is and the part where it is not. The water in this glass, for example, makes a distinction between the space where it is and the space where it is not. Now, in order to get from one of these to the other you must cross the *surface* of the water; this surface is the boundary of the space where the water is which separates it from the space where it is not. Every *thing*, considered as occupying a portion of space, has a surface which separates that space where it is from the space where it is not. But, again, a surface may be divided into parts in various ways. Part of the surface of this water is against the air, and part is against the glass. If you travel over the surface from one of these parts to the other, you have to cross the *line* which divides them; it is this circular edge where water, air, and glass meet. Every part of a surface is separated from the other parts by a line which bounds it. But now suppose, further, that this glass had been so constructed that the part towards you was blue and the part towards me was white, as it is now. Then this line, dividing two parts of the surface of the water, would itself be divided into two parts; there would be a part where it was against the blue glass, and a part where it was against the white glass. If you travel in thought along that line, so as to get from one of these two parts to the other, you have to cross a *point* which separates them, and is the boundary between them. Every part of a line is separated from the other parts by points which bound it. So we may say altogether —

The boundary of a solid (*i.e.*, of a part of space) is a surface.

The boundary of a part of a surface is a line.

The boundaries of a part of a line are points.

And we are only settling the meanings in which words are to be used. But here we may make an observation which is true of all space that we are acquainted with: it is that the process ends here. There are no parts of a point which are separated from one another by the next link in the series. This is also indicated by the reverse process.

For I shall now suppose this point — the last thing that we got to — to move round the tumbler so as to trace out the line, or edge, where air, water, and glass meet. In this way I get a series of points, one after another; a series of such a nature that, starting from any one of them, only two changes are possible that will keep it within the series: it must go forwards or it must go backwards, and each of these if perfectly definite. The line may then be regarded as an aggregate of points. Now let us imagine, further, a change to take place in this line, which is nearly a circle. Let us suppose it to contract towards the centre of the circle, until it becomes indefinitely small, and disappears. In so doing it will trace out the upper surface of the water, the part of the surface where it is in contact with the air. In this way we shall get a series of circles one after another — a series of such a nature that, starting from any one of them, only two changes are possible that will keep it within the series: it must expand or it must contract. This series, therefore, of circles, is just similar to the series of points that make one circle; and just as the line is regarded as an aggregate of points, so we may regard this surface as an aggregate of lines. But this surface is also in another sense an aggregate of point, in being an aggregate of aggregates of points. But, starting from a point in the surface, more than two changes are possible that will keep it within the surface, for it may move in any direction. The surface, then, is an aggregate of points of a different kind from the line. We speak of the line as a point-aggregate of one dimension, because, starting from one point, there are only two possible directions of change; so that the line can be traced out in one motion. In the same way, a surface is a line-aggregate of one dimension, because it can be traced out by one motion of the line; but it is a point-aggregate of two dimensions, because, in order to build it up of points, we have first to aggregate points into a line, and then lines into a surface. It requires two motions of a point to trace it out.

Lastly, let us suppose this upper surface of the water to move downwards, remaining always horizontal till it becomes the under surface. In so doing it will trace out the part of space occupied by the water. We shall thus get a series of surfaces one after another, precisely analogous to the series of points which make a line, and the series of lines which make a surface. The piece of solid space is an aggregate of surfaces, and an aggregate of the same kind as the line is of points; it is a surface-aggregate of one dimension. But at the same time it is a line-aggregate of two dimensions, and a point-aggregate of three dimensions. For if you consider a particular line which has gone to make this solid, a circle partly contracted and part of the way down, there are more than two opposite changes which it can undergo. For it can ascend or descend, or expand or contract, or do both together in any proportion. It has just as great a variety of changes as a point in a surface. And the piece of space is called a point-aggregate of three dimensions, because it takes three distinct motions to get it from a point. We must first aggregate points into a line, then lines into a surface, then surfaces into a solid. At this step it is clear, again, that the process must stop in all the space we know of. For it is not possible to move that piece of space in such a way as to change every point in it. When we moved our line or our surface, the new line or surface contained no point whatever that was in the old one; we started with one aggregate of points, and by moving it we got an entirely new aggregate, all the points of which were new. But this cannot be done with the solid; so that the process is at an end. We arrive, then, at the result that *space is of three dimensions*.

Is this, then, one of the postulates of the science of space? No; it is not. The science of space, as we have it, deals with relations of distance existing in a certain space of three dimensions, but it does not at all require us to assume that no relations of distance are possible in aggregates of more than three dimensions. The fact that there are only three dimensions does regulate the number of books that we write, and the parts of the subject that we study: but it is not itself a postulate of the science. We investigate a certain space of three dimensions, on the hypothesis that it has certain elementary properties; and it is the assumptions of these elementary properties that are the real postulates of the science of space. To these I now proceed.

The first of them is concerned with *points*, and with the relation of space to them. We spoke of a line as an aggregate of points. Now there are two kinds of aggregates, which are called respectively continuous and discrete. If you consider this line, the boundary of part of the surface of the water, you will find yourself believing that between any two points of it you can put more points of division, and between any two of these more again, and so on; and you do not believe there can be any end to the process. We may express that by saying you believe that between any two points of the line there is an infinite number of other points. But now here is an aggregate of marbles, which, regarded as an aggregate, has many characters of resemblance with the aggregate of points. It is a series of marbles, one after another; and if we take into account the relations of nextness or contiguity which they possess, then there are only two changes possible from one of them as we travel along the series: we must go to the next in front, or to the next behind. But yet it is not true that between any two of them here is an infinite number of other marbles; between these two, for example, there are only three. There, then, is a distinction at once between the two kinds of aggregates. But there is another, which was pointed out by Aristotle in his Physics and made the basis of a definition of continuity. I have here a row of two different kinds of marbles, some white and some black. This aggregate is divided into two parts, as we formerly supposed the line to be. In the case of the line the boundary between the two parts is a point which is the element of which the line is an aggregate. In this case before us, a marble is the element; but here we cannot say that the boundary between the two parts is a marble. The boundary of the white parts is a white marble, and the boundary of the black parts is a black marble; these two adjacent parts have different boundaries. Similarly, if instead of arranging my marbles in a series, I spread them out on a surface, I may have this aggregate divided into two portions — a white portion and a black portion; but the boundary of the white portion is a row of white marbles, and the boundary of the black portion is a row of black marbles. And lastly, if I made a heap of white marbles, and put black marbles on the top of them, I should have a discrete aggregate of three dimensions divided into two parts: the boundary of the white part would be a layer of white marbles, and the boundary of the black part would be a layer of black marbles. In all these cases of discrete aggregates, when they are divided into two parts, the two adjacent parts have different boundaries. But if you come to consider an aggregate that you believe to be continuous, you will see that you think of two adjacent parts as having the *same* boundary. What is the boundary between water and air here? Is it water? No; for there would still have to be a boundary to divide that water from the air. For the same reason it cannot be air. I do not want you at present to think of the actual physical facts by the aid of any molecular theories; I want you only to think of what appears to be, in order to understand clearly a conception that we all have. Suppose the things actual in contact. If, however much we magnified them, they still appeared to be thoroughly homogeneous, the water filling up a certain space, the air an adjacent space; if this held good indefinitely through all degrees of conceivable magnifying, then we could not say that the surface of the water was a layer of water and the surface of air a layer of air; we should have to say that the same surface was the surface of both of them, and was itself neither one nor the other—that this surface occupied *no* space at all. Accordingly, Aristotle defined the continuous as that of which two adjacent parts have the same boundary; and the discontinuous or discrete as that of which two adjacent parts have direct boundaries.

Now the first postulate of the science of space is that space in a continuous aggregate of points, and not a discrete aggregate. And this postulate—which I shall call the postulate of continuity—is really involved in those three of the six postulates of Euclid for which Robert Simson has retained the name of postulate. You will see, on a little reflection, that a discrete aggregate of points could not be so arranged that any two of them should be relatively situated to one another in exactly the same manner, so that any two points might be joined by a straight line which should always bear the same definite relation to them. And the same difficulty occurs in regard to the other two postulates. But perhaps the most conclusive way of showing that this postulate is really assumed by Euclid is to adduce the proposition he probes, that every finite straight line may be bisected. Now this could not be the case if it consisted of an odd number of separate points. As the first of the postulates of the science of space, the, we must reckon this postulate of Continuity; according to which two adjacent portions of space, or of a surface, or of a line, have the *same* boundary, *viz*.— a surface, a line, or a point; and between every two points on a line there is an infinite number of intermediate points.

The next postulate is that of Elementary Flatness. You know that if you get hold of a small piece of a very large circle, it seems to you nearly straight. So, if you were to take any curved line, and magnify it very much, confining your attention to a small piece of it, that piece would seem straighter to you than the curve did before it was magnified. At least, you can easily conceive a curve possessing this property, that the more you magnify it, the straighter it gets. Such a curve would possess the property of elementary flatness. In the same way, if you perceive a portion of the surface of a very large sphere, such as the earth, it appears to you to be flat. If, then, you take a sphere of say a foot diameter, and magnify it more and more, you will find that the more you magnify it the flatter it gets. And you may easily suppose that this process would go on indefinitely; that the curvature would become less and less the more the surface was magnified. Any curved surface which is such that the more you magnify it the flatter it gets, is said to possess the property of elementary flatness. But if every succeeding power of our imaginary microscope disclosed new wrinkles and inequalities without end, then we should say that the surface did not possess the property of elementary flatness.

But how am I to explain how solid space can have this property of elementary flatness? Shall I leave it as a mere analogy, and say that it is the same kind of property as this of the curve and surface, only in three dimensions instead of one or two? I think I can get a little nearer to it than that; at all events I will try.

If we start to go out from a point on a surface, there is a certain choice of directions in which we may go. These directions make certain angles with one another. We may suppose a certain direction to start with, and then gradually alter that by turning it round the point: we find thus a single series of directions in which we may start from the point. According to our first postulate, it is a continuous series of directions. Now when I speak of a direction from the point, I mean a direction of starting; I say nothing about the subsequent path. Two different paths may have the same direction at starting; in this case they will touch at the point; and there is an obvious difference between two paths which touch and two paths which meet and form an angle. Here, then, is an aggregate of directions, and they can be changed into one another. Moreover, the changes by which they pass into one another have magnitude, they constitute distance–relations; and the amount of change necessary to turn one of them into another is called the angle between them. It is involved in this postulate that we are considering, that angles can be compared in respect of magnitude. But this is not all. If we go on changing a direction of start, it will, after a certain amount of turning, come round into itself again, and be the same direction. On every surface which has the property of elementary flatness, the amount of turning necessary to take a direction all round into its first position is the same for all points of the surface. I will now show you a surface which at one point of it has not this property. I take this circle of paper from which a sector has been cut out, and bend it round so as to join the edges; in this way I form a surface which is called a *cone*. Now on all points of this surface but one, the law of elementary flatness holds good. At the vertex of the cone, however, notwithstanding that there is an aggregate of directions in which you may start, such that by continuously changing one of them you may get it round into its original position, yet the whole amount of change necessary to effect this is not the same at the vertex as it is at any other point of the surface. And this you can see at once when I unroll it; for only part of the directions in the plane have been included in the cone. At this point of the cone, then, it does not possess the property of elementary flatness; and no amount of magnifying would ever make a cone seem flat at its vertex.

To apply this to solid space, we must notice that here also there is a choice of directions in which you may go out from any point; but it is a much greater choice than a surface gives you. Whereas in a surface the aggregate of directions is only of one dimension, in solid space it is of two dimensions. But here also there are distance–relations, and the aggregate of directions may be divided into parts which have quantity. For example, the directions which start from the vertex of this cone are divided into those which go inside the cone, and those which go outside the cone. The part of the aggregate which is inside the cone is called a solid angle. Now in those spaces of three dimensions which have the property of elementary flatness, the whole amount of solid angle round one point is equal to the whole amount round another point. Although the space need not be exactly similar to itself in all parts, yet the aggregate of directions round one point is exactly similar to the aggregate of directions round another point, if the space has the property of elementary flatness.

How does Euclid assume this postulate of Elementary Flatness? In his fourth postulate he has expressed it so simply and clearly that you will wonder how anybody could make all this fuss. He says, 'All right angles are equal.'

Why could I not have adopted this at once, and saved a great deal of trouble? Because it assumes the knowledge of a surface possessing the property of elementary flatness in all its points. Unless such a surface is first made out to exist, and the definition of a right angle is restricted to lines drawn upon it—for there is no necessity for the word *straight* in that definition—the postulate in Euclid's form is obviously not true. I can make two lines cross at the vertex of a cone so that the four adjacent angles shall be equal, and yet not one of them equal to a right angle. I pass on to the third postulate of the science of space—the postulate of Superposition. According to this postulate a body can be moved about in space without altering its size or shape. This seems obvious enough, but it is worth while to examine a little more closely into the meaning of it. We must define what we mean by size and by shape. When we say that a body can be moved about without altering its size, we mean that it can be so moved as to keep unaltered the length of all the lines in it. This postulate therefore involves that lines can be compared in respect of magnitude, or that they have a length independent of position; precisely as the former one involved the comparison of angular magnitudes. And when we say that a body can be moved about without altering its shape, we mean that it can be so moved as to keep unaltered all the angles in it. It is not necessary to make mention of the motion of a body, although that is the easiest way of expressing and of conceiving this postulate; but we may, if we like, express it entirely in terms which belong to space, and that we should do in this way. Suppose a figure to have been constructed in some portion of space; say that a triangle has been drawn whose sides are the shortest distances between its angular points. Then if in any other portion of space two points are taken whose shortest distance is equal to a side of the triangle, and at one of them an angle is made equal to one of the angles adjacent to that side, and a line of shortest distance drawn equal to the corresponding side of the original triangle, the distance from the extremity of this to the other of the two points will be equal to the third side of the original triangle, and the two will be equal in all respects; or generally, if a figure has been constructed anywhere, another figure, with all its lines and all its angles equal to the corresponding lines and angles of the first, can be constructed anywhere else. Now this is exactly what is meant by the principle of superposition employed by Euclid to prove the proposition that I have just mentioned. And we may state it again in this short form—All parts of space are exactly alike.

But this postulate carries with it a most important consequence. In enables us to make a pair of most fundamental definitions—those of the plane and of the straight line. In order to explain how these come out of it when it is granted, and how they cannot be made when it is not granted, I must here say something more about the nature of the postulate itself, which might otherwise have been left until we come to criticize it.

We have stated the postulate as referring to solid space. But a similar property may exist in surfaces. Here, for instance, is part of the surface of a sphere. If I draw any figure I like upon this, I can suppose it to be moved about in any way upon the sphere, without alteration of its size or shape. If a figure has been drawn on any part of the surface of a sphere, a figure equal to it in all respects may be drawn on any other part of the surface. Now I say that this property belongs to the surface itself, is a part of its own internal economy, and does not depend in any way upon its relation to space of three dimensions. For I can pull it about and bend it in all manner of ways, so as altogether to alter its relation to solid space; and yet, if I do not stretch it or tear it, I make no difference whatever in the length of any lines upon it, or in the size of any angles upon it. I do not in any way alter the figures drawn upon it, or the possibility of drawing figures upon it, *so far as their relations with the surface itself are concerned*. This property of the surface, then, could be ascertained by people who lived entirely in it, and were absolutely ignorant of a third dimension. As a point-aggregate of two dimensions, it has in itself properties determining the distance-relations of the points upon it, which are absolutely independent of the existence of any points which are not upon it.

Now here is a surface which has not that property. You observe that it is not of the same shape all over, and that some parts of it are more curved than other parts. If you drew a figure upon this surface, and then tried to move it about, you would find that it was impossible to do so without altering the size and shape of the figure. Some parts of it would have to expand, some to contract, the lengths of the lines could not all be kept the same, the angles would not hit off together. And this property of the surface—that its parts are different from one another—is a property of the surface itself, a part of its internal economy, absolutely independent of any relations it may have with space outside of it. For, as with the other one, I can pull it about in all sorts of ways, and, so long as I do not stretch it or tear it, I make no alteration in the length of lines drawn upon it or in the size of the angles.

Here, then, is an intrinsic difference between these two surfaces, as surfaces. They are both point-aggregates of two dimensions; but the points in them have certain relations of distance (distance measured always *on* the surface), and these relations of distance are not the same in one case as they are in the other.

The supposed people living in the surface and having no idea of a third dimension might, without suspecting that third dimension at all, make a very accurate determination of the nature of their *locus in quo*. If the people who lived on the surface of the sphere were to measure the angles of a triangle, they would find them to exceed two right angles by a quantity proportional to the area of the triangle. This excess of the angles above two right angles, being divided by the area of the triangle, would be found to give exactly the same quotient at all parts of the sphere. That quotient is called the curvature of the surface; and we say that a sphere is a surface of uniform curvature. But if the people living on this irregular surface were to do the same thing, they would not find quite the same result. The sum of the angles would, indeed, differ from two right angles, but sometimes in excess, and sometimes in defect, according to the part of the surface where they were. And though for small triangles in any on neighbourhood the excess or defect would be nearly proportional to the area of the triangle, yet the quotient obtained by dividing this excess or defect by the area of the triangle would vary from one part of the surface to another. In other words, the curvature of this surface varies from point to point; it is sometimes positive, sometimes negative, sometimes nothing at all.

But now comes the important difference. When I speak of a triangle, what do I suppose the sides of that triangle to be?

If I take two points near enough together upon a surface, and stretch a string between them, that string will take up a certain definite position upon the surface, marking the line of shortest distance from one point to the other. Such a line is called a geodesic line. It is a line determined by the intrinsic properties of the surface, and not by its relations with external space. The line would still be the shortest line, however the surface were pulled about without stretching or tearing. A geodesic line may be *produced*, when a piece of it is given; for we may take one of the points, and, keeping the string stretched, make it go round in a sort of circle until the other end has turned through two right angles. The new position will then be a prolongation of the same geodesic line.

In speaking of a triangle, then, I meant a triangle whose sides are geodesic lines. But in the case of a spherical surface—or, more generally, of a surface of constant curvature—these geodesic lines have another and most important property. They are *straight*, so far as the surface is concerned. On this surface a figure may be moved about without altering its size or shape. It is possible, therefore, to draw a line which shall be of the same shape all along and on both sides. That is to say, if you take a piece of the surface on one side of such a line, you may slide it all along the line and it will fit; and you may turn it round and apply it to the other side, and it will fit there also. This is Leibniz's definition of a straight line, and, you see, it has no meaning except in the case of a surface of constant curvature, a surface all parts of which are alike.

Now let us consider the corresponding things in solid space. In this also we may have geodesic lines; namely, lines formed by stretching a string between two points. But we may also have geodesic surfaces; and they are produced in this manner. Suppose we have a point on a surface, and this surface possesses the property of elementary flatness. Then among all the directions of starting from the point, there are some which start *in the surface*, and do not make an angle with it. Let all these be prolonged into geodesics; then we may imagine one of these geodesics to travel round and coincide with all the others in turn. In so doing it will trace out a surface which is called a geodesic surface. Now in the particular case where a space of three dimensions has the property of superpositoin, or is all over alike, these geodesic surfaces are *planes*. That is to say, since the space is all over alike, these surfaces are also of the same shape all over and on both sides; which is Leibniz's definition of a plane. If you take a piece of space on one side of such a plane, partly bounded by the plane, you may slide it all over the plane, and it will fit; and you may turn it round and apply it to the other side, and it will fit there also. Now it is clear that this definition will have no meaning unless the third postulate be granted. So we may say that when the postulate of Superposition is true, then there are planes and straight lines; and they are defined as being of the same shape throughout and on both sides.

It is found that the whole geometry of a space of three dimensions is known when we know the curvature of three geodesic surfaces at every point. The third postulate requires that the curvature of all geodesic surfaces should be everywhere equal to the same quantity.

I pass to the fourth postulate, which I call the postulate of Similarity. According to this postulate, any figure may be magnified or diminished in any degree without altering its shape. If any figure has been constructed in one part of space, it may be reconstructed to any scale whatever in any other part of space, so that no one of the angles shall be altered through all the lengths of lines will of course be altered. This seems to be a sufficiently obvious induction from experience; for we have all frequently seen different sixes of the same shape; and it has the advantage of embodying the fifth and sixth of Euclid's postulates in a single principle, which bears a great resemblance in form to that of Superposition, and may be used in the same manner. It is easy to show that it involves the two postulates of Euclid: 'Two straight lines cannot enclose a space,' and 'Lines in one plane which never meet make equal angles with every other line.'

This fourth postulate is equivalent to the assumption that the constant curvature of the geodesic surfaces is zero; or the third and fourth may be put together, and we shall then say that the three curvatures of space are all of them zero at every point.

The supposition made by Lobatchewsky was, that the three first postulates were true, but not the fourth. Of the two Euclidean postulates included in this, he admitted one, *viz.*, that two straight lines cannot enclose a space, or that two lines which once diverge go on diverging for ever. But he left out the postulate about parallels, which may be stated in this form. If through a point outside of a straight line there be drawn another, indefinitely produced both ways; and if we turn this second one round so as to make the point of intersection travel along the first line, then at the very instant that this point of intersection disappears at one end it will reappear at the other, and there is only one position in which the lines do not intersect. Lobatchewsky supposed, instead, that there was a finite angle through which the second line must be turned after the point of intersection had disappeared at one end, before it reappeared at the other. For all positions of the second line within this angle there is then no intersection. In the two limiting positions, when the lines have just done meeting at one end, and when they are just going to meet at the other, they are called parallel; so that two lines can be drawn through a fixed point parallel to a given straight line. The angle between these two depends in a certain way upon the distance of the point from the line. The sum of the angles of a triangle is less than two right angles by a quantity proportional to the area of the triangle. The whole of this geometry is worked out in the style of Euclid, and the most interesting conclusions are arrived at; particularly in the theory of solid space, in which a surface turns up which is not plane relatively to that space, but which, for purposes of drawing figures upon it, is identical with the Euclidean plane.

It was Riemann, however, who first accomplished the task of analysing all the assumptions of geometry, and showing which of them were independent. This very disentangling and separation of them is sufficient to deprive them for the geometer of their exactness and necessity; for the process by which it is effected consists in showing the possibility of conceiving these suppositions one by one to be untrue; whereby it is clearly made out how much is supposed. But it may be worth while to state formally the case for and against them.

When it is maintained that we know these postulates to be universally true, in virtue of certain deliverances of our consciousness, it is implied that these deliverances could not exist, except upon the supposition that the postulates are true. If it can be shown, then, from experience that our consciousness would tell us exactly the same things if the postulates are not true, the ground of their validity will be taken away. But this is a very easy thing to show.

That same faculty which tells you that space is continuous tells you that this water is continuous, and that the motion perceived in a wheel of life is continuous. Now we happen to know that if we could magnify this water as much again as the best microscopes can magnify it, we should perceive its granular structure. And what happens in a wheel of life is discovered by stopping the machine. Even apart, then, from our knowledge of the way nerves act in carrying messages, it appears that we have no means of knowing anything more about an aggregate than that it is too fine-grained for us to perceive its discontinuity, if it has any.

Nor can we, in general, receive a conception as positive knowledge which is itself founded merely upon inaction. For the conception of a continuous thing is of that which looks just the same however much you magnify it. We may conceive the magnifying to go on to a certain extent without change, and then, as it were, leave it going on, without taking the trouble to doubt about the changes that may ensue.

In regard to the second postulate, we have merely to point to the example of polished surfaces. The smoothest surface that can be made is the one most completely covered with the minutest ruts and furrows. Yet geometrical constructions can be made with extreme accuracy upon such a surface, on the supposition that it is an exact plane. If, therefore, the sharp points, edges, and furrows of space are only small enough, there will be nothing to hinder our conviction of its elementary flatness. It has even been remarked by Riemann that we must not shrink from this supposition if it is found useful in explaining physical phenomena.

The first two postulates may therefore be doubted on the side of the very small. We may put the third and fourth together, and doubt them on the side of the very great. For if the property of elementary flatness exist on the average, the deviations from it being, as we have supposed, too small to be perceived, then, whatever were the true nature of space, we should have exactly the conceptions of it which we now have, if only the regions we can get at were small in comparison with the areas of curvature. If we suppose the curvature to vary in an irregular manner, the effect of it might be very considerable in a triangle formed by the nearest fixed stars; but if we suppose it approximately uniform to the limit of telescopic reach, it will be restricted to very much narrower limits. I cannot perhaps do better than conclude by describing to you as well as I can what is the nature of things on the supposition that the curvature of all space is nearly uniform and positive.

In this case the Universe, as known, becomes again a valid conception; for the extent of space is a finite number of cubic miles. And this comes about in a curious way. If you were to start in any direction whatever, and move in that direction in a perfect straight line according to the definition of Leibniz; after travelling a most prodigious distance, to which the parallactic unit—200,000 times the diameter of the earth's orbit—would be only a few steps, you would arrive at—this place. Only, if you had started upwards, you would appear from below. Now, one of two things would be true. Either, when you had got half-way on your journey, you came to a place that is opposite to this, and which you must have gone through, whatever direction you started in; or else all paths you could have taken diverge entirely from each other till they meet again at this place. In the former case, every two straight lines in a plane meet in two points, in the latter they meet only in one. Upon this supposition of a positive curvature, the whole of geometry is far more complete and interesting; the principle of duality, instead of half breaking down over metric relations, applies to all propositions without exception. In fact, I do no mind confessing that I personally have often found relief from the dreary infinities of homaloidal space in the consoling hope that, after all, this other may be the true state of things.

2. FULL-LENGTH ARTICLE

# BEYOND THE FRACTAL

*Sandra Lach Arlinghaus*

"I never saw a moor,
I never saw the sea;
Yet know I how the heather looks,
And what a wave must be."

*Emily Dickinson, "Chartless."*


## Abstract.

The fractal notion of self-similarity is useful for characterizing change in scale; the reason fractals are effective in the geometry of central place theory is because that geometry is hierarchical in nature. Thus, a natural place to look for other connections of this sort is to other geographical concepts that are hierarchical in nature. Within this fractal context, this chapter examines the case of spatial diffusion.

When the idea of diffusion is extended to see "adopters" of an innovation as "attractors" of new adopters, a Julia set is introduced as a possible axis against which to measure one class of geographic phenomena. Beyond the fractal context, fractal concepts, such as "compression" and "space- filling" are considered in a broader graph-theoretic context.


## Introduction.

Because a fractal may be considered as a randomly generated statistical image (Mandelbrot, 1983), one place to look for geometric fractals tailored to fit geographic concepts is within the set of ideas behind spatial configurations traditionally characterized using randomness. The spatial diffusion of an innovation is one such case; Hägerstrand characterized it using probabilistic simulation techniques (Hägerstrand, 1967). This chapter builds directly on Hägerstrand's work in order to demonstrate, in some detail, how fractals might arise in spatial diffusion. From there, and with a view of an adopter of an innovation as an "attractor" of other adopters, the connected Julia set $z = z^2 - 1$ is examined, only broadly, for its potential to serve as an axis from which to measure spatial "attraction."

More generally, it is not necessary to consider fractal- like concepts such as "attraction," "space-filling," or "compression" relative to any metric, as in the diffusion example, or relative to any axis, as in the Julia set case. These broad fractal notions are examined, in some detail, in a graph-theoretic realm, free from metric/axis encumbrance, as one step beyond the fractal. An effort has been made to explain key geographical and mathematical concepts so that much of the material, and the flow of ideas, is self- contained and accessible to readers from various disciplines.

## A fractal connection to spatial diffusion

The diffusion of the knowledge of an innovation across geographic space may be simulated numerically using Monte Carlo techniques based in probability theory (Hägerstrand, 1967). A simple example illustrates the basic mechanics of Hägerstrand's procedure.

Consider a geographic region and cover it with a grid of uniform cell size suited to the scale of the available empirical information about the innovation. Enter the number of initial adopters of the innovation in the grid: an entry of "1" means one person (household, or other set of people) knows of the innovation. Over time, this person will tell others. Assume that the spread of the news, from this person to others, decays with distance. To simulate this spread, probabilities of the likelihood of contact will be assigned to each cell surrounding each initial adopter. A table of random numbers is used in conjunction with the probabilities, as follows.

Given a gridded geographic region and a distribution of three initial adopters of an innovation (Figure 1). Assume that an initial telling occurs no more than two cells away from the initial adopters' cells. This assumption creates a five–by–five grid in which interchange can occur between an initial adopter in the central cell and others. Assign probabilities of contact to each of these twenty–five cells as a percentage likelihood that a randomly chosen four digit number falls within a given interval of numbers assigned to each cell (Figure 2). Because the intervals in Figure 2 partition the set of four digit numbers, the percentage probabilities assigned to each cell add to 100%. Pick up the five–by–five grid and center it on the original adopter in cell H3 (Figure 1). Choose the first number, 6248, in the list of random numbers (Figure 2). It falls in the interval of numbers in the central cell. Enter a "+1" in the associated cell, H3, to represent this new adopter. Move the five–by– five grid across the distribution of original adopters, stopping it and repeating this procedure with the next random number in the list each time a new original adopter is encountered. Center the five–by–five grid on H4; the next random number is 0925 which falls in the interval in the cell immediately northwest of center (Figure 2). Enter a "+1" in cell G3 (Figure 1), the cell immediately northwest of H4. Finally, center the moving grid on H5. The next random number, 4997, falls in the center cell; therefore, enter a "+1" in cell H5. Once this procedure has been applied to all original adopters, the population of adopters doubles and a "first generation" of adopters, comprising original adopters and newer adopters represented as "+1's", emerges (Figure 1). Any number of additional generations of adopters of the innovation may be simulated by iteration of this procedure.

There are numerous side issues, which are important, that may complicate this basic procedure (Hägerstrand, 1967; Haggett *et al.*, 1977). How are the percentages for the five–by–five grid chosen? Indeed, how is the dimension of "five" chosen for a side of this grid? Should the choices of percentages and of dimension be based on empirical data, on other abstract considerations, or on a mix of the two? What sorts of criteria should there be in judging suitability of empirical data? What if a random entry falls outside the given grid; what sorts of boundary/barrier considerations, both in terms of the position of new adopters relative to the regional boundary and of the symmetry of the probabilities within the five–by–five grid, should be taken into account?

Independent of how many generations are calculated using this procedure, the pattern of "filling in" of new adopters is heavily influenced by the shape of the set of original adopters. Indeed, over time, knowledge of the innovation diffuses slowly initially, picks up in speed of transmission, tapers off, and eventually the population becomes saturated with the knowledge. Typically this is characterized as a continuous phenomenon using a differential equation of inhibited growth that has as an initial supposition that the population may not exceed $M$, an upper bound, and that $P(t)$, the population $P$ at time $t$, grows at a rate proportional to the size of itself and proportional to the fraction left to grow (Haggett *et al.*, 1977; Boyce

**Figure 1.**

Three original adopters, represented as 1's. Positions are simulated for three new adopters, represented as +1's. The two sets taken together form a first generation of adopters of an innovation (grid after Hägerstrand).

North at the top.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| *A* | | | | | | | | |
| *B* | | | | | | | | |
| *C* | | | | | | | | |
| *D* | | | | | | | | |
| *E* | | | | | | | | |
| *F* | | | | | | | | |
| *G* | | | +1 | | | | | |
| *H* | | | +1 1 | 1 | +1 1 | | | |
| *I* | | | | | | | | |
| *J* | | | | | | | | |
| *K* | | | | | | | | |

and DiPrima, 1977). An equation such as

$$\frac{dP(t)}{dt} = k\,P(t)(1-(P(t)/M))$$

serves as a mathematical model for this sort of growth in which $k > 0$ is a growth constant and the fraction $(1-(P(t)/M)$ acts as a damper on the rate of growth (Boyce and DiPrima, 1977). The graph of the equation is an $S$-shaped (sigmoid) logistic curve with horizontal asymptote at $P(t) = M$ and inflection point at $P(t) = M/2$. When $dP/dt > 0$ the population shows growth; when $d^2P/dt^2 > 0$ (below $P(t) = M/2$) the rate of growth is increasing; when $d^2P/dt^2 < 0$ (above $P(t) = M/2$) the rate of growth is decreasing.

The differential equation model thus yields information concerning the rate of change of the total population and in the rate of change in growth of the total population. It does not show how to determine $M$; the choice of $M$ is given *a priori*.

Iteration of the Hägerstrand procedure gives a position for $M$ once the procedure has been run for all the generations desired.

For, it is a relatively easy matter to accumulate the distributions of adopters and stack them next to each other, creating an empirical sigmoid logistic curve based on the simulation (Haggett *et al.*, 1977). Finding the position for the asymptote (or for *an* upper bound close to the asymptotic position) is then straightforward.

Neither the Hägerstrand procedure nor the inhibited growth model provides an estimate of saturation level (horizontal asymptote position) (Haggett, *et al.*, 1977) that can be calculated early in the measurement of the growth. The fractal approach suggested below offers a means for making such a calculation when self-similar hierarchical data are involved; allometry is a special case of this procedure (Mandelbrot, 1983). The reasons for wanting to

**Figure 2.**

Five-by-five grid overlay. Numerical entries in cells show the percentage of four digit numbers associated with each cell. The given listing of cells shows which cell is associated with which range of four digit numbers.

North at the top.

