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1. WELCOME TO NEW READERS AND THANK YOU

Welcome to new subscribers! We hope you enjoy participating in this means of journal distribution. Instructions for downloading the typesetting have been repeated in this issue, near the end. They are specific to the TEX installation at The University of Michigan, but apparently they have been helpful in suggesting to others the sorts of commands that might be used on their own particular mainframe installation of TEX. New subscribers might wish to note that the electronic files are typeset files—the mathematical notation will print out as typeset notation. For example,

$$\Sigma_{i=1}^{n}$$

when properly downloaded, will print out a typeset summation as i goes from one to n, as a centered display on the page. Complex notation is no barrier to this form of journal production.

Thanks much to subscribers who have offered input. Helpful suggestions are important in trying to keep abreast, at least somewhat, of the constantly changing electronic world. Some suggestions from readers have already been implemented; others are being worked on. Indeed, it is particularly helpful when the reader making the suggestion becomes actively involved in carrying it out. We hope you continue to enjoy *Solstice*.

2. PRESS CLIPPINGS—SUMMARY

Volume 72, Number 4, October 1993 issue of *Papers in Regional Science: The Journal of the Regional Science Association* carried an article by Gunther Maier and Andreas Wildberger entitled "Wide Area Computer Networks and Scholarly Communication in Regional Science." Maier and Wildberger noted that "Only one journal in this directory can be considered to be related to Regional Science, *Solstice: An Electronic Journal of Geography and Mathematics*."

Beyond that, brief write-ups about *Solstice* have appeared in the following publications:

1. **Science**, "Online Journals" Briefings. [by Joseph Palca] 29 November 1991. Vol. 254.

2. **Science News**, "Math for all seasons" by Ivars Peterson, January 25, 1992, Vol. 141, No. 4.

3. **Newsletter of the Association of American Geographers**, June, 1992.

4. **American Mathematical Monthly**, "Telegraphic Reviews" — mentioned as "one of the World's first electronic journals using TEX," September, 1992.

5. **Harvard Technology Window**, 1993.

6. **Graduating Engineering Magazine**, 1993.

7. **Earth Surface Processes and Landforms**, 18(9), 1993, p. 874.

8. **On Internet**, 1994.

If you have read about *Solstice* elsewhere, please let us know the correct citations (and add to those above). Thanks. We are happy to share information with all and are delighted when others share with us, as well.

Publications of the Institute of Mathematical Geography have, in addition, been reviewed or noted in

1. *The Professional Geographer* published by the Association of American Geographers;

2. The *Urban Specialty Group Newsletter* of the Association of American Geographers;

3. *Mathematical Reviews* published by the American Mathematical Society;

4. *The American Mathematical Monthly* published by the Mathematical Association of America;

5. *Zentralblatt* fur Mathematik, Springer-Verlag, Berlin

6. *Mathematics Magazine*, published by the Mathematical Association of America.

7. *Newsletter* of the Association of American Geographer.

8. *Journal of The Regional Science Association.*

9. *Journal of the American Statistical Association.*

3. REPRINTS

Getting Infrastructure Built

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The profitability and success of a development project often hinge on the timely completion of improvements to adjacent highways, bridges, sewers, and transit services. In recent years, complex environmental and construction requirements have increased the lead time and costs required for many infrastructure improvements. At the same time, the public funding needed to finance road widening, interchange and bridge construction, sewer improvements, and transit development has come under severe budgetary constraints at all levels of government.

In northeast Ohio, public and private sector leaders have entered into a successful partnership to solve infrastructure development problems. The lessons learned from Cleveland's partnership can be readily translated to other communities.

Cleveland Infrastructure Team Shares the Secrets of Success

Founded in 1983 and commonly referred to as "BUGC", Build Up Greater Cleveland is a unique partnership that consists of elected and appointed officials from local, state, and federal governments as well as dedicated private sector executive volunteers from engineering, banking, investment, manufacturing, utility, accounting, and law firms. The development community has actively participated in all aspects of BUGC's activities since the program was founded. A team effort that was born of crisis but matured over a decade of wrenching economic upheaval, BUGC has earned national recognition for its ability to attract public financing needed for the improvement, repair, and construction of roads, bridges, and sewer, water, and transit facilities.

The first secret of BUGC's success is its systematic strategy for gaining the commitment of public funding through coordinated and simultaneous advocacy efforts at the local, state, and federal levels. The strategy calls for the aggressive and persistent pursuit of a fair and equitable share of state and federal infrastructure investment. Central to the strategy is the involvement of elected officials. Specifically, these officials enacted legislation that maximizes the return of tax dollars to northeast Ohio.

At the federal level, the formula used to divide highway and bridge funding has been amended in favor of Ohio and certain other states in each piece of major surface transportation legislation since 1987. The Ohio Congressional delegation's leadership for this effort was based, in large part, on technical assistance and networking support from BUGC.

For their part, private sector volunteers help develop the data that form the basis for BUGC's "fair share" advocacy efforts. Corporate expertise has been particularly useful in quantifying the public benefits of infrastructure investment. Private sector executives also play an active role in various task forces charged with solving problems and coordinating road and bridge repair work with utilities. In addition, both public and private sector members of BUGC are involved in long-term efforts to educate the public on the importance of infrastructure to the community and its economy.

The second secret of BUGC's success is its focus on improving the process by which projects move from identified need to construction. Between 1988 and 1993, the greater Cleveland area posted a 166 percent increase in the number of completed road and bridge projects. This surge resulted largely from the adoption of new procedures that reduced project completion time by 44 percent. BUGC's approach to achieving performance to achieving performance has helped reshape Cleveland's skyline and, at the same time, contributed to a major renaissance in regional economic development.

What Difference Has the Partnership Approach Made?

Over the last decade, BUGC's advocacy program has yielded more than one billion dollars in unprogrammed funds that the greater Cleveland area would not have otherwise received. Much of the credit for the funds goes to northeast Ohio's congressional delegation. The delegation orchestrated a multistate/multiyear coalition effort that has increased the return of Ohio's share of the federal gas tax from 61 cents on the dollar in 1981 to 90 cents on the dollar in 1993. Changes in the formulas used to distribute federal highway and bridge dollars have also brought more than 1.5 billion dollars in new funds to Ohio and more than two million dollars in new funds to greater Cleveland. At the state level, BUGC played a pivotal role in establishing the Ohio Public Works Commission and its program of needs-based funding for local capital assets. This effort has resulted in an estimated annual increase of 14.5 million dollars for county roads, bridges, and sewers. BUGC worked locally for legislation that increased motor vehicle license tag fees, which now generate more than 13 million dollars per year in road and bridge repair funds.

BUGC has lobbied successfully at the federal and state levels for project-specific funding for infrastructure improvements essential to Tower City Center, a new baseball stadium and arena project, the Rock and Roll Hall of Fame, and many other major development projects. BUGC has also been a key player in the cleanup of the Cuyahoga River, the overhaul and rehabilitation of Cleveland's transit system, and the completion of the Interstate highway network in northeast Ohio. BUGC's efforts have generated 396 million dollars for road and bridge improvements, 260.7 million dollars for transit development, 396 million dollars for water projects, and 341 million dollars for sewer needs. BUGC Chairman James M. Delaney of Deloitte and Touche points out, "We now are witnessing a shift in investment from rehabilitation of existing infrastructure to increased expenditures for new facilities, which support high impact economic development projects."

With the increase in funding for necessary infrastructure repairs and improvements, it became clear to city and county engineers, the Ohio Department of Transportation, the Greater Cleveland Regional Transit Authority, and the Northeast Ohio Regional Sewer District that projects were moving too slowly from design to construction. The various agencies shared many problems, particularly the burden of meeting new and complex environmental requirements while processing more and larger projects with a limited number of professional staff. To meet this challenge, the several agencies worked with private sector executives to develop, test, and implement performance improvement measures. The payoff has been a dramatic increase in the agencies' ability to complete road and bridge projects at a much faster pace.

Whenever possible, environmental impact and engineering tasks are performed simultaneously rather than sequentially. Scoping meetings that include public and private sector participants are conducted early in the project planning process. The meetings sort out which agency or entity will assume primary responsibility for each task and institute cooperative mechanisms to ensure that projects remain on schedule and that problems are addressed quickly. The application of value engineering techniques helps make certain that the right project is initiated for the right reasons within a time frame that makes sense for all involved. BUGC's recommendations for fast tracking highway projects have worked so well the Governor George Voinovich has encouraged the Ohio Department of Transportation and the Ohio Environmental Protection Agency to implement similar action on a statewide basis.

How Process Affects Products — Moving Projects Faster Means Getting More Public Investment

The fast-track approach has reduced costs, improved completion times, and helped finance infrastructure projects. Given that a great deal of federal and state financing for highway, bridge, sewer, and transit project s is distributed on a "first come, first served" basis, it is hustle — and the ability to keep the bureaucratic pipelines full of ready-to-go projects — that determines where public money is spent.

Most of the federal funding for highways and bridges is disbursed with time constraints; that is, a jurisdiction must spend federal funds within a certain number of years or lose its funding allotment to other states. Therefore, a state department of transportation must, for example, meet all federal and state planning and programming requirements while spending its allotment of federal funds within the stated time limit. Accordingly, cities and counties with ready-to-go projects consistently receive funding. On the other hand, communities that fail to put together ready-to-go projects are unable to attract anything close to their fair share of public investment.

How Can Local Communities Translate These Successes to Their Own Settings?

1. Develop a public/private infrastructure partnership team. BUS-C will be pleased to provide you with advice and written material on creating a partnership and, in return, asks only for feedback on what actions you take and what works in your community. In the meantime, even if you have a partnership in place, consider the actions outlined below.
2. Visit your metropolitan planning organization (MPO) to find out about the availability and requirements for state and federal funding. If you do not live in an urban area, visit the nearest field office of your state department of transportation. Planners and engineers at this and other agencies can assist you in determining whether federal participation is appropriate for a given project. If participation is appropriate, staff will tell you what steps are necessary to ensure federal financing.

   It is important to recognize that projects can usually be completed more quickly in the absence of assistance. The federal government conditions the receipt of funds on compliance with federal standards for planning, environmental, public participation, and programming actions. For large, expensive public works improvements, however, federal financing is typically essential.

If your project can be successfully undertaken with only state and local financing, your MPO will advise you accordingly. Your MPO can also greatly assist in guiding you through the federal and state funding processes and introducing you to the key players.

3. Identify the most appropriate public agency or government sponsor for your project and secure that party's agreement to fund your project. Early on, talk to the head of the appropriate agency about what you need, when you need it, and the rationale for the project. Ask for and follow the agency head's advice on how to "feed the agency." Identify the staff members who will be assigned to your project and get to know those individuals. In other words, find out who is responsible for your project, what they need from you, and when you need to complete key steps to ensure adherence to a mutually workable and realistic project schedule.
4. Be sure that elected officials are familiar with and support your project. Find out which public bodies must sign off on your project and what specific actions are necessary. Visit the appropriate elected officials and describe how your project will contribute to their vision of and priorities for the community. Is a consent ordinance needed from city council? Must the MPO board of directors include your project in its short- and long-term plans and project lists? Does the state need to file project-related documents with the U. S. Department of Transportation or U. S. Environmental Protection Agency? Does the state legislature assign funds for projects such as yours? Elected officials can be of immense assistance in spurring timely action by public agencies, especially if the officials are involved in the process early and consider themselves stakeholders in the project.
5. Assume 100 percent proactive responsibility for keeping your project moving. If a delay occurs, identify the reason. To the greatest extent possible, help the person who must resolve the problem get whatever he or she needs to move the project forward. Keep all involved parties informed of project progress, and alert key public agency staff to changes as soon as possible.
6. Say thank you often and keep your word; deliver on your promises. Most staff at public works agencies labor under hiring freezes and have not seen a significant pay increase in years. At the same time, they are responsible for large numbers of projects and must comply with new and confusing regulations. They are answerable to a diverse set of interests. Let them know that you appreciate their efforts.

   Encourage public agency staff to let you know if a problem arises or if you can do something to help keep your project on track. If staff members ask you for information, drawings, or legal or other information, tell them when you will submit the requested materials to them. Make sure that you deliver what you promised when you promised. Make certain that any material is delivered in a form that most readily serves agency purposes. Confirm that the right person has in fact received your information.

   If the various steps look like a lot of work — and for some projects they represent a full-time job — remember that the Cleveland experience has proven to be well worth the effort. The actions described here have led to considerable success, but the process can be expanded and improved. We look forward to hearing from you about your experiences in building public/private partnerships.

To make comments or to request further information write or phone:

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Center Here; Center There; Center, Center Everywhere!

The Geographic Center of Wisconsin and the U.S.A.: Concepts, Comments, and Misconceptions

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Abstract

Published locations of geographic centers are found to be inaccurate, inconsistently determined and in serious need of revision. The definition of geographic center is clarified. Methods of computation of two-dimensional distributions on curved surfaces are given. An accurate location for the center of Wisconsin is determined to be at latitude of 44°38′04″ N., longitude of 89°42′35″ W. The uncertainty in the geographic center of the United States is discussed. Recommendations for future further work are given.

Introduction

For more than two thirds of a century the U.S. Geological Survey has published information about the area and geographic center of the various states and the United States (Douglas, 1923, 1930; Van Zandt, 1966, 1976; and pamphlets of the U.S. Geological Survey, 1967, 1991). These publications are careful to point out the uncertainty and limitation of the data and results. For example, Douglas (1923, p. 221) states that "That exact position of the center of each State can not be determined from the data available, ..." and Van Zandt (1966, p. 265) states that "There being no generally accepted definition of 'geographic center' and no completely satisfactory method for determining it, a State or country may have as many geographic centers as there are definitions of the term." and, "Because many factors, such as the curvature of the earth, large bodies of water, and irregular surfaces, affect the determination of geographic centers, the locality of the centers should be considered as approximations only." Since first published, the information (with minor exceptions) has not been revised.