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 0.96 | 1.40 | 1.68 | 1.40 | 0.96 |
| 2 | 1.40 | 3.01 | 5.47 | 3.01 | 1.40 |
| 3 | 1.68 | 5.47 | 44.31 | 5.47 | 1.68 |
| 4 | 1.40 | 3.01 | 5.47 | 3.01 | 1.40 |
| 5 | 0.96 | 1.40 | 1.68 | 1.40 | 0.97 |

A random set of numbers (source: *CRC Handbook of Standard Mathematical Tables*):
6248, 0925, 4997, 9024, 7754
7617, 2854, 2077, 9262, 2841
9904, 9647,
and so forth.

Random number assignment to matrix cells, with cell number given as an ordered pair whose first entry refers to the reference number on the left of the matrix in this figure and whose second entry refers to the reference number at the top of that matrix.

(1,1): 0000-0095; (1,2): 0096-0235; (1,3): 0236-0403
(1,4): 0404-0543; (1,5): 0544-0639

(2,1): 0640-0779; (2,2): 0780-1080; (2,3): 1081-1627
(2,4): 1628-1928; (2,5): 1929-2068

(3,1): 2069-2236; (3,2): 2237-2783; (3,3): 2784-7214
(3,4): 7215-7761; (3,5): 7762-7929

(4,1): 7930-8069; (4,2): 8070-8370; (4,3): 8371-8917
(4,4): 8918-9218; (4,5): 9219-9358

(5,1): 9359-9454; (5,2): 9455-9594; (5,3): 9595-9762
(5,4): 9763-9902; (5,5): 9903-9999

make such a calculation might be to determine where to position adopter "seeds" in order to produce various levels of innovation saturation.

As is well-known, not all innovations diffuse in a uniform manner; Paris fashions readily available in major U. S. cities up and down each coast might seldom be seen in rural midwestern towns. To determine how the ideas of fractal "space-filling" and this sort of diffusion-related "space- filling" might be aligned, consider the following example.

Given a distribution of three original adopters occupying cells H3, H4, and H5 in a linear pattern (Figure 3.A). The probabilities for positions for new adopters are encoded within each cell surrounding each of these (as determined from the five-by-five grid of Figure 2).

Figure 3.A.

The simulation is run on three original adopters with positions given below. Numerical entries show the likelihood, out of 300, that a new adopter will fall into a given cell. Zones of interaction between overlapping five-by-five grids are outlined by a heavy line (begin at the upper left-hand corner (ulhc) of cell F2; move horizontally to the upper right-hand corner (urhc) of cell F6; vertically to lower right-hand corner (lrhc) of cell J6; horizontally to lower left-hand corner (llhc) of cell J2; vertically to ulhc of F2 — should be a rectangular enclosure that you have added to this figure). **Original adopters are in cells H3, H4, H5.**

North at the top.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | *Totals* |
|---|---|---|---|---|---|---|---|---|---|
| *A* | | | | | | | | | |
| *B* | | | | | | | | | |
| *C* | | | | | | | | | |
| *D* | | | | | | | | | |
| *E* | | | | | | | | | |
| *F* | 0.96 | 2.36 | 4.04 | 4.48 | 4.04 | 2.36 | 0.96 | | 19.20 |
| *G* | 1.40 | 4.41 | 9.88 | 11.49 | 9.88 | 4.41 | 1.40 | | 42.87 |
| *H* | 1.68 | 7.15 | 51.46 | 55.25 | 51.46 | 7.15 | 1.68 | | 175.83 |
| *I* | 1.40 | 4.41 | 9.88 | 11.49 | 9.88 | 4.41 | 1.40 | | 42.87 |
| *J* | 0.96 | 2.36 | 4.04 | 4.48 | 4.04 | 2.36 | 0.96 | | 19.20 |
| *K* | | | | | | | | | |
| | 6.40 | 20.69 | 79.30 | 87.19 | 79.31 | 20.70 | 6.41 | | 300 |

Thus, for example, when the grid of Figure 2 is superimposed and centered on the original adopter in cell H3, a probability of 3.01% is assigned to the likelihood for contact from H3 to G4; when it is superimposed and centered on the original adopter in H4, there is a 5.47% likelihood for contact from H4 to G4; and, when it is superimposed and centered on the original adopter in H5, there is a 3.01% likelihood for contact from H5 to G4. Therefore, the percentage likelihood of a new first-generation adopter in cell G4, given this initial configuration of adopters, is the sum of the percentages divided by the number of initial adopters, or 11.49/3. For ease in inserting fractions into the grid, only the numerator, 11.49, is shown as the entry (Figure 3.A). It would be useful, for purposes of comparison of this distribution to those with sets of initial adopters of sizes other than 3, to divide by the number of initial adopters in order to derive a percentage that is independent of the size of the initial distribution (*i.e.*, to normalize the numerical entries).

It is easy to see that the values in the cells of Figure 3.A must add to a total of 300 if one views them as derived from each of three five-by-five grids centered on each original adopter. A "zone of interaction" of entries from two or more five-by-five grids is outlined by a heavy line; 25 cells are enclosed in it in Figure 3.A. The pattern of numbers exhibits bilateral symmetry, insofar as is possible (allowing for the "appendix" of .01 required to make the numerical partition associated with Figure 2 complete) with respect to both North-South and East-West axes (with the origin in cell H4). Sum and column totals are calculated; as

**Figure 3.B.**

The simulation is run on three original adopters with positions given below. Numerical entries show the likelihood, out of 300, that a new adopter will fall into a given cell. Zones of interaction between overlapping five-by-five grids are outlined by a heavy line (begin ulhc of F2; horizontally to urhc of F6; vertically to lrhc of I6; horizontally to lrhc of I5; vertically to lrhc of J5; horizontally to llhc of J3; vertically to llhc of I3; horizontally to llhc of I2; vertically to ulhc of F2 — should be a "fat" T-shaped enclosure that you have added to this figure). **Original adopters are in cells H3, G4, H5.**

North at the top.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | *Totals* |
|---|---|---|---|---|---|---|---|---|---|
| *A* | | | | | | | | | |
| *B* | | | | | | | | | |
| *C* | | | | | | | | | |
| *D* | | | | | | | | | |
| *E* | | 0.96 | 1.40 | 1.68 | 1.40 | 0.96 | | | 6.40 |
| *F* | 0.96 | 2.80 | 5.65 | 8.27 | 5.65 | 2.80 | 0.96 | | 27.09 |
| *G* | 1.40 | 4.69 | 12.34 | 50.33 | 12.34 | 4.69 | 1.40 | | 87.19 |
| *H* | 1.68 | 6.87 | 49.00 | 16.41 | 49.00 | 6.87 | 1.68 | | 131.51 |
| *I* | 1.40 | 3.97 | 8.27 | 7.70 | 8.27 | 3.98 | 1.40 | | 34.99 |
| *J* | 0.96 | 1.40 | 2.64 | 2.80 | 2.65 | 1.40 | 0.97 | | 12.82 |
| *K* | | | | | | | | | |
| | 6.40 | 20.69 | 79.30 | 87.19 | 79.31 | 20.70 | 6.41 | | 300 |

the shape of the distribution of initial adopters is altered (below), these totals will tag sets of cells to demonstrate how changes in the zone of interaction are occurring.

Next consider a distribution of three initial adopters derived from the linear one by moving the middle adopter one unit to the North (Figure 3.B). When interaction values are calculated as they were for the initial distribution in Figure 3.A, a comparable, but different numerical pattern emerges (Figure 3.B). Here, the column totals are the same as those in Figure 3.A, but the row totals are different. The zone of interaction contains 23 cells; the highest individual cell value of 50.33 is less than that of the highest cell value, 55.25, in Figure 3.A. Because both sets of values are partitions of the number 300, and because there are more cells with potential for contact in Figure 3.B than in Figure 3.A, the concentration of entries in Figure 3.B is not as compressed as in Figure 3.A. This is reflected in the row totals; a visual device useful for tracking this compression is to think of the five-by-five grid centered on the middle adopter being gradually pulled, to the North, from under the set of entries in Figure 3.A. In Figure 3.B the top of this middle grid slips out from under, failing to intersect the bottom row, J, of the grid. With this view, it is easy to understand why only the row totals, and not the column totals, change.

Naturally, as the middle initial adopter is pulled successively one unit to the north in the configuration of original adopters, the middle five-by-five grid is also pulled one unit to the north (Figures 3.C, 3.D, 3.E, and 3.F). The numerical consequence is to reduce the

**Figure 3.C.**

The simulation is run on three original adopters with positions given below. Numerical entries show the likelihood, out of 300, that a new adopter will fall into a given cell. Zones of interaction between overlapping five-by-five grids are outlined by a heavy line (begin ulhc of F2; horizontally to urhc of F6; vertically to lrhc of H6; horizontally to lrhc of H5; vertically to lrhc of J5; horizontally to llhc of J3; vertically to llhc of H3; horizontally to llhc of H2; vertically to ulhc of F2 — should be a "less-fat" T-shaped enclosure that you have added to this figure). **Original adopters are in cells H3, F4, H5.**

North at the top.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | *Totals* |
|---|---|---|---|---|---|---|---|---|---|
| *A* | | | | | | | | | |
| *B* | | | | | | | | | |
| *C* | | | | | | | | | |
| *D* | | 0.96 | 1.40 | 1.68 | 1.40 | 0.96 | | | 6.40 |
| *E* | | 1.40 | 3.01 | 5.47 | 3.01 | 1.40 | | | 14.29 |
| *F* | 0.96 | 3.08 | 8.11 | 47.11 | 8.11 | 3.08 | 0.96 | | 71.41 |
| *G* | 1.40 | 4.41 | 9.88 | 11.49 | 9.88 | 4.41 | 1.40 | | 42.87 |
| *H* | 1.68 | 6.43 | 47.39 | 12.62 | 47.39 | 6.44 | 1.68 | | 123.63 |
| *I* | 1.40 | 3.01 | 6.87 | 6.02 | 6.87 | 3.01 | 1.40 | | 28.58 |
| *J* | 0.96 | 1.40 | 2.64 | 2.80 | 2.65 | 1.40 | 0.97 | | 12.82 |
| *K* | | | | | | | | | |
| | 6.40 | 20.69 | 79.30 | 87.19 | 79.31 | 20.70 | 6.41 | | 300 |

size of the zone of interaction among the initial adopters and to spread the range of cells over which the value of 300 is partitioned. This implies less concentration near the original adopters and less "filling in" around them as one proceeds from Figure 3.A to Figure 3.F. Thus, in Figure 3.C the zone of interaction shrinks to 21 cells with a largest individual cell entry of 47.39. At the stage shown in Figure 3.D, the largest cell entry is 45.99; because the cells associated with this value are not overlapped by the five-by- five grid centered on the middle adopter, this largest value will not change as the middle adopter is pulled more to the north. Table 1 shows the sizes of the zones of interaction of the largest individual cell entry for each of Figures 3.A to 3.F. No new information arises from moving the middle cell to the north beyond the position in Figure 3.F; the five- by-five grid is revealed and no longer intersects the two overlapping grids associated with the other two initial adopters.

The example depicted in Figure 3 shows that even as early as the first generation, the pattern of the positions of the initial adopters affects significantly the configuration of the later adopters. Figure 3.A with the heaviest possible filling of space using three initial adopters represents a most saturated case, which, taken together with an underlying symmetry that is bilateral relative to mutually perpendicular axes, suggests that an associated space-filling curve should have dimension 2, should have a rectilinear appearance, and should be formed from a generator whose shape is related to the pattern of placement of the original adopters. One space-filling curve that meets these requirements is the rectilinear curve of Figure 4.A.

**Figure 3.D.**

The simulation is run on three original adopters with positions given below. Numerical entries show the likelihood, out of 300, that a new adopter will fall into a given cell. Zones of interaction between overlapping five–by–five grids are outlined by a heavy line (begin ulhc of F2; horizontally to urhc of F6; vertically to lrhc of G6; horizontally to lrhc of G5; vertically to lrhc of J5; horizontally to llhc of J3; vertically to llhc of G3; horizontally to llhc of G2; vertically to ulhc of F2 — should be a "less–fat" T–shaped enclosure that you have added to this figure). **Original adopters are in cells H3, E4, H5.**

North at the top.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | *Totals* |
|---|---|---|---|---|---|---|---|---|---|
| *A* | | | | | | | | | |
| *B* | | | | | | | | | |
| *C* | | 0.96 | 1.40 | 1.68 | 1.40 | 0.96 | | | 6.40 |
| *D* | | 1.40 | 3.01 | 5.47 | 3.01 | 1.40 | | | 14.29 |
| *E* | | 1.68 | 5.47 | 44.31 | 5.47 | 1.68 | | | 58.61 |
| *F* | 0.96 | 2.80 | 5.65 | 8.27 | 5.65 | 2.80 | 0.96 | | 27.09 |
| *G* | 1.40 | 3.97 | 8.27 | 7.70 | 8.27 | 3.98 | 1.40 | | 34.99 |
| *H* | 1.68 | 5.47 | 45.99 | 10.94 | 45.99 | 5.47 | 1.68 | | 117.22 |
| *I* | 1.40 | 3.01 | 6.87 | 6.02 | 6.87 | 3.01 | 1.40 | | 28.58 |
| *J* | 0.96 | 1.40 | 2.64 | 2.80 | 2.65 | 1.40 | 0.97 | | 12.82 |
| *K* | | | | | | | | | |
| | 6.40 | 20.69 | 79.30 | 87.19 | 79.31 | 20.70 | 6.41 | | 300 |

The generator is composed of three nodes hooked together by two edges in a straight path. This is scaled–down, by a factor of 1/2, and hooked to the endpoints of the original generator. Iteration of this procedure leads to a rectilinear tree with the desired properties. The approach of looking for a geometric form to fit a given set of conditions is like the calculus approach of looking for a differential equation to fit a given set of conditions. The difference here is that the shape of the generator and other information from early stages may be used to estimate the relative saturation or space–filling level.

The spatial position of the original adopters in Figure 3.B suggests a fractal generator in the shape of a "V" with an interbranch angle, $\theta$, of 90 degrees, while the V in Figure 3.C suggests a generator with $\theta \approx 53°$, that of Figure 3.D one with $\theta \approx 37°$, that of Figure 3.E one with $\theta \approx 28°$, and that of Figure 3.F one with $\theta \approx 23°$. Figures 4.B, 4.C, 4.D, 4.E, and 4.F suggest trees that can be generated using these values for $\theta$.

A rough measure of how much space each one "fills" may be calculated using Mandelbrot's formula for fractal dimension, D, as,

$$D = \frac{\ln N}{\ln(1/r)}$$

where $N$ represents the number of sides in the generator, which in all cases here is the value 2, and where $r$ is some sort of scaling value that remains constant independent of scale

**Figure 3.E.**

The simulation is run on three original adopters with positions given below. Numerical entries show the likelihood, out of 300, that a new adopter will fall into a given cell. Zones of interaction between overlapping five-by-five grids are outlined by a heavy line (begin ulhc of F2; horizontally to urhc of F6; vertically to lrhc of F6; horizontally to lrhc of F5; vertically to lrhc of J5; horizontally to llhc of J3; vertically to llhc of F3; horizontally to llhc of F2; vertically to ulhc of F2 — should be a "less-fat" T-shaped enclosure that you have added to this figure). **Original adopters are in cells H3, D4, H5.**

North at the top.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | *Totals* |
|---|---|---|---|---|---|---|---|---|---|
| *A* | | | | | | | | | |
| *B* | | 0.96 | 1.40 | 1.68 | 1.40 | 0.96 | | | 6.40 |
| *C* | | 1.40 | 3.01 | 5.47 | 3.01 | 1.40 | | | 14.29 |
| *D* | | 1.68 | 5.47 | 44.31 | 5.47 | 1.68 | | | 58.61 |
| *E* | | 1.40 | 3.01 | 5.47 | 3.01 | 1.40 | | | 14.29 |
| *F* | 0.96 | 2.36 | 4.04 | 4.48 | 4.04 | 2.37 | 0.96 | | 19.21 |
| *G* | 1.40 | 3.01 | 6.87 | 6.02 | 6.87 | 3.01 | 1.40 | | 28.58 |
| *H* | 1.68 | 5.47 | 45.99 | 10.94 | 45.99 | 5.47 | 1.68 | | 117.22 |
| *I* | 1.40 | 3.01 | 6.87 | 6.02 | 6.87 | 3.01 | 1.40 | | 28.58 |
| *J* | 0.96 | 1.40 | 2.64 | 2.80 | 2.65 | 1.40 | 0.97 | | 12.82 |
| *K* | | | | | | | | | |
| | 6.40 | 20.69 | 79.30 | 87.19 | 79.31 | 20.70 | 6.41 | | 300 |

(Mandelbrot, 1977). The difficulty in the case of trees, deriving from the complication of intersecting branches, is to select a suitable description for $r$. One angle, $\phi$, that remains constant throughout the iteration, and that produces the desired effect for the case in which the diffusion is the most saturated, is the base angle of the isoceles triangle with apex angle $\theta/2$ whose equal sides have the length of the equal sides of the two branches of the generator (Figure 5). When $r$ is taken as the cosine of $\phi$, then $D = 2$ in the case of Figure 4.A and it decreases dramatically as the trees generated by the distribution of original adopters fill less space (Table 2).

This decreasing sequence of $D$-values corresponds only loosely to Mandelbrot's measurements of fractal dimensions of trees (Mandelbrot, 1983); here, however, when $D = 1$ the corresponding tree is one with an interbranch angle of 120°. This has some appeal if one notes that then the tree associated with $D = 1$ might therefore represent a Steiner network (tree of shortest total length under certain circumstances) or part of a central place net. The numerical unit $D$-value would thus correspond to optimal forms for transport networks or for urban arrangements in abstract geographic space (in which Hägerstrand's diffusion procedure also exists).

One use for these $D$-values, which measure the relative space-filling by trees, might be as units fundamental to developing an algebraic structure for planning the eventual saturation level to arise in communities into which an innovation is introduced to selected adopters. By

**Figure 3.F.**

The simulation is run on three original adopters with positions given below. Numerical entries show the likelihood, out of 300, that a new adopter will fall into a given cell. Zones of interaction between overlapping five-by-five grids are outlined by a heavy line (begin at ulhc of F3; horizontally to urhc of F5; vertically to lrhc of J5; horizontally to llhc of J3; vertically to ulhc of F3 — should be a rectangular enclosure that you have added to this figure). **Original adopters are in cells H3, C4, H5.**

North at the top.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | *Totals* |
|---|---|---|---|---|---|---|---|---|---|
| *A* | | 0.96 | 1.40 | 1.68 | 1.40 | 0.96 | | | 6.40 |
| *B* | | 1.40 | 3.01 | 5.47 | 3.01 | 1.40 | | | 14.29 |
| *C* | | 1.68 | 5.47 | 44.31 | 5.47 | 1.68 | | | 58.61 |
| *D* | | 1.40 | 3.01 | 5.47 | 3.01 | 1.40 | | | 14.29 |
| *E* | | 0.96 | 1.40 | 1.68 | 1.40 | 0.97 | | | 6.41 |
| *F* | 0.96 | 1.40 | 2.64 | 2.80 | 2.64 | 1.40 | 0.96 | | 12.80 |
| *G* | 1.40 | 3.01 | 6.87 | 6.02 | 6.87 | 3.01 | 1.40 | | 28.58 |
| *H* | 1.68 | 5.47 | 45.99 | 10.94 | 45.99 | 5.47 | 1.68 | | 117.22 |
| *I* | 1.40 | 3.01 | 6.87 | 6.02 | 6.87 | 3.01 | 1.40 | | 28.58 |
| *J* | 0.96 | 1.40 | 2.64 | 2.80 | 2.65 | 1.40 | 0.97 | | 12.82 |
| *K* | | | | | | | | | |
| | 6.40 | 20.69 | 79.30 | 87.19 | 79.31 | 20.70 | 6.41 | | 300 |

TABLE 1

Sizes of zones of interaction and of largest individual cell value for each of the distributions of initial adopters in Figure 3.

| Figure number: Position of three original adopters. | Number of cells in interaction zone. | Largest value (out of 300) in individual cell. |
|---|---|---|
| Figure 3.A: linear arrangement | 25 | 55.25 |
| Figure 3.B: middle cell 1 unit north | 23 | 50.33 |
| Figure 3.C: middle cell 2 units north | 21 | 47.39 |
| Figure 3.D: middle cell 3 units north | 19 | 45.99 |
| Figure 3.E: middle cell 4 units north | 17 | 45.99 |
| Figure 3.F: middle cell 5 units north | 15 | 45.99 |

choosing judiciously the pattern of initial adopters, the relative space-filling of associated trees might be guided by local municipal authorities so as not to conflict with, or to interfere with, other issues of local concern. The $D$-values associated with triads of original adopters (as in Table 2) might serve as irreducible elements of this algebra, into which larger sets could be decomposed (much as positive integers ($>1$) can be decomposed into a product of powers of prime numbers). The manner in which the decomposition is to take place would

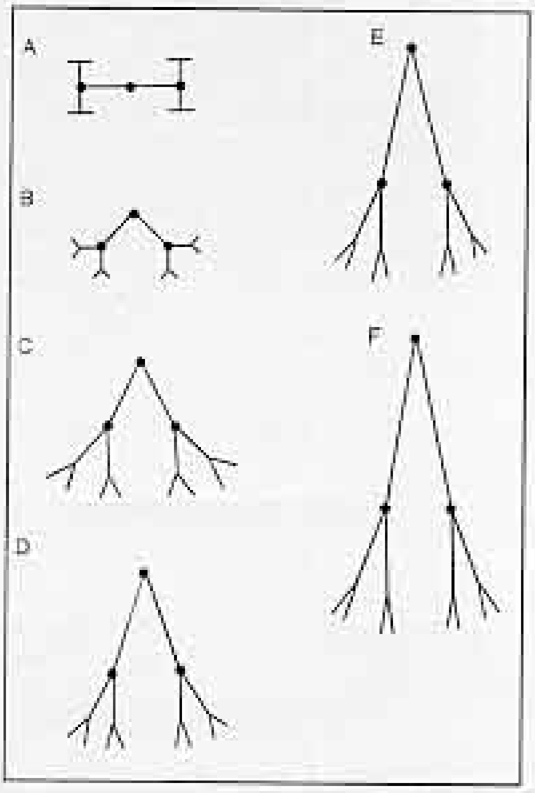


**Figure 4.**

Fractal trees derived from the diffusion grids of Figure 3; labels A through F correspond in the two Figures. The position of the distribution of original adopters in Figure 3 determines the positions for generators for fractal trees. The interbranch angle, $\theta$, is constant within a tree; values of $\theta$ decrease from A. to F. as does the fractal dimension, $D$.
A. $\theta = 180°$, $D = 2$.
B. $\theta = 90°$, $D \approx 0.72$.
C. $\theta \approx 53.13°$, $D \approx 0.47$.
D. $\theta \approx 36.87°$, $D \approx 0.38$.
E. $\theta \approx 28.07°$, $D \approx 0.33$.
F. $\theta \approx 22.62°$, $D \approx 0.30$.

likely be an issue of considerable algebraic difficulty, no doubt requiring the use of geographic constraints to limit it. (For, unlike the parallel with integer decomposition, this one would seem not to be unique.) An initial direction for such a diffusion-algebra might therefore be to exploit the parallel with the Fundamental Theorem of Arithmetic.

Another use might involve a self-study by the National Center for Geographic Information and Analysis (NCGIA) in order to monitor the diffusion of Geographic Information

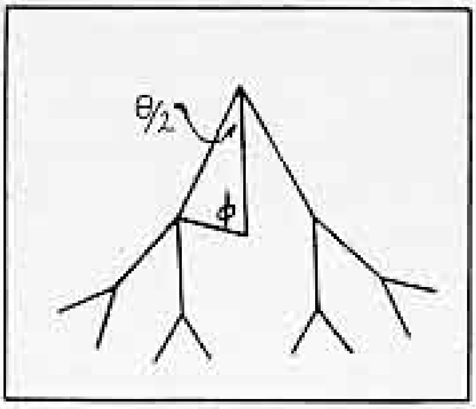


**Figure 5**. The construction of the angle $\phi$ used in the calculation of the fractal dimension, $D$, of the trees in Figure 4.

TABLE 2

$D$-values, which suggest extent of space-filling, for the trees (Figure 4) representing the patterns of initial adopters in Figure 3.

| Figure number: Position of three original adopters. | Size of interbranch angle, $\theta$, in associated tree. | Size of $\phi = (180 - (\theta/2))$ | $D$-value: $D = (\ln 2)/(\ln(1/\cos\phi))$ |
|---|---|---|---|
| Figure 3.A: linear arrangement | Figure 4.A: $\theta = 180°$ | 45° | 2 |
| Figure 3.B: middle cell 1 unit north | Figure 4.B: $\theta = 90°$ | 67.5° | 0.721617 |
| Figure 3.C: middle cell 2 units north | Figure 4.C: $\theta \approx 53.13°$ | 76.78 | 0.471288 |
| Figure 3.D: middle cell 3 units north | Figure 4.D: $\theta \approx 36.87°$ | 80.78 | 0.378471 |
| Figure 3.E: middle cell 4 units north | Figure 4.E: $\theta \approx 28.07°$ | 82.98 | 0.32971 |
| Figure 3.F: middle cell 5 units north | Figure 4.F: $\theta \approx 22.62°$ | 84.35 | 0.299116 |

System (GIS) technology through the various programs designed to promote this technology in the academic arena. University test-sites for the materials of the NCGIA, for example, might be selected as "seeds" with deliberate plans for using a diffusion structure based on these seeds to bring later adopters up to date.

Another use might involve the determination of sites for locally unwanted land uses such as waste sites, prisons, and so forth. Regions expected to experience high concentrations of population coming from the totality of innovations already introduced, or to be introduced, might be overburdened by such a landuse. When relative fractal saturation estimates are run on a computer in conjunction with a GIS, local municipal authorities might examine issues such as this for themselves.

**Attraction: the Julia set $z = z^2 - 1$**

A different way to view the space-filling characteristics of the diffusion example is to consider each initial adopter as an "attractor" of other adopters, once again suggesting a fractal connection. Viewed broadly, the diffusion example sees adopters attracted to points within an abstract geographic space. The fractal connection is to describe space-filling rather than to describe the pattern or the direction of the attraction. The material below suggests a means of viewing the broad class of spiral geographic phenomena as repelled away from a Julia set toward points of attraction within and beyond the "fractal": hence, pattern and direction of attraction.

The familiar Mandelbrot set, comprising a large central cardioid and circles tangent to the cardioid, along with points interior and exterior to this boundary, is associated with $z = z^2 + c$, where "$z$" is a complex variable and "$c$" is a complex constant (Mandelbrot, 1977; Peitgen and Saupe, 1988). When constant values for $c$ are chosen, Julia sets fall out of the Mandelbrot set (Peitgen and Saupe, 1988).

When $c = 0$, the corresponding Julia set is the unit circle centered at the origin. The boundary itself is fixed, as a whole, under the transformation $z \mapsto z^2$, although only the individual point $(1,0)$ is itself fixed. Points interior to the boundary are attracted to the origin: for them, iteration of the transformation leads eventually to a value of 0. Points outside the circle are attracted toward infinity; the boundary repels points not on it (Peitgen and Saupe, 1988). Various natural associations might be made between this simple Julia set and astronomical phenomena such as orbits or compression within black holes.

When $c = -1$, the corresponding Julia set is described by $z = z^2 - 1$ (Figure 6). The attractive fixed points are $0$, $-1$, and infinity. The repulsive fixed points on the Julia set, found using the "quadratic" formula on $z^2 - z - 1 = 0$, are at distances of $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$ units from the origin along the real axis (distinguished on Figure 6). Points within the Julia set are attracted alternately to 0 and to $-1$ as attractive "two-cycle" fixed points; points outside it are attracted to infinity. To see the "two-cycle" effect, iterate the transformation using $z = 1.59$ (located within the Julia set) as the initial value.

$$
\begin{aligned}
1.59 &\mapsto 1.5281 \mapsto 1.3350896 \mapsto 0.7824643 \mapsto -0.3877497 \\
&\mapsto -0.849650 \mapsto -0.2780946 \mapsto -0.9226634 \mapsto -0.1486922 \\
&\mapsto -0.9778906 \mapsto -0.0437299 \mapsto -0.9980877 \mapsto -0.003821 \\
&\mapsto -0.9999854 \mapsto -0.0000292 \mapsto -1 \mapsto -0.00000000016 \\
&\mapsto -1 \mapsto 0.
\end{aligned}
$$

This value of $z$ is attracted to $-1$ faster than it is to 0. In this case, iteration strings close down on points of at traction; this is not the case for all Julia sets. The choice of the value of $c$ determines whether or not such strings can escape (Peitgen and Saupe, 1988).

The movement of an initial point toward an attractor, and away from a fixed boundary (as above), suggests a view of this Julia set as an axis: lines from which the movement of points are measured are "axes." Indeed, the repulsive fixed points on this set, located at $((1 + \sqrt{5})/2, 0)$ and $((1 - \sqrt{5})/2, 0)$, might serve as "units." They are the non- zero terms of the coefficients in the generating function for the Fibonacci numbers (thanks to W. Arlinghaus for suggesting this connection to the Fibonacci generating function; Rosen, 1988). For, the $n$th Fibonacci number, $a_n = a_{n-1} + a_{n-2}$, $a_0 = 0$, $a_1 = 1$, is generated

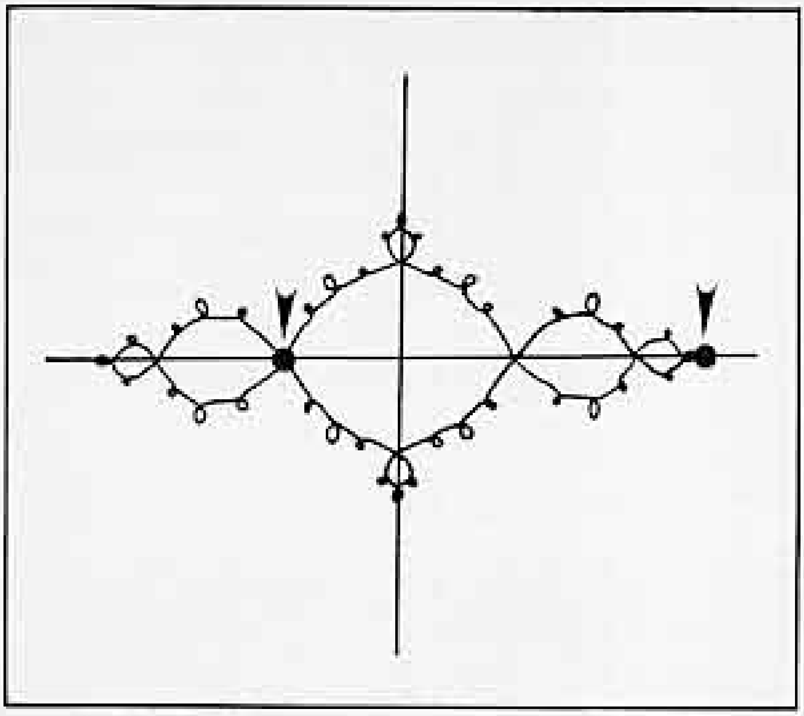

**Figure 6.**

The Julia set $z = z^2 - 1$. Fixed points $((1 \pm \sqrt{5})/2, 0)$ are distinguished on the boundary.

by

$$a_n = \frac{1}{\sqrt{5}}((1+\sqrt{5})/2)^n - \frac{1}{\sqrt{5}}((1-\sqrt{5})/2)^n.$$

Because the Fibonacci sequence can be expressed using the logarithmic spiral, this particular Julia set with these values as "units" might therefore serve as an axis from which to measure spiral phenomena at various scales ranging from the global to the local: from, for example, the climatological to the meteorological.

The mechanics of using this curve as an axis might involve an approach different from that customarily employed. The curve might, for example, be mounted as an equator on the globe partitioning the earth into two pieces in much the way that a seam serves as an equatorial line to partition the hide on a baseball. In this circumstance, there would be freedom to choose how the equator partitions the earth's landmass. It might be located in such a way that exactly half of the earth's water and half of the earth's land lie on either side of the Julia set (using theorems from algebraic topology (Lefschetz, 1949; Dugundji, 1966; Spanier, 1966)).

**Beyond the fractal: a graph theoretic connection.**

The notions of "attraction" and "repulsion" have also been expressed in the physical world, using graph theory (Harary, 1969; Uhlenbeck, 1960). Fractals rely on distance, angle, or some other quantifier; graphs do not, and in that respect, are more general than are fractals. Fractal-like concepts, such as space–filling and the associated image compression (Barnsley, 1988), may be characterized using graphs, as below (Arlinghaus, 1977; 1985).

This strategy will be expressed in terms of cubic trees (all nodes are of degree three, unless they are at the tip of a branch) of shortest total length (Steiner trees) of maximal branching. It could be expressed in terms of graphs of various linkage patterns; what is important is to begin with some systematic process for forming graphs.

Given a geographic region whose periphery is outlined by landmark positions at $P_1$, $P_2$, $P_3$, $P_4$, and $P_5$ (Figure 7.A). View the landmarks as the nodes of a graph and the peripheral outline as the edges linking these nodes (Figure 7.A). A "global" network within the entire pentagonal region might lie along lines of a Steiner (shortest total distance) tree (Figure 7.A) (Arlinghaus, 1977; 1985) attached to the pentagonal hull joining neighboring branch tips (Balaban, *et al.*, 1970).

Figure 7.B will be used as an initial figure from which to produce a network that penetrates triangular geographic subregions (introducing edges $P_2P_5$ and $P_2P_4$) more deeply than does the global network of Figure 7.A, yet retains the Steiner characteristic locally within each geographic subregion. An iterative process using Steiner trees (as a "Steiner transformation") will be applied to Figure 7.B (Arlinghaus, 1977; 1983), as follows.

Introduce Steiner networks into each of the three triangular regions and remove the edges $P_2P_5$ and $P_2P_4$ so that a new network, containing two quadrangular cells, is hooked into the pentagon $P_1P_2P_3P_4P_5$ (Figure 7.C). Repeat this procedure in the network of Figure 7.C, introducing Steiner networks into all circuits that do not have an edge in common with the pentagon $P_1P_2P_3P_4P_5$. Thus, the two four–sided circuits, $P_5S_1P_2S_2$; $P_2S_2P_4S_3$, in Figure 7.C are replaced with the lines of the network, $P_5S'_1$, $S_1S'_1$, $S'_1S'_2$, $P_2S'_2$, $S'_2S_2$; $S_2S'_3$, $P_2S'_3$, $S'_3S'_4$, $S'_4P_4$, $S'_4S_3$, shown in Figure 7.D. Repeat this process in Figure 7.D, using a Steiner tree, $S_2S''_1$, $S'_2S''_1$, $S''_1S''_2$, $S''_2P_2$, $S''_2S'_3$, to replace the single four–sided cell, $P_2S'_2S_2S'_3$, not sharing an edge with $P_1P_2P_3P_4P_5$. The result, shown in Figure 7.E, is a tree which cannot be further reduced using the Steiner transformation. It satisfies the initial conditions of generating a tree more local than the Steiner network of maximal branching on $P_1P_2P_3P_4P_5$ (but with local Steiner characteristics), while retaining the global structure of a graph–theoretic tree hooked into $P_1P_2P_3P_4P_5$ in a pattern similar to that of the global Steiner tree (with only local variation as along the edge $S_2S''_1$). This process attempts to integrate local with global concerns. In this case, the process terminates after a finite number of steps; were it to continue, greater space–filling of the geographic region by lines of the network would occur (Arlinghaus, 1977; 1985).

A natural question to ask is whether or not this process necessarily terminates; do successive applications generate a finite reduction sequence of the "cellular" structure into a "tree" structure within $P_1P_2P_3P_4P_5$? Or, is it possible that this transformation, applied iteratively, might fill enough space to choke the entire region with an infinite regeneration of cells and of lines bounding those cells (Arlinghaus, 1977; 1985)?

In this vein, take Figure 7.B and add one edge to it, creating four triangular geographic

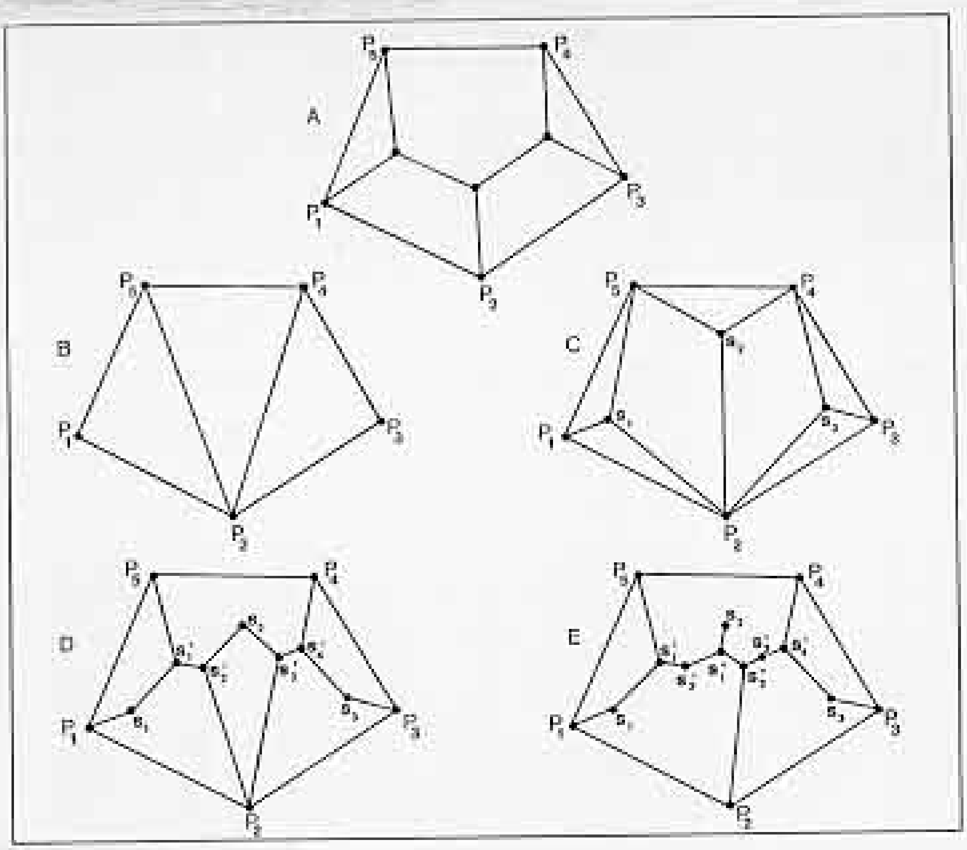

**Figure 7.**

Network location within geographic regions. Points of the pentagonal hull have "P" as a notational base; Steiner points have "S" as a notational base. A. A Steiner (shortest total distance) tree linking five locations. B. Partition into three distinct, contiguous geographic regions. C. Steiner networks in each geographic region; boundaries separating regions are removed. D. Steiner networks in two quadrangular circuits; circuit boundaries removed. E. Process repeated on remaining quadrangular cell; the result is a tree with local Steiner characteristics that provides global linkage following the basic pattern of the global Steiner tree (Figure 7.A)).

regions (Figure 8.A). Apply the same process to it as above, producing the networks shown in Figures 8.B and 8.C. Clearly, further iteration would simply produce a greater number of polygonal cells, tightly compressed around the node $P_2$. Discovering a means to calculate

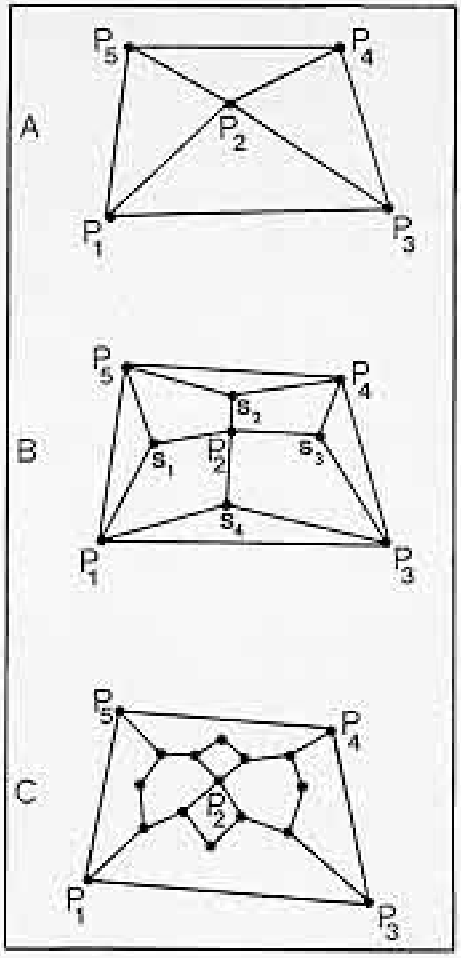


**Figure 8.**

A modification of Figure 7. An extra edge is added to Figure 7.A, creating a graph–theoretic "wheel." When the procedure displayed in Figure 7 is applied to this initial configuration, cells are added within the hull (B. and C.), rather than removed.

the dimension of this compression is an open issue. It is not difficult, however, to understand under what conditions this sequence might, or might not, terminate (Comments (based on material in Arlinghaus, 1977; 1985) below).

Definition (Harary, 1969; Tutte, 1966).

A wheel $W_n$ of order $n$, $n > 3$, is a graph obtained from an $n$-gon by inserting one new vertex, the hub, and by joining the hub to at least two of the vertices of the $n$-gon by a finite sequence of edges ($P_2$ is the hub of a wheel formed in Figure 8.A).

Comment 1

Hubs of wheels are invariant, as hubs of wheels, under a sequence of successive applications of the Steiner transformation described above.

Comment 2

Suppose that there exists a finite set of contiguous triangles, $T$. If $T$ contains a wheel, then a sequence of successive applications of the Steiner transformation to $T$ fails to produce an irreducible tree. The sequence fails to terminate (as long as the Steiner trees produced at each stage are not degenerate).

Comment 3

Suppose that there exists a finite set of contiguous triangles $T = \{L_1 \ldots L_m\}$ with vertex set $V = \{P_1 \ldots P_n\}$, $n > m$ (as in Figure 7.B, $m = 3$, $n = 5$). Suppose that $T$ does not contain a wheel. The number of steps $M$, in the sequence of successive applications of the Steiner transformation to $T$ required to reduce $T$ to a tree is

$$M = (\max(\text{degree}(P_i))) - 1.$$

Since $T$ does not contain a wheel, it follows from Comment 2 that the reduction sequence is finite. The actual size of $M$ might be found using mathematical induction on the number of cells in $T$ and on the graph-theoretic degree of $P_i$.

The examples shown in Figures 7 and 8, together with the Comments above, suggest that a sequence of successive applications of the Steiner transformation to such "geo- graphs" resolves scale problems in the same manner as fractals. A natural next step beyond the fractal might be to note that a graph is a simplicial complex of dimension 0 or 1 (Harary, 1969). Thus, similar strategy might be applied there: the triangles of Figure 7.B might represent simplexes of arbitrary dimension in a simplicial complex of higher dimension. Theorems from algebraic topology might then be turned back on the mapping of geographic information using a computer. This notion is already in evidence: because "point," "line," and "area" translate into the topological notions of "0-cell," "1-cell," and "2-cell" in a Geographic Information System, cells in the underlying computerized "sim-pixel" complex can then be colored as "inside" or "outside" a given data set. This follows from the Jordan Curve Theorem (of algebraic topology).

Independent of choice of theoretical tool—from fractal to graph to simplicial complex—the resolution of scale is achieved by uniting local and global mathematical structures: within fractal geometry as well as beyond it.

> "In nature, parts clearly do fit together into real structures, and the parts are affected by their environment. The problem is largely one of understanding. The mystery that remains lies largely in the nature of structural hierarchy, for the human mind can examine nature on many different scales sequentially, but not simultaneously."
>
> *C. S. Smith, in Arthur L. Loeb, 1976.*

**References.**

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3. SHORT ARTICLE

# GROUPS, GRAPHS, AND GOD

*William C. Arlinghaus*

**Abstract**

The fact that almost all graphs are rigid (have trivial automorphism groups) is exploited to argue probabilistically for the existence of God. This is presented in the context that applications of mathematics need not be limited to scientific ones.

Recently I was teaching some elementary graph theory to a class studying finite mathematics when, inevitably, someone asked the question, "But what is all this good for?" This question is posed often, and the answer rarely satisfies either the poser or the responder.

Usually the responder is a little annoyed at the question, for often a deeper look by the poser would have yielded some insight into the question. But also the responder is irritated on account of inability to give a satisfactory answer. Two obvious choices present themselves:

1. Most mathematicians find the process of discovery in mathematics rewarding in itself. An elegantly concocted proof of a pleasingly stated theorem gives a sense of satisfaction and a joy in the appreciation of beauty that makes real-world application unnecessary. But the questioner usually lacks the mathematical maturity necessary to appreciate this answer.

2. The most readily available sources of application are in the physical sciences, although there is an increasing use of mathematics in the social sciences. But often the mathematician lacks confidence in the extent of his knowledge of the appropriate science. This makes response somewhat tentative, and again the response fails to satisfy the questioner.

On this occasion, a third alternative presented itself. Being human, all people have some interest in philosophy, varying from formal study to informal discussion. What better place to find a meeting ground to answer the above question?

**Definition 1** Let $G$ be a finite graph. Then the automorphism group of $G$, Aut $G$, is the set of all edge– preserving 1–1 maps of the vertex set $V(G)$ onto itself, with composition the binary operation.

Informally, one can view the size of Aut $G$ as a measure of the amount of symmetry that $G$ possesses, the structure of Aut $G$ as a measure of the way in which the symmetry occurs.

**Definition 2** Let $g(n)$ be the number of $n$–point graphs which have non–identity automorphism group, $h(n)$ the number of $n$–point graphs. Define $f(n) = (g(n))/(h(n))$.

It is well–known [2, 3, 4, 6] that

$$\lim_{n \to \infty} f(n) = 0.$$

In other words, almost all graphs have identity automorphism group.

Viewed from a philosophical perspective, this says that the probability of symmetry existing in a complex world is virtually zero. Yet symmetry abounds in our own complex world. This provides plausibility for the view that the world did not evolve randomly, that some force shaped it; *i.e.*, it may be taken as a "proof" for the existence of God.

One might point out at this point that many other proofs for the existence of God rely on mathematical foundations. Causality depends on the belief that infinite regress through successive causes must eventually reach an infinite First Cause. Anselm's ontological argument involves the idea of being able to abstract the idea of perfection and then posit its existence. Pascal's view that one should behave as if God exists on the basis of expected value of reward if He does is surely a probabilistic view.

Since there is a whole first-order class of logical sentences about graphs [1] each of which is either almost always true or almost never true, further examples of this nature should be easy to find. Indeed, to close with one, observe that [3] almost every tree has non-trivial automorphisms. Thus even a random tree has some symmetry. This might lead one to question Joyce Kilmer's statement that "Only God can make a tree."

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The author is Associate Professor and Chairperson, Department of Mathematics and Computer Science, Lawrence Technological University, 21000 West Ten Mile Road, Southfield, MI 48075. This material was presented as a paper to the MIchigan GrapH TheorY (MIGHTY) meeting, Saturday, October 29, 1988 at Oakland University, Rochester, Michigan.

## 4. REGULAR FEATURES

### Theorem Museum —

One purpose of a museum is to display to the public concepts of an enduring character in some sort of hands-on manner that will promote grasp and retention of that concept. When the display also piques the interest of the observer, so much the better.

This particular feature is motivated by a variety of sources. About ten years ago, William E. Arlinghaus and I submitted a proposal to *The Mathematical Intelligencer* for a museum exhibit, based on constructing a giant Rubik's (trademarked name) Cube, to teach people elements of group theory by carrying them physically (in Ferris wheel fashion) through group theoretic motions while riding inside the cube. At the same time, I also submitted another proposal to the same journal for another museum exhibit to be called "The Garden of Shadows." This was to be an outdoor display based on using the sun as a point source of light at "infinite" distance to physically demonstrate a number of theorems from projective geometry.

A number of years later, I came to know fine artist David Barr who specializes in large outdoor sculpture. Bill Arlinghaus and John Nystuen are continuing participants at my IMaGe meetings; over the years others have joined us, and one of the most regular is David Barr. Often, we have, as a group, discussed various aspects of using outdoor sculpture to educate the public as well as colleagues. John Nystuen suggested that we build an actual, physical Theorem Museum, dedicated to Theorems that could be portrayed in sculpture (similar to the *Intelligencer* proposals). Barr informs us that interest in this sort of idea is well-established in the world of Art: Swiss artist Max Bill, and other Western European painters and sculptors, create art determined by mathematical equations of various sorts. Here, we are suggesting that it is the theorem, itself, that is art. This feature is therefore the written groundwork for such a museum. If you have a favorite theorem, and can suggest how to express it physically using artistic media, you might want to consider submitting it to *Solstice* for this section. Theorems that can be so envisioned may also be ones that are easiest to mold to fit other real- world phenomena. Projective geometry is a highly general geometry that is perfectly symmetric in its statements. The reason for this is that "parallel" lines meet in "ideal" points, lying on an "ideal" line, at infinity. Thus, in the projective plane, as in the Euclidean plane, two points determine a line; however, in the projective plane a dual statement (that is NOT true in the Euclidean plane) that two lines determine a point is also true. Duality in language results in symmetry of form. Here is a remarkable theorem from projective geometry (see reference for proof).

**Desargues's Two Triangle Theorem.**

Given two triangles, $PQR$ and $P'Q'R'$ such that $PP'$, $QQ'$, and $RR'$ are concurrent at point O. It follows that the intersection points of corresponding sides of the two triangles are collinear. That is, suppose that corresponding sides $PQ$ and $P'Q'$ intersect at point L, that $QR$ and $Q'R'$ intersect at point M, and that $PR$ and $P'R'$ intersect at point N. Then, the points L, M, and N all lie along a single straight line (please draw your own figure from these directions).

From a geographic viewpoint, this says that if a rigid tetrahedron were built of metal rods with apex at point O, that any two triangles that fit perfectly inside this structure would have this property. One triangle "projects" from a point (as for example in gnomonic

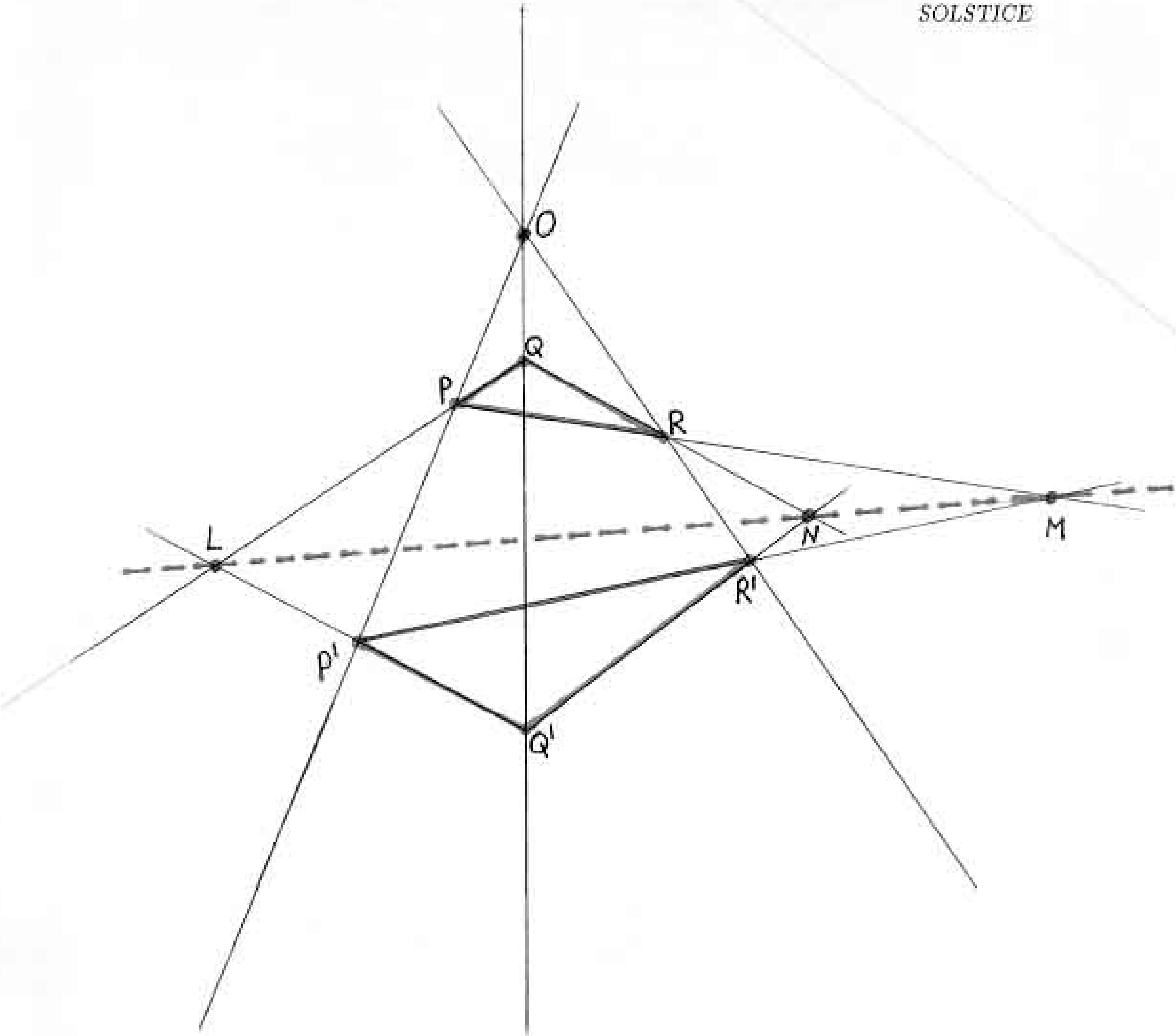


Figure to accompany Desargues's Two Triangle Theorem

or stereographic map projection) to the other. This might suggest a way to deform cells of a triangulation of a region of the earth into one another in such a way that this Desargues's line serves as some sort of an invariant of the deformation.

This observation might then make one wonder what sorts of geometries exist that do not obey Desargues's Theorem. There is a whole class of "Combinatorial geometries" or finite projective planes that do not.

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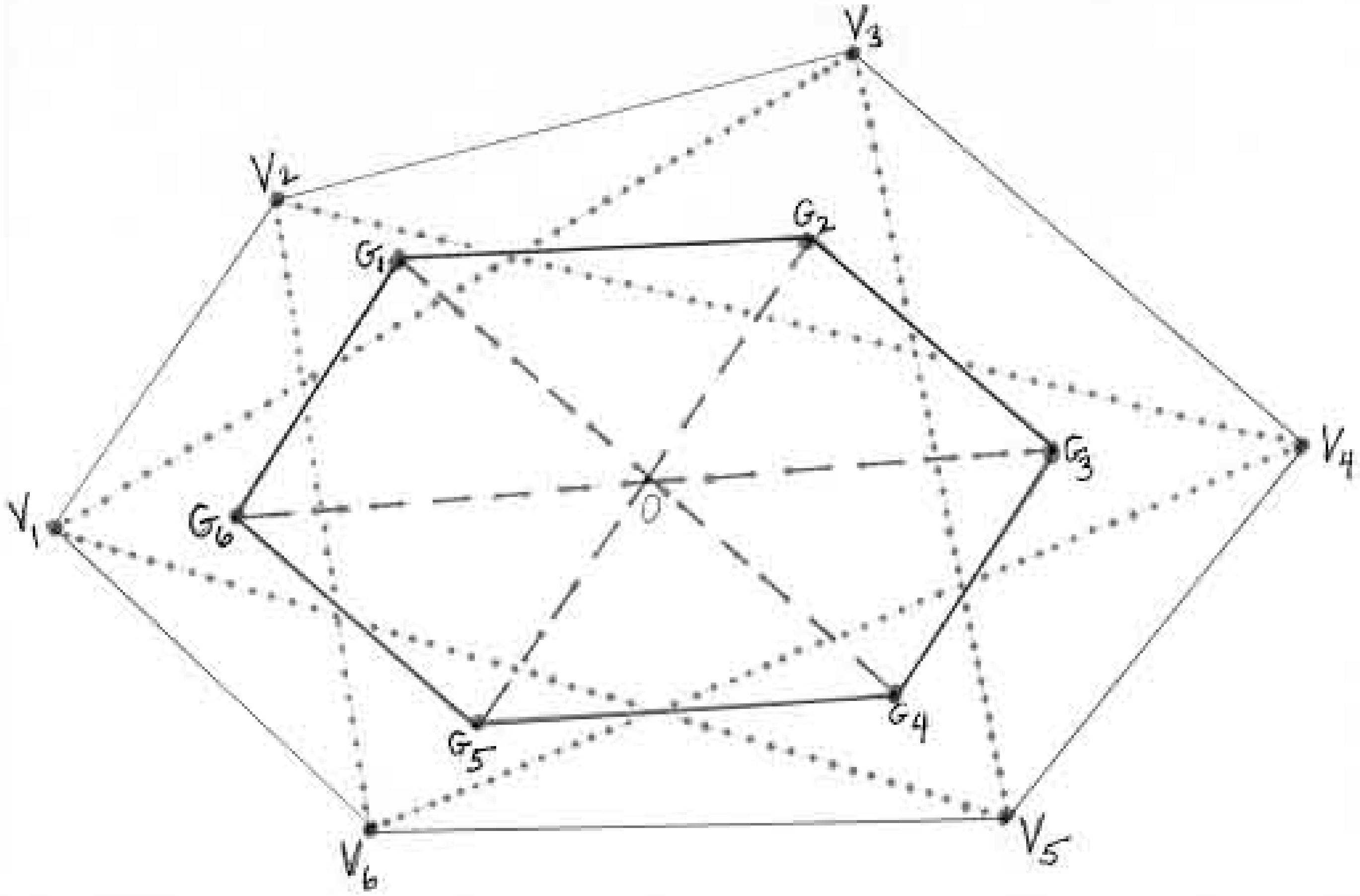


Figure to accompany construction of centrally symmetric hexagon.

**Construction Zone** —

One possible direction for application of Desargues's Theorem is to deform one tesselation of a region into another, leaving something invariant. Another related issue with tesselations is to try to regularize a tesselation composed of irregularly shaped cells. The following construction shows how to derive a centrally symmetric hexagon from an arbitrary convex hexagon. Given an arbitrary convex hexagon, $V_1V_2V_3V_4V_5V_6$. Join alternate vertices to inscribe a six-pointed star within this hexagon—that is, draw lines $V_1V_3$, $V_2V_4$, $V_3V_5$, $V_4V_6$, $V_5V_1$, $V_6V_2$ (it is suggested that you do so on a separate sheet of paper, at this point).

This produces six distinct triangles (of interest here–of course there are more):

$$\Delta V_1V_2V_3;\quad \Delta V_2V_3V_4;\quad \Delta V_3V_4V_5;\quad \Delta V_4V_5V_6;\quad \Delta V_5V_6V_1;\quad \Delta V_6V_1V_2.$$

To find the center of gravity of any triangle, find the point at which the medians are concurrent (the median is the line joining a vertex to the midpoint of the opposite side). This point is the center of gravity. Find the centers of gravity

$$G_1,\quad G_2,\quad G_3,\quad G_4,\quad G_5,\quad G_6$$

of each of the triangles distinguished above (in the order suggested). The hexagon determined by these centers of gravity will be centrally symmetric. That is, opposite sides will be equal in length and parallel to each other:

$$G_1G_2 \parallel G_4G_5; \quad |G_1G_2| = |G_4G_5|;$$

$$G_2G_3 \parallel G_5G_6; \quad |G_2G_3| = |G_5G_6|;$$

$$G_3G_4 \parallel G_6G_1; \quad |G_3G_4| = |G_6G_1|.$$

Another way of visualizing the symmetry is to observe that the three lines joining $G_1G_4$, $G_2G_5$, $G_3G_6$ are concurrent at a single point (call it $O$). In this way, one might also determine a "center" for this symmetric hexagon which might then serve as a point to which a reference value might be attached for the arbitrary hexagon from which it was derived. This centrally symmetric hexagon is called the Dirichlet region of the arbitrary convex hexagon. This construction can be proved using Euclidean geometry (if requests come in, I'll put it in a later issue).

This feature is based on discussions in

Kasner, Edward, and Newman, James R. "New names for old," in *The World of Mathematics*, edited by James R. Newman, Volume III, 1996-2010. New York: Simon and Schuster, 1956.

Coxeter, H. S. M. *Introduction to Geometry*, New York: Wiley, 1961.

**Reference Corner —**

Point set theory and topology. A recent pleasant evening spent with Hal Moellering had him questioning me and Bill Arlinghaus as to what might be reasonable, or useful, references from which graduate students in geography could get some sort of grasp of the elements of point set topology. A few references are listed below; send in your favorites and they will be printed next time. Hope that mathematicians as well as geographers will do so. Future topics to include graph theory and number theory as well as others suggested by reader input. Thanks Hal for the idea (generated by your questions) of doing this feature!

Some long-time favorites and classics:

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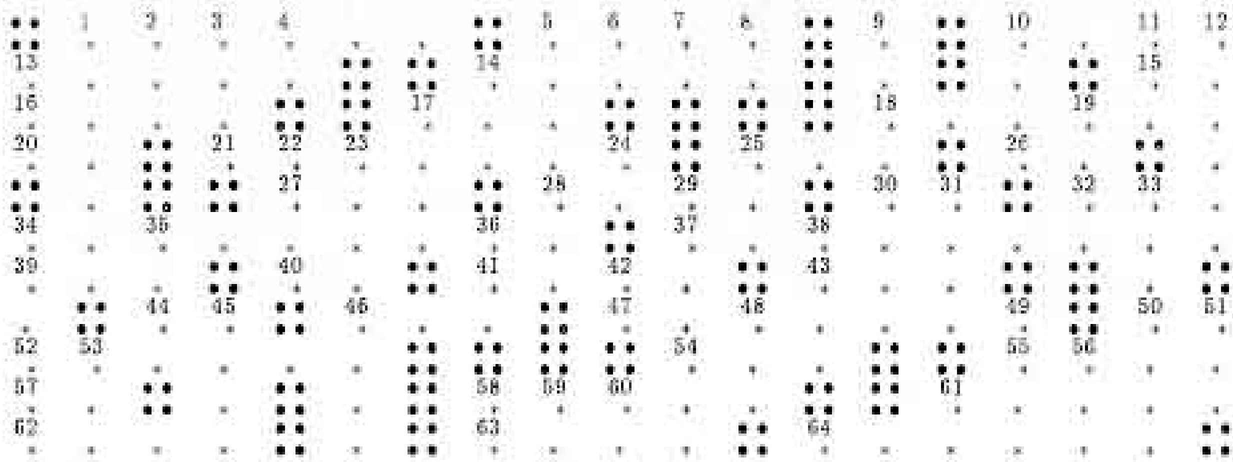

**Games and other educational features —**

### Crossword puzzle.

The focus of this puzzle is on herbs and spices. Spice trade has helped to shape many geographic alignments and spices such as pepper, known from its preservative characteristic, helped make long voyages possible. Puzzles should be fun; they should also stimulate thought and offer some sort of educational value. If you think that this puzzle might be of use to your students in this capacity, feel free to copy it from this page. Think of the asterisks as the blank squares, or as tiles with letter on the other side. Each set of four bullets represents a black square.

ACROSS

1. Plant of the Capsicum family, native to the Americas. Good source of vitamins A and C. Some varieties are native to Tabasco in Mexico.
5. Fruit native to the Americas is the prickly —.
10. Powder made from young sassafras leaves that is essential in making creole gumbo.
13. The "royal" herb – often the dominant herb in Pesto.
14. Bush–bud often seen in Tartare sauce or with an anchovy coiled around it.
15. Hour – abbreviation.
16. College of Liberal —.
17. A fundamental tool of the geographer and of the mathematician.
18. U.S. state – remove one letter from the spice in 47 across to form an anagram of this state name.
20. United States – abbreviation.
21. Jumble of letters in "another."
25. Black, sticky substance.
26. Eastern Uganda – abbreviation.
27. The bran of this grain is much in vogue.
28. Unit on a ruler.

30. He, she, —.
32. Herb sometimes used in fruit cup. Can cause severe allergic reactions. Also: French for street.
34. Along with coriander or cumin, this is a dominant ingredient in many curries.
37. A plant extract from which candies can be made.
39. In humans, the color blue, for this, is a recessive genetic trait.
40. Senior – abbreviation.
41. Spice with flavor close to nutmeg.
43. Chronological or mental —.
44. Association of American Geographers: —G.
46. "A poem should be palpable and mute; As a globed fruit," from Archibald MacLeish's "— Poetica."
47. Often found in Italian sauces.
50. Fifth and sixth letters of the alphabet used in English.
52. Spice often ground freshly and sprinkled on eggnog.
54. Eau de —.
55. Noise a lion might make.
57. First two letters of Spanish for United States.
58. Jumble of the letters in the name of an herb with a licorice flavor.
61. Word that might describe the flavor of a julep (adjectival form).
62. This broadleaf "big onion" is a key ingredient in Vichyssoise.
63. Herb used in many pickled cucumbers.
64. "Spiced-up" multiplication tables might be called "—" tables.

DOWN

1. This herb supposedly has the power to destroy the scent of garlic and onion.
2. East, in French.
3. Italian city – home to Fibonacci.
4. Postal letter (abbreviation)
5. Orangish powder often association with Hungarian dishes.
6. East Prussia (abbreviation).
7. Almost everywhere (mathematical term – abbreviation).
8. Railroad (abbreviation).
9. First initial and last name of former Panamanian leader.
10. A complimentary copy is a — one.
11. Left hand opponent (duplicate bridge term, abbreviation).
12. Jumble of the word "neared."
17. "—s and bounds."
19. Spiritual guide in Hinduism.

22. Poland China is a variety of these.
23. This herb is often held in vinegar because its leaf veins stiffen when dried and do not resoften when cooked. "Estragon" in French.
24. "— A Clear Day"
25. "Though" – some newspapers spell that word in this way.
29. This herb loses most of its flavor when dried: "Pluches de cerfeuil" refers to sprigs of this herb.
31. If/"—": typical manner in which a theorem is stated.
33. Removes from political office.
34. One variety of this herb, often used in conjunction with fat fish and lentils, is known as Florence —.
35. Tidy.
36. Paramedic vans are often marked with these three letters.
38. Uncontrolled anger.
42. Company (abbreviation)
45. Running — (Malay word). To be in a violently frenzied state.
48. Fine German white wine made from grapes harvested after frost: —wein.
49. Oyster Research Institute of Michigan, might be abbreviated thus.
51. Popular description of wok cookery: stir—.
53. Employ.
56. Identity element of the integers under multiplication.
58. Anno Domini (abbreviation)
59. National income (abbreviation)
60. Elevated train (abbreviation) – forms "Loop" in Chicago.
61. Prefix meaning "muscle."