Some things are getting better. Adequate data are now available. There are satisfactory definitions. Computers and powerful software are now widely available. There are analytical means of taking into account the Earth's surface curvature. Large bodies of water are just as much a part of the whole as is the land and should be included. As a result, it is now possible to determine "geographic centers" to high accuracy. This paper will discuss these points and their impact on the determination of the geographic centers of Wisconsin and the United States.

Definition of Geographic Center

The lack of agreement on a definition of geographic center (center of area) is unfortunately true. Opinions range from despair of any suitable solution existing, expressed by Adams (1932), to enthusiasm over the existence of an infinity of centers, all equally valid (if not equally popular), outlined by Neft (1966, p. 21). I suggest that a reading of the literature will show that an intermediate view is widely held and a single definition of "center" is agreed on: **The center of any distribution of things is the average location**

of those things. It corresponds to the "center of gravity" or "balance point" of the distribution. In Euclidean spaces the average location is most easily calculated by taking the weighted vector sum of the location vectors (vectors whose magnitude and direction are the distance and direction of the various things in the distribution) and dividing by the total weight or total population of the things. Such a center has the additional property that the sum of the squares of the distances between the center and the location of the various things in the distribution is minimum. This definition is equally suitable for distributions in one-, two- three-, or higher-dimensional Euclidean spaces.

Almost a century ago, Hayford (1902) convincingly argued that the average location was the most appropriate center. D. I. Mendeleev (1907 and before) used formulæ which may be derived from the "balance point" concept (derived by his son I. D. Mendeleev) for finding the geographic center and population center of Russia. Deetz (1918, p. 57) states that the " 'Geographic center of the United States' is here considered as a point analogous to the center of gravity of a spherical surface equally weighted (per unit area) and of the outline of the country, and hence it may be found by means similar to those employed to find the center of gravity." All six Geological Survey publications, cited in the Introduction, appeal to the "balance point" concept. For more than a century the U. S. Bureau of the Census has used the concept of a "center of gravity" or "balance point" as defining the U.S.A. population center (Barmore, 1991).

In spite of this long tradition, there are still dissenters. Kumler and Goodchild (1992, p. 278) recommend that the point of minimum aggregate travel (M.A.T.) is the best measure of center of population. I find it hard to accept some of their reasons for this recommendation. First, they say that when calculating the mean or average location, "the points, or people, are effectively weighted proportionally to their distance from the center — more distant people have greater influence on the location of the mean center than people nearby." But, it is **location** that is being averaged (weighted by population), **not people** being averaged (weighted by distance). Each individual has exactly the same weight in finding the average location. Second, they believe the M.A.T. "point does have one flaw — it is insensitive to radial movement: If a person moves 1,000 kilometers directly toward or away from the mat [M.A.T.], the point will not move; if that same person, however moves only a few kilometers in any other direction the [M.A.T.] point will move accordingly." And this shortcoming "is the least severe" shortcoming of the various measures of center of population they discuss. I disagree. Are we to have preferred or elite directions? Shouldn't the center of a distribution be equally sensitive to the motion of its component parts in any direction?

I suggest that the term, center, should be reserved for the average (arithmetic mean) location. Other statistical concepts that are found to be useful should be labeled with names (other than center) that are descriptive of what they represent. For example, "the point of minimum aggregate travel," is just that; it should not be called the center. To do otherwise is to invite a return to the confusion that existed earlier in this century when the point of minimum aggregate travel, the center (or average) location and the median latitude (and/or longitude) of an area were often and incorrectly thought to be the same (Eells, 1930).

Geographic Center of a Curved Surface

As mentioned in the Introduction, one difficulty that must be dealt with is the curvature of the Earth's surface. If the Earth's surface were flat, or if there existed a flat map projection which left area, distance, and direction undistorted, the determination of geographic center of portions of the Earth's surface would be much simplified. However, distributions on the Earth's curved surface are spread over a two-dimensional non-Euclidean space. Traditionally there have been two different ways of responding to this problem.

One response is to find a higher dimension space that is Euclidean in which to embed the non-Euclidean space. Then the necessary calculations can be carried out using the familiar Euclidean geometry. Thus, one can embed the two-dimensional Earth's surface in a three-dimensional Earth's surface in a three-dimensional Euclidean space and calculate the three-dimensional average location, balance point, or "center of gravity". This three-dimensional approach is equivalent to the one sentence definition given by Deetz (1918) and results in the formulæ given and used by Mendeleev (1907) for population and geographic centers on a spherical Earth. The method can easily be extended for distributions on the surface of an ellipsoid of revolution representing the Earth, though the formulæ are more complex. The resulting centers are below the surface and I find this distasteful.

The second response is to adapt and restrict the calculations to the two-dimensional non-Euclidean space. As I have previously described in some detail (Barmore, 1991, 1992) this second solution is preferable. The result is a method that restricts the computations of average location and the outcome to the surface of a sphere or an ellipsoid of revolution which very closely approximates the Earth's surface.

Geographic Center of Wisconsin

There exists, several hundred feet south of the geometric center of the City of Pittsville, Wood Co., Wisconsin, a monument with the following text:

Center of the State of Wisconsin

In the early 1950's Governor Walter J. Kohler, Jr.
frequently visited the Pittsville area.
On one such trip he Proclaimed Pittsville to be
the exact center of the State by Official Proclamation
on the 27th of June, 1952.
Professional Land Surveyors established the corner
lying 250 feet North of where you are now standing.

This monument donated by the Central Chapter
of the Wisconsin Society of Land Surveyors
Erected July 1987

Wayside construction donated by Cedar Corporation, Marshfield.
Dale Decker Surveying; Esser Trucking, Arpin;
Mid State Associates; People's State Bank, Pittsville.

The text of the proclamation (Kohler, 1952) gives no hint of how or when it was determined that the center of Wisconsin was at Pittsville. The Geological Survey places the Wisconsin geographic center at "9 miles southeast of Marshfield." This point is 16 km from the Pittsville monument.

The geographic centers of the various states were first published by the U.S. Geological Survey (Douglas, 1923, p. 221-222). Since then, and until as recently as 1991, the centers for most of the States and particularly for Wisconsin have remained unrevised. Thus, the most recently published center of Wisconsin reflects the boundaries and geographic data quality as of 1923 or earlier. Also, according to the very brief definition accompanying the list of centers and Adams' (1932) lament that no analytical process was available, the outcome is only approximate. Thus, the results are of low accuracy. Third, the Great Lakes and some islands were not included when determining the centers. Thus, significant portions of Wisconsin were not included. Clearly, these centers are ripe for revising.

It is now possible to calculate the geographic center of Wisconsin to much higher accuracy. I have determined the geographic center of Wisconsin with an uncertainty of less than 0.1 km. The determination was done for the center of all land and water areas including those portions of the Great Lakes within Wisconsin. The center is in the east central portion of Sec. 19, R 7 E, T 25 N, in the Town of Eau Pleine, Portage Co. A second determination was done for the center of the land area and "inland waters" for comparison with the previous determination given by the Geological Survey. This "center" is near the center of Sec. 23, R 4 E, T 25 N, a little northeast of the northeast corner of the city of Auburndale in Wood Co. and is about 8 km from the point published by the Geological Survey. Based on these results, it would be reasonable to assume that one could expect similar errors in the existing published locations of the other state centers and they are also in need of revision. These and previous results for Wisconsin are given in Table 1 and displayed on a map in Figure 1. (The computational details and assumptions are given in Appendix A).

Geographic Center of the Conterminous United States

The geographic center of the "Conterminous" United States (48 States and the District of Columbia) is widely published on maps, in atlases and in government documents, as being near Lebanon, Smith County, Kansas, at latitude of 39°50′ N and at longitude of 98°35′ W.

All sources for this and similar statements that can be traced, ultimately refer to a one sentence statement with a brief footnote published by Deetz (1918, p. 57) that reads:

> "The Geographic center (*) of the United States is approximately in latitude 39°50′ and longitude 98°35′.
>
> (*) 'Geographic center of the United States' is here considered as a point analogous to the center of gravity of a spherical surface equally weighted (per unit area) and of the outline of the country, and hence it may be found by means similar to those employed to find the center of gravity."

There is a hint as to how this might have been determined in the melancholy paper by Adams (1932) which states:

> "A method that was used in the Coast and Geodetic Survey a number of years ago was the following: An equal-area map of the United States was constructed on thin cardboard and then the outline map was cut out along the various boundaries. The center of gravity of this outline map was then determined."

As this was done in an analogue way (on what must have been a map of modest scale) rather than calculated in a precise way, the result is probably of modest accuracy. Note that:

Table 1. Wisconsin Geographic Center According to Various Sources

Description of Computation	Source	N. lat.	W. long.
Center of all of Wisconsin	this work	44.6344°	89.7098°
Center of land and "inland waters"	this work	44.6351°	89.9923°
9 miles southeast of Marshfield, WI	USGS 1923	44.5728°	90.0441°
On the Pittsville, WI monument	Gov. 1952	44.4384°	90.1301°

Table 2. Geographic Center of the Conterminous United States

Description of Computation	Source	N. lat.	W. long.
On Clarke's (1866) ellipsoid surface			
a) land & inland waters only	this work	39.7872°	98.9830°
b) all land & water areas	this work	39.9074°	98.6843°
In three dimensions	this work	39.9020°	98.6909°
On a Lambert Azimuthal Equal Area map	this work	39.8785°	98.6593°
On Albers Equal Area Conic projection	this work	39.8352°	98.6896°
Analogue: Balancing flat map (??)	Deetz 1918	39.8333°	98.5833°

(a) It is a flat (and therefore distorted) map not a spherical map whose center was found. (b) It is not stated which map projection was used to produce the map. (c) It is not stated what boundaries were used.

In an attempt to reproduce Deetz's result, this geographic center was recomputed in a variety of ways. If only the areas and centers of the land and "inland waters" of the various states were used the agreement was very poor. However, if the list of areas and centers used was expanded to include the portions of the Great Lakes within the United States and to include the various sounds, straits, bays and coastal waters that are not part of the "inland waters" of the various states, then modest agreement could be achieved (see Appendix B for details of these calculations). The results are summarized in Table 2 and displayed on a map in Figure 2. Because of the low quality of the data used in the computation, these results should not be considered accurate.

Geographic Center of the United States

Apparently, the geographic center of The United States (50 States and the District of Columbia) was determined by the U.S. Coast and Geodetic Survey (ca. 1959) in a manner described, if nowhere else, in several news releases. The accuracy of this result is questionable for reasons outlined below.

The center of all 50 states was apparently determined, piecemeal, as follows: The 48 states were represented as being 3,022,400 square miles in area at the previously determined location given by Deetz (1918) at latitude 39°50′ N., longitude 98°35′. Alaska's land and

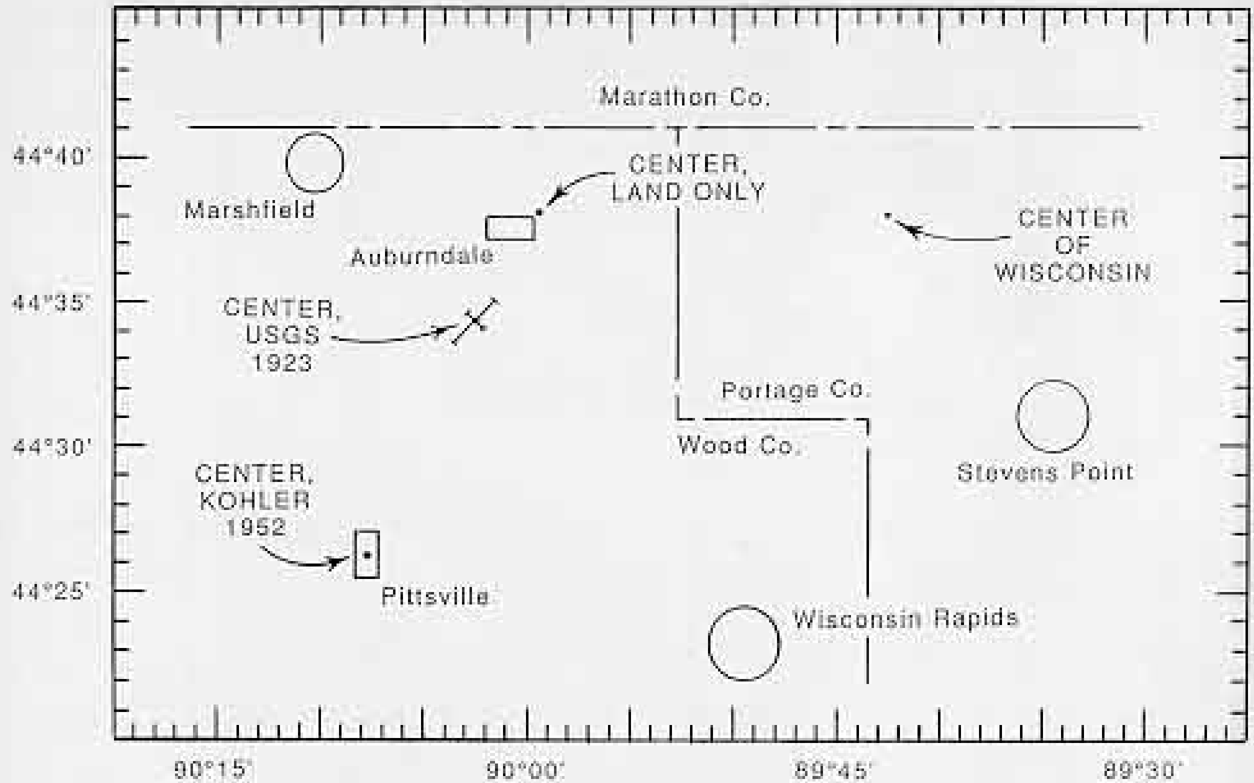

Figure 1. Wisconsin Geographic Centers according to various sources. The point labeled "CENTER OF WISCONSIN" is the center calculated for all the land and water area within the boundaries of Wisconsin. The location uncertainty of the point is not noticeable on a map of this scale. The point labeled "CENTER, LAND ONLY" is the center calculated for all the land and "inland waters" but excluding the portions of the Great Lakes lying within Wisconsin. The location uncertainty of this point is not noticeable on a map of this scale. The point labeled "CENTER, USGS, 1923" is the center published by the U.S. Geological Survey since 1923. The "error bars" indicate the probable uncertainty implied by the manner in which the various State center locations were stated. The point labeled "CENTER, KOHLER, 1952" inside the boundaries of Pittsville is the result of Governor Kohler's 1952 Official Proclamation. The location uncertainty and method of determination of this point are unknown.