**Coming attractions —**

Feigenbaum's number
Pascal's theorem from projective geometry
Braikenridge–MacLaurin construction for a conic in the projective plane.

| | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ■ | 1 P | 2 E | 3 P | 4 P | E | R | ■ | 5 P | 6 E | 7 A | 8 R | ■ | 9 M | ■ | 10 F | I | 11 L | 12 E |
| 13 B | A | S | I | L | ■ | ■ | 14 C | A | P | E | R | ■ | N | ■ | R | ■ | 15 H | R |
| 16 A | R | T | S | ■ | ■ | 17 M | A | P | ■ | ■ | ■ | ■ | 18 O | R | E | 19 G | O | N |
| 20 U | S | ■ | 21 A | 22 B | 23 T | E | N | R | 24 O | ■ | 25 T | A | R | ■ | 26 E | U | ■ | A |
| ■ | L | ■ | ■ | 27 O | A | T | ■ | 28 I | N | 29 C | H | ■ | 30 I | 31 T | ■ | 32 R | 33 U | E |
| 34 F | E | 35 N | U | G | R | E | 36 E | K | ■ | 37 H | O | 38 R | E | H | O | U | N | D |
| 39 E | Y | E | ■ | 40 S | R | ■ | 41 M | A | 42 C | E | ■ | 43 A | G | E | ■ | ■ | S | ■ |
| N | ■ | 44 A | 45 A | ■ | 46 A | R | S | ■ | 47 O | R | 48 E | G | A | N | 49 O | ■ | 50 E | 51 F |
| 52 N | 53 U | T | M | E | G | ■ | ■ | ■ | ■ | 54 V | I | E | ■ | ■ | 55 R | 56 O | A | R |
| 57 E | S | ■ | O | ■ | O | ■ | 58 A | 59 N | 60 E | I | S | ■ | ■ | 61 M | I | N | T | Y |
| 62 L | E | E | K | ■ | N | ■ | 63 D | I | L | L | ■ | 64 T | H | Y | M | E | S | ■ |

**Crossword puzzle solution**

5. SAMPLE OF HOW TO DOWNLOAD THE ELECTRONIC FILE

This section shows the exact set of commands that work to download this file on The University of Michigan's Xerox 9700. Because different universities will have different installations of TEX, this is only a rough guideline which *might* be of use to the reader.

This document prints out to be about 50 pages; on UM equipment, there are varying rates at varying times of day. At the minimum rate, the cost to print this out, using TEX , is about six dollars.

ASSUME YOU HAVE SIGNED ON AND ARE AT THE SYSTEM PROMPT, #. # create −t.tex

# percent-sign t from pc c:backslash words backslash solstice.tex to mts −t.tex char notab

[this command sends my file, solstice.tex, which I did as a WordStar (subdirectory, "words") ASCII file to the mainframe]

# run *tex par=−t.tex

# run *dvixer par=−t.dvi

# control *print* onesided

# run *pagepr scards=−t.xer, par=paper=plain

# SOLSTICE

Institute of Mathematical Geography

Journal of the
Institute of Mathematical Geography

Winter, 1990

# SOLSTICE:

## AN ELECTRONIC JOURNAL OF GEOGRAPHY AND MATHEMATICS

WINTER, 1990

Volume I, Number 2
Institute of Mathematical Geography
Ann Arbor, Michigan

The purpose of *Solstice* is to promote interaction between geography and mathematics. Articles in which elements of one discipline are used to shed light on the other are particularly sought. Also welcome, are original contributions that are purely geographical or purely mathematical. These may be prefaced (by editor or author) with commentary suggesting directions that might lead toward the desired interaction. Individuals wishing to submit articles, either short or full- length, as well as contributions for regular features, should send them, in triplicate, directly to the Editor-in-Chief. Contributed articles will be refereed by geographers and/or mathematicians. Invited articles will be screened by suitable members of the editorial board. IMaGe is open to having authors suggest, and furnish material for, new regular features.

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ISBN: 1-877751-44-8

## SUMMARY OF CONTENT

Numbering given below corresponds to the number of the electronically transmitted file.

1. Typesetting code; file of TeX commands that may be inserted at the beginning of each file (or in front of the whole set run at once) in order to typeset the document.

2. File of front matter, including this material!

3 and 4. Reprint of John D. Nystuen from 1974. *A city of strangers: Spatial aspects of alienation in the Detroit metropolitan region.*

Examines urban shift from "people space" to "machine space" (see R. Horvath, *Geographical Review* April, 1974) in the context of the Detroit metropolitan region of 1974. As with Clifford's *Postulates of the Science of Space*, reprinted in the last issue of *Solstice*, note the timely quality of many of the observations.

5. Sandra Lach Arlinghaus. *Scale and dimension: Their logical harmony*

Linkage between scale and dimension is made using the Fallacy of Division and the Fallacy of Composition in a fractal setting.

6 and 7. Sandra Lach Arlinghaus. *Parallels between parallels.* A manuscript originally accepted by the now–defunct interdisciplinary journal, *Symmetry.*

The earth's sun introduces a symmetry in the perception of its trajectory in the sky that naturally partitions the earth's surface into zones of affine and hyperbolic geometry. The affine zones, with single geometric parallels, are located north and south of the geographic tropical parallels. The hyperbolic zone, with multiple geometric parallels, is located between the geographic tropical parallels. Evidence of this geometric partition is suggested in the geographic environment—in the design of houses and of gameboards.

8. Sandra L. Arlinghaus, William C. Arlinghaus, and John D. Nystuen. *The Hedetniemi matrix sum: A real–world application.*

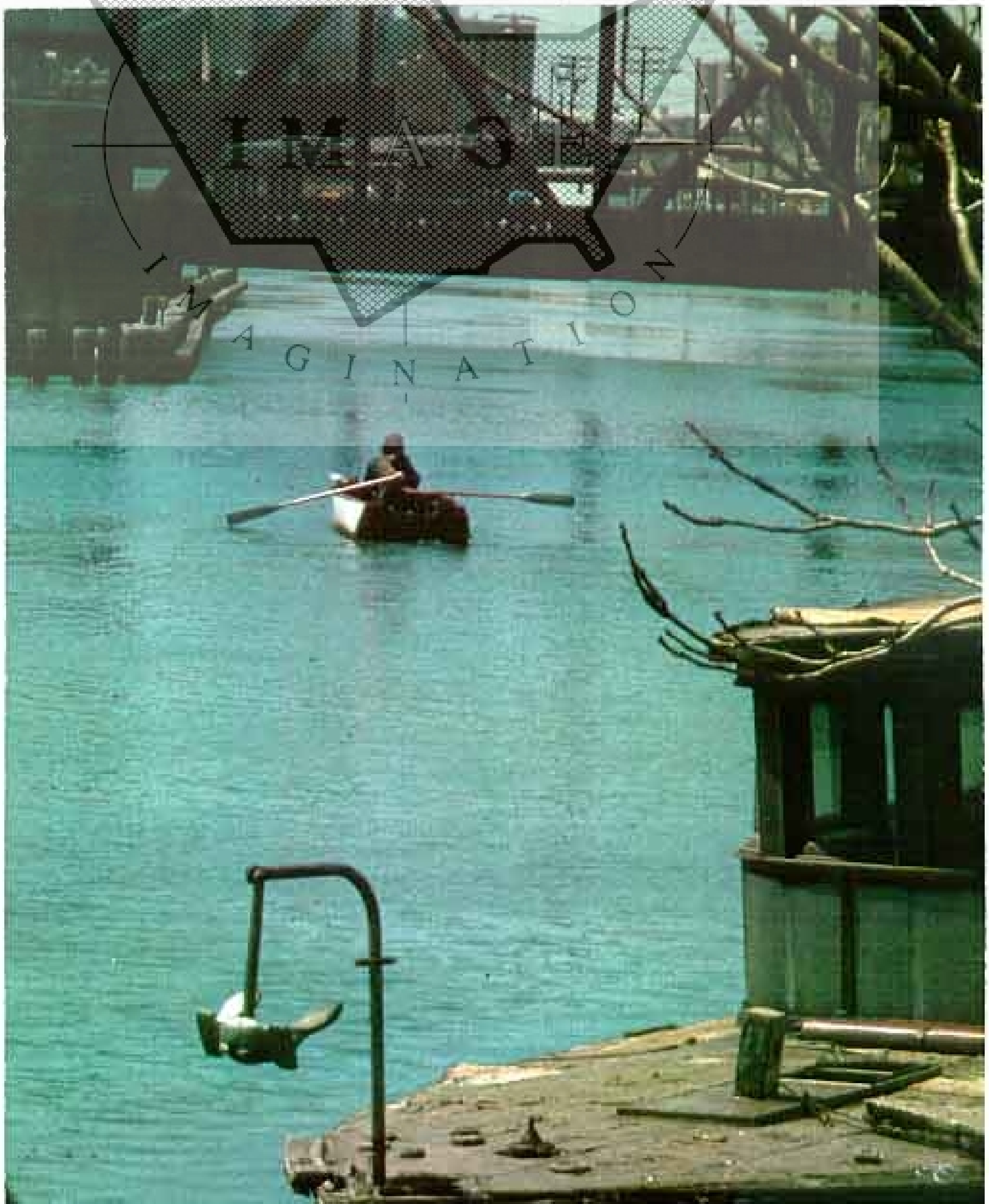
In a recent paper, we presented an algorithm for finding the shortest distance between any two nodes in a network of $n$ nodes when given only distances between adjacent nodes [Arlinghaus, Arlinghaus, Nystuen, *Geographical Analysis, 1990*]. In that previous research, we applied the algorithm to the generalized road network graph surrounding San Francisco Bay. Here, we examine consequent changes in matrix entries when the underlying adjacency pattern of the road network was altered by the 1989 earthquake that closed the San Francisco–Oakland Bay Bridge.

9. Sandra Lach Arlinghaus. *Fractal geometry of infinite pixel sequences: "Super–definition" resolution?*

Comparison of space–filling qualities of square and hexagonal pixels.

10. *Construction Zone.* Feigenbaum's number; a triangular coordinatization of the Euclidean plane.

INDUSTRIAL WASTELAND RIVER
Photograph by John D. Nystuen; Rouge River, Detroit, 1974.
FRONTISPIECE: A City of Strangers.



# A CITY OF STRANGERS: SPATIAL ASPECTS OF ALIENATION IN THE DETROIT METROPOLITAN REGION.

*John D. Nystuen*
The University of Michigan, Ann Arbor

An invited address given in the conference:
*Detroit Metropolitan Politics: Decisions and Decision Makers*
Conference held at Henry Ford Community College
April 29, 1974
Dearborn, Michigan
Comments added, 1990

Suburbanization at the edge of the metropolitan region and the destruction of homes in the inner city through "urban renewal" or expressway construction are the results of uncoordinated and decentralized decisions made by people remote from those directly affected. Unwanted transportation burdens are forced on us by changes in the location of population and jobs. There has been a shift, still continuing, from "people space" to "machine space" [5] in our cities which we seem powerless to stem. "Machine spaces" are those spaces dedicated to machines or to inter-regional facilities which present larger than human, impersonal and often hostile, aspects of society. We are alienated from our urban environment to the degree it has become machine space. We are alienated from land controlled by strangers. These strangers may be decision makers in institutions with metropolitan-wide jurisdictions such as transportation planning authorities, mortgage and banking firms, and the regional power company. The interests of people of this type are at least focused on the metropolis. Other decision makers affecting local land use are outlanders whose concerns are not exclusively local. One type of outlander is the decision maker at state and federal level, concerned with and responsible for general policy of some aspect of urban life but whose vision cannot be expected to distinguish variations in every neighborhood within his/her broad jurisdiction. Other outlanders are decision makers in multi-state or international corporations and institutions whose structures extend horizontally across many communities or even continents. Their aspirations and understanding of urban life are often incommensurate with local community objectives. Misunderstanding, alienation, and conflict easily result.

## The Cost of Victory over the "Tyranny of Space"

From the geographical point of view these disturbing aspects or urban life today are the result of our victory over the "tyranny of space [7]." Much of the technological achievement of our society has been improvement in transportation and communication. We made the oceans routes not barriers; achieved air and space flight; built power transmission lines to move energy, and sewer lines to carry off wastes. Innovations in communication are equally important. The invention of the alphabet was a great achievement in ancient times (history begins); the printing press followed in medieval times (information widely shared); today we have mini-computers made of inexpensive printed circuits. Electronic data processing (embracing complexity) is as revolutionary as the alphabet and the printing press. The change which will be forthcoming can be only dimly perceived. These inventions affect society by radically changing spatial and temporal limits within which we are confined. This freedom over space and linear time, while closely linked to the rise in our standard of living, now threatens us in other ways. Previously, local community organization and control processes developed relatively free of outside interference because of the friction of distance. Decisions about local land uses and activities had to be made locally because control at a distance was too inefficient. Freedom from the tyranny of space has made us subject to other tyrannies which may be worse. The opportunity to control at a distance which technology offers us may be seized by those who are indifferent to others' needs, selfish and unscrupulous in their quest for power. Too often one man's gain is another man's loss. The unscrupulous become anonymous and unreachable by being hidden in vast institutional hierarchies. Traditional mechanisms of social control and the means to draw people to act for the good of the community are lost. The community is lost in the old geographical sense. We are a city of strangers. I do not advocate giving up our victory over space. Instead we must consider new means of association and control that will humanize the space around us once again.

### Alienated Space

Alienated land in the sense I am using it has two meanings. It is any place where humans are not welcome or may be in real danger; lands dedicated to machines are of this type. But it is also space controlled by strangers, perhaps pleasant places from which we are excluded by fences and "no trespassing" signs, or places we may enjoy but over which we have no control as to how they are to be used or changed; state and federal parks are examples. We may find ourselves excluded from many places, subject to regulations in others and even in that kingdom, our own home, denied the right to modify it as we see fit. Little wonder we feel a certain detachment and alienation. Loss of sense of community is the price for our victory over the tyranny of space. Machine space and control of community or neighborhood by strangers are the consequences.

### Machine space

Ron Horvath, in an article in the *Geographical Review* entitled "Machine Space," classified land parcels as "machine space" rather than "people space" depending upon "who or what is given priority of use in the event of a conflict" [Horvath, p. 169]. He then pointed out how much of our cities we have given up to machines, especially the automobile. He characterized this machine as the "sacred cow" in American culture. He said

> In the minds of many Westerners, India's sacred cow has come to symbolize the lengths to which people will go to preserve a nonfunctional cultural trait. But India's sacred cow is downright rational in comparison to ours. Could an Indian imagine devoting 70 percent of downtown Delhi to cow trails and pasturage as we do for our automobiles in Detroit and Los Angeles. Every year nationally we sacrifice more than 50,000 Americans to our sacred cow in traffic accident fatalities (Figure 1) [2, p. 168].

Something like 20 percent of our gross national product is tied directly to manufacturing, servicing and fueling the automobile—twice the amount we spend on war machines, another more sinister genre of sacred cow machine to which we seem addicted.

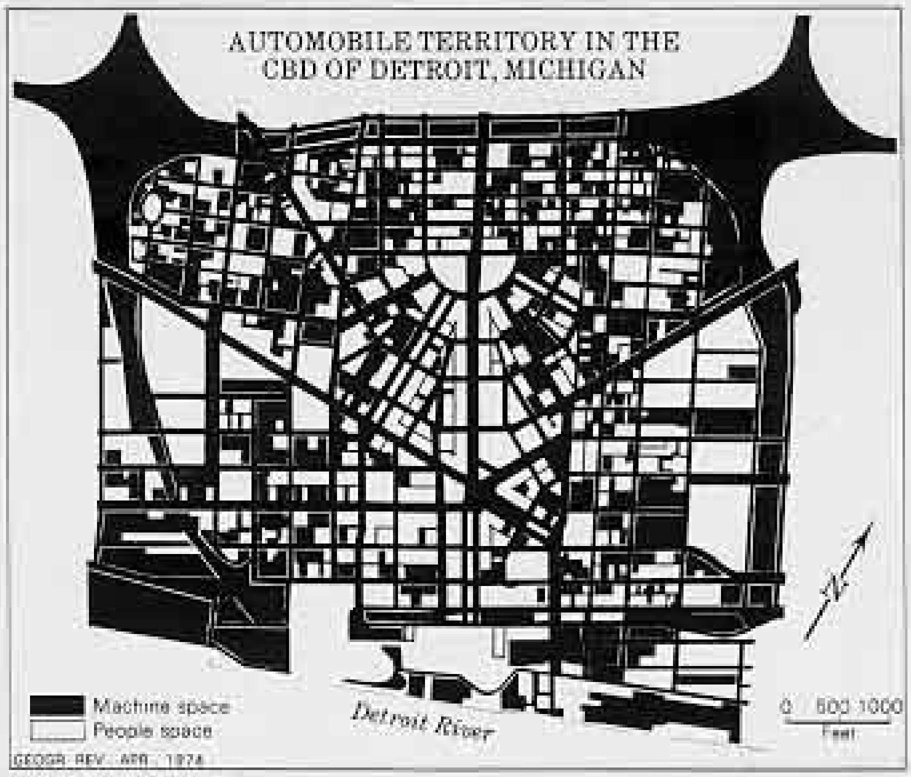



**Figure 1.** "Machine Space" in downtown Detroit, ground level, 1971, by R. Horvath. Map reprinted with permission of The American Geographical Society, from "Machine Space," R. Horvath, *The Geographical Review*, April, 1974, p. 171.

### Vertical Control or Scale Transforms.

There are signs of a reaction setting in. Ralph Nader effectively pointed out that automobiles are "unsafe at any speed." The solution called for is not crash proof cars. It is reduction of exposure by reducing passenger miles traveled by private automobiles. We can accomplish this in two very general ways: by developing mass transit systems and by reducing the number and length of trips taken. The latter calls for re– ordering land use patterns or changing our life style by giving up some of our triumphs over space. Trends in the Detroit Metropolitan Area suggest otherwise. We are still in the process of completing an expressway system. The state has authorized one–half cent of the nine cent gasoline tax to be devoted to mass transit systems: a significant step but hardly a major re– allocation of priorities. SEMTA, the state transportation authority for Southeast Michigan, has recently released its mass transportation plan calling for a 1990 completion date. If the experience of systems such as the San Francisco Bay Area's BART can be taken as an example, significant delays due to the operation of political processes will set that date further into the future, if indeed, the system is ever built. [As of 1990, the Southeastern Michigan Transportation Authority (SEMTA) is defunct. Their mass transit plan, released in 1975, called for a 1990 completion date (Figure 2). All that came of this plan was the elevated downtown Detroit People Mover, delayed, over budget, and out–of–control as the rest of the mass transportation plan was never implemented and doomed to go out of business. Too massive to tear down without great expense, it will remain a bizarre monument to inadequate planning and fragmented action. On the other hand, the Detroit expressway system is largely completed. A final link in the circumferential network, I-696, opened in 1989, twenty-five years after it was proposed. This stretch of expressway was met with determined opposition from an upper-middle class, politically effective neighborhood. The final links were modified to lessen impact on adjacent residents. Neighborhoods near downtown locations succumbed to the huge concrete corridors years ago. The expressways created huge barriers and the livable spaces between them proved too fragmented to sustain and are now abandoned.]

Multi-million dollar transportation projects greatly affect land use patterns and are once-and-for-all investments. They come infrequently and permanently affect the geography of the region. The massive water and interceptor plan of the Detroit Water Board is a similar large scale project with more benign consequences. This brought water from Lake Huron via tunnel and aqueduct to a large portion of the metropolitan region. [It was also a planning error. In retrospect we see it was overbuilt due to the decline in heavy industry in the city and the exodus of people to the suburbs.]

Decisions associated with large scale projects are examples of factors which are out of the hands of the ordinary citizen or even the large land developers working in the region. They impose important constraints on land use possibilities. They are decisions made by strangers and represent a loss of private or small community freedom of choice. Many gross forms in the Detroit metropolitan region are the consequence of decisions made many decades ago. Some individuals and communities try to resist the pressures of single large scale commitments. In the case of water procurement, this can be done by using local ground water wells and septic tanks or small municipal sewage plants. At low population densities these local devices may work fine and a decentralized system is probably best. At high densities, however, local environmental capacities are exceeded. Other public agencies, such as the County Health Departments, may then operate to pressure communities into the larger system. It is this hierarchical ordering of systems that removes local control from one aspect after another of urban life. When the problem condition in the environment enlarges previously separate problems begin to merge, the best institutional response we have yet devised is to establish a hierarchically ordered social process to address the larger problem. This change in scale may result in qualitatively different situations. Institutions operating at metropolitan levels may appear very inflexible and arbitrary from the point of view of a local authority, municipality, or private home owner. The need for standardization and routinization is absolutely crucial for such organizations. Alienation may develop between parties who view things at different scales without anyone being at fault.

Politically, a metropolitan region is hierarchically organized by spatial jurisdictions. Local problems are most appropriately dealt with by local authority and regional problems by regional authorities. We have yet to devise a means of graciously transferring jurisdiction up or down the hierarchy to correspond to changes in scale in the nature of the problems. Our greatly increased capacity to overcome transportation and communication costs has led to changes in population density and locations of jobs which have often exacerbated local problems and called forth a scale transfer. The local community, no longer able to perform the service, loses jurisdiction over the problem to higher authorities. At a higher level, much of the loss of state power to the federal government has been a change of this sort. [To some extent deregulation efforts of recent years prior to 1990 have shifted responsibility back to

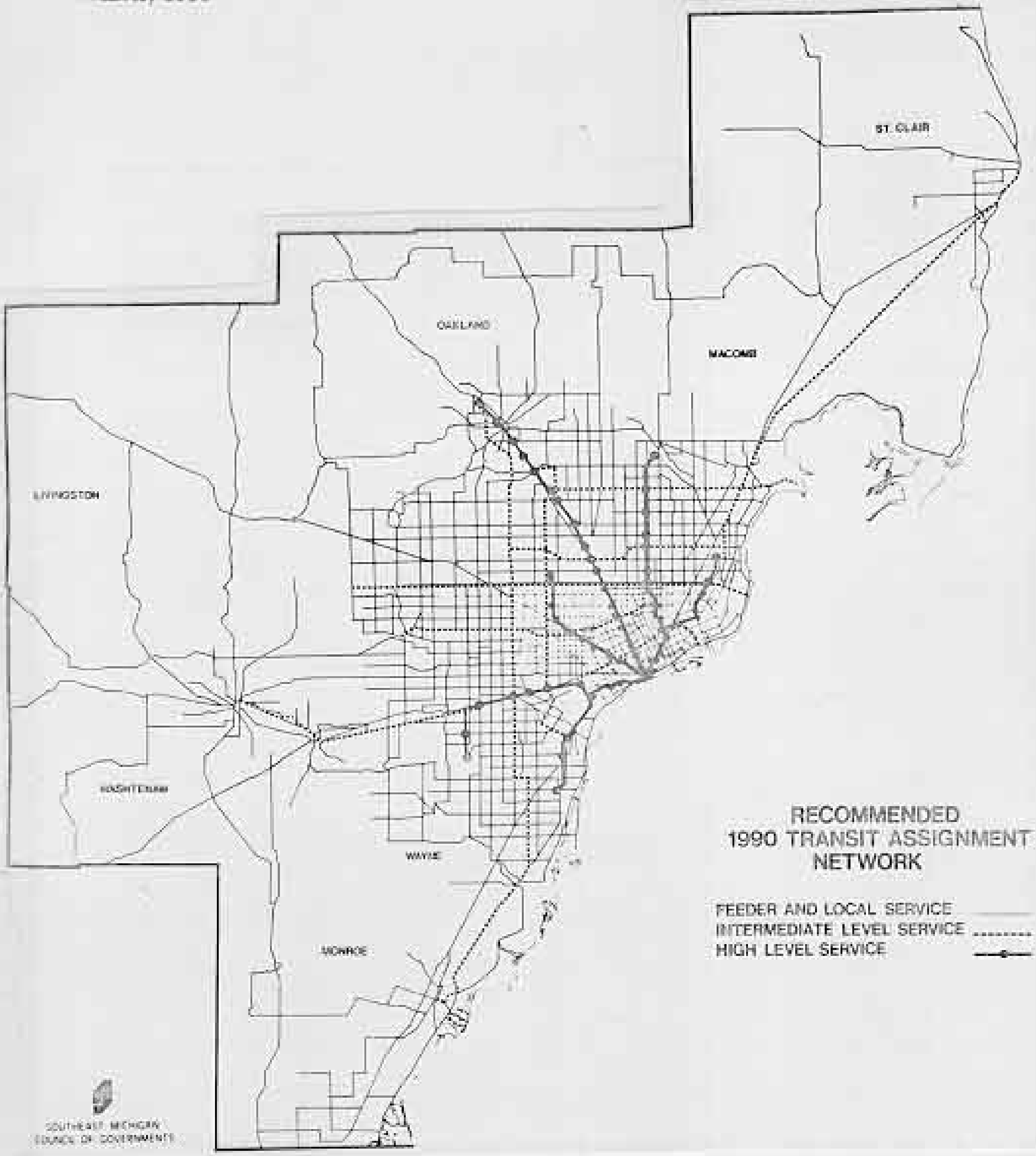



**Figure 2.** Map from 1974 suggests a network that was never built (as of 1990).

local authorities, especially from Federal to State levels. Hierarchies need to be designed that set limits or levels of acceptable performance but remain tolerant of variation in local actions. State rules regarding equalization of county property taxes and local school performance are examples.]

### Horizontal Control.

Some institutions and corporations are cross-threaded in the fabric of society. Their interests and actions are uncoupled from the local community because they are interested in a single category of phenomena and not in the mix of all spatial categories at one location. The decision makers in these organizations are very likely to be outlanders; people who live in entirely different communities or even other nations, yet whose decisions may be controlling factors in a local situation. The ability of multi-plant firms to make long distance decisions is closely tied to the effectiveness of channels of control via communication and transportation facilities. As communication improves the management has the option to centralize decision making, thereby reducing the autonomy of each plant manager. In times of poorer communication major decisions regarding enlargement or closing of plants would have been made at the headquarters of the central management. A local community finds its fortunes very much in the hands of outlanders. Three subtle and disturbing aspects may characterize such a relationship. In the first place the central management may act in what it believes to be rational and moral purposes in closing least profitable facilities in favor of expansion in areas which promise higher returns. The overall result may be pernicious. A supermarket chain operating under such rules may end up closing all its stores in the inner city in favor of suburban stores. The internal firm reasons may make complete sense; close the oldest facilities on lots too small to accommodate the latest technologies, in neighborhoods which have declining populations and which do not yield high returns because of general low income levels. Inner city neighborhoods with older retired people and poverty stricken ethnic groups, losing population to urban renewal or expressway construction end up losing their local supermarket. They are the least able to afford the loss. The decision may be made in another city by outlanders unresponsive to the local peoples' problems and with no court of appeals available.

A second difficulty for the local community with a plant owned by an international corporation is the policy of the corporation to keep its young and most talented management moving from place to place in order that they can learn the business and eventually be able to assume roles higher up in the corporate hierarchy. It is a perfectly reasonable policy with respect to the internal firm requirements. The consequence, however, is a cadre of talented nomads who show little or no interest in the local welfare of the community in which they are temporarily located. Nor would the community want to commit political resources to such people if they expressed an interest. They are simply removed from making a local community contribution which they might easily have pursued had they been permanently in the community. The only loyalty that makes sense to them is company loyalty. Higher corporate management is certainly not going to discourage this. A third tendency of horizontal cross-community control in society is the homogeneity of facilities and company policy. Hierarchies work best under standard operating procedures. Economies of scale are possible, substitution of material and personnel from one locality to another are facilitated if the installations are all the same. If disciplined standardization and routinization has been enforced top management can make broad, basic decisions secure in the knowledge that countless local exceptions will not subvert their intent during the implementation phase. But what happens when accommodation to local situations is required. You may get a machine answer, "that request will not compute!" or more likely the local manager will say, "I sure would like to help you but my hands are tied by company policy." He may not be telling the truth. The impersonal corporate presence is an easy way to solve a problem by defining oneself out of any concern or responsibility. Of course, he may be telling the truth but be as powerless to change corporate policy as the outsider seeking accommodation.

### We Are the Enemy

Pogo said, "We have met the enemy, and he is us" [Kelly, 1972]. All metropolitan areas are complex. The Detroit region is no exception. There is no one to blame for the mess. We are the enemy; we are the city of strangers. There is no single leader or group, either evil or benign to blame. The land use pattern grows from our decentralized decision processes. The decisions which actually affect local land use extend over time and space well beyond the here and now. It is true the channels of control could be in the hands of evil doers and we could improve our lot by exposing and removing them. But I think we are not generally in the hands of the unscrupulous; not even in the hands of the stupid and insensitive. It just appears that way. Each decision or action is contingent upon conditions that are beyond the control of the individual or group making a particular choice. There is rarely an instance where these constraints are not present. The outcome often seems stupid or callous. Most deleterious outcomes are probably unanticipated. They are indirect effects not thought of by the decision makers. We need to understand our urban processes well enough to take action to avoid effects which cause discomfort or inequity to others. Constraints on decisions may be classed into three groups. There are institutional and legal policies. There are physical and natural environmental limitations which have to do with laws of nature and the technological capacities with which we may accommodate to those laws. And finally, there are limitations to our aspirations and goals, the imagined conditions that motivate our actions. These aspirations are not hampered by any finiteness of imagination in any single pursuit, for we all know flights of imagination are boundless. Rather limits appear because we harbor multiple needs which are often in conflict. We choose to restrain our objectives in one pursuit in order to achieve goals in other pursuits. For example we find it hard to have large lots and big lawns which provide us with seclusion and status and at the same time have many close and friendly neighbors which make available to us the pleasures and security of sharing a close community. Under most circumstances to gain one value is to lose the other.

### Scale Attributes of Value Systems

A definition of values is that they are an individual's feelings about and identification with things and people in his environment. Values have scale attributes. Another three fold classification is convenient. There are *individual/familial identification*, a commitment to proxemic space — the space within which one touches, tastes and smells things. Secondly there is *community identification*, embracing the individual's feelings and concern for those with whom he or she lives and interacts, not in the same house, but in the vicinity or neighborhood. This is local space generally recognizable by sight and smell. Finally there is *political-cultural identification* which refers to ideals and concerns extending beyond the people and community with which the person has daily contact. This realm must be dealt with abstractly and through instruments, either mechanical or institutional for it is too large to be perceived by the senses directly. This is national or global space. Machine space and control by outlanders may be viewed as intrusions into our community space by organizations and facilities of this larger domain. How they look, sound or smell has not been taken into account in the design of such facilities. Examples include Edison power stations, the Lodge and Ford expressways, and Detroit Metropolitan Airport. We give up local community values for the benefits of the global mobility and interaction. Metropolitan life pushes us to scale extremes. We value individual rights and perogatives and mainline connections with the global culture over familial and community concerns. Intermediate spatial scale values suffer and the community declines along with them. The consequences are visual blight, noise pollution, reduced security, and injustice. Community values include concern for our fellow man, a sense of equity and humaneness. The mechanisms for enforcing a community code of ethics are ostracism, social pressure and the use of sense of humor to keep people responding to others as human beings. These mechanisms do not work well in a city of strangers and are not followed. They are particularly ineffective in those large impersonal machine spaces, the streets and expressways, bus stations, terminals and warehouse and factory districts. The urban code of ethics carefully preserves the privacy of individuals and tolerates eccentrics. A person has functional but fragmented value and is valued for specific tasks he or she can do. A major problem with the dehumanization and anonymity of urban life is that the unscrupulous are freed from social control along with the rest of us. We have distinct evidence that we are being "ripped off" at both ends of the spatial scale of involvement. Corporations manipulate markets through advertisements thereby creating artificial shortages and rapid obsolescence of their products without fear of being called to account. Radical monopolies in the words of Ivan Illich. At the other extreme individuals, free of local control, satisfy their wants by committing violent criminal acts against others and then disappearing into the crowd. Ostracism and social pressure work between friends. They are meaningless to the corporate manipulator and street criminal.

We are in a crisis of conflicting values when we attempt to reform the structure of society to eliminate these problems. We tend to throw the baby out with the bath water. Action against crime in the streets and the home is moving toward hardening our shelters, walling up windows, barring doors, hiring guards and guard dogs, and restricting access. Security guards in Detroit are big business. Even entering the Federal District Court in downtown Detroit now requires a personal search. These actions are destructive of community spirit. They are a falling back to greater individual isolation. Burglar proof apartments are more effective against neighbors than against burglars (Figure 3).

We have barely recognized the assault on our well being through manipulation by national corporations, let alone having devised counter measures. The major instruments of global firms are standardization and routinization. And Detroit is a symbol of giant multinational corporations and the Henry Ford–perfected assembly line. A defensive action of sorts is uncoupling part of one's life from the national distribution system. Making and using homemade products are countermeasures. The great rise in home crafts, community garden projects, potters' guilds, art fairs and galleries and counter-culture craft shops provide some vehicles for humanizing city space and reestablishing a sense of community. College youth are showing the way. Wearing old work clothes everywhere, worn and patched (whether needed or not) is a symbol of a society moving beyond mass consumption. Of course, as soon as old work clothes become *de rigueur* the agents of mass production can reassert themselves by selling pre-patched garments. Community values benefit most by seeking simple





Figure 3. Photographs of Detroit scenes by John D. Nystuen, c. 1974.

handmade products. The craft shop and modern craft guilds should be valued for their local community effect and should be supported because of their community value (Table 1).