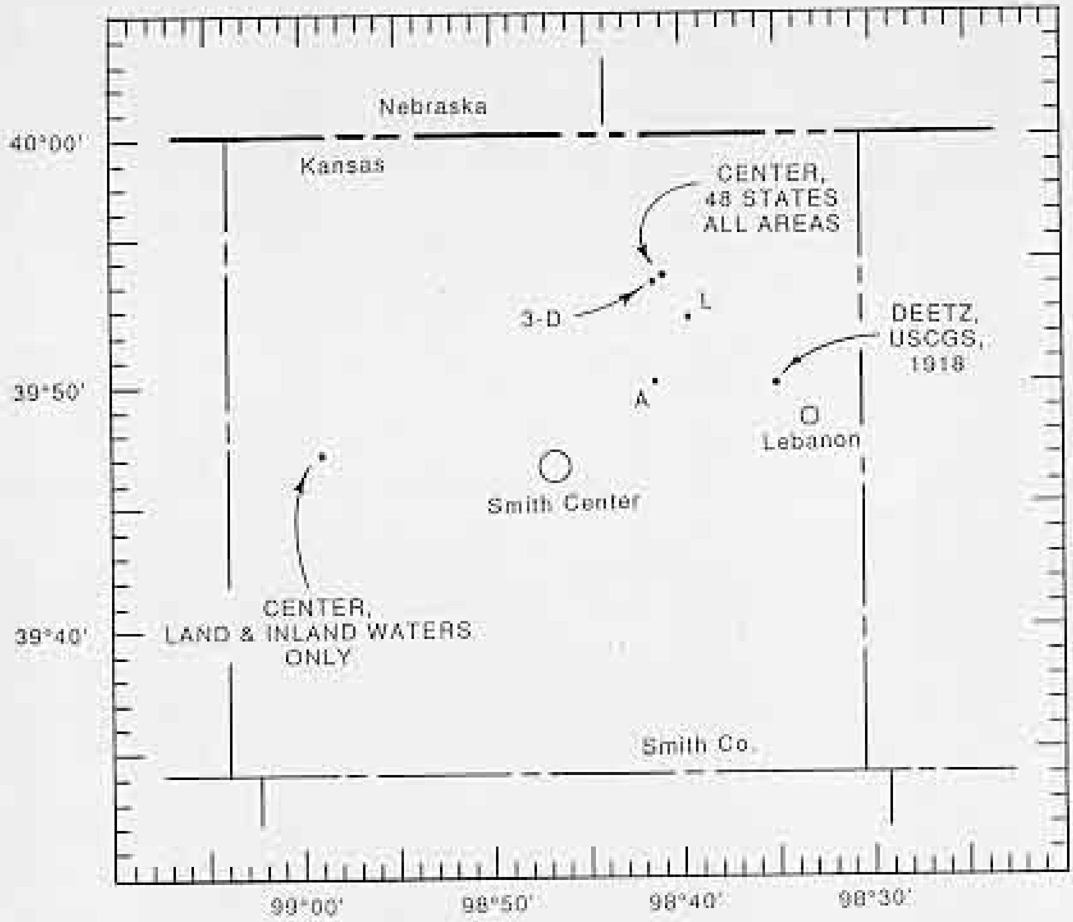

Figure 2. Geographic Center of the Conterminous United States determined by various computational methods. The point labeled "CENTER, 48 STATES, ALL AREAS" is the average location of all the various areas that make up the conterminous United States calculated on the surface of Clarke's (1866) ellipsoid using the preferred method (Barmore, 1991, 1992). The point labeled "CENTER, LAND & INLAND WATERS ONLY" is the average location of the various areas (the Great Lakes and other "non-inland waters" being excluded) that make up the conterminous United States calculated in the same manner. Because of the limited accuracy of the data used, neither of these locations nor the other center locations displayed in this figure should be considered as accurate. The point labeled "3-D" is the three-dimensional average location, projected onto the surface, of all the areas that make up the conterminous United States. The point labeled "DEETZ, USCGS, 1918" is the widely quoted result. The points labeled "L" and "A" are the centers determined by an analytical computation that is equivalent to finding the balance point of a Lambert Azimuthal Equal Area map and an Albers Conical Equal map, respectively, of all areas of the conterminous United States.

"inland waters" area were represented as being 586,400 square miles at latitude 63°50′ N., longitude 152°00′ W. The balance point of these two areas was found to be at latitude 44°59′ N., longitude 103°38′ W. on the (presumed great circle) arc between them. Then when Hawaii joined the Union, the process was repeated. The 49 states were represented as the sum of the previous two areas (3,608,800 square miles) located at their balance point. Hawaii's land and "inland waters" area were represented as 6424 square miles at latitude 20°15′ N., longitude 156°20′ W. The balance point of these two areas was found to be at latitude 44°58′ N., longitude 103°46′ W.

If the Earth's surface were flat, this procedure would be as accurate as the data used would allow. However, the surface is not flat, but curved. When the distances are as large as those between the various states of the United States, ignoring the curvature can result in a substantial error (Barmore, 1991, 1992). If the center is to be determined with distances measured on the curved surface of the Earth, it must be redone from the beginning with each addition.

Another difficulty has to do with using data of mixed consistency. The 3,022,400 square mile figure for the 48 States is the land plus "inland waters" only. The location used for this area is apparently the center of a different area — the land, "inland waters," **and a substantial area of "non-inland waters."** (These "non-inland waters" have an area of about 74,364 square miles, 2.4% of the 3,022,400 square mile figure [U.S. Bureau of the Census, 1940].)

In order to illustrate the differences that can result, the geographic center for the entire United States was calculated various ways. The results are summarized in Table 3 and displayed on a map in Figure 3. the same methods and data were used that were used in the preceding example with the exception that the total of all land and all water areas for Alaska and Hawaii were those given most recently (U.S. Bureau of the Census, 1992, table 340). Because the centers and areas used have not been revised (with the exception of the Alaskan and Hawaiian areas) the results should not be taken as accurate.

Summary and Recommendations

The geographic centers and areas of the various States and The United States are in serious need of revision for several reasons. In the seventy years that have passed since the centers were determined, much has happened. Mapping of the United States is much improved. Computational capability is now widely available —it should no longer be necessary to make compromises for computational reasons. Data on land and water area are much improved. It should now be possible to compute the location of the various centers to an accuracy of ca. 10 m. The following recommendations are made for this revision and any similar sort of statistical analysis.

I. The term **center** of spatial distributions should be reserved for the average (arithmetic mean) location. Other statistics of spatial distributions that are found to be useful should be given other names to avoid confusion.

Table 3. Geographic Center of the United States (All 50 States)

Description of Computation	Source	N. lat.	W. long.
On Clarke's (1866) ellipsoid surface	this work	45.4344°	104.3524°
In three dimensions	this work	45.2517°	104.1776°
On ellipsoid surface. Land and "inland waters" only. For comparison.	this work	44.9482°	104.1189°
U.S. Coast and Geodetic Survey	news release	44.9667°	103.7667°

Table 4. Wisconsin Geographic Center Calculated Two Ways

Description of Computation	N. lat.	W. long.	depth
In a three-dimensional Euclidean volume	44.6343739°	89.7097544°	2.4 km
On a two-dimensional non-Euclidean surface	44.6343818°	89.7097566°	0.0 km

II. If the distribution covers enough of a curved surface for the curvature of the surface to be noticeable, then special care must be taken. Unless appropriate compensation is made for the Earth's surface curvature, these calculations may not be properly done using any flat map projection. There is no flat map of the Earth's curved surface that leaves area, distance, and direction undistorted. For distributions on the surface of the earth, the computations of average location should be carried out on the surface and the results restricted to the surface. The method of doing this is outlined in some detail elsewhere (Barmore, 1991, 1992). Alternately, the computations can be carried out in three dimensions using more familiar procedures, but the computation of two-dimensional distribution statistics in two dimensions is preferable.

III. If geographic centers of hierarchical sets of areas are presented, they should be done in a consistent way so that comparisons are easy within a level and between levels. It should be possible at any level to find the average of the larger group by averaging over its component parts. In particular, if centers at one level for separate land and water areas are given, the centers for the subdivisions should be separated in the same manner. If "non-inland waters" are excluded at one level they should be excluded at all levels.

IV. What is included (or excluded) should be clearly stated. The absence of any discussion of what is meant by the term "North America" makes meaningless the statements concerning the center of North America published by the U.S. Geological Survey (Douglas, 1930; and pamphlets by the U. S. Geological Survey, 1967 and 1991). Is Greenland included? Are "non-inland waters" included? Are off-shore islands included?

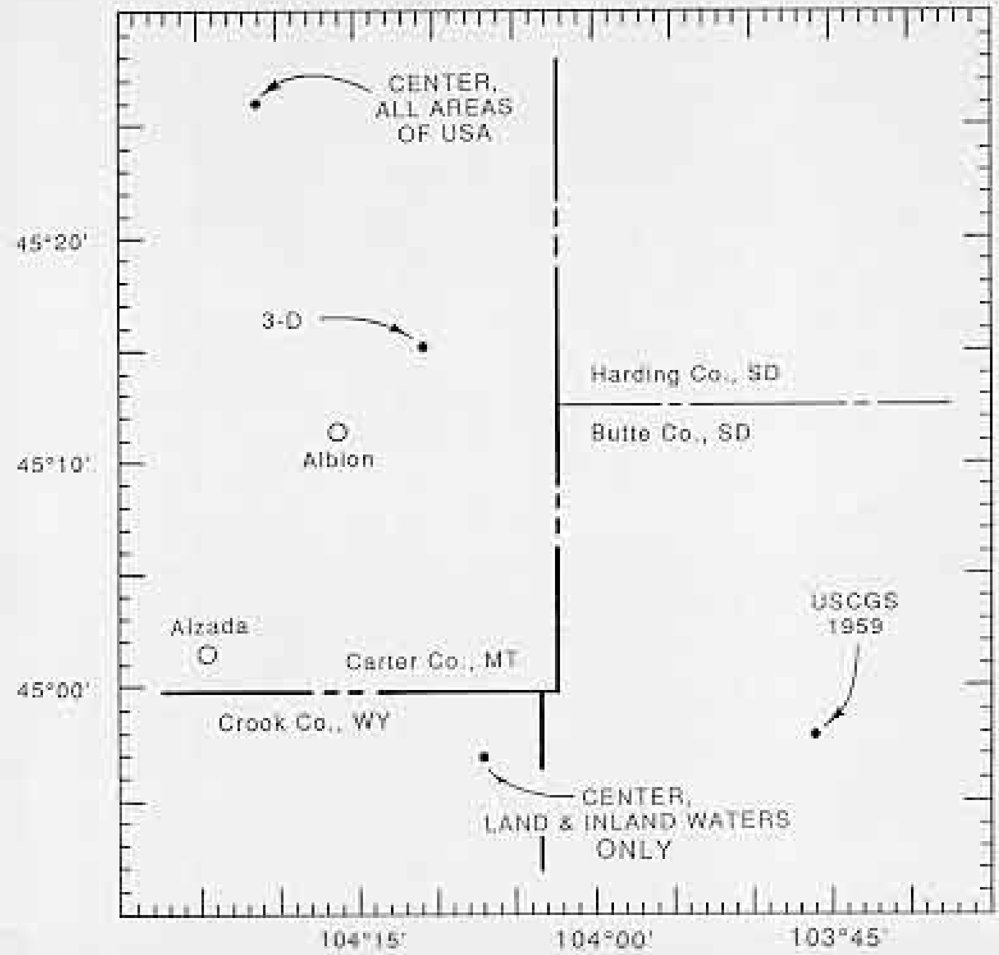

Figure 3. Geographic Center of the United States (all land and water areas of all 50 States and the District of Columbia) determined various ways. The point labeled "CENTER, ALL AREAS OF USA" is the average location of all the various land and water areas that make up the United States calculated on the surface of Clarke's (1866) ellipsoid using the preferred method (Barmore, 1991, 1992). Because of the limited accuracy and limited internal consistency of the data used, neither this location nor the other center locations displayed in this figure should be considered as accurate. The point labeled "3-D" is the three-dimensional average location, projected onto the surface, of the same areas, that make up the United States. The point labeled "USCGS, 1959" is the widely quoted result. The point labeled "CENTER, LAND & INLAND WATERS ONLY" is the center of all land area combined with only the "inland waters" area, the Great Lakes and "non-inland waters" being excluded. This center, calculated on the Earth's curved surface, corresponds most closely to the U.S. Coast and Geodetic Survey procedure for determining the geographic center. It is presented here for comparison.