**TABLE 1.**
HUMAN VALUES CLASSED BY SPATIAL SCALE

| Value | Space | How Sensed |
|---|---|---|
| individual–familial | proxemic | see, hear, touch, smell |
| communal | local | see, hear |
| political–cultural | global (national) | abstract via instruments and institutions |

Human values are an individual's feelings and sense of identification with people and things in the surrounding environment.

## Card Carrying Americans

My standard sized dictionary has a dozen meanings listed for the word *trust*. The first meaning of trust is that it is a confident reliance on the integrity, honesty, veracity or justice of another. It used to be that credit was a local community relationship. When you moved to a new town or new neighborhood you could gain credit by managing to buy some clothes or furniture on time and then making sure that you payed up in a timely fashion according to the agreed–upon terms. It was a way to establish trust with local merchants. Today large financial institutions and other multinational corporations such as petroleum companies have taken advantage of innovations in communication and information handling to make a space adjustment in extending credit which better fits their scale of operations. Credit cards make trust an abstract, formal relationship which operates nationwide or globally and which can be entrusted to machines for monitoring. But as with other abstractions, not all the original meaning of the word transfers to the new use. Justice fades. The new scale of operation provides a marvelous freedom for those who carry cards. Unfortunately it is easier for some people to get credit cards than it is for others. The poor and the young are often prevented from obtaining them at all. We have created two classes of Americans — card carrying Americans and second class citizens who must pay cash. There is every reason to believe that in the future consumer exchanges will be increasingly handled by some type of credit transaction. The effect is pernicious in poor neighborhoods. In the past the local grocer or merchant often provided credit to local people whom they had come to trust. This service has become less common and the range of goods obtainable through local credit is shrinking as large corporations capture greater and greater share of the market. They deal in cash only or with credit cards. They do not maintain personal charge accounts.

Typically in an urban renewal process a poor, ghettoed family is forced to move because their house is condemned by the "improvement." They move to a new neighborhood where likely as not they must pay more for housing than they did previously and simultaneously they lose the credit relationship they had built with local merchants in the old neighborhood.

Credit cards are typical of space adjusting developments which accomplish their purpose through abstracting and depersonalizing relations. Accounting for the full circumstances of an individual and making a judgment about his or her trustworthiness is not possible.

Justice is lost in the transform and the word trust begins to mean something else.

## Mainlining Fantasy with the Television Tube

Just as surely as the automobile is the dominant anti– neighborhood transportation device, television is the dominant anti–community communication device. Think of the products sold on television: standardized balms and salves for our bodies, stomachs and minds; automobiles to speed us into exotic landscapes; miracle materials to clean our homes without effort; and corporate images to make us all like the firms which deliver these products. Television is a device for mainlining messages directly from national and global organizations to individuals: to millions of individuals. The messages must necessarily be abstract, standardized and unreal. There is a certain lack of trust in the transmission. Value priorities and the meaning of common English words used in ads do not resemble the values and common usage used in face to face communications. The verbiage is exaggerated; hyperbole employed to describe mundane products. Cliches are strung together one after another. If one of these advertising images came alive in our living room and we tried to have a conversation we would find the person indeed odd.

From the point of view of community values television messages have several bad features. First and foremost there is no way to clarify or challenge a point because the communication is one way. Secondly it is difficult to compete with the siren songs of the national product distributors. A message meant for millions is worth purchasing the best possible creative talent to deliver it. Corporations that can afford national TV time are selling standardization and routinization nationwide. They gain economies of scale in doing so. This often means they have a price advantage over local competition or worse, they convince people the national product is a superior albeit more expensive item than a local one. Countermeasures for this assault are to substitute handmade items for mass produced ones. Another step is to consume less. Seeking satisfaction in other than materialistic pursuits will often mean turning to local, community–level activities.

It hardly need be said that the images projected by television are fantasies that mirror reality through very strange glasses. They glorify individualism and vilify community forces. Nature is also often depicted as implacable, hostile and competitive. This view requires that the individual seek some inner strength in order to prevail when threatened by the environment. Other views in which nature and society are more benign and cooperative are possible but they do not provide the excitement which seem to attract viewers. This hostile approach to the fantasy environment apparently affects people's evaluation of the real environment. There is evidence that people who watch television extensively are more fearful of crime than people who seldom watch it.

Large communication systems affect perception apart from the fantasy content. In reporting news in a metropolitan area the size of Detroit with nearly five million people in the "community" many bizarre crimes are avidly reported by telecasters and other media sources. Upon hearing such reports people think, "What a terrible thing right here in our city." The populace of metropolitan areas of half a million will not hear such stories about their town with nearly the same frequency because there is an order of magnitude difference in the base population. This is not to make light of the crime rate in Detroit which is large on a *per capita* basis or by almost any measure. But the scale effect is present in addition to the hard facts of the high crime rates in Detroit.

Further technological innovation may deliver us from some of the worst effects of the current revolution in transportation and communications devices. It is becoming more feasible to handle great complexity in large systems through information control. The likely consequence is greater individual freedom of choice while still permitting participation in a large system. The automobile assembly line is again an example. Henry Ford provided Model T and Model A Fords in the colors of your choice — so long as that choice was black. Modern auto manufacturers now deliver autos of many styles, in scores of colors, streaming from assembly lines in a complex sequence which matches the week by week flow of customer orders coming in from throughout the country. This is achieved through computer control of parts scheduling on the assembly line. Cable TV promises multiple channels, possible two way communication, and tapes and libraries of past broadcasts, and narrow casting in which programs and exchanges are limited to specified audiences. These developments might provide such a great range of choices to the viewer that the current monopolizing of television by outlander interest, as with major news networks, could be weakened. Capacity to handle an order of magnitude greater complexity through effective information processing could serve a broader range of values. But, as with credit cards, who will be served by the greater freedom? Freedom will go to those with the knowledge and money to use the services. Justice need not be served. Community values could regain some lost ground under such developments but only if concerted and careful efforts in support of local values is brought to bear on decisions as to how the new technology is to be used.

## Strategies for Local Control

Our message is that the decline in quality of urban life is due in part to loss of community values in competition with individual and outlander values which were better served by advances in transportation and communication. Our goal should be to restore balance in our lives by restoring some community commitments. In general, as temporal and spatial constraints are lifted institutional and legal parameters need to be erected to avoid abuse and pathologies in our social processes. This is easier said than done.

The first problem is to recognize a problem when we see it. We have been slow to see that the automobile is actually taking over the spaces of our cities as if it were becoming a biologically dominant species. Bunge and Bordessa suggest that we concentrate on improving and enlarging the spaces devoted to children in our cities as a first priority in ordering city space. They show that much benefit flows to the entire society through such strategies. People space gains at the expense of machine space. If the long distance transportation facilities and other sinews of the large metropolitan systems are channelized and confined to corridors and special locations the spatial cells created will be available for local uses. But priorities must be correct. We live in the local cells. We only temporarily exist in the transportation channels at which times we suspend normal civilities and common courtesy. The life cells (neighborhoods) should be the objects, not the residuals, of the urban form. Bunge and Bordessa [3] suggest mapping local and non-local land use in urban neighborhoods. The simple facts of that division will reveal the extent of outlander control of a community. I repeat, you have to see a problem before you can deal with it. Professional planners, academics and citizen groups should develop the concepts and generate the data which highlight the areas that are directly and humanly used rather than those spaces that are indirectly, abstractly used through machines.

Hierarchies are necessary for the operation of large systems but the tendency for imposing standardization and routinization in control hierarchies should be resisted. This can be done by incorporating the rapidly increasing capacity to handle complex information flows. Great metropolitan–wide hierarchies to deal with water supply, traffic control and crime suppression are possible if these large structures are robust enough to allow local variation and still retain an overall integrity. The goals should be always to allow maximum freedom of choice at local levels but with that choice constrained by considerations of equity relative to other elements in the system. Promoting local initiative, self–respect and autonomy would tend to create a heterogeneous urban landscape. But freedom and equity can be conflicting values.

We must strive to make the heterogeneity healthy. We would do well to give first consideration to local people space rather than to machine space. Once our attention is so directed we should make certain that no living space in the city is mere residual left from the process of carving the urban landscape into machine space and space for the outlander and the powerful. I wager that the reader is probably viewing the metropolis at full regional scales. I will close with a word of advice. If you are active in trying to make Detroit a better place in which to live you may well be viewed as an outlander by most of those with whom you interact. There may be a conflict of interest between local community and regional views. I believe your strategy should be to encourage local initiative to enlarge and to improve the quality of neighborhood people–space while at the same time being careful that such actions are not at the expense of other neighborhoods. The achieving of equity is the responsibility of those with regionwide vision. Value, understand, and encourage heterogeneity in living spaces but strive to prevent any living area from falling too far behind in the quest for quality neighborhoods. That will insure integrity of the whole while affording maximum freedom to the parts.

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# SCALE AND DIMENSION: THEIR LOGICAL HARMONY

*Sandra Lach Arlinghaus*

*"Large streams from little fountains flow,*
*Tall oaks from little acorns grow."*
David Everett, *Lines Written for a School Declamation.*

## Introduction.

Until recently, the concept of "dimension" was one that brought "integers" to mind to all but a handful of mathematicians [Mandelbrot, 1983]; a point has dimension 0, a line dimension 1, an area dimension 2, and a volume dimension 3 [Nystuen, 1963]. When a fourth dimension is added to these usual spatial dimensions, time can be included, as well. Indeed, much "pure" mathematics takes place in abstract $n$-dimensional hypercubes, where $n$ is an integer. Geographic maps, globes (and other representations of part or of all of the earth), are traditionally bounded by these integral dimensions, as well; map scale is expressed in discrete, integral units. Often, however, it is the case in geography as it is in mathematics, that a change in scale, or in dimension, runs across a continuum of possible values. In either case, discrete regular steps are usual as benchmarks at which to consider what the continuing process looks like at varying stages of evolution. As fractal geometry suggests, however, this need not be the case.

Within an integral view of scale or dimension, there are logical and perceptual difficulties in jumping from one integral vantage point to another: Edwin Abbott [1955] has commented on this in his classic abstract essay on "Flatland," and more recently, Edward Tufte has done so in the real-world context of "envisioning information" [1989].

Methods for dealing with these dimensional-jump difficulties abound, particularly in the arts [Barratt, 1980]. In a musical context Charles Wuorinen sees composition as a process of fitting "large" musical forms with scaled-down, self-similar, equivalents of these larger components in order to introduce richness of detail to the theme [NY Times, 1990]. Maurits Escher, in his "Circle Limit" series of tilings of the non- Euclidean hyperbolic plane, uses tiles of successively smaller size to suggest a direction of movement—that of falling off an edge or of being engulfed in a central vortex. A gastronomic leap sees a Savarin as self-similar to a Baba au Rhum [Lach, 1974]; indeed, even more broadly, Savarin himself is purported to have said, "You are what you eat." Rupert Brooke (in "The Soldier") captured this notion poetically, in commenting on the possible fate of a soldier in a distant land:

"If I should die, think only this of me;
that there is some corner of a foreign field
that is forever England."

In the end, Brooke's "Soldier" becomes 'place'.

The fractal concept of self-similarity can be employed to suggest one way to resolve difficulties in scale changes as one moves from dimension to dimension. At the theoretical level, symbolic logic classifies logical fallacies that may, or may not, emerge from scale shifts. When self-similarity is viewed in this sort of logic context, the outcome is a "Scale Shift Law." What is presented here are the abstract arguments; it remains to test empirical content against these arguments.

## Logical fallacies.

A question of enduring interest in geography, and in other social sciences, is to consider what can be said about information concerning individuals of a group when given information only about characteristics of the group as a whole. When an attribute of the whole is **erroneously** assigned to one or more of its parts, the logic of this assignment falters. In the social scientific literature, this is generally referred to as commission of the so-called "ecological" fallacy; because the symphony played poorly does not necessarily mean that each, or indeed that any, individual musician did so. In this circumstance, it is simply not possible to assign any truth value, derived from principles of symbolic logic, to the quality of the performance of any subset of musicians (based only on the quality of the performance of the whole orchestra) [Engel, 1982].

It is natural, however, to look for a cause for the poor performance, and indeed to consider some "middle" position that asks to what extent the performance of the orchestra is related to the performance of its individual members. It is this sort of search for finding and measuring the extent of relationship that is the hallmark of quantitative social scientific effort, much of which appears to have been guided [Upton, 1990], in varying degree, by an early effort to determine the extent to which race and literacy are related [Robinson, 1950].

A fallacy, in a lexicographic sense might be "a false idea" or it might be of "erroneous character" or "an argument failing to satisfy the conditions of valid or correct inference" [Webster, 1965]. In a formal logic sense, a fallacy is "a 'natural' mistake in reasoning" [Copi, 1986, p. 4] or it is an argument that fails because its premisses do not imply its conclusion; it is an argument whose conclusion **could** be (but is not necessarily) false even if all of its premisses are true [Copi, 1986, p. 90].

Viewed in this manner, the so-called "ecological" fallacy is nothing different; it is merely a restatement of the "fallacy of division" of classical elementary symbolic logic. The fallacy of division is committed by assigning, **erroneously**, the attributes of the whole to one or more of its parts [Copi]. Thus, it may or may not be valid to make an inference about the nature of a part based on the nature of the whole. That is, sometimes the assignment of truth value from whole to part, in jumping across the dimensional scale from whole to part, is a reasonable practice, and sometimes it is not. The key is to determine when this practice is reasonable, when it is not, and when it simply does not apply. Commission of this fallacy is frequently the result of confusing terminology which refers to the whole ("collective" terms) with those which refer only to the parts ("distributive" terms) [Copi, 1986].

The fallacy of division exists within an abstract human system of reasoning based on the Law of the Excluded Middle: in this Law, a statement is true or false—not some of each. There is "black" and "white," but no "gray" in this system. Statistical work that stems from this fallacy seeks, when it rests on finding correlations, relations that blend "black" and "white"—the foundation in "logic" is thus ignored. This fallacy is examined, here, with an eye to understanding the logical circumstances under which such assignment might, or might not, be erroneous (when it applies).

## Scale and dimension.

To understand when the assignment of characteristics from whole to part (division), or from part to whole (the fallacy of composition—the string sections played well, therefore the symphony played well), might be erroneous, it is useful to consider what are the fundamental components composing these fallacies. The notion of scale is involved in the consideration of "whole" and "part." When is the individual a "scaled-down" orchestra; or, when is the orchestra a "scaled-up" individual? The notion of dimension is also involved. When does the zero-dimensional musician- point spread out to fill the two-dimensional (or three-or more-dimensional) orchestra; or, when does the higher dimensional orchestra collapse, black-hole-like, into the single performer. The performing soloist can dominate the orchestra; the conductor perhaps does dominate the orchestra; yet, the orchestra itself is composed of numerous single performers who do not dominate.

## Self-similarity and scale shift.

Integral dimensions, with discrete spacing separating them, might be viewed as simply a set of positions marking intervals along a continuum of fractional dimensions [Mandelbrot, 1983].. When the discrete set of integral dimensions is replaced by the "dense" set of fractional dimensions (between any two fractional dimensions there is another one), what happens to our various relative vantage points and to scale problems associated with them?

Abstractly, the relationship is not difficult to tie to logic, under the following fundamental assumption.

**Fundamental Assumption.**

When two views of the same phenomenon at different scales are self-similar one can properly divide or compose these views to shift scale.

The whole can be divided "continuously" through a "dense" stream of fractional dimensions until the part is reached (and in reverse). Self-similarity suggests a sort of dimensional stability of the characteristic or phenomenon in question. One commits the Fallacy of Division ("Ecological" Fallacy) when the attributes (terminological or otherwise) of the whole are assigned to the parts that are **not** self-similar to the whole. One commits the Fallacy of Composition when the attributes of the parts are assigned to a whole that is **not** self-similar to these parts. This notion is evident in the many animated graphic displays of the Mandelbrot (and other) sets in which zooming in on some detail presents some sort of repetitive sequence of views (in the case of self-similarity, this sequence has length 1). More formally, this idea may be cast as a "Law."

**Scale Shift Law**

Suppose that the attributes of the whole (part) are assigned to the part (whole).

1. If the whole and the part **are not** self- similar, then that assignment **is** erroneous; and, conversely (inversely, actually),
2. If the whole and the part **are** self-similar, then that assignment **is not** erroneous.

This is one way to look at the "part-whole" dichotomy; physicists wonder about splitting the latest "fundamental" particle; philosophers search for fundamental units of the self [Leibniz, monadology, in Thompson, 1956; Nicod, 1969]; topologists worry about what properties a topological subspace can inherit from its containing topological space [Kelley, 1955].

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# PARALLELS BETWEEN PARALLELS

*Sandra Lach Arlinghaus*

*"I have a little shadow that goes in and out with me,
And what can be the use of him is more than I can see."*
*Robert Louis Stevenson*
"My Shadow" in *A Child's Garden of Verses*

**Abstract:**

The earth's sun introduces a symmetry in the perception of its trajectory in the sky that naturally partitions the earth's surface into zones of affine and hyperbolic geometry. The affine zones, with single geometric parallels, are located north and south of the geographic tropical parallels. The hyperbolic zone, with multiple geometric parallels, is located between the geographic tropical parallels. Evidence of this geometric partition is suggested in the geographic environment—in the design of houses and of gameboards.

## 1. Introduction.

Subtle influences shape our perceptions of the world. The breadth of a world-view is a function not only of "real"-world experience, but also of the "abstract"-world context within which that experience can be structured. As William Kingdon Clifford asked in his Postulates of the Science of Space [3], how can one recognize flatness when magnification of the landscape merely reveals new wrinkles to traverse?

Geometry is a "source of form" not only in mathematics [10], but also in the "real" world [2]. Street patterns are geometric; architectural designs are geometric; and, diffusion patterns are geometric. In this study, the geometric notion of parallelism is examined in relation to the manner in which the sun's trajectory in the earth's sky is observed by inhabitants at various latitudinal positions: from north and south of the tropics to between the tropical parallels of latitude. A fundamental geometrical notion is thus aligned with fundamental geographical and astronomical relationships; this alignment is interpreted in cultural contexts ranging from the design of rooflines to the design of board games.

## 2. Basic Geometric Background.

To understand how geometry might guide the perception of form, it is therefore important to understand what "geometry" might be. Projective geometry is totally symmetric and possesses a completely "dual" vocabulary: "points" and "lines," "collinear" and "concurrent," and a host of others, are interchangeable terms [6]. Indeed, a Principle of Duality serves as a linguistic axis, or mirror, halving the difficulty of proving theorems. Thus, because "two points determine a line" is true, it follows, dually, that "two lines determine a point" is also true. The corresponding situation does not hold in the Euclidean plane: two lines do not necessarily determine a point because parallel lines do not determine a point [6].

Coxeter classifies other geometries as specializations of projective geometry based on the notion of parallelism, depending on whether a geometry admits zero, one, or more than one lines parallel to a given line, through a point not on the given line [6]. In the "elliptic" geometry of Riemann, there are no parallel lines, much as there are none in the geometry

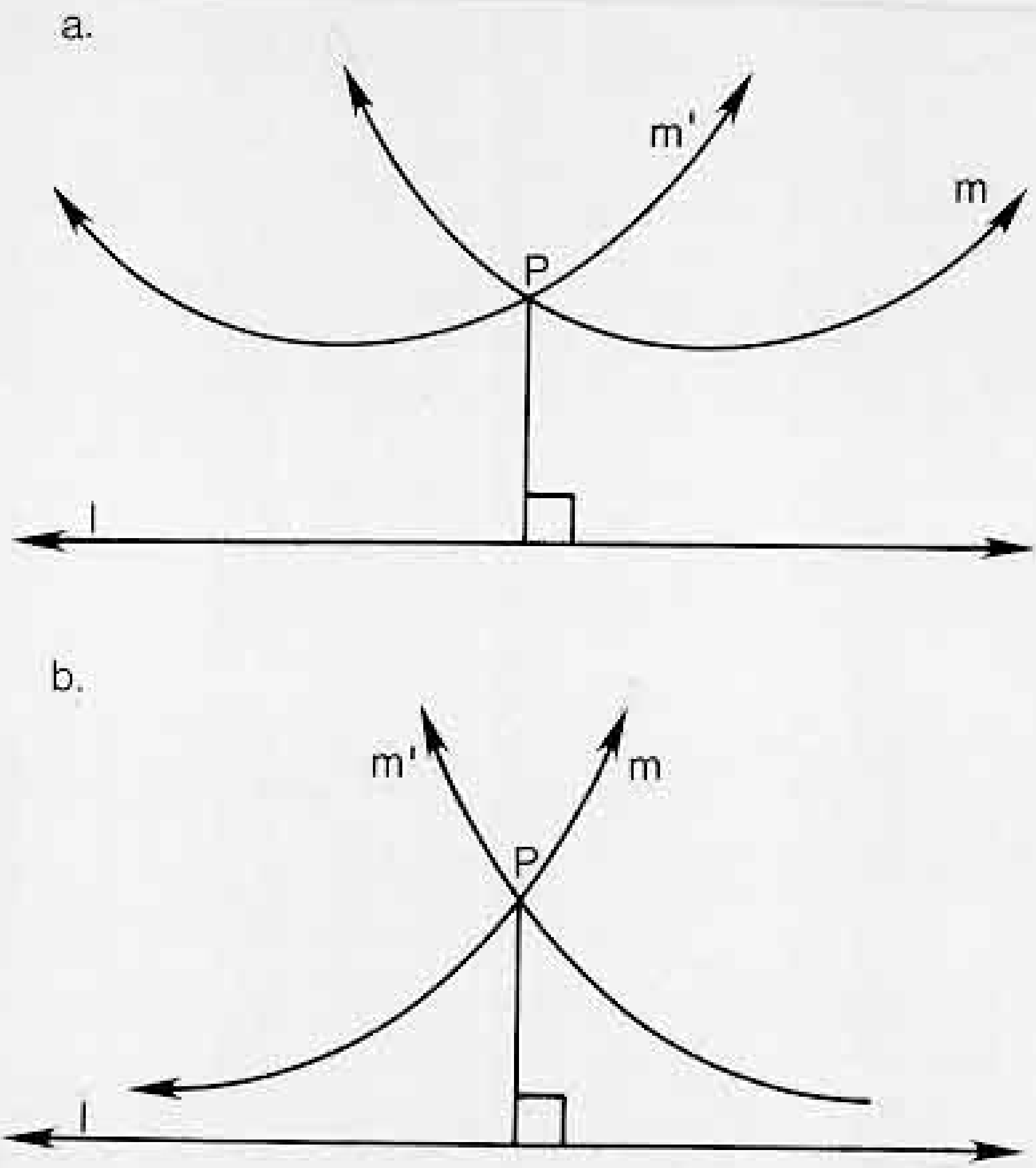


**Figure 1.** The hyperbolic plane.
a. Two lines $m$ and $m'$ (passing through $P$) are divergently parallel to line $\ell$.
b. Two lines $m$ and $m'$ (passing through $P$) are asymptotically parallel to line $\ell$.

of the sphere that includes great circles as the only lines, any two of which intersect at antipodal points. In "affine" geometry, there is exactly one line parallel to a given line, through a point not on that line. Affine geometry is further subdivided into Euclidean and Minkowskian geometries. Finally, in the "hyperbolic" geometry of Lobachevsky, there are at least two lines parallel to a given line through a point not on that line.

To visualize, intuitively, the possibility of more than one line parallel to a given line it is helpful to bend the lines, sacrificing "straightness" in order to retain the non- intersecting character of parallel lines. Thus, two upward- bending lines $m$ and $m'$ passing through a point $P$ not on a given line $\ell$ never intersect $\ell$; they are divergently parallel to $\ell$ (Figure 1.a). Or, one might imagine lines $m$ and $m'$ that are asymptotically parallel to $\ell$ (Figure 1.b) [8].

Elliptic geometry, with no parallels, and associated great-circle charts and maps have long been used as the basis for finding routes to traverse the surface of the earth. The suggestion here is that affine geometry, with single geometric parallels, captures fundamental elements of the earth-sun system outside the tropical parallels of latitude, and that hyperbolic geometry, with multiple geometric parallels does so between the tropical parallels of latitude.

## 3. Geographic and Geometric "Parallels".

As the Principle of Duality is a "meta" concept about symmetry in relation to projective geometry, so too is the earth-sun system in relation to terrestrial space. The changing seasons and the passing from daylight into darkness are straightforward facts of life on earth, often taken for granted. Some individuals appear to be more sensitive to observing this broad relationship, and to deriving information from it, than do others. Shadows may serve as markers of orientation as well as of the passing of time.

### 3.1 North and south of the tropical parallels.

Individuals north of 23.5° N. latitude and those south of 23.5° S. latitude always look in the same direction for the path of the sun: either to the south, or to the north (not both). Shadows give them linear information only, as to whether it is before or after noon; shadows never lie on the south side of an object north of the Tropic of Cancer. The perceived path of the sun in the sky does not intersect the expanse of the observer's habitat, from horizon to horizon. Thus, it is "parallel" to that habitat. North and South of the tropics there is but one such parallel, corresponding to the one basic direction an individual must look to follow the sun's trajectory across the sky.

### 3.2 Between the tropical parallels.

Between the tropics, however, the situation is entirely different. On the equator, for example, one must look half the year to the north and half the year to the south to follow the path of the sun. Thus, there are two distinct (asymptotic) parallels for the path of the sun through the observer's point of perception. Shadows can lie in any direction, providing a full compass-rose of straightforward information as to time of day as well as to time of year: apparently a broader "use" of shadow than Stevenson envisioned!

This population is thus surrounded, in its perception of the external environment of earth-sun relations, by the multiple parallel notion. (Those accustomed to primarily an Euclidean earth-sun trajectory might find this disconcerting.) This hyperbolic "vision" of the earth-sun system, suggests a consistency, for tropical inhabitants only, established in a natural correspondence of the perception of the external environment and the internal environment of the brain. For, it is the contention of R. K. Luneberg that hyperbolic geometry is the natural geometry of the mapping of visual images onto the brain [9].

## 4. The Poincaré Model of the Hyperbolic Plane.

To see how this variation in perception of the earth-sun system might be reflected in real-world settings, and to compare such settings between and outside the tropical parallels, it is necessary to understand one of these geometries in terms of the other. Both Euclidean and hyperbolic geometries are single, complete mathematical systems. They are not, themselves, composed of multiple subgeometries, nor can one of them be deduced from the other: they have the mathematical attributes of being categorical and consistent [6]. A mathematical system is categorical if all possible (mathematical) models of the system are structurally equivalent to one another (isomorphic) [13]; these models are, by definition, Euclidean and are therefore useful as tools of visualization. Because the hyperbolic plane is a categorical system, all models of it are isomorphic. Therefore, it will suffice to understand but a single one, and that one will then serve as an Euclidean model of the hyperbolic plane.

Henri Poincaré's conformal disk model (in the Euclidean plane) of the hyperbolic plane [8], was inspired by considering the path of a light ray (in a circle) whose velocity at an arbitrary point in the circle is equal to the distance of the point from the circular perimeter [4]. To understand how the model works, a "dictionary" that aligns basic shapes in the hyperbolic plane with corresponding Euclidean objects is useful (Table 1, Figure 2) [8].

The hyperbolic plane is represented as the disk, $D$, interior to an Euclidean circle $C$. Because the bounding circle, $C$, is not included, the notion of infinity is suggested by choosing points of $D$ closer and closer to this unreachable boundary. Points in the hyperbolic plane correspond to points in $D$. Lines in the hyperbolic plane correspond to diameters of $D$ or to arcs of circles orthogonal to $C$. These arcs and diameters are referred to as "Poincaré" lines. Because $C$ is not included in the model, the endpoints of the Poincaré lines are not included, suggesting the notion of two points at infinity. Two Poincaré lines $\ell$ and $m$ are parallel if and only if they have no common point. Thus, the disk diameter $\ell$ and the circular arc, $m$, orthogonal to $C$ are parallel because they do not intersect; however, the disk diameter $\ell$ and the circular arc, $m'$, orthogonal to $C$ are not parallel because they do intersect (Figure 2a).

## 5. Hyperbolic Triangles and Quadrilaterals.

Any triangle in the hyperbolic plane is such that the sum of its angles is less than $180°$. When a triangle is drawn in the Poincaré model this becomes quite believable; draw Poincaré lines $\ell$ and $m$ as disk diameters and draw Poincaré line $n$ as an arc of a circle orthogonal to the disk boundary (Figure 2b) [8]. The triangle formed in this manner has one side that has "caved-in" suggesting how it happens that the angle sum can be less than $180°$ (note that three diameters cannot intersect in a triangle because all diameters are concurrent at the center of the disk). Triangles formed from more than one Poincaré line that is an arc of a circle would become even more concave.

Because all triangles have angle sum less than $180°$, there can be no rectangles (quadrilaterals with four right angles) in the hyperbolic plane. The idea that corresponds to that of a rectangle is a quadrilateral with three right angles, one acute angle, and pairs of opposite sides parallel (in the hyperbolic sense). The sides, $OP$, $OQ$, $PR$, and $RQ$, of this quadrilateral are drawn on Poincaré lines that are segments of disk diameters or arcs of circles orthogonal to the outer circle (Figure 2c; $OQ$ is parallel to $PR$ and $RQ$ is parallel to $PO$). This quadrilateral is called a Lambert quadrilateral after Johann Heinrich Lambert [8], creator of the "Lambert" azimuthal equal area map projection (among others) [12]. When such a quadrilateral is drawn in the Poincaré model, the acute angle at $R$ can be drawn to suggest that its sides are divergent, asymptotic, or intersecting. Here, these sides have been drawn to intersect (Figure 2c) and to evidently compress the angle at $R$ as a suggestion of the angular compression [12] present in azimuthal map projections (including those of Lambert)

**Table 1:**
The Poincaré conformal model of the hyperbolic plane
(referenced to Figure 2—after Greenberg)

| Term in hyperbolic geometry | Corresponding term in the Poincaré model in the Euclidean plane |
|---|---|
| Hyperbolic plane | A disk, $D$, interior to a Euclidean circle, $C$ |
| Point | Point, $P$, in the disk, $D$. |
| Line | 1. Disk diameter, $\ell$, not including endpoints on $C$); or 2. Arcs, $m$, $m'$, in $D$ of circles orthogonal to $C$ (tangent lines at points of intersection are mutually perpendicular). |

around the projection center.

## 6. Tiling the Hyperbolic Plane.

If one views a map grid as a tiling by quadrilaterals of a portion of the Euclidean plane, then it might be instructive to consider a tiling of the "map" of the Poincaré disk model

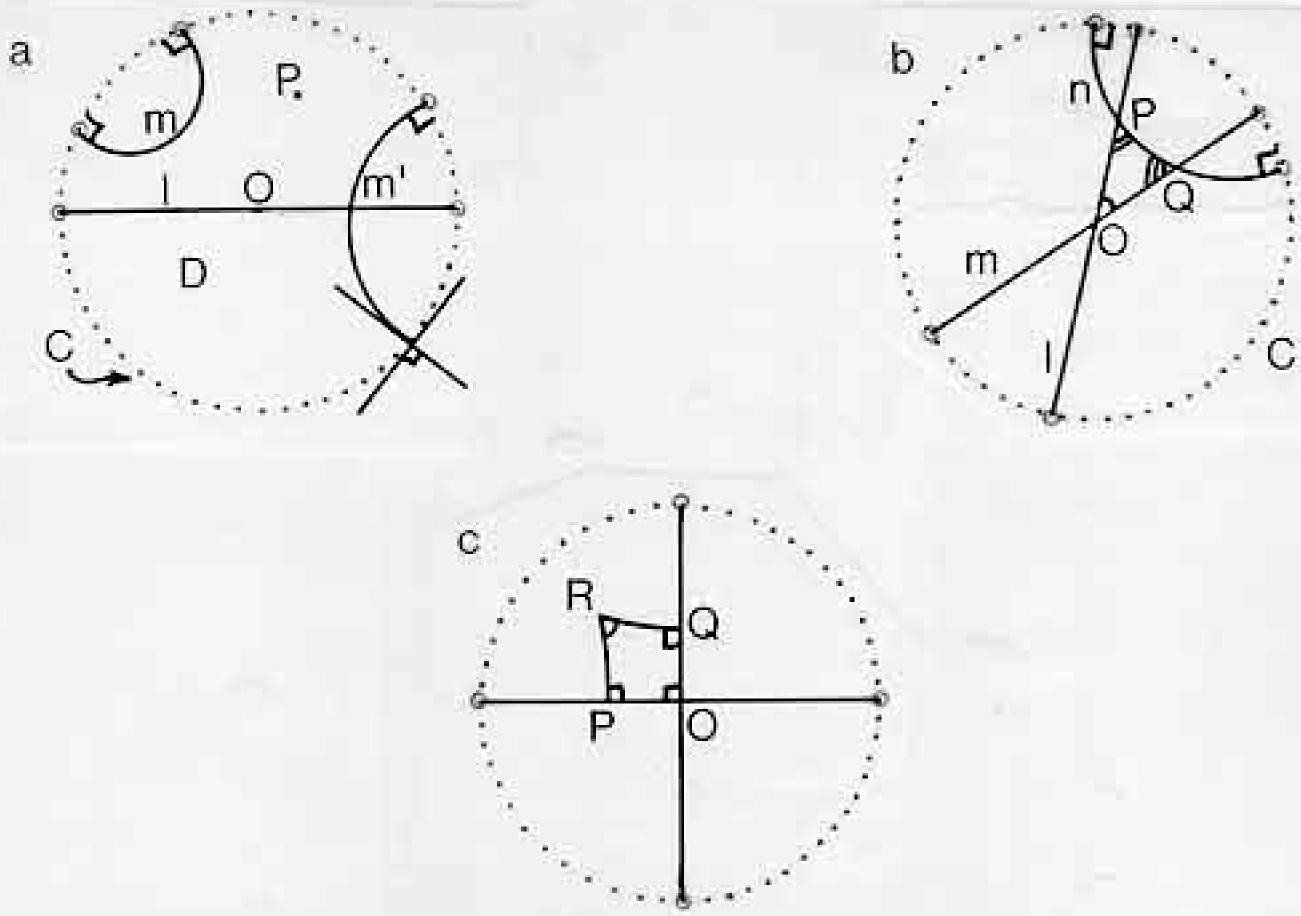


**Figure 2.** The Poincaré Disk Model of the hyperbolic plane.

a. The diameter, $\ell$, is a Poincaré line of the model, as are arcs $m$ and $m'$ which are orthogonal to the boundary $C$. The Poincaré lines $\ell$ and $m$ are parallel (do not intersect); the lines $\ell$ and $m'$ are not parallel (do intersect).

b. The sum of the angles of $\Delta OPQ$ is less than $180°$. The triangle is formed by sides $\ell$, $m$, $n$; the Poincaré lines $\ell$ and $m$ are diameters, and the Poincaré line $n$ is an arc of a circle orthogonal to C.

c. A Lambert quadrilateral with three right angles and one acute angle $(PRQ)$. Pairs of opposite sides are parallel.

by Lambert and other quadrilaterals [5]. Gluing quadrilaterals together along Poincaré lines produces a variety of quadrilaterals (Figure 3). All have pairs of opposite sides parallel; Poincaré lines represented as arcs are orthogonal to the outer circle. Naturally, the tiling can never completely cover the disk, because the disk boundary is not included. Thus, tilings of this map have quadrilaterals of shrinking dimensions as the outer circle is approached.

This permits hyperbolic "tilings" to suggest the infinite; indeed, they have served as artistic inspiration for the "limitless" art of M. C. Escher [7].

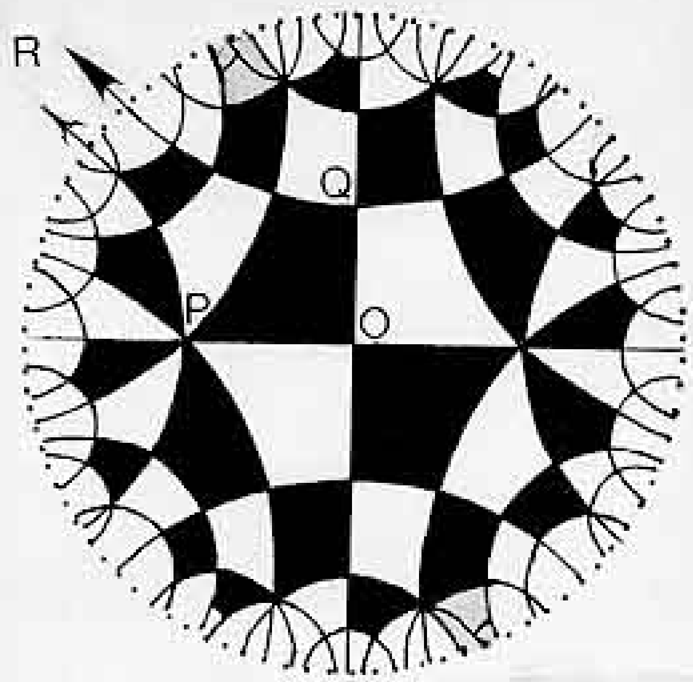


**Figure 3.** A partial tiling of the Poincaré Disk Model by quadrilaterals bounded by Poincaré lines. Quadrilateral $(OPQR)$ is a Lambert quadrilateral with two sides drawn asymptotic to each other.

### 7. Triangles, Quadrilaterals, and Tilings Between the Tropics.

Concern with home and family are universal human values. Typical American houses exhibit Euclidean cross sections: a rectangular one from a side view and a pentagonal one, as a triangular roofline atop a square base, from a head-on view. Western Sumatran Minangkabau house- types fit more naturally into a non-Euclidean framework than they do into the Euclidean one, exhibiting hyperbolic cross sections as a Saccheri quadrilateral (two Lambert quadrilaterals glued together along a "straight" edge (Figure 4a) [8]) when viewed from the side, and as a concave, hyperbolic, triangle atop a (possibly Euclidean) quadrilateral when viewed from the front (Figure 4b).

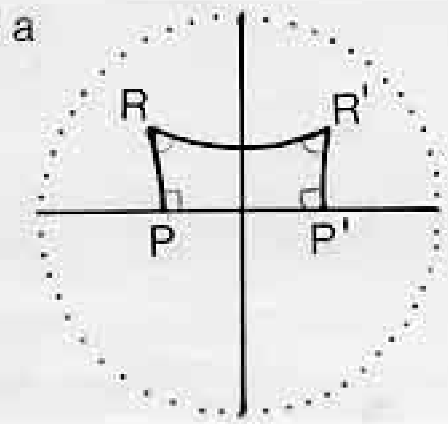



**Figure 4.**

a. A Saccheri quadrilateral, formed from two Lambert quadrilaterals. It has two right angles and two acute angles. Pairs of opposite sides are parallel, as drawn in the Poincaré Disk Model.

b. West Sumatran Minangkabau house. Roofline is suggestive of a Saccheri quadrilateral. Photograph by John D. Nystuen.

Games children play often reveal deeper traditions of an entire society. As the sun moves through its entire range of possible positions, shadows dance across the full range of

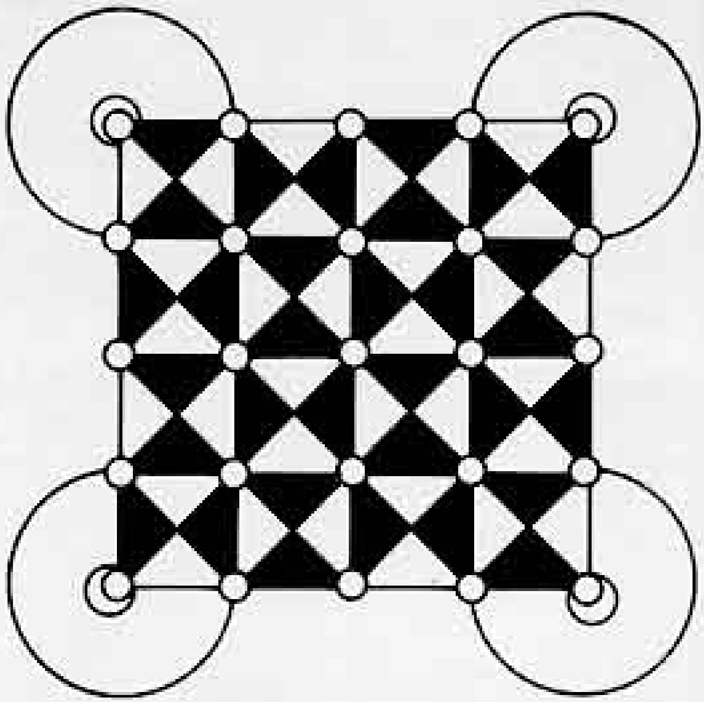

**Figure 5.**

a. Sodokan game board in Euclidean space. Markers travel along lines separating regions of contrasting color and along circular loops at the corners.

compass positions on Indonesian soil and come alive, as "shadow puppets," in Indonesian theatrical productions. Elegant cut-outs traced on goat skins and other hides are mounted on sticks and dance in a plane of light between a single point-source and a screen, casting their filigreed, shadowy outlines high enough for all to see. The motions of the Indonesian puppetteer are regulated by the world of projective geometry, with shadows stretching out diffuse arms toward the infinite.

A commonly played Indonesian board game is "Sodokan," a variant of checkers [1]. Two people play until all of an opponent's ten pieces, arranged initially on the intersection points of the last two lines of a $5 \times 5$ board (Figure 5a), have been captured. Pieces move across the board horizontally, vertically, or diagonally, one square at a time. What is unusual is the method of capture; to take an opponent's marker requires a "surprise" attack along the loops outside the apparent natural grid of the gameboard.

For example, with just two pieces remaining (so that there are no intervening pieces), black may capture white (Figure 5b). To do so, black must traverse at least one loop; in the act of capture, black can slide across as many open grid intersections as required to gain entry to a loop. Then, still in the same turn, black slides around the loop, re-enters the game board, and continues to slide across grid intersections and loops until an opponent's marker is reached, and therefore captured.

The name, "Sodokan," means "push out." Its name seems to apply only loosely to the

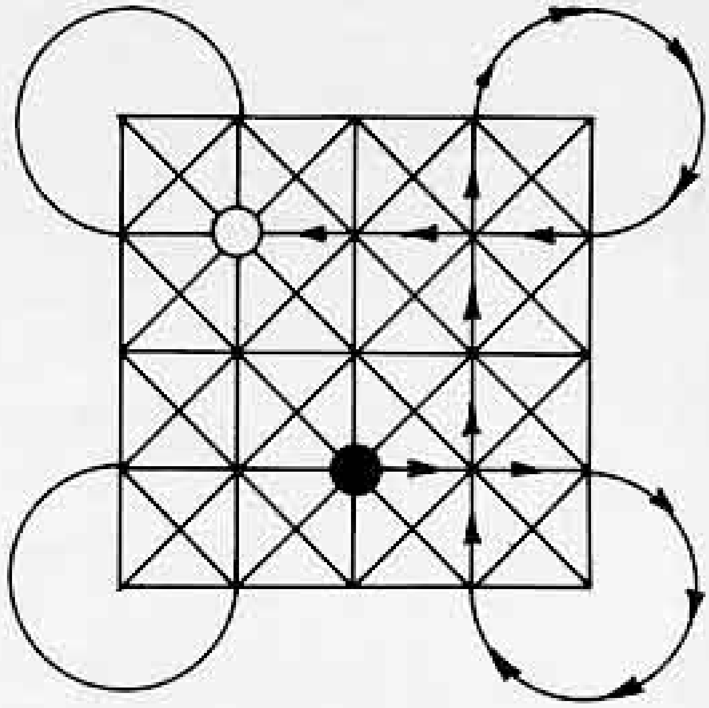

**Figure 5.**
b. Sample of capture. Black captures white—a single move.

$5 \times 5$ Euclidean game board (Figure 5a) because the loops are not, themselves, "pushed out" from the natural gameboard grid. If they were, the corners of the Euclidean grid would disappear. However, when the game board is drawn on a grid in the Poincaré disk model of the hyperbolic plane (Figure 5c), the loops appear naturally from grid intersections outside the circular boundary. A marker engaged in a capture on this non–Euclidean (hyperbolic) board traverses the entire hyperbolic plane ("universe"), passes across the infinite and is provided a natural avenue within the system for return to the universe. The loops are naturally "pushed out" of the underlying grid, tiled partially by Lambert quadrilaterals; they might suggest paths along which gods [11], skipping across space, interrupt (sacrifice) elements within the predictable universe of the life–space in the disk. However, independent of speculation as to what such paths might mean, the fact remains that it is within the hyperbolic geometric framework, only, that this game board emerges as a part of a natural grid system. Thus, capture is no longer a mysterious event from "outside" the system; the change in theoretical framework, from an Euclidean to an hyperbolic viewpoint, made it a logical occurence.

A change in the underlying symmetry introduced order. The "meta" earth–sun system, when viewed as that which introduces a symmetric partition of the earth according to bands of sun–delivered affine and hyperbolic geometry, offered order in understanding roofline and gameboard shape where none had been apparent.

Sources of evidence for other similar interpretations are plentiful: from Indonesian calen-

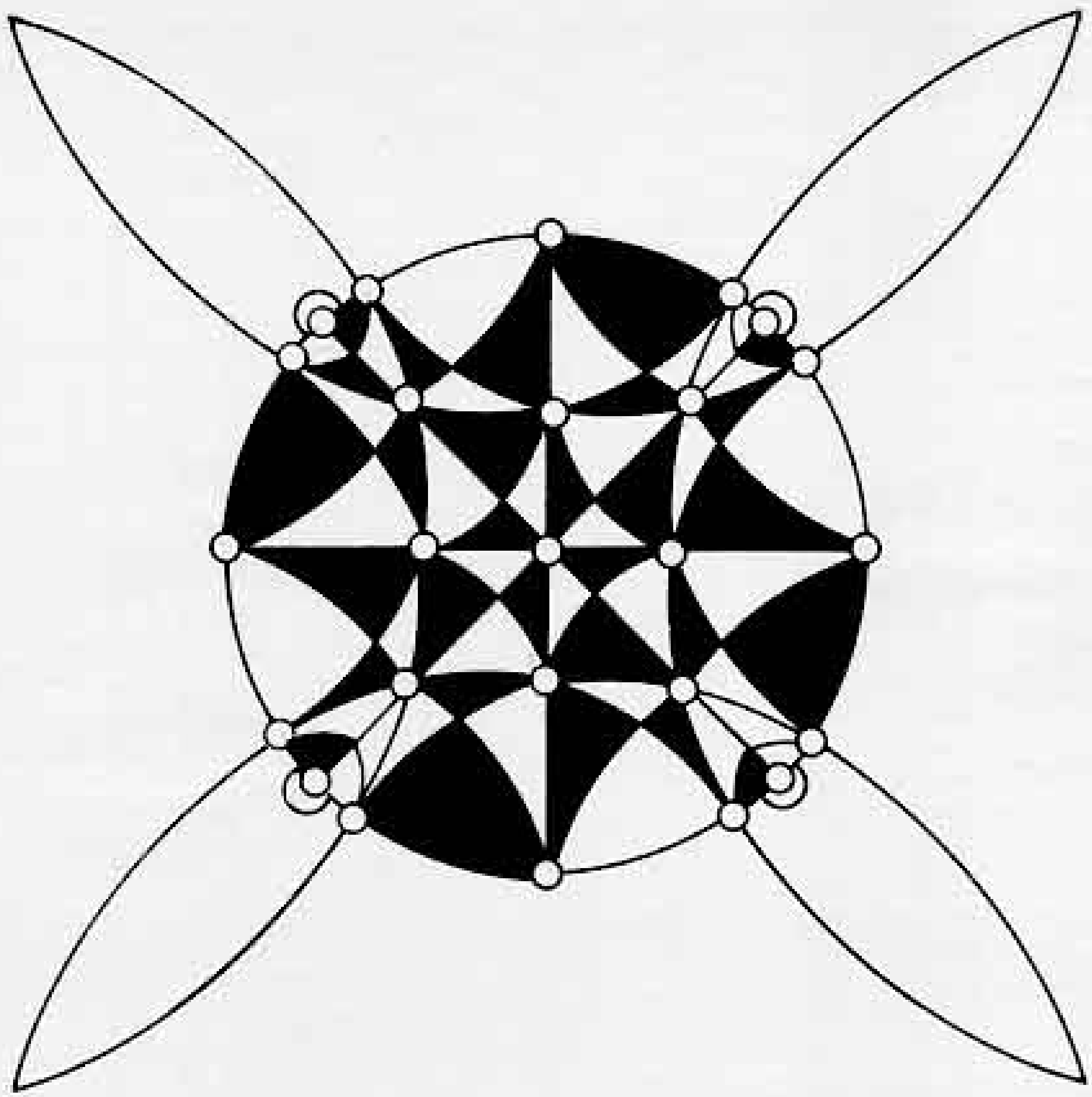

**Figure 5.**

c. Sodokan game board drawn on the Poincaré Disk Model of the hyperbolic plane. The four central quadrilaterals are Lambert quadrilaterals—the intersecting versions of quadrilateral $(OPQR)$ in Figure 3. When their sides are extended, the gameboard loops are formed naturally by these grid lines and their intersection points.

dars based on a nested hierarchy of cycles, to the loops within loops creating the syncopated forms characteristic of Indonesian gamelan music. Perhaps Indonesians and other between-the-parallels dwellers have escaped the asymmetric confines of Euclidean thought, enabling them to include a comfortable vision of infinity as part of the underlying symmetry of their daily circle of life.

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### Acknowledgment

The author wishes to thank John D. Nystuen for his kindness in sharing information, concerning various aspects of Indonesian culture, gathered in field work. Nystuen pointed out the connection between West Sumatran, Minangkabau house-types and Saccheri quadrilaterals, and taught the author and others to play the board game he had learned of in Indonesia. The photograph of the West Sumatran house was taken by Nystuen and appears here with his permission. She also wishes to thank István Hargittai of the Hungarian Academy of Sciences and Arthur Loeb of Harvard University for earlier efforts with this manuscript; this paper was originally accepted by *Symmetry*—Dr. Hargittai was Editor of that journal and Professor Loeb was the Board member of that now defunct journal who communicated this work to Hargittai. The paper appears here exactly as it was communicated to *Symmetry*.

# THE HEDETNIEMI MATRIX SUM: A REAL–WORLD APPLICATION

*Sandra L. Arlinghaus, William C. Arlinghaus, John D. Nystuen*

In a recent paper, we presented an algorithm for finding the shortest distance between any two nodes in a network of $n$ nodes when given only distances between adjacent nodes [Arlinghaus, Arlinghaus, Nystuen, 1990(b)]. In that previous research, we applied the algorithm to the generalized road network graph surrounding San Francisco Bay. The resulting matrices are repeated here (Figure 1), in order to examine consequent changes in matrix entries when the underlying adjacency pattern of the road network was altered by the 1989 earthquake that closed the San Francisco–Oakland Bay Bridge. Thus, we test the algorithm against a changed adjacency configuration and interpret the results with the benefit of hindsight from an actual event. Figure 1 shows a graph, with edges weighted with time–distances, representing the general expressway linkage pattern joining selected cities surrounding San Francisco Bay. The matrix $A$ displays these time–distances in tabular form; an asterisk indicates that there is no direct linkage between corresponding entries. Thus, an asterisk in entry $a_{13}$ indicates that there is no single edge of the graph linking San Francisco and San Jose (all paths have 2 or more edges). Higher powers of the matrix $A$ count numbers of paths of longer length—$A^2$ counts paths of 2 edges as well as those of one edge. Thus, one expects in $A^2$ to see a number measuring time–distance between San Francisco and San Jose; indeed, there are two such paths, one of length 30+50=80, and one of length 30+25=55. The Hedetniemi matrix operator always selects the shortest. Readers wishing to understand the mechanics of this algorithm should refer to the other references related to this topic in the list at the end [Arlinghaus, Arlinghaus, and Nystuen; W. Arlinghaus]. It is sufficieint here simply to understand generally how the procedure works, as described above.

When a recent earthquake caused a disastrous collapse of a span on the San Francisco–Oakland Bay Bridge, forcing the closing of the bridge, municipal authorities managed to keep the city moving using a well–balanced combination of added ferry boats, media messages urging people to stay off the roads, and dispersal of information concerning alternate route strategies. National telecasts showed a city on the move, albeit slowly, although outside forecasters of doom were predicting a massive grid–lock that never occured. What would the Hedetniemi algorithm have forecast in this situation?

To find out, we compare the matrices of Figure 1 to those of Figure 2, derived from the graph of Figure 1 with the link between San Francisco and Oakland removed; that is, the edge linking vertex 4 to vertex 1 is removed — the results show in the matrix entries $a_{14}$ and $a_{41}$. Thus in Figure 2, the adjacency matrix $A$, describing 1–step edge linkages differs from that of Figure 1 only in the $a_{14}$ ($a_{41}$) position. The value of * replaces the time–distance of 30 minutes in that graph because the bridge connection was destroyed. When 2–edge paths are counted, there is spread of increased time–distances across these paths, as well. What used to take 30 minutes, under conditions of normal traffic, to go from San Francisco to Oakland now takes 70 minutes, under conditions of normal traffic, going by way of San Mateo. The trip from San Francisco to Walnut Creek had been possible along a 2–edge path passing through Oakland (and taking a total of 60 minutes); the asterisk in $A^2$ in the $a_{15}$ entry indicates that that path no longer exists. The journey from San Francisco to Richmond, along a 2–edge path, increased in time–distance from 50 to 60 minutes—going around the "longer" side of the rectangle. Note that what is being evaluated here is change in trip–time under "normal" circumstances, according to whether or not routing exists; congestion fluctuates but actual road lengths do not (once in place). These values therefore form a set of benchmarks against which to measure time–distance changes resulting from more variable quantities, such as increased congestion.

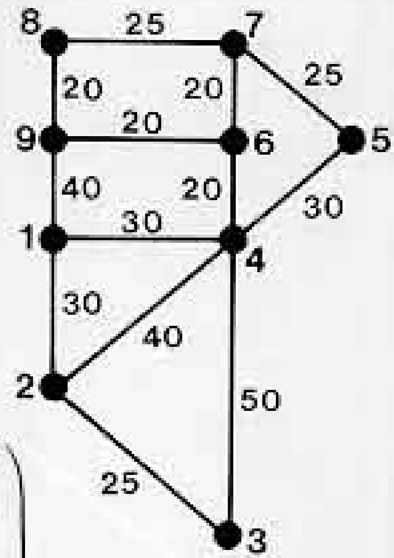
When three–edged paths are brought into the system, in $A^3$ (Figure 2), the trip from San Francisco to Walnut Creek now becomes possible, but takes 100 rather than 60 minutes. Also, the trip from San Francisco to Vallejo now becomes possible (in both pre- and post-earthquake systems) although it takes 10 minutes longer with removal of the bridge. When paths of length four are introduced, no changes occur in these entries; the system is stable and the effects are confined to locations "close" to the bridge that was removed. The relatively small number of changes in the basic underlying route choices, forced by the removal of the Bay Bridge, suggest **why** it was possible, with swift action by municipal authorities and citizens to control congestion, to avert a situation that appeared destined to lead to gridlock.

What if the Golden Gate Bridge had been removed rather than the San Francisco–Oakland Bay Bridge? Figure 3 shows that the same sort of clustered, localized results follow. When both bridges are removed (Figure 4), the position of affected matrix entries is identical to the union of the positions of entries in Figures 1 and 2, but the magnitude of time–distances has been magnified by the combined removal.

With hindsight, the test seems to be reasonable. One direction for a larger application might therefore be to consider historical evidence in which bridge bombing (or some such) was critical to associated circulation patterns. When large data sets are entered into a computer, and manipulated using the Hedetniemi matrix algorithm, previously unnoticed historical associations might emerge and maps showing alternate possibilities could be produced. In short, this might serve as a tool useful in historical discovery. Other important directions for application of the Hedetniemi algorithm involve those in a discrete mathematical setting that focus on tracing actual paths [W. Arlinghaus, 1990—includes program for algorithm], and those using the Hedetniemi algorithm in the computer architecture of parallel processing [Romeijn and Smith].

SAN FRANCISCO BAY AREA; GRAPH OF TIME-DISTANCES
(in minutes)
LEGEND: numeral attached to city is its node number in the corresponding, underlying, graph.

1. SAN FRANCISCO
2. SAN MATEO COUNTY
3. SAN JOSE
4. OAKLAND
5. WALNUT CREEK
6. RICHMOND
7. VALLEJO
8. NOVATO
9. SAN RAFAEL (MARIN COUNTY)




$$A = \begin{pmatrix} 0 & 30 & * & 30 & * & * & * & * & 40 \\ 30 & 0 & 25 & 40 & * & * & * & * & * \\ * & 25 & 0 & 50 & * & * & * & * & * \\ 30 & 40 & 50 & 0 & 30 & 20 & * & * & * \\ * & * & * & 30 & 0 & * & 25 & * & * \\ * & * & * & 20 & * & 0 & 20 & * & 20 \\ * & * & * & * & 25 & 20 & 0 & 25 & * \\ * & * & * & * & * & * & 25 & 0 & 20 \\ 40 & * & * & * & * & 20 & * & 20 & 0 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 0 & 30 & 55 & 30 & 60 & 50 & * & 60 & 40 \\ 30 & 0 & 25 & 40 & 70 & 60 & * & * & 70 \\ 55 & 25 & 0 & 50 & 80 & 70 & * & * & * \\ 30 & 40 & 50 & 0 & 30 & 20 & 40 & * & 40 \\ 60 & 70 & 80 & 30 & 0 & 45 & 25 & 50 & * \\ 50 & 60 & 70 & 20 & 45 & 0 & 20 & 40 & 20 \\ * & * & * & 40 & 25 & 20 & 0 & 25 & 40 \\ 60 & * & * & * & 50 & 40 & 25 & 0 & 20 \\ 40 & 70 & * & 40 & * & 20 & 40 & 20 & 0 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 0 & 30 & 55 & 30 & 60 & 50 & 70 & 60 & 40 \\ 30 & 0 & 25 & 40 & 70 & 60 & 80 & 90 & 70 \\ 55 & 25 & 0 & 50 & 80 & 70 & 90 & * & 90 \\ 30 & 40 & 50 & 0 & 30 & 20 & 40 & 60 & 40 \\ 60 & 70 & 80 & 30 & 0 & 45 & 25 & 50 & 65 \\ 50 & 60 & 70 & 20 & 45 & 0 & 20 & 40 & 20 \\ 70 & 80 & 90 & 40 & 25 & 20 & 0 & 25 & 40 \\ 60 & 90 & * & 60 & 50 & 40 & 25 & 0 & 20 \\ 40 & 70 & 90 & 40 & 65 & 20 & 40 & 20 & 0 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 0 & 30 & 55 & 30 & 60 & 50 & 70 & 60 & 40 \\ 30 & 0 & 25 & 40 & 70 & 60 & 80 & 90 & 70 \\ 55 & 25 & 0 & 50 & 80 & 70 & 90 & 110 & 90 \\ 30 & 40 & 50 & 0 & 30 & 20 & 40 & 60 & 40 \\ 60 & 70 & 80 & 30 & 0 & 45 & 25 & 50 & 65 \\ 50 & 60 & 70 & 20 & 45 & 0 & 20 & 40 & 20 \\ 70 & 80 & 90 & 40 & 25 & 20 & 0 & 25 & 40 \\ 60 & 90 & 110 & 60 & 50 & 40 & 25 & 0 & 20 \\ 40 & 70 & 90 & 40 & 65 & 20 & 40 & 20 & 0 \end{pmatrix}$$

$$A^5 = \begin{pmatrix} 0 & 30 & 55 & 30 & 60 & 50 & 70 & 60 & 40 \\ 30 & 0 & 25 & 40 & 70 & 60 & 80 & 90 & 70 \\ 55 & 25 & 0 & 50 & 80 & 70 & 90 & 110 & 90 \\ 30 & 40 & 50 & 0 & 30 & 20 & 40 & 60 & 40 \\ 60 & 70 & 80 & 30 & 0 & 45 & 25 & 50 & 65 \\ 50 & 60 & 70 & 20 & 45 & 0 & 20 & 40 & 20 \\ 70 & 80 & 90 & 40 & 25 & 20 & 0 & 25 & 40 \\ 60 & 90 & 110 & 60 & 50 & 40 & 25 & 0 & 20 \\ 40 & 70 & 90 & 40 & 65 & 20 & 40 & 20 & 0 \end{pmatrix}$$

$$A^4 = A^5 = \ldots = A^9$$

**Figure 1**. Pre–earthquake matrix sequence.

POST–EARTHQUAKE SAN FRANCISCO BAY AREA
SAN FRANCISCO–OAKLAND BAY BRIDGE IS REMOVED.
GRAPH OF TIME–DISTANCES (in minutes)
Adjustment is made for change in time–distance
in a "normal" situation–not for resultant fluctuation in congestion

$$A = \begin{pmatrix} 0 & 30 & * & * & * & * & * & * & 40 \\ 30 & 0 & 25 & 40 & * & * & * & * & * \\ * & 25 & 0 & 50 & * & * & * & * & * \\ * & 40 & 50 & 0 & 30 & 20 & * & * & * \\ * & * & * & 30 & 0 & * & 25 & * & * \\ * & * & * & 20 & * & 0 & 20 & * & 20 \\ * & * & * & * & 25 & 20 & 0 & 25 & * \\ * & * & * & * & * & * & 25 & 0 & 20 \\ 40 & * & * & * & * & 20 & * & 20 & 0 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 0 & 30 & 55 & 70 & * & 60 & * & 60 & 40 \\ 30 & 0 & 25 & 40 & 70 & 60 & * & * & 70 \\ 55 & 25 & 0 & 50 & 80 & 70 & * & * & * \\ 70 & 40 & 50 & 0 & 30 & 20 & 40 & * & 40 \\ * & 70 & 80 & 30 & 0 & 45 & 25 & 50 & * \\ 60 & 60 & 70 & 20 & 45 & 0 & 20 & 40 & 20 \\ * & * & * & 40 & 25 & 20 & 0 & 25 & 40 \\ 60 & * & * & * & 50 & 40 & 25 & 0 & 20 \\ 40 & 70 & * & 40 & * & 20 & 40 & 20 & 0 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 0 & 30 & 55 & 70 & 100 & 60 & 80 & 60 & 40 \\ 30 & 0 & 25 & 40 & 70 & 60 & 80 & 90 & 70 \\ 55 & 25 & 0 & 50 & 80 & 70 & 90 & * & 90 \\ 70 & 40 & 50 & 0 & 30 & 20 & 40 & 60 & 40 \\ 100 & 70 & 80 & 30 & 0 & 45 & 25 & 50 & 65 \\ 60 & 60 & 70 & 20 & 45 & 0 & 20 & 40 & 20 \\ 80 & 80 & 90 & 40 & 25 & 20 & 0 & 25 & 40 \\ 60 & 90 & * & 60 & 50 & 40 & 25 & 0 & 20 \\ 40 & 70 & 90 & 40 & 65 & 20 & 40 & 20 & 0 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 0 & 30 & 55 & 70 & 100 & 60 & 80 & 60 & 40 \\ 30 & 0 & 25 & 40 & 70 & 60 & 80 & 90 & 70 \\ 55 & 25 & 0 & 50 & 80 & 70 & 90 & 110 & 90 \\ 70 & 40 & 50 & 0 & 30 & 20 & 40 & 60 & 40 \\ 100 & 70 & 80 & 30 & 0 & 45 & 25 & 50 & 65 \\ 60 & 60 & 70 & 20 & 45 & 0 & 20 & 40 & 20 \\ 80 & 80 & 90 & 40 & 25 & 20 & 0 & 25 & 40 \\ 60 & 90 & 110 & 60 & 50 & 40 & 25 & 0 & 20 \\ 40 & 70 & 90 & 40 & 65 & 20 & 40 & 20 & 0 \end{pmatrix}$$

$$A^5 = \begin{pmatrix} 0 & 30 & 55 & 70 & 100 & 60 & 80 & 60 & 40 \\ 30 & 0 & 25 & 40 & 70 & 60 & 80 & 90 & 70 \\ 55 & 25 & 0 & 50 & 80 & 70 & 90 & 110 & 90 \\ 70 & 40 & 50 & 0 & 30 & 20 & 40 & 60 & 40 \\ 100 & 70 & 80 & 30 & 0 & 45 & 25 & 50 & 65 \\ 60 & 60 & 70 & 20 & 45 & 0 & 20 & 40 & 20 \\ 80 & 80 & 90 & 40 & 25 & 20 & 0 & 25 & 40 \\ 60 & 90 & 110 & 60 & 50 & 40 & 25 & 0 & 20 \\ 40 & 70 & 90 & 40 & 65 & 20 & 40 & 20 & 0 \end{pmatrix}$$

$$A^4 = A^5 = \ldots = A^9$$

**Figure 2**. Matrix sequence with San Francisco–Oakland Bay Bridge removed.

POST–EARTHQUAKE SAN FRANCISCO BAY AREA
GOLDEN GATE BRIDGE IS REMOVED.
GRAPH OF TIME–DISTANCES (in minutes)
Adjustment is made for change in time–distance
in a "normal" situation—not for resultant fluctuation in congestion

$$A = \begin{pmatrix} 0 & 30 & \ast & 30 & \ast & \ast & \ast & \ast & \ast \\ 30 & 0 & 25 & 40 & \ast & \ast & \ast & \ast & \ast \\ \ast & 25 & 0 & 50 & \ast & \ast & \ast & \ast & \ast \\ 30 & 40 & 50 & 0 & 30 & 20 & \ast & \ast & \ast \\ \ast & \ast & \ast & 30 & 0 & \ast & 25 & \ast & \ast \\ \ast & \ast & \ast & 20 & \ast & 0 & 20 & \ast & 20 \\ \ast & \ast & \ast & \ast & 25 & 20 & 0 & 25 & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast & 25 & 0 & 20 \\ \ast & \ast & \ast & \ast & \ast & 20 & \ast & 20 & 0 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 0 & 30 & 55 & 30 & 60 & 50 & \ast & \ast & \ast \\ 30 & 0 & 25 & 40 & 70 & 60 & \ast & \ast & \ast \\ 55 & 25 & 0 & 50 & 80 & 70 & \ast & \ast & \ast \\ 30 & 40 & 50 & 0 & 30 & 20 & 40 & \ast & 40 \\ 60 & 70 & 80 & 30 & 0 & 45 & 25 & 50 & \ast \\ 50 & 60 & 70 & 20 & 45 & 0 & 20 & 40 & 20 \\ \ast & \ast & \ast & 40 & 25 & 20 & 0 & 25 & 40 \\ \ast & \ast & \ast & \ast & 50 & 40 & 25 & 0 & 20 \\ \ast & \ast & \ast & 40 & \ast & 20 & 40 & 20 & 0 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 0 & 30 & 55 & 30 & 60 & 50 & 70 & \ast & 70 \\ 30 & 0 & 25 & 40 & 70 & 60 & 80 & \ast & 80 \\ 55 & 25 & 0 & 50 & 80 & 70 & 90 & \ast & 90 \\ 30 & 40 & 50 & 0 & 30 & 20 & 40 & 60 & 40 \\ 60 & 70 & 80 & 30 & 0 & 45 & 25 & 50 & 65 \\ 50 & 60 & 70 & 20 & 45 & 0 & 20 & 40 & 20 \\ 70 & 80 & 90 & 40 & 25 & 20 & 0 & 25 & 40 \\ \ast & \ast & \ast & 60 & 50 & 40 & 25 & 0 & 20 \\ 70 & 80 & 90 & 40 & 65 & 20 & 40 & 20 & 0 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 0 & 30 & 55 & 30 & 60 & 50 & 70 & 90 & 70 \\ 30 & 0 & 25 & 40 & 70 & 60 & 80 & 100 & 80 \\ 55 & 25 & 0 & 50 & 80 & 70 & 90 & 110 & 90 \\ 30 & 40 & 50 & 0 & 30 & 20 & 40 & 60 & 40 \\ 60 & 70 & 80 & 30 & 0 & 45 & 25 & 50 & 65 \\ 50 & 60 & 70 & 20 & 45 & 0 & 20 & 40 & 20 \\ 70 & 80 & 90 & 40 & 25 & 20 & 0 & 25 & 40 \\ 90 & 100 & 110 & 60 & 50 & 40 & 25 & 0 & 20 \\ 70 & 80 & 90 & 40 & 65 & 20 & 40 & 20 & 0 \end{pmatrix}$$

$$A^5 = \begin{pmatrix} 0 & 30 & 55 & 30 & 60 & 50 & 70 & 90 & 70 \\ 30 & 0 & 25 & 40 & 70 & 60 & 80 & 100 & 80 \\ 55 & 25 & 0 & 50 & 80 & 70 & 90 & 110 & 90 \\ 30 & 40 & 50 & 0 & 30 & 20 & 40 & 60 & 40 \\ 60 & 70 & 80 & 30 & 0 & 45 & 25 & 50 & 65 \\ 50 & 60 & 70 & 20 & 45 & 0 & 20 & 40 & 20 \\ 70 & 80 & 90 & 40 & 25 & 20 & 0 & 25 & 40 \\ 90 & 100 & 110 & 60 & 50 & 40 & 25 & 0 & 20 \\ 70 & 80 & 90 & 40 & 65 & 20 & 40 & 20 & 0 \end{pmatrix}$$

$$A^4 = A^5 = \ldots = A^9$$

Figure 3. Matrix sequence with the Golden Gate Bridge removed.