Appendix A: Calculation of Wisconsin's Geographic Center

All areas and centers were determined assuming they lay on the surface of Clarke's (1866) ellipsoid ($a = 6378.2064$ km and $e = 0.08227185$).

The State's surface and adjacent areas were divided into 30×60 minute quadrangles. For 30×60 minute quadrangles that lay completely inside the State boundaries (or had more than half their area within the boundaries) the areas and centers were calculated using the ellipsoid geometry found in Bomford (1977). These results are very accurate. Wherever the boundary cut a quadrangle, the areas and centers were determined from the 1:100000, 30×60 minute quadrangle maps published by the Geological Survey. If less than half the quadrangle's area was within Wisconsin, only the portion within the State was considered. If more than half the quadrangle's area was within the State, the area and center of the portion to be excluded were determined and subtracted from the previously calculated values for the entire quadrangle. This process minimizes the areas that had to be measured rather than calculated.

The areas and centers that had to be measured were done as follows: a) If the areas were composed of quadrilaterals or triangles, the areas and centers were calculated from measurements taken directly from the map. b) If the areas were irregular, they were carefully traced onto a uniform sheet whose areal density had been previously determined with the aid of an electronic "balance," cut out, reweighed to determine their area and suspended from several points to determine their centers. c) The latitude and longitude of the centers were then determined directly from the geographic grid on the map. d) The areas were then corrected for scale changes. The scale changes have two causes: First, very small variations in scale resulting from the Universal Transverse Mercator projection (Snyder, 1987, p. 58-64). Second, scale changes due to expansion or shrinkage of the map paper caused by humidity changes (determined from measurements of the 10000 m grid on the map).

This process created a collection of 111 area elements representing the State. Over 87% of the area (represented by the 37 full 30×60 minute quadrangles) in the calculations of center have calculated areas and centers for which the accuracy is very high. For the remaining 13% of the area (represented by 74 fractional areas averaging 325 sq. km) the accuracy of the areas is probably limited by how well the areas were corrected for scale changes caused by humidity changes. As a check, the total area of land and "inland waters" was found to be 145435.166 sq. km = 56152.8 sq. miles. This compares favorably with the 56153 sq. miles listed as the area of Wisconsin in the 1980 Census (U.S. Bureau of the Census, 1983). Also, the total area of Wisconsin (including the portion of the Great Lakes falling within Wisconsin) was found to be 169609.8 sq. km. The Bureau of the Census (1992) reports the total area of Wisconsin to be 169653 sq. km. The difference of 43 sq. km may be due to disagreement about the boundaries of the State in Lake Michigan. I have used the boundaries shown on the 1:100000 scale; 30×60 minute series maps published by the Geological Survey. These boundaries, in turn, are in agreement with those given in Van Zandt (1976) and further clarified in the 1948 Compact between Michigan, Wisconsin, and Minnesota which finally settled the boundary (U.S. Statutes at Large, 1948). Other sources show a different boundary — *The National Atlas* (U.S. Geological Survey, 1970, p. 17, 19, 313) or the Geological Survey map, *State of Wisconsin*, 1:500,000 scale, 1966 comp., 1968 ed., for example. In the worst possible case an error of this magnitude would shift the center of Wisconsin two or three seconds of arc or about 50 m on the surface.

The State center was then calculated by finding the average location of the 111 area elements. This calculation was done two ways: first as a three-dimensional volume distribution and second as a two-dimensional surface distribution (Barmore, 1991, 1992). For areas the size of Wisconsin, there is little difference between the two results except for depth. For example, see Table 4. The difference is only a few hundredths of a second of arc, and corresponds to a distance of one or two meters on the surface.

In order to provide a comparison for the Center of Wisconsin given by Douglas (1923), that included the land and "inland waters" only, this center was also redetermined. Therefore, the process, outlined above, was repeated for a somewhat different collection of 111 area elements (30 full 30 × 60 minute quadrangles and 81 fractional areas averaging 197 sq. km; representing 82% and 18%, respectively, of the areas used in the calculation). These area elements represent the area of the land and "inland waters", but not the Great Lakes, within the boundaries of Wisconsin.

Appendix B: Calculation of the Geographical Center of the Conterminous United States

The geographic center of the conterminous United States was calculated using methods previously described. The centers calculated on the curved surface in two dimensions or when treating the areas as a three-dimensional volume distribution assumed Clarke's (1866) ellipsoid (though the data quality hardly justifies such accuracy). The centers calculated by distributing the areas on the surface of various flat maps used equations for the projections given by Snyder (1987, p. 100-101, 185-187) for a spherical earth. The Lambert Azimuthal Equal Area map was centered at 38° N. latitude, 95° W. longitude, following Deetz (1918, p. 57) and the Albers Equal Area map used two standard parallels at 29°30′ and 45°30′ N. latitude as suggested by Deetz and Adams (1945, p. 94).

The data used consisted of two parts. The first part was the areas of land and "inland waters" and centers as given by Douglas (1923, p. 219, 222) for the 48 States and the District of Columbia. If the example of Wisconsin is typical, the accuracy of this data is not high. More recent and probably better data were not used because the 1923 data for area are nearly identical to that given by Gannet (1906, p. 7, 8) and thus more characteristic of the data available to Deetz than more modern material. The second part of the data was for the "non-inland waters". The areas included are those delineated earlier (U.S. Bureau of the Census, 1942, Map I, and Table IV). The approximate centers for these "non-inland water" areas were determined from maps in *The National Atlas* (U.S. Geological Survey, 1970).

Because of the uncertainty in the areas and centers of the area elements whose locations were averaged to get these results, they should not be considered accurate.

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4. ARTICLES

Equal-Area Venn Diagrams of Two Circles: Their Use with Real-World Data.

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General Problem

We are concerned with populations whose members have two discrete characteristics; that is, characteristics which are either present or absent in each member. In populations having two characteristics, the populations can be described by the proportion of the population that has both characteristics present, one or the other (but not both) characteristics present, or both characteristics absent. One well-known way to present this data is with a two circle Venn Diagram in which each circle represents one of the characteristics, the intersection of the circles represents the members with both characteristics, and the region outside both circles but within the universe of discourse (depicted as a bounded figure surrounding the circles — often a rectangle) represents the members with neither characteristic. Appropriate regions might then be labelled with suitable percentages, whether or not the geometric intersection pattern is suggestive of the numeric partition of the sample. At this point, it may be useful to the reader to draw a two-circle Venn diagram.

For example, consider a population of countries with national child vaccination programs. Some of the countries in the population use a campaign strategy, some a clinic-based strategy, some use both strategies, and a few use neither strategy. Construct a Venn diagram to represent this grouping of mixed strategies. Draw Circle α on the left and draw an intersecting Circle β on the right. Draw a rectangle that is large enough to easily contain all of the intersecting circle configuration. In this Venn diagram, Circle α could represent countries using a campaign strategy and Circle β could represent countries with a clinic-based strategy. Partition each of these circles according to their intersection pattern using the following notation.

Notation

Let the symbol A denote the area of Circle α that does NOT also lie within the Circle β.
Let the symbol B denote the area of Circle β that does NOT also lie within the Circle α.
Let the symbol AB denote the area of the intersection of Circles α and β.

In the vaccination program interpretation of the two circle Venn diagram, Area A is proportional to the countries using only a campaign strategy, Area B to the countries using only a clinic-based strategy, and Area AB to the countries using both a campaign and a clinic-based strategy. Countries using neither strategy are not represented; indeed, only participant populations will be considered for the remainder of this analysis, although we do note the existence of the logical category of the "neither" class.

Data such as this is sometimes illustrated with bar or pie charts. Such illustrations are inadequate because they do not allow easy portrayal on a single diagram of both the intersecting areas and the total percentage of each characteristic. The advantage of using a diagram of intersecting circles is that it portrays all the data clearly in a single diagram.

The objective here is to draw the intersecting circles so that the different areas are exactly proportional to the data, in much the way that an equal area map is drawn so that different areas are exactly proportional to the size of the landmass. Equal area graphic displays, be they maps or diagrams, are critical in making accurate visual comparisons of mapped or plotted data.

The remainder of this paper is devoted to displaying the detail of the calculations required to construct equal-area Venn diagrams from real-world data. Fundamentally, these calculations rest on the problem of finding the radii of the circles and the location of their centers, given the various common (intersecting) and non-common areas.

Definition of the two-circle problem

Given two intersecting circles, α and β, and given their common and non-common areas, AB, A and B, respectively. Find the radii of both circles, r_A and r_B, and the distance between the centers of the two circles d_{AB} such that the centers can be located and the circles drawn. It is clear that if the center of Circle α is at the origin, the Cartesian coordinates of the center of Circle β are $(d_{AB}, 0)$.

We can transform the three areas into percentages of the total area covered by the two intersecting circles by noting that the total area covered is $A + B + AB$.

$$A\% = 100 \times (A/(A + B + AB)) \tag{1}$$

gives the A-only area percent;

$$B\% = 100 \times (B/(A + B + AB)) \tag{2}$$

gives the B-only area percent;

$$AB\% = 100 \times (AB/(A + B + AB)) \tag{3}$$

gives the AB area percent;

No generality is lost by requiring Circle α to be the larger circle and by standardizing the size of Circle α by setting its radius equal to 1: $r_A = 1$. (Naturally, this assumes that $\alpha > \beta$ and that the bigger real-world characteristic is assigned to Circle α.) As a result, the area $A + AB$ of Circle α is π. The problem can now be restated, more simply, as follows.

Given: $A\%$, $B\%$, and $AB\%$, where $A\% + B\% + AB\% = 100$.

Find: r_B, the radius of Circle β, and d_{AB}, the distance between the centers.

Analytic Strategy

In order to solve the two-circle problem, define the *chord* of the intersection to be the straight line joining the two points where the perimeters of the two circles intersect –assuming here, and throughout the remainder of the text, that one of the two circles is not fully contained within the other. There are two situations that arise: one in which the chord lies between the two centers and a second in which the chord lies to one side of both centers. To visualize this relationship, draw one pair of circles with a relatively small area of intersection; in this case the chord lies between the centers. In what follows, this configuration will be referred to as one of type Case I. Alternatively, draw two circles with a relatively large area of overlap; in this case the chord lies on one side of both centers. In what follows, this configuration will be referred to as one of type Case II.

Starting with $A\%$, $B\%$, and $AB\%$, it is straightforward to derive r_B, but not to derive d_{AB}. Therefore, we reverse the situation and seek the function that yields $A\%$, $B\%$, and $AB\%$ given r_A, r_B, and d_{AB}. Several derivations are possible. The simplest one (not using integral calculus) is presented below to maximize accessibility of content.

Functions for $A\%$, $B\%$, and $AB\%$ in terms of r_A, r_B, and d_{AB} were obtained for Case I and for Case II. These functions are sufficiently complex to obstruct the derivation of an inverse function that would yield d_{AB} in terms of $A\%$, $B\%$, and $AB\%$. Consequently, a numerical approach was used in which d_{AB} was calculated for a grid of values of $B\%$ and $AB\%$. The results are presented in Tables 1 and 2. The value r_A is assumed equal to one and r_B is readily calculated from $A\%$, $B\%$, and $AB\%$.

With these results, several options are available to estimate d_{AB} from $A\%$, $B\%$, and $AB\%$. The preferred option depends on the accuracy desired. Option 1 entails interpolating from the data in Tables 1 and 2. Option 2 entails using a polynomial in $B\%$ and $AB\%$ (obtained via regression) to estimate d_{AB}. Options 1 and 2 are the least accurate, both giving answers within one percent accuracy relative to the radius of the largest circle (Circle α). Option 3, which will yield d_{AB} to any desired accuracy, entails searching by trial and error using the functions $B\% = f_1(d_{AB}, r_B)$ and $AB\% = f_2(d_{AB}, r_B)$. The trial and error search is greatly simplified by the fact that r_B can be calculated directly from $B\%$ and $AB\%$.

Derivation of $B\%$ and $AB\%$ as a function of r_B and d_{AB}

General formulæ

Heron's formula for the area of a triangle is based on the lengths, a, b, and c, of its sides. Let $S = (1/2) \times (a + b + c)$. Then the area of the triangle is:

(4) $$(S \times (S - a) \times (S - b) \times (S - c))^{1/2}$$

A sector of a circle is the pie-shaped wedge cut from the center of the circle out to the edge. The region of overlap of two intersecting circles is called a "lune." A sector can be decomposed into a triangle and a lune split longitudinally. We refer to the triangular portion as the "triangle" of a sector and to the lunar portion as the "segment" of a sector. The formula for the area of a sector of a circle with central angle Q (measured in radians) and radius r is:

(5) $$(1/2) \times (Q/\pi) \times (\pi \times r^2) = (1/2) \times (Q \times r^2).$$

The formula for the area of the corresponding triangle of a sector is:

(6) $$(1/2) \times r^2 \times \sin Q.$$

The formula for the area of the corresponding segment of a sector is:

(7) $$(1/2) \times (Q \times r^2) - (1/2) \times r^2 \times \sin Q = (1/2) \times r^2 \times (Q - \sin Q).$$

Case I: Chord lies between the two centers

In Case I, the chord of the lune separates the centers of circles α and β. The distance d_{AB} is the distance between the two centers, measured along the line of centers. Form a triangle using the line of centers as one side of length d_{AB}. The second side is formed by joining the center of circle α to the top intersection point of the lune; the acute angle enclosed by the line of centers and this side has measure Q_A which is 1/2 of the central angle subtending the chord of the lune from the center of circle α. In a similar fashion, join the center of circle β to the same third vertex to complete the triangle. The acute angle enclosed between the line of centers and this side has measure Q_B which is 1/2 of the central angle subtending the chord of the lune from the center of circle β. Let h denote the altitude of this triangle from the vertex of the lune to the line of centers. Let X denote the horizontal distance from the center of Circle α to the intersection with the chord. Let Z denote the area of the triangle with sides of lengths $r_A = 1$, r_B, and d_{AB}. Let K_A and K_B denote areas of the sectors in circles α and β subtended by the chord. Let L_A and L_B denote the areas of the triangles of these two sectors. Finally, let M_A and M_B denote the areas of the segments of the two sectors. Then given r_A, r_B, and d_{AB}, find h, $B\%$, $AB\%$, and $A\%$ as follows.

From equation (4),

(8) $$S = (1 + r_B + d_{AB})/2.$$

From equations (4) and (8), we get the area of the triangle as

(9) $$Z = (S \times (S-1) \times (S - r_B) \times (S - d_{AB}))^{1/2}.$$

From equation (9),

(10) $$h = 2 \times Z/d_{AB},$$

because $Z = (1/2) \times h \times d_{AB}$.

From equation (10),

(11) $$Q_A = \text{Arcsin}\,(h/r_A),$$

because $\text{Sin}\, Q_A = h/r_A$; also from equation (10),

(12) $$Q_B = \text{Arcsin}\,(h/r_B),$$

because $\text{Sin}\, Q_B = h/r_B$.

From equations (5) and (11), the sector A area is

(13) $$K_A = r_A{}^2 \times (2 \times Q_A)/2 = Q_A,$$

and from equations (5) and (12), the sector B area is

$$K_B = {r_B}^2 \times (2 \times Q_B)/2 = Q_B. \tag{14}$$

From equation (9), the sum of the areas of the two sectors is

$$L_A + L_B = 2 \times Z. \tag{15}$$

To find the area AB of the intersecting area (lune), view it as the sum of the two segments of the two sectors. From equation (7): $AB = M_A + M_B = (K_A - L_A) + (K_B - L_B) = (K_A + K_B) - (L_A + L_B)$ so that from equations (9), (13), (14), (15), $AB = Q_A + Q_B \times {r_B}^2 - 2 \times Z$. Using equations (11) and (12), it follows that $AB = \text{Arcsin}(h/r_A) + \text{Arcsin}(h/rB) \times {r_B}^2 - 2 \times Z$ and finally, noting that r_A=1, that

$$AB = \text{Arcsin}(h) + \text{Arcsin}(h/r_B) \times {r_B}^2 - 2 \times Z. \tag{16}$$

The B-only area is found by subtracting the area of the lune from the area of the whole circle as

$$B = (\pi \times {r_B}^2) - (AB). \tag{17}$$

Subtracting out the extra intersection, the total area covered by the circles, denoted TOTAL, is (from equation (16))

$$\text{TOTAL} = (\pi) + (\pi \times {r_B}^2) - AB. \tag{18}$$

[Some may recognize the formula in (18) as one form of the Principle of Inclusion and Exclusion–ed.]

From equations (16) and (18) it follows that

$$AB\% = 100 \times AB/\text{TOTAL}; \tag{19}$$

from equations (17) and (18) it follows that

$$B\% = 100 \times B/\text{TOTAL}; \tag{20}$$

and from equations (19) and (20) it follows that

$$A\% = 100 - AB\% - B\%. \tag{21}$$

These results hold for d_{AB} greater than or equal to X, the distance from the origin to the chord, but not greater than $r_A + r_B$, that is:

$$X \le d_{AB} \le r_A + r_B, \tag{22}$$

where, from equation (11), $X = \text{Cos}(Q_A/2)$.

Case II: Chord to one side of both centers

The same definitions apply as in the previous section, except in relation to the following situation. Draw two intersecting circles and associated lines, labelling them as follows. Draw the larger of the two circles on the left. Insert the center of the large circle as a distinguished dot. Draw a smaller circle intersecting the larger one in such a way that the center of the large circle is contained within the smaller circle. Much of the small circle is therefore necessarily contained within the large circle. Note the center of the small circle as a dot. Draw the chord joining the two intersection points of the small and large circles; half of it has length h. Draw the line segment joining the two circle centers, of length d_{AB} and extend the segment to intersect the chord. The small circle now contains a right triangle which in turn contains a triangle with an obtuse angle. Label the radius of the larger circle as r_A; label the radius of the smaller circle as r_B. Label the constructed central angle in the larger circle as Q_A and the constructed central angle in the smaller circle as Q_B. The area of the obtuse triangle is Z_1 and the area of the difference between the right triangle and the obtuse triangle is Z_2.

The following formulæ can then be readily deduced. From equation (4),

(23) $$S_1 = (r_A + r_B + d_{AB})/2 = (1 + r_B + d_{AB})/2;$$

from equations (5) and (23),

(24) $$Z_1 = (S_1 \times (S_1 - 1) \times (S_1 - r_B) \times (S_1 - d_{AB}))^{1/2};$$

from equation (24),

(25) $$h = 2 \times Z_1/d_{AB}$$

because $Z_1 = (1/2) \times d_{AB} \times h$; from equation (25)

(26) $$Q_A = \text{Arcsin}\,(h),$$

because $\text{Sin}\,(Q_A) = h/r_A = h$; from equation (26)

(27) $$Q_B = \text{Arcsin}\,(h/r_B),$$

because $\text{Sin}\,(Q_B) = h/r_B$; from equations (25) and (27)

(28) $$Z_2 = (1/2) \times h \times r_B \times \text{Cos}\,(Q_B));$$

from equations (5) and (26)

(29) $$K_A = \text{Sector } A \text{ area} = (1/2) \times (2 \times Q_A) \times {r_A}^2 = Q_A;$$

from equations (24) and (28)

(30) $$L_A = \text{Triangle Area of Sector } A = 2 \times (Z_1 + Z_2);$$

from equations (7), (29), (30)

(31) $$M_A = \text{Segment Area of Sector } A = K_A - L_A;$$

from equations (5) and (27),

(32) $$K_B = \text{Sector } B \text{ Area} = (1/2) \times (2 \times Q_B) \times {r_B}^2 = Q_B \times {r_B}^2;$$

from equation (28)

(33) $$L_B = \text{Triangle Area of Sector } B = 2 \times Z_2;$$

from equations (32) and (33)

(34) $$M_B = \text{Segment Area of Sector } B = K_B - L_B;$$

from equations (31) and (34)

(35) $$\text{Area } W = M_B - M_A;$$

from equation (35)

(36) $$AB = \text{Area of Circle } B - \text{Area } W = \pi \times {r_B}^2 - W;$$

thus,

(37) $$B = W;$$

from equation (35)

TOTAL, the total area covered by the circles

(38) $$= \text{Area of Circle } A + \text{Area } W = \pi + W;$$

from equations (36) and (38)

(39) $$AB\% = 100 \times AB/\text{TOTAL};$$

from equations (37) and (38)

(40) $$B\% = 100 \times B/\text{TOTAL};$$

and from equations (39) and (40)

(41) $$A\% = 100 - AB\% - B\%.$$

These results hold for d_{AB} greater than or equal to zero, but not greater than X, the distance from the origin to the chord, that is:

(42) $$0 \leq d_{AB} \leq X,$$

where from equation (26) $X = \cos(Q_A)$.

Methods for Computing r_B and d_{AB}.

The computation of r_B given $A\%$, $B\%$, and $AB\%$ is straightforward. However, this is not the case for d_{AB} in light of the fact that we did not obtain a function for d_{AB} in terms of $A\%$, $B\%$, and $AB\%$. We present three numerical methods for estimating d_{AB}, each with a different level of accuracy. First, however, we derive $AB\%$ and $B\%$ as a function of A, B, and AB, and r_B as a function of $AB\%$ and $B\%$. These derivations of $AB\%$, $B\%$ and r_B are the same for all three methods of estimating d_{AB}.

r_B as a Function of $B\%$ and $AB\%$

Let A, B, and AB be the B-circle only area, the A-circle only area, and the area of intersection, respectively, and let $TA = A + B + AB$; then,

(43) $$A\% = 100 \times A/TA, \quad A = A\% \times TA/100;$$

(44) $$B\% = 100 \times B/TA, \quad B = B\% \times TA/100;$$

(45) $$AB\% = 100 \times AB/TA, \quad AB = AB\% \times TA/100;$$

(46) $$\text{Circle } A \text{ area} = A + AB = \pi;$$

(47) $$\text{Circle } B \text{ area} = B + AB = \pi \times {r_B}^2.$$

Substituting equation (46) in equation (47): $B + AB = \pi \times {r_B}^2 = (A + AB) \times {r_B}^2$,

(48) $${r_B}^2 = (B + AB)/(A + AB).$$

Substituting equations (43), (44), and (45) in equation (48):

(49) $${r_B}^2 = \frac{(B\% \times TA)/100 + (AB\% \times TA)/100}{(A\% \times TA)/100 + (AB\% \times TA)/100} = \frac{B\% + AB\%}{A\% + AB\%}.$$

(50) $$A\% = 100 - B\% - AB\%,$$

because $A\% + B\% + AB\% = 100$.

Substituting equation (50) into equation (49):

$${r_B}^2 = \frac{B\% + AB\%}{(100 - B\% - AB\%) + AB\%} = \frac{B\% + AB\%}{100 - B\%}.$$

Thus,

(51) $$r_B = ((B\% + AB\%)/(100 - B\%))^{1/2}.$$

Look-up Table Method for Estimating d_{AB}

Table 1 (at end of article) gives the value of d_{AB} to 6 decimal places for all values of $AB\%$ from 0 to 100 and for $B\%$ from 0 to 50 in 5 percentage point increments for values of d_{AB} from 0 to $r_A + r_B$. Note from Table 1 that some of the values are calculated using the procedure for Case I and some using the Case II procedure.

The procedure used to obtain the values in Table 1 is summarized here. For each combination of $B\%$ and $AB\%$ in Table 1, calculate d_{AB} as follows. First calculate r_B using equation (51). Then guess a value for d_{AB} that is approximately correct, and guess whether Case I or Case II applies. (In most areas of the table this is obvious.) Then calculate the values of $B\%$ and $AB\%$ using the guessed value of d_{AB}, the calculated value of r_B and either equations (8) through (20) in Case I, or equations (23) through (40) in Case II. Then adjust the guessed value of d_{AB} up or down and recalculate until the resulting values of $B\%$ and $AB\%$ approximate the desired values as closely as desired (six decimal points) in Table 1. Check the final value of d_{AB} to be sure the correct calculation procedure was used (Case I or II) with the inequalities (22) or (42).

Table 2 contains values of d_{AB} for values of $AB\%$ in the 0 to 10 range. Between 0 and 5, $AB\%$ is in increments of 1. This table was produced because of the large and non-linear increments in d_{AB} in this range of $AB\%$.

If the given values of $B\%$ and $AB\%$ are one of the combinations found in Table 1 or Table 2, then the value of d_{AB} can be obtained directly from the tables to six decimal point accuracy. If the exact values of $B\%$ and $AB\%$ are not in either table, then an interpolation procedure can be used. In Table 1, the procedure would be as follows.

(a) Assume the given values of $B\%$ and $AB\%$ are not along the lower diagonal of the table, so that they are bounded by table values of $B\%$ and $AB\%$ at four corners forming a rectangle within the table. Let $d(i,j)$ be the value of d_{AB} for any values of $AB\%$ (or i) and $B\%$ (or j), within the defined range. Then $d(AB\%, B\%)$ is the value of d_{AB} at the given values of $B\%$ and $AB\%$. If e is the value of $AB\%$ just less than the given $AB\%$, f is the value of $AB\%$ just greater than the given $AB\%$, g is the value of $B\%$ just less than the given $B\%$, and h is the value of $B\%$ just greater than the given $B\%$, then,

$$d_K = \text{estimated value of } d \text{ at point } K$$

$$= d(e,g) + (d(f,g) - d(e,g)) \times (AB\% - e)/(f - e), \tag{52}$$

where K is the intersection point of a horizontal through $d(i,j)$ with the vertical line through g.

$$d_L = \text{estimated value of } d \text{ at point } L$$

$$= d(e,h) + (d(f,h) - d(e,h)) \times (AB\% - e)/(f - e), \tag{53}$$

where L is the intersection point of a horizontal through $d(i,j)$ with the vertical line through h.

$$\text{Estimate of } d(AB\%, B\%) = d_K + (d_L - d_K) \times (B\% - g)/(h - g). \tag{54}$$

(b) Assume the given values of $AB\%$ and $B\%$ are near the lower diagonal of the defined range such that the location is bounded by only three table values (rather than by four table values, as in case (a) above). Use the same labelling as in case (a) above for the bounding table entries; note, however, that d(f,h) is not defined in this case (because of the nearness of the table entry to the lower diagonal). At the boundaries where $B\% = 0$ or $AB\% = 0$, equation (54) holds. Otherwise, we have:

$$d_K = \text{estimated value of } d \text{ at point } K$$

$$= d(e,g) + (d(f,g) - d(e,g)) \times (AB\% - e)/(f - e), \tag{55}$$

where K is the intersection point of a horizontal through $d(i,j)$ with the vertical line through g.

$$d_L = \text{estimated value of } d \text{ at point } L$$

$$= d(e,h) + (d(e,h) - d(e,g)) \times (B\% - g)/(h - g), \tag{56}$$

where L is the intersection point of a vertical through $d(i,j)$ with the horizontal line through e.

$$d(AB\%, B\%) = d(e,g) + (d_K - d(e,g)) + (k_L - d(e,g)) = d_K + d_L - d(e,g)$$

$$= d(e,g) + (d(f,g) - d(e,g)) \times (AB\% - e)/(f - e) + (d(e,h) - d(e,g)) \times (B\% - g)/(h - g). \tag{57}$$

The use of formulas (54) and (57) in conjunction with Table 1 will generally produce answers for d_{AB} within 0.01 of the correct figures, with the exception of the range for $AB\%$ from 0 to 5. In some areas of this range, particularly for $B\%$ greater than 45, the error can be over 0.05. For example, this method produces an estimated value for $d(2.5, 47.5) = 1.7394$, compared to the correct value of 1.7927, an error of 0.053. (This error is 5.3% of the radius of circle A, which is 1, and is $100 * 0.053/1.7927 = 3\%$ of the correct value of d_{AB}.)