POST–EARTHQUAKE SAN FRANCISCO BAY AREA
BAY BRIDGE AND GOLDEN GATE BRIDGE ARE BOTH REMOVED.
GRAPH OF TIME–DISTANCES (in minutes)
Adjustment is made for change in time–distance
in a "normal" situation—not for resultant fluctuation in congestion

$$A = \begin{pmatrix} 0 & 30 & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\ 30 & 0 & 25 & 40 & \ast & \ast & \ast & \ast & \ast \\ \ast & 25 & 0 & 50 & \ast & \ast & \ast & \ast & \ast \\ \ast & 40 & 50 & 0 & 30 & 20 & \ast & \ast & \ast \\ \ast & \ast & \ast & 30 & 0 & \ast & 25 & \ast & \ast \\ \ast & \ast & \ast & 20 & \ast & 0 & 20 & \ast & 20 \\ \ast & \ast & \ast & \ast & 25 & 20 & 0 & 25 & \ast \\ \ast & \ast & \ast & \ast & \ast & \ast & 25 & 0 & 20 \\ \ast & \ast & \ast & \ast & \ast & 20 & \ast & 20 & 0 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 0 & 30 & 55 & 70 & \ast & \ast & \ast & \ast & \ast \\ 30 & 0 & 25 & 40 & 70 & 60 & \ast & \ast & \ast \\ 55 & 25 & 0 & 50 & 80 & 70 & \ast & \ast & \ast \\ 70 & 40 & 50 & 0 & 30 & 20 & 40 & \ast & 40 \\ \ast & 70 & 80 & 30 & 0 & 45 & 25 & 50 & \ast \\ \ast & 60 & 70 & 20 & 45 & 0 & 20 & 40 & 20 \\ \ast & \ast & \ast & 40 & 25 & 20 & 0 & 25 & 40 \\ \ast & \ast & \ast & \ast & 50 & 40 & 25 & 0 & 20 \\ \ast & \ast & \ast & 40 & \ast & 20 & 40 & 20 & 0 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 0 & 30 & 55 & 70 & 100 & 90 & \ast & \ast & \ast \\ 30 & 0 & 25 & 40 & 70 & 60 & 80 & \ast & 80 \\ 55 & 25 & 0 & 50 & 80 & 70 & 90 & \ast & 90 \\ 70 & 40 & 50 & 0 & 30 & 20 & 40 & 60 & 40 \\ 100 & 70 & 80 & 30 & 0 & 45 & 25 & 50 & 65 \\ 90 & 60 & 70 & 20 & 45 & 0 & 20 & 40 & 20 \\ \ast & 80 & 90 & 40 & 25 & 20 & 0 & 25 & 40 \\ \ast & 80 & \ast & 60 & 50 & 40 & 25 & 0 & 20 \\ \ast & \ast & 90 & 40 & 65 & 20 & 40 & 20 & 0 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 0 & 30 & 55 & 70 & 100 & 90 & 110 & \ast & 110 \\ 30 & 0 & 25 & 40 & 70 & 60 & 80 & 100 & 80 \\ 55 & 25 & 0 & 50 & 80 & 70 & 90 & 110 & 90 \\ 70 & 40 & 50 & 0 & 30 & 20 & 40 & 60 & 40 \\ 100 & 70 & 80 & 30 & 0 & 45 & 25 & 50 & 65 \\ 90 & 60 & 70 & 20 & 45 & 0 & 20 & 40 & 20 \\ 110 & 80 & 90 & 40 & 25 & 20 & 0 & 25 & 40 \\ \ast & 100 & 110 & 60 & 50 & 40 & 25 & 0 & 20 \\ 110 & 80 & 90 & 40 & 65 & 20 & 40 & 20 & 0 \end{pmatrix}$$

$$A^5 = \begin{pmatrix} 0 & 30 & 55 & 70 & 100 & 90 & 110 & 130 & 110 \\ 30 & 0 & 25 & 40 & 70 & 60 & 80 & 100 & 80 \\ 55 & 25 & 0 & 50 & 80 & 70 & 90 & 110 & 90 \\ 70 & 40 & 50 & 0 & 30 & 20 & 40 & 60 & 40 \\ 100 & 70 & 80 & 30 & 0 & 45 & 25 & 50 & 65 \\ 90 & 60 & 70 & 20 & 45 & 0 & 20 & 40 & 20 \\ 110 & 80 & 90 & 40 & 25 & 20 & 0 & 25 & 40 \\ 130 & 100 & 110 & 60 & 50 & 40 & 25 & 0 & 20 \\ 110 & 80 & 90 & 40 & 65 & 20 & 40 & 20 & 0 \end{pmatrix}$$

$$A^4 = A^5 = \ldots = A^9$$

**Figure 4.** Matrix sequence with both the Golden Gate and the Bay bridges removed.

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# FRACTAL GEOMETRY OF INFINITE PIXEL SEQUENCES: "SUPER-DEFINITION" RESOLUTION?

*Sandra Lach Arlinghaus*

## Introduction

The fractal approach to the geometry of central place theory is particularly powerful because, among other things, it provides numerical proof that the subjective labels of "marketing," "transportation," and "administration" for the $K = 3$, $K = 4$, and $K = 7$ hierarchies are indeed correct [Arlinghaus, 1985] and because it enables solution of all open geometric questions identified by Dacey, Marshall, and others in earlier research [Dacey; Marshall; Arlinghaus and Arlinghaus]. When the problem is wrapped back on itself and the nature of the original, underlying environment is altered—from urban to electronic—the same results, recast in a different light, suggest the degree of improvement in picture resolution that can come from decreasing pixel size.

Curves on cathode ray tubes are formed from a sequence of pixels hooked together at their corners; font designers in word processors offer an easy opportunity to observe these pixel formations (Horstmann, 1986). The pixel sequence merely suggests the curve; it does not actually produce a "correct" curve. Reducing the size of the pixel can improve the resolution of the image representing the curve. The material below uses established results from fractal geometry to evaluate the degree of success, in improving resolution in a raster environment, that results from decreasing pixel size.

## Manhattan pixel arrangement

When a square pixel is the fundamental unit, a sequence of pixels has boundaries separating pixels in Manhattan, "city- block" space. When smaller square pixels are introduced, more lines separating pixels are also introduced. The interior of the pixel is what carries the content—not the boundary of the pixel. Thus, it is significant to know what proportion of the space filled with pixels is filled with pixel boundary.

Suppose that, in an effort to produce "high- definition" resolution, the number of square pixels used to cover a fixed area (a cathode ray tube) is substantially increased. One might be tempted to use even more pixels to produce even better resolution and even more beyond that. If the process is carried out infinitely, using a Manhattan grid, the pixel mesh has arbitrarily small cell size and the entire plane region is "filled" with pixel boundary, only; the scale transformation of superimposing finer and finer square mesh on a fixed area has dimension $D = 2$ (Mandelbrot, p. 63, 1983). In this situation, all pixel content is therefore lost. Clearly then, improvement in resolution does not continue, ad infinitum; there is some point at which the tradeoff between fineness in resolution and loss of information content is at its peak. Determining this point is an issue of difficulty and significance. Is this dilemma a universal situation that exists independent of the shape of the fundamental pixel unit?

## Hexagonal pixel arrangement

Consider instead an electronic environment in which the fundamental picture element is hexagonal in shape (Rosenfeld; Gibson and Lucas). Such a geometric environment has a number of well-documented advantages, centering on close-packing characteristics (Gibson and Lucas). This environment is examined here along the lines suggested above—to see if improvement in resolution can be carried out infinitely through pixel subdivision.

When a bounded lattice of regular hexagons of uniform cell diameter (on a CRT) is refined as a similar lattice of smaller uniform cell diameter, improvement in resolution results. There are an infinite number of ways in which the lattice of smaller cell-size might be superimposed on the lattice of larger cell size. The geometry of central place theory describes these relative positions of layers. Independent of the orientation selected, when this transformation from larger to smaller cell lattice is iterated infinitely, the bounded space is once again filled (as in the rectangular pixel case) with hexagonal pixel boundary. Thus, in both the case of the rectangular pixel and the hexagonal pixel environments, infinite "improvement" in resolution, brought about by decreasing pixel size, causes a black-hole-like collapse of the original, entire image. However, is this characteristic of the whole necessarily inherited by each of its parts? Any part that does not inherit this collapsing, space-filling characteristic is capable of infinite, "super-definition" resolution. Such a part is invariant (to some extent) under scale transformation.

The fractal approach to central place theory shows that there do exist shapes in the hexagonal pixel environment which, when refined infinitely, do not fill a bounded piece of two dimensional space. Figure 1 shows a hexagon to which a fractal generator has been applied to produce a $K = 4$ hierarchy. Infinite iteration of this self-similarity transformation produces a highly crenulated replacement which **does not** fill a bounded two-dimensional space; in fact, it fills only 1.585 of a two-dimensional space. When the corresponding self-similarity transformation is applied to a square pixel a highly crenulated shape is again the result of infinite iteration; this shape **does** fill a bounded two- dimensional space (Figure 2). The two fractal generators selected are parallel in structure: each is half of the boundary of the fundamental pixel shape.

If both geometric environments are then viewed as composed of these highly-crenulated elements (which do fit together to cover the plane), then the hexagonal environment is the one that permits infinite iteration without loss of all pixel content. This approach is akin to that of Barnsley, which stores sets of transformations that are used to drive image production. What is suggested here is a possible way to vastly improve image resolution corresponding,

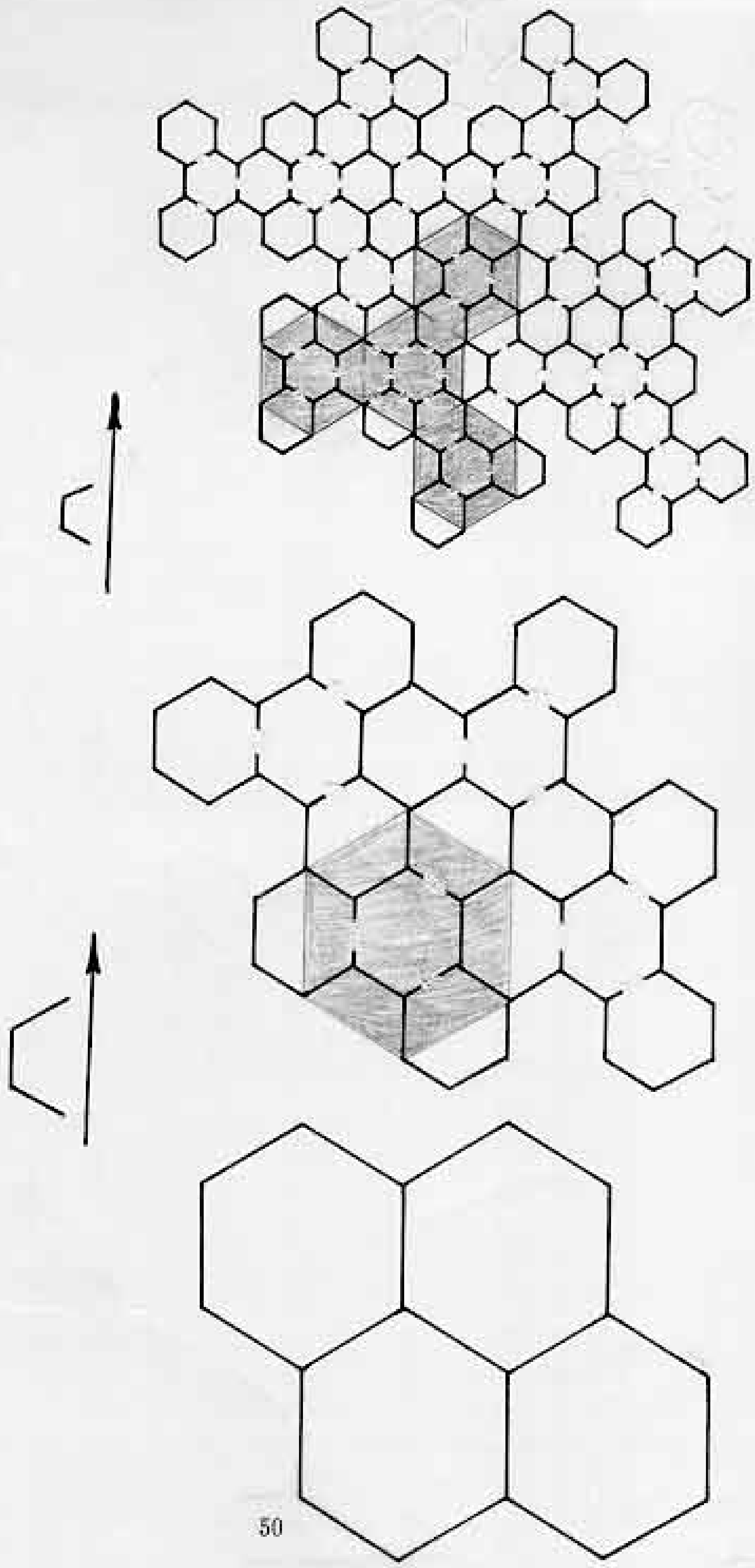

**Figure 1.** K=4 hierarchy of hexagonal pixels generated fractally.

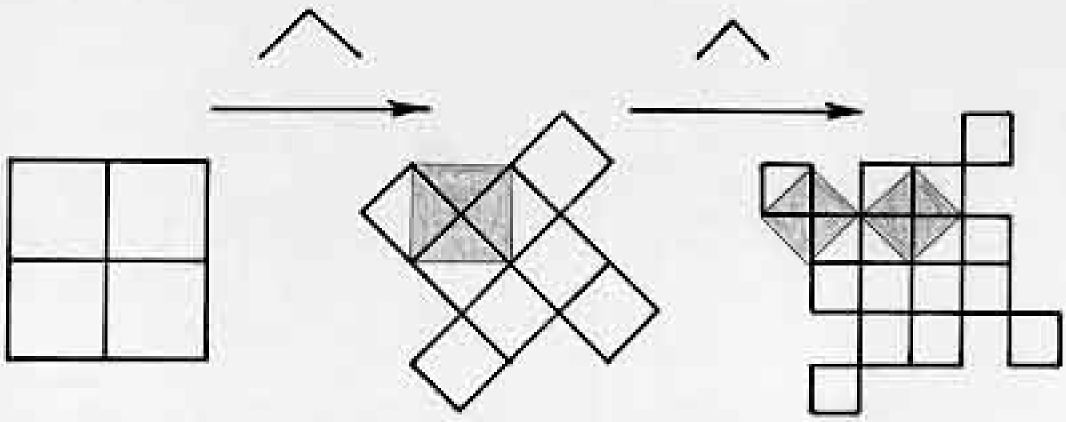

**Figure 2.** K=4 type of hierarchy generated fractally from square initiators.

to some extent, to Barnsley's successful strategy to improve data compression (Barnsley).

This approach is also similar, in general strategy to that employed by Hall and Gökmen; both seek transformations, applied in an electronic environment, under which some properties are preserved. Hall and Gökmen focus on transformations linking hexagonal and rectangular pixel space whereas the transformations employed here function entirely within a single type of geometric environment (using one on the other appears to be of interest). Additionally, this approach offers a systematic characterization, in the infinite, for the aggregate 7-kernels of hexagons, at various levels of aggregation, suggested only as finite sequences in Gibson and Lucas. Finally, Tobler's maps of Swiss migration patterns at three levels of spatial resolution suggest a methodological handle of an attractivity function to implement ideas involving spatial resolution in an electronic environment. Deeper analysis, of the sort represented in the works mentioned here, is beyond the scope of this particular short piece.

Table 1 shows a set of fractal dimensions for selected Löschian numbers.

**Table 1**

(derived from a Table in Arlinghaus and Arlinghaus, 1989)

| | | | | |
|---|---|---|---|---|
| K=3, D=1.262; | K=12, D=1.116; | K=27, D=1.087; | K=48, D=1.074; | ... |
| K=7, D=1.129; | K=19, D=1.093; | K=37, D=1.078; | K=61, D=1.069; | ... |
| K=4, D=1.585; | K=13, D=1.255; | K=28, D=1.168; | K=49, D=1.129; | ... |

The line of Löschian numbers that begins with $K = 4$, those that are organized according to an "transportation" principle, are the ones that fill two dimensional space most thickly. Thus, when introducing smaller and smaller hexagonal cells to improve resolution in the quality of curve representation, or when "zooming in," it would appear appropriate to let the orientation of successive layers of smaller and smaller cells correspond to the $K = 4$ type of hierarchy. Clutter would not enter as fast as in the Manhattan environment, even in this densest arrangement. "Super," rather than "high," definition of resolution could therefore fall naturally from an underlying hexagonal pixel geometry with measures of clutter and information content determined using fractal dimensions.

### Shortest paths

At an even broader scale, one might also look for this sort of application in hooking computers together as parallel processing units. When "central places" are thought of as central processing units, not of urban information, but rather of electronic information, then an underlying geometry for finding "shortest" paths through networks linking multiple points might emerge. For in an electronic environment with the hexagonal pixel as the fundamental unit, the 120° intersection points would correspond exactly to the requirements for finding Steiner networks, as "shortest" networks linking multiple locations. Steiner points in an electronic configuration might then correspond to locations at which to "jump" from one hexagonal lattice of fixed cell-size to another of different cell size (from one machine to another), where cell size is prescribed by "lengths" (in whatever metric) between "transmission times" between adjacent Steiner points.

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* The author wishes to thank Michael Goodchild for constructive comments on a 1989 version of this paper. Much of this content content has been presented previously: before national meetings of the American Mathematical Society in August of 1990; before national

meetings of the Association of American Geographers in April of 1990; and, before a classroom audience at The University of Michigan in the Winter Semester of 1989/90.

## CONSTRUCTION ZONE

FIRST CONSTRUCTION;
readers might wish to construct figures to accompany
the electronic text as they read

### Feigenbaum's number: exposition of one case

Motivated by queries from Michael Woldenberg,
Department of Geography, SUNY Buffalo,
during his visit to Ann Arbor, Summer, 1990.

Here is a description of how Feigenbaum's number arises from a graphical analysis of a simple geometric system [1]. Feigenbaum's original paper is clear and straightforward [1]; this construction is presented to serve as exposure prior to reading Feigenbaum's longer paper [1]. The construction is complicated although individual steps are not generally difficult. Following the construction, a suggestion will be offered as to how to select mathematical constraints within which to choose geographical systems for Feigenbaum-type analysis.

1. Consider the family of parabolas $y = x^2 + c$, where $c$ is an integral constant. This is just the set of parabolas that are like $y = x^2$, slid up or down the $y$-axis. The smaller the value of $c$, the more the parabola opens up (otherwise a lower one would intersect a higher one, creating an algebraic impossibility such as $-1 = 0$) (Figure 1).

2. To begin, consider the particular parabola, $y = x^2 - 1$, obtained by setting $c = -1$. Graph this (Figure 2). Also draw the line $y = x$ on this graph. Now we're going to look at the "orbit" of the value $x = 1/2$ with respect to this parabola (function). By "orbit" is meant simply the iteration string obtained by using $x = 1/2$ as input into $y = x^2 - 1$, then using that output as a new input into $y = x^2 - 1$, then using that output as a new input ... and so forth. In this case, the orbit of $x = 1/2$ is represented as follows, numerically. (Use $.5 \mapsto -0.75$ to mean that the input of .5 is mapped to the output value of $-0.75$ by the function $y = x^2 - 1$.)

$$0.5 \mapsto -0.75 \mapsto -0.4375 \mapsto -0.8085938$$

$$\mapsto -0.3461761 \mapsto -0.8801621 \mapsto -0.2253147$$

$$\mapsto -0.9492333 \mapsto -0.0989562 \mapsto -0.9902077$$

$$\mapsto -0.019488 \mapsto -0.9996202 \mapsto -0.0007595$$

$$\mapsto -0.9999994 \mapsto -0.0000012 \mapsto -1$$

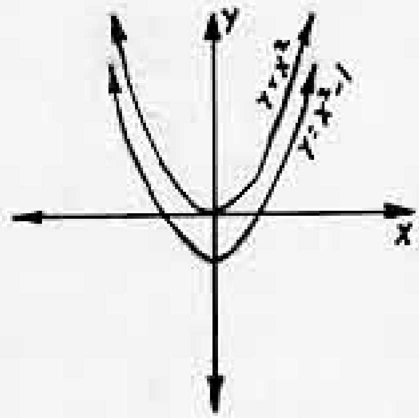
$$\mapsto 0 \mapsto -1 \mapsto 0 \mapsto \ldots$$

Clearly the values bounce around for awhile, and then eventually settle down to the values, $-1$ and $0$.

3. Let's see what this particular iteration string means geometrically (Figure 3). Locate $x = 0.5$ on the $x$-axis. Drop down to the parabola to read off the corresponding $y$-value (in the usual manner) $-0.75$. Now it is this $y$-value that is to be used as the next input in the iteration string. We could go back up to the $x$-axis and find it and drop back to the parabola, but we won't. Instead execute the following, equivalent transformation—THIS IS THE KEY POINT. Assume your penpoint is on the $y$-value $-0.75$; now slide

horizontally over to the line $y = x$—you want to use the $y$-value in the role of the $x$-value. Thus, treat this point as the new input and drop to the parabola from it as you did in moving from the $x$-axis to the parabola. Then, with your penpoint on the parabola, slide horizontally back to the line $y = x$ and use this as the input; drop to the parabola and keep going. A glance at Figure 2 suggests why economists call this a "cobweb" diagram (presumably looking at fluctuating supply and demand). Follow this diagram long enough, and you will see that eventually values for $x$ fluctuate between 0 and $-1$, around a stationary square cycle. Looking at the "dynamics" of a value, with respect to a function, in this geometrical manner is referred to as (Feigenbaum's) "graphical analysis" [1].

1

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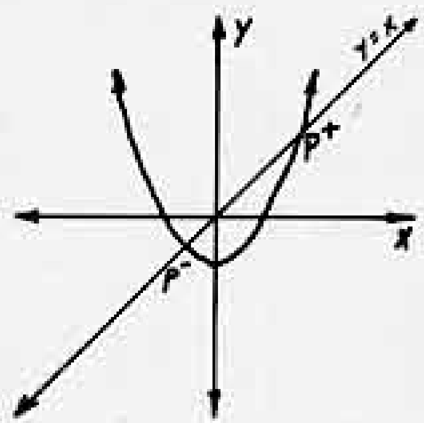

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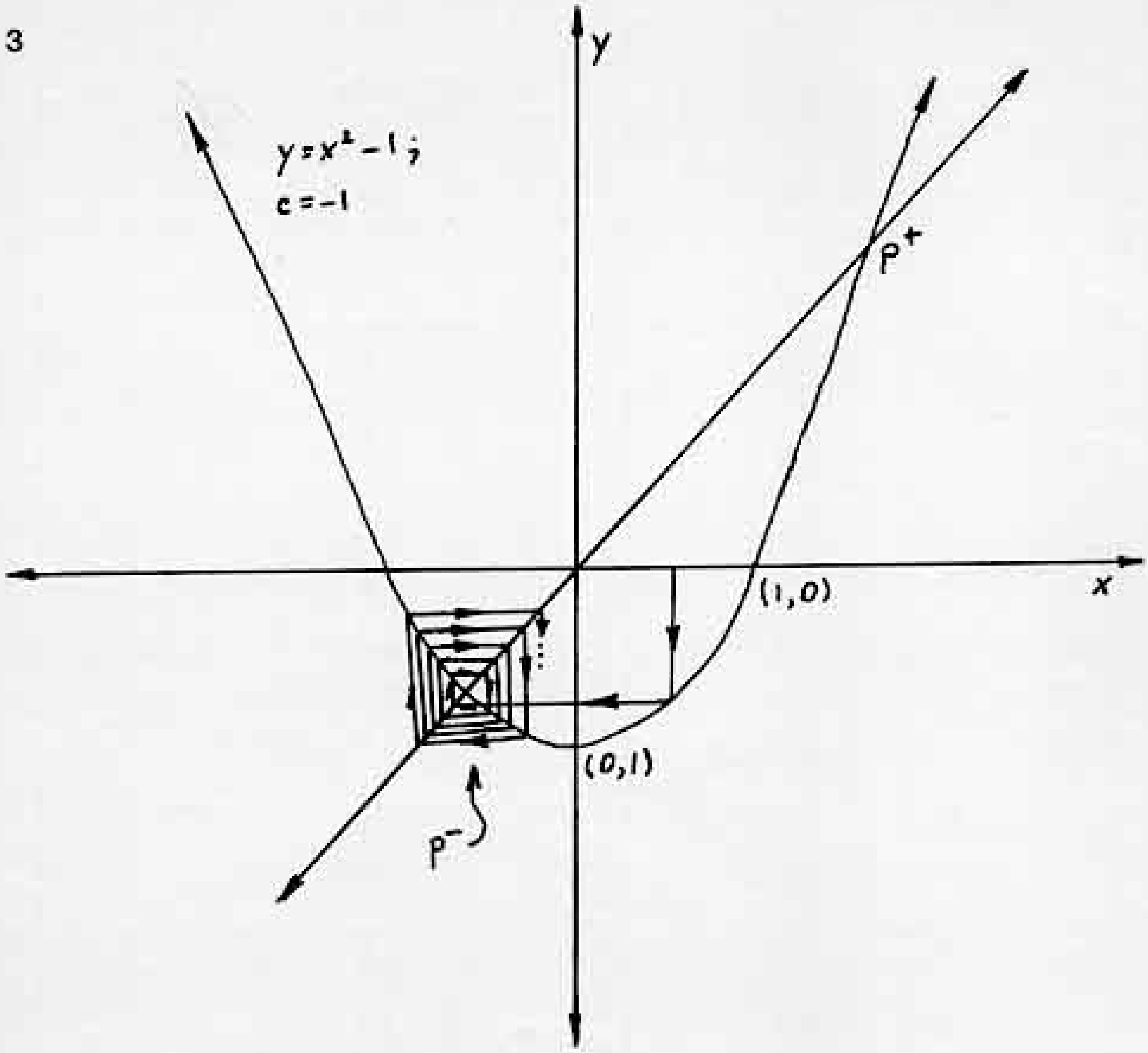



**Figure 1.** Parabolas of the form $y = x^2 + c$. **Figure 2.** The parabola $y = x^2 - 1$ and $y = x$. **Figure 3.** Graphical analysis of $y = x^2 - 1$.

4. So, we have the numerical orbit and the graphical analysis for the value $x = 0.5$ with respect to the function $y = x^2 - 1$. What about calculating these values for starting values of $x$ other than $x = 0.5$. Consider $x = 1.6$. Its orbit is as below, and the

corresponding graphical analysis is given in Figure 4.

$$1.6 \mapsto 1.56 \mapsto 1.4336 \mapsto 1.055209$$

$$\mapsto 0.1134659 \mapsto -0.9871255 \mapsto -0.0255833$$

$$\mapsto -0.9993455 \mapsto -0.0013086 \mapsto -0.9999983$$

$$\mapsto -0.0000034 \mapsto -1 \mapsto 0 \mapsto -1 \mapsto 0 \mapsto \ldots$$

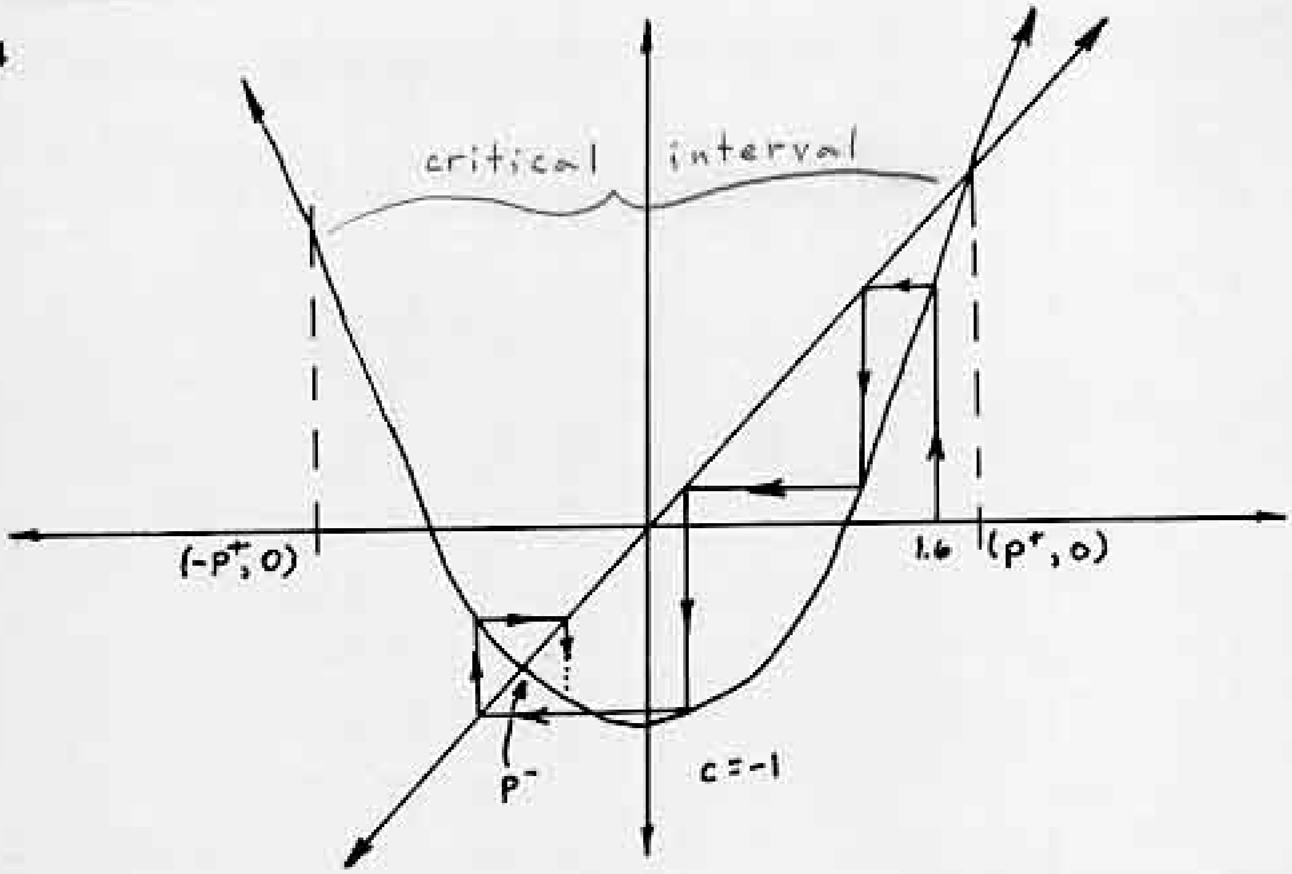
The dynamics of $x = 1.6$ are really very much the same as for $x = 0.5$ with respect to the given function. Let's look at $x = 1.7$.

$$1.7 \mapsto 1.89 \mapsto 2.5721 \mapsto 5.6156984$$

$$\mapsto 30.536069 \mapsto 931.45149 \mapsto 867600.87 \mapsto \ldots \mathit{to}\ \infty.$$

Graphical analysis shows this clearly, geometrically, too (Figure 5). This shooting off to infinity is not "interesting" in the way that the cobweb dynamics are. So, for what values of $x$ do you get "interesting" dynamics?