If Table 2 is used for values of $AB\%$ between 0 and 5, the error can be reduced to less than 0.02 in the worst cases. For example, the use of Table 2 produces an estimated value for $d(0.5, 49.5) = 1.93016$, compared to the correct value of 1.91299, an error of 0.01717. (This is 1.7% of the radius of circle A and 0.9% of the correct value of d_{AB}.)

Polynomial Estimation of d_{AB}

The regression formulæ were used to obtain polynomials in $AB\%$ and $B\%$ that estimated d_{AB}, in effect interpolating for values of $AB\%$ and $B\%$ between the grid points in Table 1. The range of $AB\%$ and $B\%$ was separated into three subranges and a polynomial was obtained for each subrange. The three subranges are specified below and also denoted graphically, using a variety of typefaces, in Table 3.

$$\text{Subrange 1}: AB\% > 5 \text{ and } B\% \geq 5. \tag{58}$$

$$\text{Subrange 2}: 0 \leq AB\% \leq 5 \text{ and } 0 \leq B\% \leq 50. \tag{59}$$

$$\text{Subrange 3}: 0 \leq AB\% \leq 100 \text{ and } 0 \leq B\% < 5. \tag{60}$$

The polynomials obtained for each subrange are given below.
Polynomial 1 for subrange 1:

$$\text{est}\, {d_{AB}}^1 = c_0 + c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4 + c_5X_5 + c_6X_6 + c_7X_7 + c_8X_8, \tag{61}$$

where

$c_0 = \ \ 0.994388189$
$c_1 = \ \ 0.003790799, \ X_1 = AB\%$
$c_2 = -0.001818030, \ X_2 = B\%$
$c_3 = -0.129148003, \ X_3 = (AB\%)^{1/2}$
$c_4 = \ \ 0.130891455, \ X_4 = (B\%)^{1/2}$
$c_5 = -0.000147200, \ X_5 = (AB\%) \times (B\%)$
$c_6 = -0.000017449, \ X_6 = (AB\%)^2$
$c_7 = \ \ 0.000081024, \ X_7 = (B\%)^2$
$c_8 = -0.004913375, \ X_8 = (AB\% \times B\%)^{1/2}.$

Polynomial 2 for subrange 2:

$$\text{est}\, {d_{AB}}^2 = c_0 + c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4 + c_5X_5 + c_6X_6 + c_7X_7, \tag{62}$$

where

$c_0 = \ \ 1.003584849$
$c_1 = -0.009650203, \ X_1 = AB\%$
$c_2 = \ \ 0.002712922, \ X_2 = B\%$
$c_3 = -0.089520075, \ X_3 = (AB\%)^{1/2}$
$c_4 = \ \ 0.093223275, \ X_4 = (B\%)^{1/2}$
$c_5 = -0.000366121, \ X_5 = (AB\%) \times (B\%)$
$c_6 = -0.000521608, \ X_6 = (AB\%)^2$
$c_7 = \ \ 0.000075950, \ X_7 = (B\%)^2.$

Polynomial 3 for subrange 3:

$$\text{est}\, {d_{AB}}^3 = c_0 + c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4 + c_5X_5 + c_6X_6 + c_7X_7 + c_8X_8, \tag{63}$$

where

$c_0 = \ \ 1.009984781$
$c_1 = \ \ 0.001043059, \ X_1 = AB\%$
$c_2 = \ \ 0.009094475, \ X_2 = B\%$
$c_3 = -0.106641306, \ X_3 = (AB\%)^{1/2}$
$c_4 = \ \ 0.088497298, \ X_4 = (B\%)^{1/2}$
$c_5 = -0.000104701, \ X_5 = (AB\%) \times (B\%)$
$c_6 = -0.000052845, \ X_6 = (AB\%)^2$
$c_7 = -0.000084727, \ X_7 = (B\%)^2$

$c_8 = -0.007209240$; $X_8 = (AB\% \times B\%)^{1/2}$.

Numerical Search on the Inverse Function

The value of d_{AB} can be obtained to any desired accuracy for any combination of $AB\%$ and $B\%$ in the defined range using the same procedure as was used to derive Table 1. We summarize the procedure below.

(1) Given A, B, and AB.

(2) Calculate $B\%$ using equation (44).

(3) Calculate $AB\%$ using equation (45).

(4) Calculate r_B using equation (51).

(5) Estimate an approximate value for d_{AB} using Table 1.

(6) Estimate whether Case I or Case II applies for the calculated values of $B\%$ and $AB\%$ using Tables 1 and 2.

(7) Calculate estimated values of $AB\%$ and $B\%$ using the calculated value of r_B from step 4, the estimated value of d_{AB} from step 5 and equations (19) and (20) for Case I or equations (39) and (40) for Case II.

(8) If the estimated value of $B\%$ obtained in step 7 is too small, increase the estimated value of d_{AB} and recalculate $AB\%$ and $B\%$ by recycling through step 7; if $B\%$ is too big, reduce the estimated value of d_{AB} and recycle through step 7. The size of the adjustment depends on the approximate slope in the region of concern. For example, if we were in a region of Table 1 where d_{AB} increased 0.025 while $B\%$ was increasing by 5 (as is approximately the case for $B\%$ between 10 and 15 and $AB\%$=60), then adjust d_{AB} by 0.005 for each error increment of 1 in $B\%$. In this way, the error in $B\%$ can be made as small as desired by continued recycling. The value of $AB\%$ converges to the desired value along with $B\%$.

(9) Check to be sure that the proper case (I or II) was used by applying the inequalities (22) or (42).

Editor's Note:

In the original submission the author also considered Venn diagrams of three circles, noting that the three-circle Venn diagram contains insufficient degrees of freedom to provide a general solution to a three characteristic situation. The reader interested in generalizations of the two-circle case might wish to examine the literature of Boolean algebra, particularly Karnaugh maps used in the minimization of switching circuits.

More detail is presented in this presentation than would be in traditional publications, suggesting yet another avenue to explore in the dissemination of information across disciplinary boundaries and one way to offer detail that might be required by engineers in the field to implement abstract ideas presented in journals. The increase in cost, to present extra detail that may not be necessary to all, is minuscule in an electronic format.

Author's Note:

The author wishes to thank anonymous referees for suggesting the viewpoint of "equal area" Venn diagrams, and for substantial help in making the context of the problem reflect this viewpoint.

TABLE 1
TWO INTERSECTING CIRCLES PROBLEM
Distance between centers (d), given *AB*% and *B*-ONLY %

	B-Only %										
AB%	00	05	10	15	20	25	30	35	40	45	50
00	1	1.229399	1.33333	1.42008	1.5	1.57735	1.65464	1.7337	1.8163	1.90453	2
05	**0.776393**	0.982242	1.082793	1.164457	1.23779	1.307136	1.374962	1.443	1.5127	1.58547	
10	**0.683772**	**0.868866**	0.961981	1.037539	1.10493	1.168008	1.229156	1.289705	1.350925	1.413929	
15	**0.612702**	**0.782479**	0.86877	0.938611	1.0005	1.05796	1.112964	1.166795	1.220433		
20	**0.552787**	**0.710053**	**0.79011**	0.8548	0.9113	0.96341	1.012665	1.060145	1.106585		
25	**0.5**	**0.646459**	**0.72074**	0.78016	0.831894	0.878849	0.922537	0.963822			
30	**0.452278**	**0.589054**	**0.657608**	0.712452	0.759365	0.801269	0.839453	0.874543			
35	**0.408393**	**0.536264**	**0.599914**	0.649738	0.691905	0.728788	0.761409				
40	**0.367545**	**0.487058**	**0.545676**	**0.590844**	0.628273	0.660049	0.686956				
45	**0.32918**	**0.440711**	**0.494394**	**0.534916**	0.567539	0.594046					
50	**0.292894**	**0.399085**	**0.445404**	**0.48126**	0.508937	0.529864					
55	**0.258381**	**0.354550**	**0.398303**	0.429363	0.451767						
60	**0.225404**	**0.313984**	**0.352752**	0.378617	0.395288						
65	**0.193775**	**0.27465**	**0.308123**	0.328451							
70	**0.16334**	**0.236231**	**0.264046**	0.278098							
75	**0.133975**	**0.198480**	**0.219922**								
80	**0.105573**	**0.160016**	0.174756								
85	**0.078049**	**0.122853**									
90	**0.051317**	0.082698									
95	**0.025321**										
100	**0**										

Case I: Chord joining intersection points lies between the two centers

Case II: Chord lies to one side of both centers.

TABLE 2
DATA TABLE FOR TWO-CIRCLE PROBLEM
Distance between centers (d), given $AB\%$ and B-ONLY %
for $AB\% = 00 - 05$

	B-Only %					
$AB\%$	00	01	02	03	04	05
00	1	1.100503	1.142857	1.175863	1.204124	1.229399
01	**0.9**	0.996625	1.041666	1.076413	1.105877	1.132029
02	**0.858578**	**0.949095**	0.993166	1.027487	1.056694	1.082652
03	**0.826794**	**0.91298**	**0.95593**	0.989618	1.018366	1.044
04	**0.8**	**0.882799**	**0.924676**	**0.957696**	0.985978	1.011199
05	**0.776397**	**0.856395**	**0.897278**	**0.929644**	0.957428	0.982242

	B-Only %						
$AB\%$	10	15	20	25	30	35	40
00	1.33333	1.42008	1.5	1.57735	1.65464	1.7337	1.8163
01	1.237653	1.324053	1.402561	1.477746	1.552221	1.627878	1.706373
02	1.187417	1.272749	1.349919	1.423492	1.496065	1.569499	1.645393
03	1.147486	1.231629	1.307493	1.379587	1.450471	1.521965	1.595619
04	1.113248	1.196158	1.270744	1.341437	1.410751	1.480464	1.552075
05	1.082793	1.164457	1.23779	1.307136	1.374962	1.443	1.5127

	B-Only %					
$AB\%$	45	46	47	48	49	50
00	1.90453	1.922958	1.941696	1.960768	1.980196	2
01	1.789387	1.806694	1.824274	1.842146	1.860327	
02	1.725354	1.741988	1.758872	1.776022	1.793457	
03	1.67297	1.689029	1.705319	1.721855		
04	1.627057	1.642596	1.658348	1.674328		
05	1.58547	1.600524	1.615776			

Case I: Chord joining intersection points lies between the two centers
Case II: Chord lies to one side of both centers.

TABLE 3
TWO INTERSECTING CIRCLES PROBLEM
Error in the Estimated Distance between the two centers: (dest - d)

	B-Only %										
AB%	00	05	10	15	20	25	30	35	40	45	50
00	**0.003585**	**-0.00180**	**-0.00022**	**0.002330**	**0.005130**	**0.007043**	**0.009202**	**0.009302**	**0.006917**	**0.000295**	**-0.01170**
05	**-0.00796**	**0.000948**	**-0.00314**	**-0.00465**	**-0.00442**	**-0.00308**	**-0.00110**	**0.000865**	**0.002138**	**0.001824**	
10	*-0.00111*	-0.00320	0.000151	0.000254	-0.00029	-0.00075	-0.00096	-0.00100	-0.00117	-0.00193	
15	*-0.00126*	-0.00331	-0.00005	-0.00039	-0.00121	-0.00152	-0.00099	0.000469	0.002778		
20	*-0.00096*	-0.00263	0.000458	-0.00030	-0.00137	-0.00157	-0.00037	0.002464	0.007088		
25	*-0.00044*	-0.00166	0.001121	-0.00009	-0.00143	-0.00152	0.000319	0.004538			
30	*0.000144*	-0.00065	0.001711	0.000014	-0.00158	-0.00154	0.001003	0.006707			
35	*0.000727*	0.000235	0.002122	-0.00005	-0.00188	-0.00163	0.001762				
40	*0.001248*	0.00930	0.002310	-0.00031	-0.00229	-0.00168	0.002796				
45	*0.001670*	0.001372	0.002261	-0.00074	-0.00272	-0.00151					
50	*0.001965*	0.001534	0.001995	-0.00124	-0.00300	-0.00084					
55	*0.002114*	0.001407	0.001563	-0.00172	-0.00292						
60	*0.002102*	0.001011	0.001052	-0.00195	-0.00207						
65	*0.001914*	0.000389	0.000606	-0.00165							
70	*0.001542*	-0.00037	0.000606	-0.00165							
75	*0.000976*	-0.00113	0.001043								
80	*0.000210*	-0.00160	0.003186								
85	*-0.00076*	-0.00124									
90	*-0.00194*	0.001460									
95	*-0.00334*										
100	*-0.00496*										

Actual Distance from Table 1; Estimated Distance from Polynomials 1, 2, and 3.
Polynomial 1
Polynomial 2
Polynomial 3

Los Angeles, 1994–A Spatial Scientific View

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An algorithm discussed by Maria Hasse (Hasse, 1961; Harary, Norman, and Cartwright, 1965) offers a method for finding the shortest distance between any two nodes in a network of n nodes when given only distances between adjacent nodes. The algorithm is one that focuses on structure alone, and it is therefore spatial. The procedure is similar in form to that used to multiply matrices, given two $n \times n$ matrices A and B. To find the entries in their Hasse-sum, matrix C, take the minimum of the row-by-column sums; thus, the entry

$$c(21) = \min\{a(21) + b(12), a(22) + b(22), a(23) + b(32), ..., a(2n) + b(n2)\}.$$

The results below show the outcome of applying this tool from theoretical spatial science to the real-world: to one change in the Los Angeles freeway pattern following the recent devastating earthquake (January 17, 1994).

Los Angeles, 1994.

When a recent earthquake caused a disastrous collapse of a span of the Santa Monica freeway, according to television reports the world's busiest freeway (carrying an estimated 300,000 vehicles per weekday), municipal authorities managed to keep the city moving. They employed a well-balanced combination of alternate routing using intelligent vehicle highway systems (IVHS) in which traffic lights along surface routes paralleling freeways were coordinated in response to user demand, together with media messages urging people to stay off roadways and the effective dispersal of information concerning alternate routes. Outside forecasters of doom predicted massive gridlock that did not occur in regions where alternate routing was available.

In the analysis below, we test Hasse's algorithm against a changed adjacency configuration and interpret the results. Indeed, what would a forecaster using the Hasse algorithm have predicted in this situation?

The map in Figure 1 shows a portion of the Los Angeles freeway system, and nearby major surface arterials, linking Los Angeles International Airport (LAX) to the Central City (CC). We tightened our focus to consider what sort of impact the partial closing of the Santa Monica freeway might have on travel times to and from the airport and the downtown region. The routing in Figure 1 is along freeways, only, that form a square envelope around the direct diagonal route (that does not exist in the real world) linking LAX to the CC. Any rupture along this circuit will completely destroy one of the two possible routes, sending all the traffic along one path only. Thus, when the Santa Monica freeway was ruptured (Figure 2)–cross-hatched area on Figure 1 – all the traffic would have been forced due east and then north, if only freeway linkages were employed.

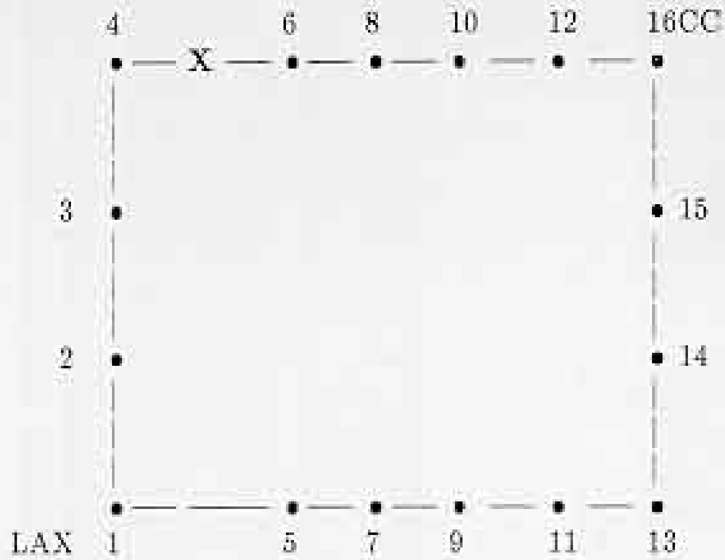

Figure 1. LAX denotes Los Angeles International Airport. CC denotes the Central City. Routes are along major expressways. The X indicates the rupture in the Santa Monica freeway caused by the January 17, 1994 earthquake. Consider that all lines, whether dashed or solid, represent continuous graphical linkage between adjacent nodes. The only break in the freeway is at the X.

To overcome this apparently disastrous traffic situation, it is natural to introduce alternate routes along roads that are already present. Indeed, in earlier mathematical references there is consideration of this sort of rerouting problem after some edges of a network have been deleted (Menger, 1927; Ford and Fulkerson, 1962). One set of major surface routes is added to the map in Figure 1 to offer a number of different routes (Figure 3). The matrix A (Figure 4) displays time-distances in tabular form across the network shown in Figure 3. The entry of 12 in the first row, second column indicates that it takes 12 quarter-minutes to travel from the node labelled 1 to the node labelled 2. A zero in this matrix indicates that there are zero quarter-minutes required as travel time–thus, zeroes appear in this matrix only along the main diagonal. Nodes are treated as points within which no travel is possible. An asterisk indicates that there is no direct linkage between corresponding entries– an asterisk in the (1,3) position indicates that there is no single edge of the graph linking node 1 to node 3. All numerical entries are expressed in quarter-minutes; the Pascal program (Figure 5), was written to display integral results. (Use of a spreadsheet is possible but is far more time-consuming.) Travel times were calculated from distances in the 1993 Rand McNally Road Atlas, assuming (from field experience) an average speed of 40 mph.

Higher powers of the matrix A count numbers of paths of longer length. The matrix A^2 counts paths of 2 edges as well as those of one edge. Thus, in A^2 one would expect to find an entry indicating the total time to travel from node 1 to node 3, as well as entries representing travel times across single graphical edges from node 1 to node 2 and from node

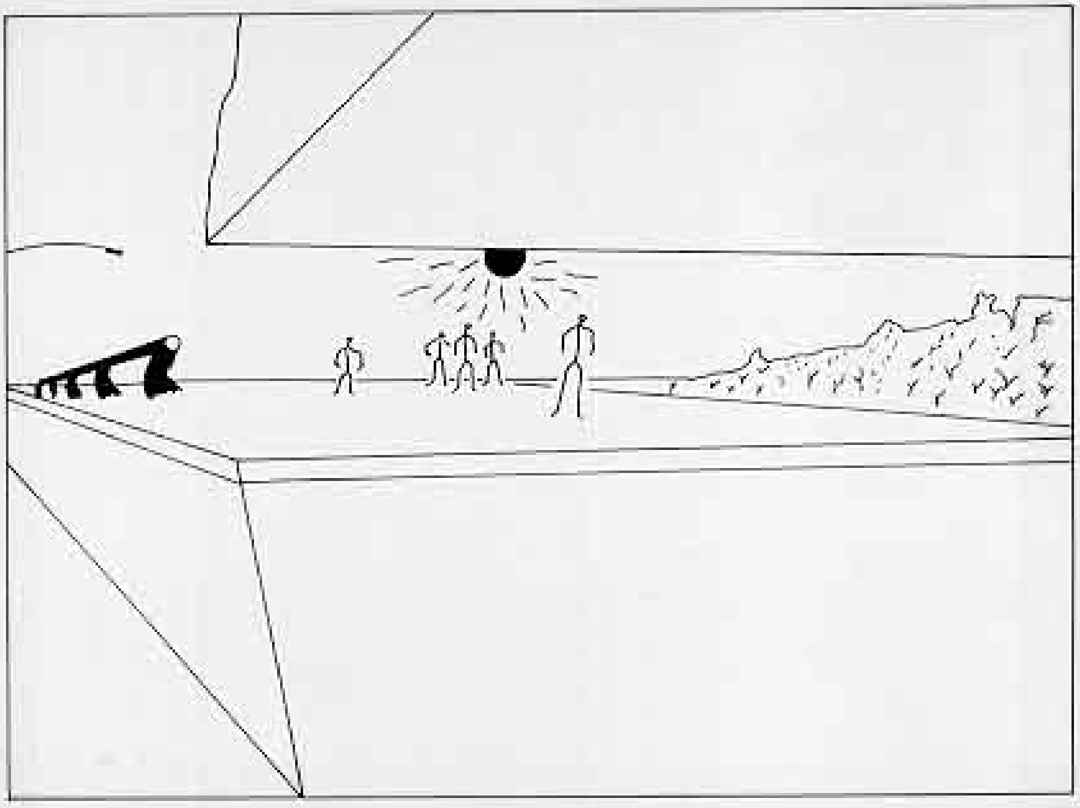

Figure 2. Drawing based on a photo, showing damage to the Los Angeles freeways, from the *New York Times*, Tuesday, January 18, 1994.

2 to node 3; indeed, as would be hoped the travel time of 30 quarter-minutes from nodes 1 to 3 is the sum of the travel times from 1 to 2 (12 quarter-minutes) and from 2 to 3 (18 quarter-minutes).

The Hasse operator (erroneously referred to as the Hedetniemi operator in some earlier work, corrected by F. Harary who also notes that this procedure may also be present in literature earlier than Hasse's 1961 article) always selects the shortest path if more than one is available. Other algorithms execute similar calculations; however, Floyd's algorithm provides easy display of lengths only (and not the components that compose them), while Dijkstra's algorithm is not designed for easy display of results but does permit the determination of the actual position of the shortest path. Both of these algorithms require the same number of steps independent of the actual data; Hasse's does not – it stops shorter than would Floyd's or Dijkstra's in many situations. Further detail has been published elsewhere (Harary, Norman, and Cartwright, 1960; Arlinghaus, Arlinghaus, and Nystuen, 1990).

The matrices A through A^8 show travel times across paths of varying length for the

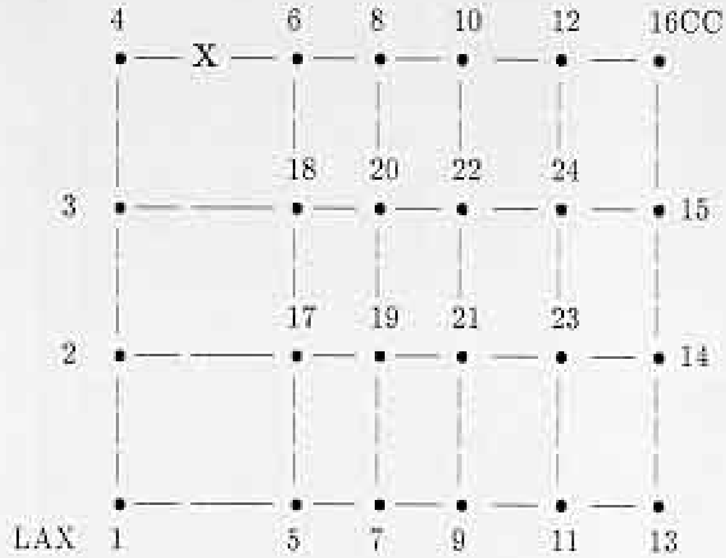

Figure 3. Same basic pattern as Figure 1, with surface routes added, and intersections of surface routes added as nodes in the graph. LAX denotes Los Angeles International Airport. CC denotes the Central City. Routes are along major expressways. The X indicates the rupture in the Santa Monica freeway caused by the January 17, 1994 earthquake. Consider that all lines, whether dashed or solid, represent continuous graphical linkage between adjacent nodes. The only break in the freeway is at the X.

freeway system prior to the earthquake (Figure 4a). The algorithm stops when $A^{n+1} = A^n$; in this case, therefore, the last matrix with new entries is A^8–the matrix A^9 is calculated to know when to stop the iteration. A different initial matrix is required to capture the linkage pattern between LAX and CC following the 1994 earthquake (Figure 4b)–the Santa Monica freeway was shattered between nodes 4 and 6 on the graph in Figure 3. The matrix B in Figure 6 indicates a new adjacency pattern; in A, the 4th row, 6th column contained a value of 30 to represent the direct linkage between nodes 4 and 6. The corresponding entry in matrix B is an asterisk – that is, there is no path, of a single graphical edge, available between nodes 4 and 6. When Hasse's algorithm is run using B (Figure 4b), instead of A (Figure 4a), as the initial matrix, the iteration requires the same number of stages; however, some of the entries are larger in B than in A, reflecting the need for longer paths to provide alternate routes around the earthquake-altered freeway. In the eighth iteration, the B-iterate contains entries in the (4,6), (4,8), (4,10), (4,12), and (4,16) positions that are about 30 quarter-minutes larger than are the entries in the corresponding eighth A-iterate. This increase in the structural model comes purely from spatial pattern–it does not address the natural increase in congestion that one would also expect.

The surface route pattern that was introduced permitted all turns at each of the surface route intersections; this sort of strategy appears desirable, but because turns (especially U.S. left-hand turns onto two way streets) generally force additional slowing of the traffic

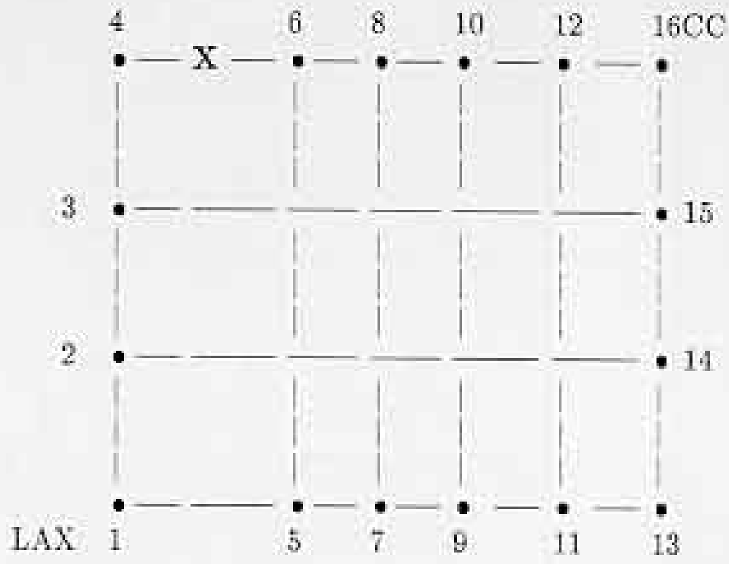

Figure 6. Same basic pattern as Figure 1, with surface routes added; unlike Figure 2, in this case intersections of surface routes are NOT added as nodes in the graph. The surface routes have limited access. LAX denotes Los Angeles International Airport. CC denotes the Central City. Routes are along major expressways. The X indicates the rupture in the Santa Monica freeway caused by the January 17, 1994 earthquake. Consider that all lines, whether dashed or solid, represent continuous graphical linkage between adjacent nodes. The only break in the freeway is at the X.

one might consider further alteration of the structural model.