4



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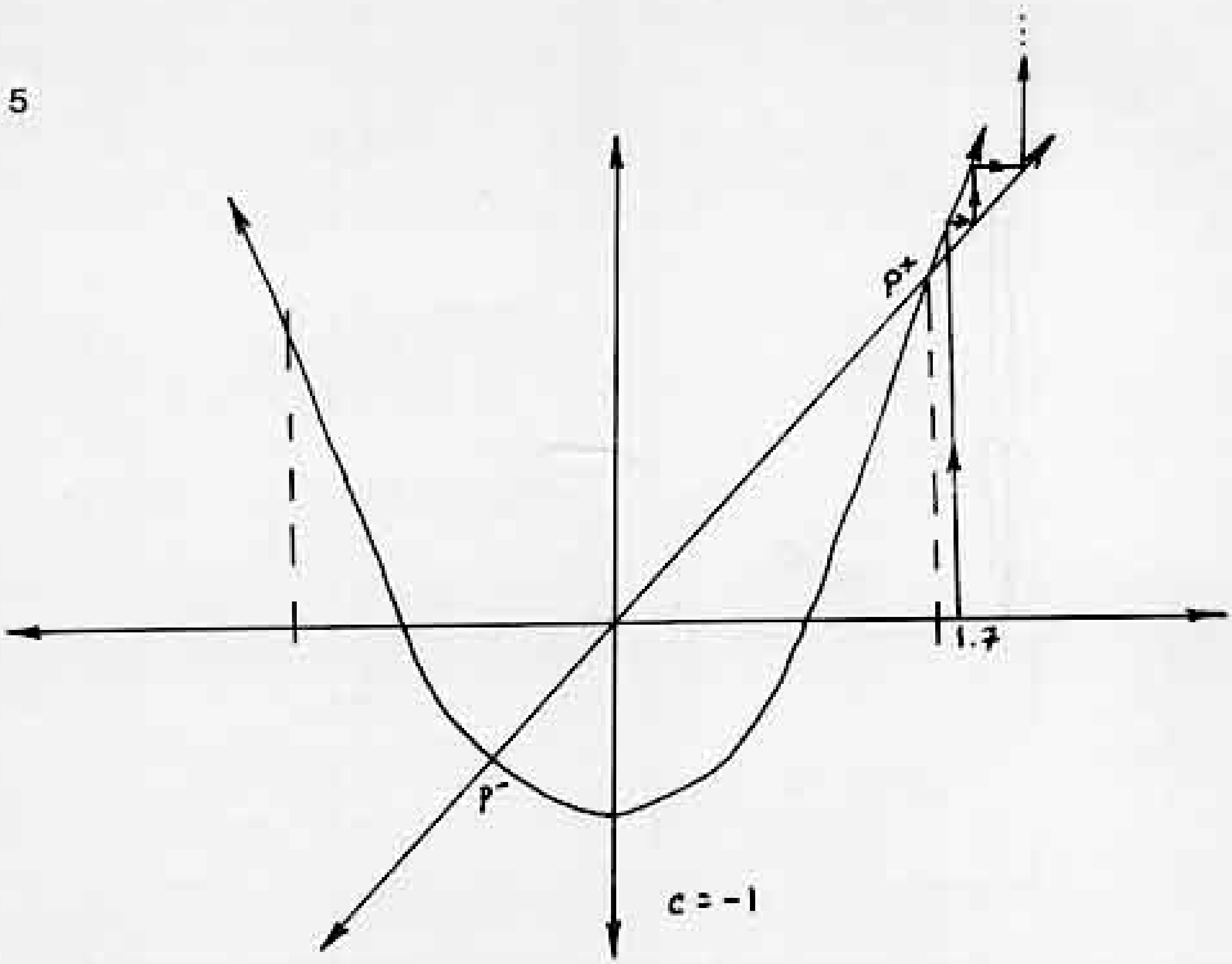



**Figure 4.** Orbit of $x = 1.6$. **Figure 5.** Orbit of $x = 1.7$.

5. No doubt you will have noted from the graphical analyses in Figures 4 and 5 that the reason one iteration closes down into a cobweb and the other goes to infinity is that one initial value of $x$ lies to the left of the intersection point of the parabola and the line $y = x$, and the other lies to the right of that intersection point. You might therefore be tempted to guess that all initial values of $x$ that lie between the right hand intersection

point (call it $p^+$) of the parabola and the line and the left hand intersection point (call it $p^-$) of the parabola and the line $y = x$, produce interesting dynamics. (The $x$–coordinates for $p^+$ and $p^-$ are found by solving $y = x$ and $y = x^2 - 1$ simultaneously—that is by solving $x^2 - x - 1 = 0$—the quadratic formula yields $x = (1 \pm \sqrt{5})/2$, or $x = 1.618034$, $x = -0.618034$). Indeed, if you try a number of values intermediate between these you will find that to be the case. However, consider a value of $x$ to the left of $x = -0.62$. Try $x = -1.6$.

$$-1.6 \mapsto 1.56 \mapsto 1.4336 \mapsto 1.055209$$

$$\mapsto 0.1134659 \mapsto -0.9871255 \mapsto -0.0255833$$

$$\mapsto -0.9993455 \mapsto -0.0013086 \mapsto -0.9999983$$

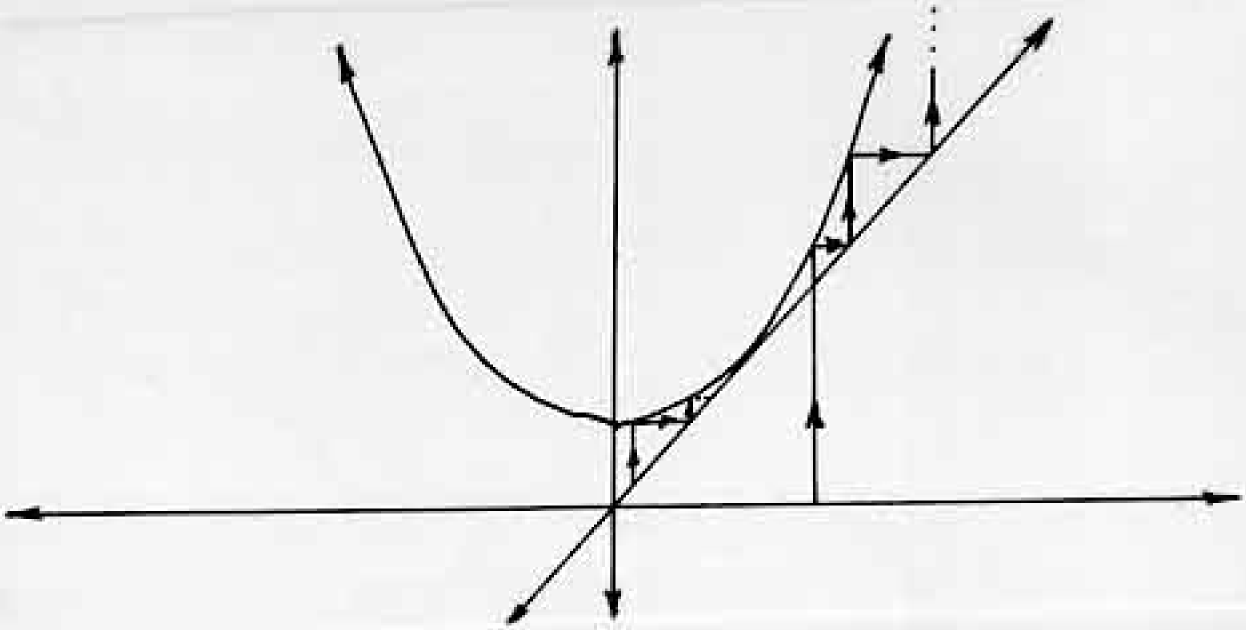
$$\mapsto -0.000003 \mapsto -1 \mapsto 0 \mapsto -1 \mapsto 0 \mapsto \ldots$$

There is obvious bilateral (about the $y$–axis) symmetry in the iteration string, produced by squaring inputs. Clearly, the initial value of $-1.7$ will go to positive infinity, as above. So, the interval of values of $x$ that will produce interesting dynamics is NOT $[p^-, p^+]$, but rather $[-p^+, p^+]$. You might want to draw graphical analyses for $x = -1.6$ and $x = -1.7$ with respect to this function. Call the interval, $[-p^+, p^+]$ the "critical" interval for any given system of parabola and $y = x$. In the case of the system $y = x$ and $y = x^2 - 1$ the critical interval has length 3.236068.

So, now we know something general about the dynamics of input values with respect to the function $y = x^2 - 1$. Recall that we got this function by picking one value, $c = -1$, from the family of parabolas $y = x^2 + c$. Let's see what happens for different values of $c$.

6. Consider $c = 0.25$. For this value of $c$, the line $y = x$ and the parabola $y = x^2 + 0.25$ are tangent to each other. Values of $x$ to the left of the point of tangency (at (0.5, 0.25)) have orbits that converge to 0.5 (Figure 6) while values of $x$ to the right of the point of tangency have orbits that go to positive infinity. Initial inputs to the left of the point of tangency have orbits that are "attracted" to the point of tangency, while initial inputs to the right of the point of tangency have orbits that are "repelled" from the point of tangency. Here, you might view it that $p^+ = p^-$. When $c > 0.25$, the line $y = x$ and the corresponding parabola do not intersect, and so all orbits go to infinity—the dynamics are not interesting (Figure 7). So, we should be looking at parabolas with $c$ less than or equal to 0.25. Let's look at some, in regard to the notions of "attracting" and "repelling."

**6**



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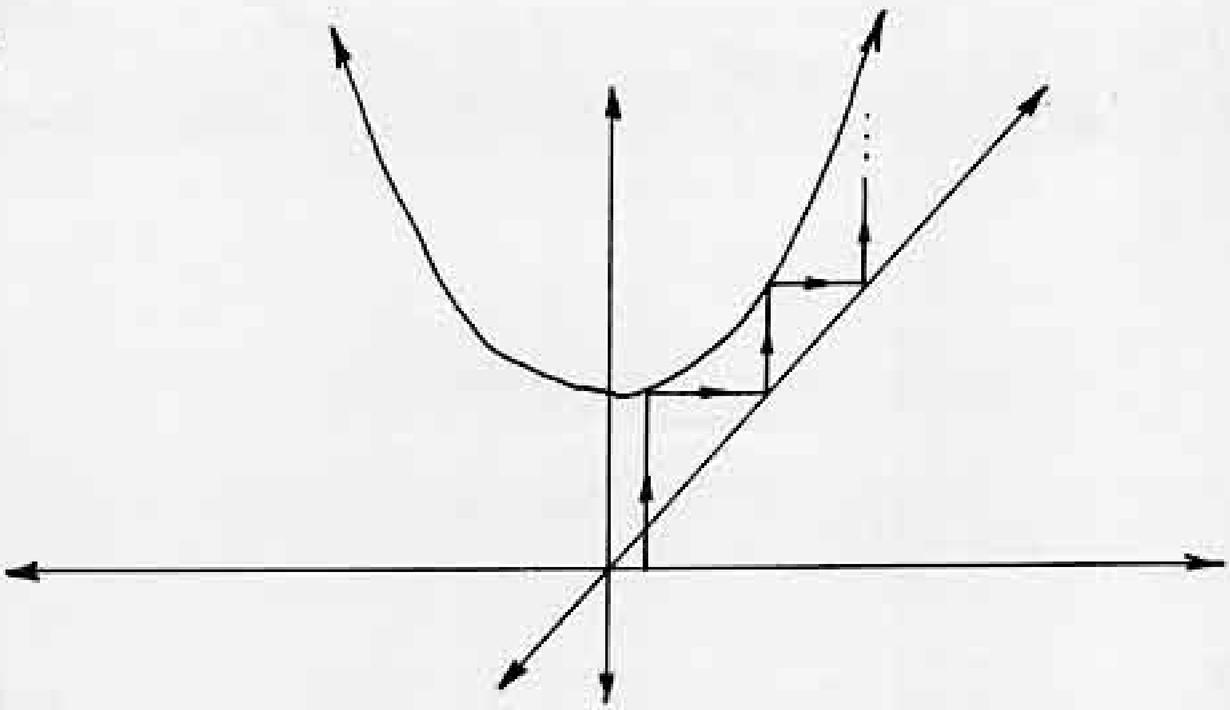


**Figure 6.** The case for $c = 1/4$. **Figure 7.** The case for $c > 1/4$.

7. Consider $c = 0.24$—system: $y = x$, $y = x^2 + 0.24$ (Figure 8). Use graphical analysis to study the dynamics (Figure 8). An orbit of 0.5 is

$$0.5 \mapsto .3025 \mapsto .3315063 \mapsto .3498964$$

$$\mapsto .362427 \mapsto .3713537 \mapsto .3779036$$

$$\mapsto .3828111 \mapsto .3865443 \mapsto .3894165$$

$$\mapsto .3916452 \mapsto .393386 \mapsto .3947525 \mapsto \ldots \mapsto 0.4.$$

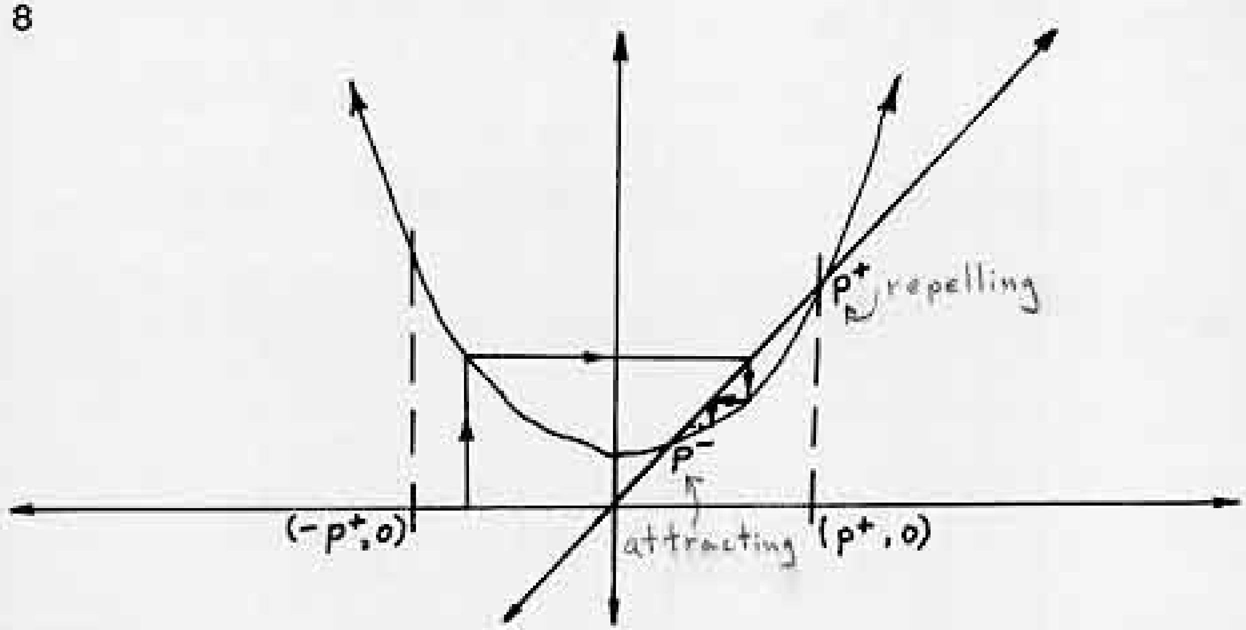
The orbit converges to the $x$-value of $p^-$ which is found as 0.4 by solving the system using the quadratic formula. Here, $p^-$ is an attracting fixed point of the system, and $p^+$ is a repelling fixed point of the system. There is convergence of orbits to a single value within the zone $[-p^+, p^+]$. Notice a kind of doubling effect as one moves from the system with $c = 0.25$ to the one with $c = 0.26$ (period-doubling).

8. Consider $c = -0.74$. The system is: $y = x$, $y = x^2 - 0.74$. Graphical analysis (Figure 9) shows that this system behaves similarly to the one for $c = 0.24$; $p^-$ is attracting and $p^+$ is repelling for all $x$ in $[-p^+, p^+]$. The values of $p^-$ and $p^+$ are respectively $-0.4949874$ and $1.4949874$. Look at the orbit of 0.5, for example.

$$0.5 \mapsto -0.49 \mapsto -0.4999 \mapsto -0.4901$$

$$\mapsto -0.499802 \mapsto -0.490198 \mapsto \ldots \mapsto -0.4949874$$

8



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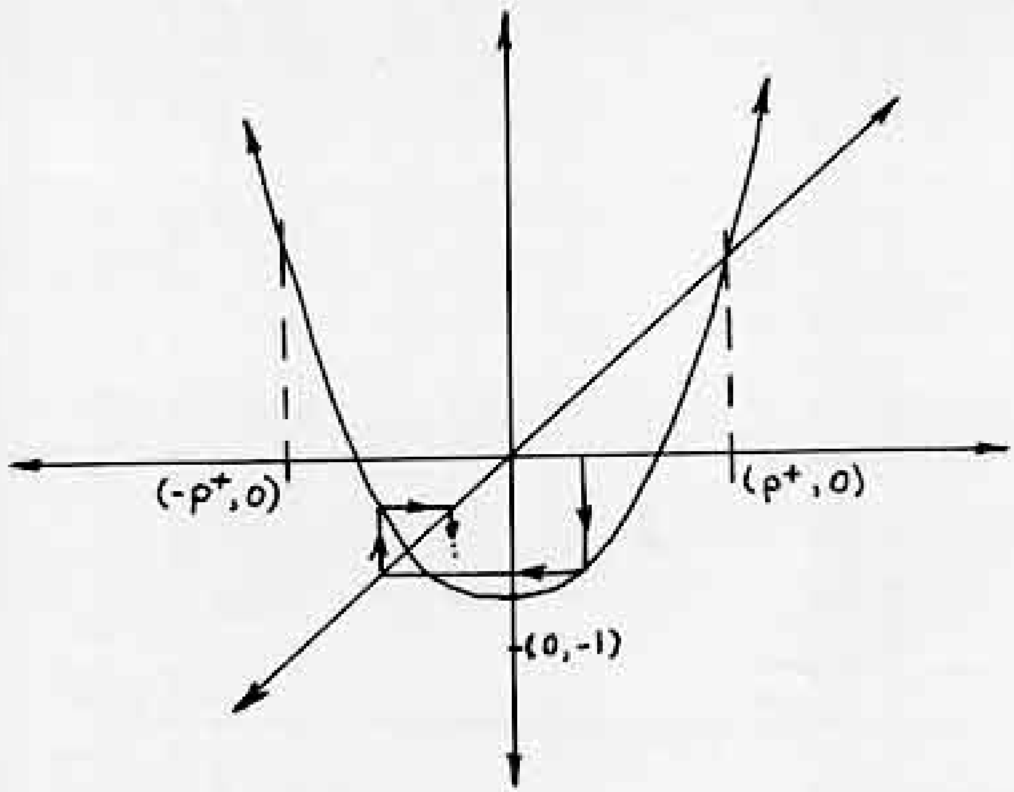


**Figure 8.** The case for $c = 0.24$. **Figure 9.** The case for $c = -0.74$.

9. Consider $c = -0.75$. The system is: $y = x$, $y = x^2 - 0.75$. This is not at all the same sort of system as those in 7 and 8 above. Here, $p^-$ and $p^+$ are respectively $-0.5$ and $1.5$. Consider the orbit of $0.5$.

$$0.5 \mapsto -0.5 \mapsto -0.5 \mapsto -0.5 \mapsto \ldots$$

Consider the orbit of 0.1:

$$0.1 \mapsto -0.74 \mapsto -0.2024 \mapsto -0.7090342$$

$$\mapsto -0.2472704 \mapsto -.6888573 \mapsto -.2754756$$

$$\mapsto -.6741132 \mapsto -.2955714 \mapsto -.6626376 \mapsto -.3109115 \mapsto \ldots$$

here, one might see this closing in, from above and below, very slowly on $-0.5$. Or, there might be two points the orbit is fluctuating toward getting close to. Consider the orbit of 1.4:

$$1.4 \mapsto 1.21 \mapsto .7141 \mapsto -.2400612 \mapsto -.6923706 \mapsto \ldots$$

Again, the same sort of thing as above. The behavior of this system is suggestive of that of the tangent case when $c = 0.25$.

10. So, we might suspect some sort of shift in the dynamics for values of $c$ less than $-0.75$. Indeed, we have already looked at the case $c = -1$. In that case, the point $p^-$ is repelling, rather than attracting (as it was for $0.25 < c < -0.75$). Also, the length of the period over which an orbit stabilizes has doubled — lands on two values, instead of converging to one. Again, there is a sort of bifurcation of dynamical process at $c = -0.75$, much as there was at $c = 0.25$. The next value of c at which there is bifurcation of process is at $c = -1.25$ (analysis not shown). Values of $c$ slightly less than $-1.25$ produce systems with orbits for initial $x$-values in the critical interval that settle down to fluctuating among four values; the point $p^-$, which had been repelling for $-0.75 < c < -1.25$ now becomes attracting. And so this continues— another bifurcation near 1.37, and another somewhere near 1.4. The values for $c$ at which successive bifurcations occur come faster and faster.
11. A summary of this material appears below.

Bifurcation values, $b$:

$$c = 0.25 - - - b = 1$$

$$c = -0.75 - - - b = 2$$

$$c = -1.25 - - - b = 3$$

$$c = -1.37 - - - b = 4$$

derived from empirical evidence of examining the orbit dynamics of the corresponding systems of parabolas and $y = x$. Lengths of critical intervals, $I_b$, $[-p^+, p^+]$, associated with the system corresponding to each bifurcation value, $b$.

$c = 0.25$; Solve: $y = x$, $y = x^2 + .25$; use quadratic formula—
$x = (1 \pm \sqrt{(1 - 4 \times 0.25)})/2 = 0.5$. Thus, $p^+ = 0.5$ so

$$I_1 = 2 \times 0.5 = 1.0$$

$c = -0.75$. Solve: $y = x$, $y = x^2 - .75$. $x = (1 \pm \sqrt{(1 + 4 \times 0.75)})/2 = 1.5$ or $-0.5$. Thus, $p^+ = 1.5$ so

$$I_2 = 2 \times 1.5 = 3.0$$

$c = -1.25$. Solve: $y = x$, $y = x^2 - 1.25$. $x = (1 \pm \sqrt{(1 + 4 \times 1.25)})/2 = 1.7247449$ or $-0.7247449$. So,

$$I_3 = 3.4494898$$

$c = -1.37$. Solve: $y = x$, $y = x^2 - 1.37$. $x = (1 \pm \sqrt{(1 + 4 \times 1.37)})/2 = 1.7727922$ or $-0.7727922$. So,

$$I_4 = 3.5455844$$

Now, suppose we find the successive differences between these interval lengths:

$$D_1 = I_2 - I_1 = 3 - 1 = 2$$

$$D_2 = I_3 - I_2 = 3.4494898 - 3 = 0.4494898$$

$$D_3 = I_4 - I_3 = 3.5455844 - 3.4494898 = 0.0960946$$

Then, form successive ratios of these differences, larger over smaller:

$$D_1/D_2 = 2/0.4494898 = 4.4494892$$

$$D_2/D_3 = .4494898/.0960946 = 4.6775761$$

This set of ratios converges to Feigenbaum's number, 4.6692016...

12. Apparently, empirical evidence suggests that any parabola–like system exhibits the same sorts of dynamics and the corresponding sets of ratios converge to Feigenbaum's number. For example, this appears to be the case, from literature, for the system $y = x$ and $y = c(sinx)$ and for the system involving the logistic curve, $y = x$ and $y = cx(1 - x)$ [1].

13. However, when the curved piece of the system is not parabola–like, different constants may occur. (A different curve might be a parabola with the vertex squared off— singularities are introduced—where the derivative is undefined) [1].

14. Obviously, many geographical systems can be characterized by a curve with fluctuations that are somewhat parabolic. Of course, we often do not know the equation of the curve. But, Simpson's rule from calculus, that pieces together parabolic slabs to approximate the area under a curve, generally gives a good approximation to the area of such curves. Thus, geographic systems that give rise to curves for which Simpson's rule provides a good areal approximation are ones that might be reasonable to explore in connection with Feigenbaum's number.

15. Steps 1 to 11 show how Feigenbaum's "universal" number can be generated. Steps 12 to 14 give a systematic way to select geographical systems to examine with respect to this constant.

## REFERENCE

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SECOND CONSTRUCTION

A three-axis coordinatization of the plane

Motivated by a question from Richard Weinand

Department of Computer Science, Wayne State University

1. Triangulate the plane using equilateral triangles. Then, choose any triangle as a triangle of reference—this triangle is to serve as an "origin" for a coordinate system (an area-origin rather than a conventional point-origin—this is like homogeneous coordinates in projective geometry *e.g.* H. S. M. Coxeter, *The Real Projective Plane*). Each side of the triangle is an axis—$x = 0$, $y = 0$, $z = 0$ (Figure 10-draw to match text).

2. Each vertex of a triangle has unique representation as an ordered triple with reference to the origin-triangle (but, not every ordered triple of integers corresponds to a lattice point— there is no point $(x, x, x)$) (Figure 10).

3. Assign an orientation (clockwise or counterclockwise) to the origin-triangle, and mark the edges of the triangle with arrowheads to correspond to this orientation. This then determines the orientation of all the remaining triangles.

4. Now suppose that a triangle is picked out at random. Suppose it has orientation the same as the reference triangle (clockwise, say). The coordinates of its vertices, in general, will be (choosing $(x, y, z)$ to be the lower left-hand corner):

$$(x, y, z); (x+1, y, z-1); (x, y+1, z-1)$$

and those of triangles sharing a common edge with it (and of opposite orientation to it) will have coordinates:

$$\text{left}: (x, y, z); (x+1, y, z-1); (x+1, y-1, z)$$

$$\text{right}: (x+1, y, z-1); (x, y+1, z-1); (x+1, y+1, z-2)$$

$$\text{bottom}: (x, y+1, z-1); (x, y, z); (x-1, y+1, z)$$

Suppose the arbitrarily selected triangle has orientation opposite that of the reference triangle (counterclockwise). The coordinates of its vertices, in general, will be (choosing $(x, y, z)$ to be the upper left-hand corner):

$$(x, y, z); (x-1, y+1, z); (x, y+1, z-1)$$

and those of triangles sharing a common edge with it (and of opposite orientation to it (clockwise)) will have coordinates:

$$\text{left}: (x, y, z); (x-1, y+1, z); (x-1, y, z+1)$$

$$\text{right}: (x-1, y+1, z); (x, y+1, z-1); (x-1, y+2, z-1)$$

$$\text{top}: (x, y, z); (x+1, y, z-1); (x, y+1, z-1)$$

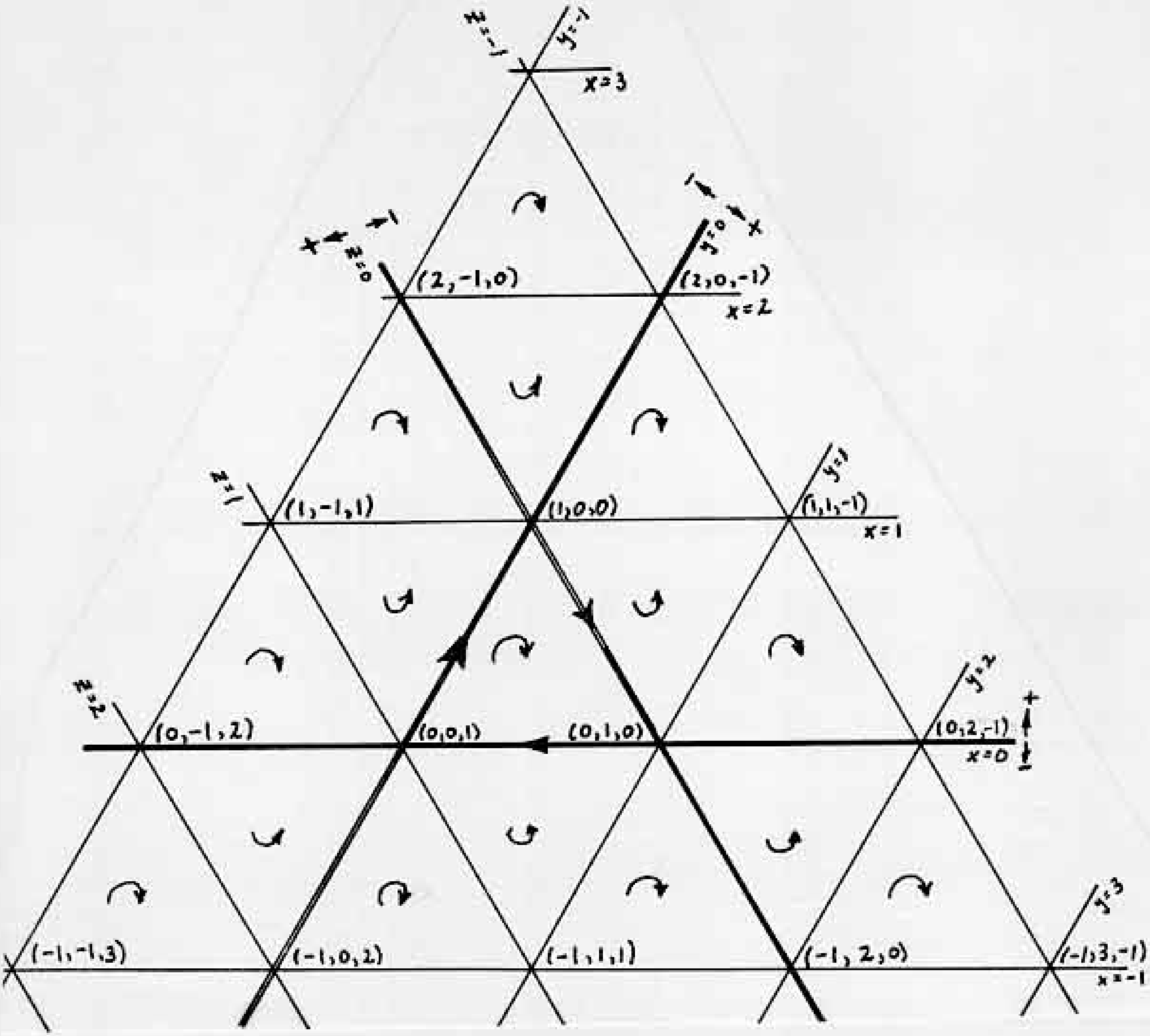


**Figure 10.** Three-axis coordinate system for the plane.

5. Coordinates of triangles sharing a point-boundary (and of the same orientation as the arbitrarily selected triangle) might also be read off in a similar fashion.

6. Naturally, six of these triangles form a hexagon. So, this could be considered from the viewpoint of an hexagonal tesselation, as well. Choose an arbitrary hexagon and read off coordinates of adjacent hexagonal regions in a similar manner.

7. In a current *College Mathematics Journal*, Vol 21, No. 4, September, 1990, there is an article by David Singmaster (of Rubik's Cube fame) which also employs triangular coordinates of the sort mentioned above (pages 278-285— "Triangles with integer sides and sharing barrels").

8. This strategy would seem to work for any developable surface (cylinder, torus, Möbius strip, Klein bottle—all can be cut apart into a plane). Triangles were chosen because procedure involving them might be extended to simplicial complexes (triangle=simplex).

9. One way to triangulate a sphere is to project an icosahedron, inscribed in the sphere, onto the surface of the sphere (conversation with Jerrold Grossman, Dep't. of Mathematics, Oakland University). This procedure will produce 20 triangular regions of equal size (under suitable transformation). But, more triangles may be desirable. Alternately, one might subdivide the triangular faces of the icosahedron into, say, three triangles of equal area, and project the point that produces this subdivision (a barycentric subdivision, for example) onto the sphere (using gnomonic projection (from the sphere's center)). (Subdividing all of them a second time would produce 180 triangles of equal area and shape covering the sphere.) Subdivision centers on opposite sides of the icosahedron appear to lie on a single diameter of the sphere; therefore, when their images are projected onto the sphere they will be antipodal points. In that event, a coordinate system similar to the one described for developable surfaces might work.