Figure 6 shows a modified form of the map in Figure 3; in it, the nodes 17 through 24 have been omitted. This graphical omission corresponds to the real-world notion of preventing intersecting traffic flows within the interchanges. One way to reduce congestion is to prohibit all turns. Another is to use traffic lights in a manner that responds to the traffic itself, rather that to estimates of traffic. The structural model in Figure 6 represents this sort of approach; the north-south route from node 5 to node 6 does not intersect any of the east-west surface vehicular flows.

Figure 7a shows the initial matrix C representing this particular structural model that permits restricted pre-earthquake travel across surface routes. Figure 7b shows the initial matrix D representing the model with the rupture in the Santa Monica freeway. When Hasse's algorithm is run, there are clearly once again a number of locations that stand out: the (4,6) entry, for example, goes from 30 quarter-minutes to 114 quarter-minutes in this case. Indeed, there is not even any path available of length less than 5 for this entry: there is an asterisk in this position in D^4 –the only asterisk for this entry in the C iteration sequence, with the bridge in, is in the first matrix. The last entries to come into the D sequence iteration are (4,8) and (4,9)–this situation tallies with the relationships shown on the map in Figure 3. Traffic engineers might choose this latter model during times of the day when volumes are not high at the nodes showing large increases, or some other strategy that responds to traffic history.

The path structure from node 1 (LAX) to node 16 (CC) is not altered; the Santa Monica freeway was not the shortest route from LAX to CC although its closure no doubt adds to the congestion along shorter routes. Most of the entries in the fourth row of D^7 to the right of, and including, the sixth column show increases in time – some only slight and some substantial. Only the fourth row and the fourth column show altered time patterns, pre- and post-earthquake – Hasse's algorithm shows that the underlying spatial structure of the road network is sufficient to provide alternate routing to and from LAX to CC and between many of the intervening locations. This finding matches what has apparently happened in the actual post-earthquake environment.

Policy Implications

In order to turn the elegant theoretical tool of Hasse into one a traffic engineer might actually employ, there are a number of policy implications to consider – policy changes can put real-world teeth into theory.

1. No turns except onto expressways means maximum flow; however, this strategy is awkward for those living in the area. Indeed, even if it is assumed that people can turn off onto minor streets but cannot turn at major intersections, these local turns cause a lower average speed.
2. Permit right hand turns only –not too disruptive of flow so speed is maintained. The algorithm still holds, even with an asymmetric matrix.
3. Permit all turns – there are a number of engineering strategies that might have corresponding structural components in a graphical model. Left hand turns slow the system. Insert different average speeds or times on the edges of the structural model.
4. Use one-way streets–this strategy equalizes left and right turns; it, too, produces asymmetric adjacency matrices.

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Figures containing tables

00	12	*	*	18	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
12	00	18	*	*	*	*	*	*	*	*	*	*	*	*	*	6	*	*	*	*	*	*	*
*	18	00	24	*	*	*	*	*	*	*	*	*	*	*	*	*	18	*	*	*	*	*	*
*	*	24	00	*	30	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
18	*	*	*	00	*	12	*	*	*	*	*	*	*	*	*	12	*	*	*	*	*	*	*
*	*	*	30	*	00	*	12	*	*	*	*	*	*	*	*	*	21	*	*	*	*	*	*
*	*	*	*	12	*	00	*	6	*	*	*	*	*	*	*	*	*	15	*	*	*	*	*
*	*	*	*	*	12	*	00	*	9	*	*	*	*	*	*	*	*	*	21	*	*	*	*
*	*	*	*	*	*	6	*	00	*	6	*	*	*	*	*	*	*	*	*	15	*	*	*
*	*	*	*	*	*	*	9	*	00	*	6	*	*	*	*	*	*	*	*	*	21	*	*
*	*	*	*	*	*	*	*	6	*	00	*	6	*	*	*	*	*	*	*	*	*	15	*
*	*	*	*	*	*	*	*	*	6	*	00	*	*	*	6	*	*	*	*	*	*	*	21
*	*	*	*	*	*	*	*	*	*	6	*	00	12	*	*	*	*	*	*	*	*	*	*
*	*	*	*	*	*	*	*	*	*	*	*	12	00	12	*	*	*	*	*	*	*	3	*
*	*	*	*	*	*	*	*	*	*	*	*	*	12	00	21	*	*	*	*	*	*	*	3
*	*	*	*	*	*	*	*	*	*	*	6	*	*	21	00	*	*	*	*	*	*	*	*
*	6	*	*	12	*	*	*	*	*	*	*	*	*	*	*	00	12	9	*	*	*	*	*
*	*	18	*	*	21	*	*	*	*	*	*	*	*	*	*	12	00	*	9	*	*	*	*
*	*	*	*	*	*	15	*	*	*	*	*	*	*	*	*	9	*	00	12	9	*	*	*
*	*	*	*	*	*	*	21	*	*	*	*	*	*	*	*	*	9	12	00	*	9	*	*
*	*	*	*	*	*	*	*	15	*	*	*	*	*	*	*	*	*	9	*	00	12	6	*
*	*	*	*	*	*	*	*	*	21	*	*	*	*	*	*	*	*	*	9	12	00	*	12
*	*	*	*	*	*	*	*	*	*	15	*	*	3	*	*	*	*	*	*	6	*	00	12
*	*	*	*	*	*	*	*	*	*	*	21	*	*	3	*	*	*	*	*	*	12	12	00

Figure 4a.i This is the initial matrix, A. Figure 4a contains a set of nine tables (i to ix) illustrating the use of Hasse's algorithm on part of the LA freeway/surface route system, shown in Figure 3, prior to the earthquake of January 17, 1994. Travel times are in one-quarter minutes. An asterisk indicates that the travel time between locations is too large to enter the matrix. A double-zero indicates an entry of 0.

00	12	30	*	18	*	30	*	*	*	*	*	*	*	*	*	18	*	*	*	*	*	*	*
12	00	18	42	18	*	*	*	*	*	*	*	*	*	*	*	6	18	15	*	*	*	*	*
30	18	00	24	*	39	*	*	*	*	*	*	*	*	*	*	24	18	*	27	*	*	*	*
*	42	24	00	*	30	*	42	*	*	*	*	*	*	*	*	*	42	*	*	*	*	*	*
18	18	*	*	00	*	12	*	18	*	*	*	*	*	*	*	12	24	21	*	*	*	*	*
*	*	39	30	*	00	*	12	*	21	*	*	*	*	*	*	33	21	*	30	*	*	*	*
30	*	*	*	12	*	00	*	6	*	12	*	*	*	*	*	24	*	15	27	21	*	*	*
*	*	*	42	*	12	*	00	*	9	*	15	*	*	*	*	*	30	33	21	*	30	*	*
*	*	*	*	18	*	6	*	00	*	6	*	12	*	*	*	*	*	21	*	15	27	21	*
*	*	*	*	*	21	*	9	*	00	*	6	*	*	*	12	*	*	*	30	33	21	*	27
*	*	*	*	*	*	12	*	6	*	00	*	6	18	*	*	*	*	*	*	21	*	15	27
*	*	*	*	*	*	*	15	*	6	*	00	*	*	24	6	*	*	*	*	*	27	33	21
*	*	*	*	*	*	*	*	12	*	6	*	00	12	24	*	*	*	*	*	*	*	15	*
*	*	*	*	*	*	*	*	*	*	18	*	12	00	12	33	*	*	*	*	9	*	3	15
*	*	*	*	*	*	*	*	*	*	*	24	24	12	00	21	*	*	*	*	*	15	15	3
*	*	*	*	*	*	*	*	*	12	*	6	*	33	21	00	*	*	*	*	*	*	*	24
18	6	24	*	12	33	24	*	*	*	*	*	*	*	*	*	00	12	9	21	18	*	*	*
*	18	18	42	24	21	*	30	*	*	*	*	*	*	*	*	12	00	21	9	*	18	*	*
*	15	*	*	21	*	15	33	21	*	*	*	*	*	*	*	9	21	00	12	9	21	15	*
*	*	27	*	*	30	27	21	*	30	*	*	*	*	*	*	21	9	12	00	21	9	*	21
*	*	*	*	*	*	21	*	15	33	21	*	*	9	*	*	18	*	9	21	00	12	6	18
*	*	*	*	*	*	*	30	27	21	*	27	*	*	15	*	*	18	21	9	12	00	18	12
*	*	*	*	*	*	*	*	21	*	15	33	15	3	15	*	*	*	15	*	6	18	00	12
*	*	*	*	*	*	*	*	*	27	27	21	*	15	3	24	*	*	*	21	18	12	12	00

Figure 4a.ii. This is the power 2 matrix, A^2.

00	12	30	54	18	*	30	*	36	*	*	*	*	*	*	*	18	30	27	*	*	*	*	*
12	00	18	42	18	39	30	*	*	*	*	*	*	*	*	*	6	18	15	27	24	*	*	*
30	18	00	24	36	39	*	48	*	*	*	*	*	*	*	*	24	18	33	27	*	36	*	*
54	42	24	00	*	30	*	42	*	51	*	*	*	*	*	*	48	42	*	51	*	*	*	*
18	18	36	*	00	45	12	*	18	*	24	*	*	*	*	*	12	24	21	33	30	*	*	*
*	39	39	30	45	00	*	12	*	21	*	27	*	*	*	*	33	21	42	30	*	39	*	*
30	30	*	*	12	*	00	48	6	*	12	*	18	*	*	*	24	36	15	27	21	33	27	*
*	*	48	42	*	12	48	00	*	9	*	15	*	*	*	21	42	30	33	21	42	30	*	36
36	*	*	*	18	*	6	*	00	48	6	*	12	24	*	*	30	*	21	33	15	27	21	33
*	*	*	51	*	21	*	9	48	00	*	6	*	*	30	12	*	39	42	30	33	21	39	27
*	*	*	*	24	*	12	*	6	*	00	48	6	18	30	*	*	*	27	*	21	33	15	27
*	*	*	*	*	27	*	15	*	6	48	00	*	36	24	6	*	*	*	36	39	27	33	21
*	*	*	*	*	*	18	*	12	*	6	*	00	12	24	45	*	*	*	*	21	*	15	27
*	*	*	*	*	*	*	*	24	*	18	36	12	00	12	33	*	*	18	*	9	21	3	15
*	*	*	*	*	*	*	*	*	30	30	24	24	12	00	21	*	*	*	24	21	15	15	3
*	*	*	*	*	*	*	21	*	12	*	6	45	33	21	00	*	*	*	*	*	33	36	24
18	6	24	48	12	33	24	42	30	*	*	*	*	*	*	*	00	12	9	21	18	30	24	*
30	18	18	42	24	21	36	30	*	39	*	*	*	*	*	*	12	00	21	9	30	18	*	30
27	15	33	*	21	42	15	33	21	42	27	*	*	18	*	*	9	21	00	12	9	21	15	27
*	27	27	51	33	30	27	21	33	30	*	36	*	*	24	*	21	9	12	00	21	9	27	21
*	24	*	*	30	*	21	42	15	33	21	39	21	9	21	*	18	30	9	21	00	12	6	18
*	*	36	*	*	39	33	30	27	21	*	27	*	21	15	33	30	18	21	9	12	00	18	12
*	*	*	*	*	*	27	*	21	39	15	33	15	3	15	36	24	*	15	27	6	18	00	12
*	*	*	*	*	*	*	36	33	27	27	21	27	15	3	24	*	30	27	21	18	12	12	00

Figure 4a.iii This is the power 3 matrix, A^3.

00	12	30	54	18	51	30	*	36	*	42	*	*	*	*	*	18	30	27	39	36	*	*	*
12	00	18	42	18	39	30	48	36	*	*	*	*	*	*	*	6	18	15	27	24	36	30	*
30	18	00	24	36	39	48	48	*	57	*	*	*	*	*	*	24	18	33	27	42	36	*	48
54	42	24	00	60	30	*	42	*	51	*	57	*	*	*	*	48	42	57	51	*	60	*	*
18	18	36	60	00	45	12	54	18	*	24	*	30	*	*	*	12	24	21	33	30	42	36	*
51	39	39	30	45	00	57	12	*	21	*	27	*	*	*	33	33	21	42	30	51	39	*	48
30	30	48	*	12	57	00	48	6	54	12	*	18	30	*	*	24	36	15	27	21	33	27	39
*	48	48	42	54	12	48	00	54	9	*	15	*	*	39	21	42	30	33	21	42	30	48	36
36	36	*	*	18	*	6	54	00	48	6	54	12	24	36	*	30	42	21	33	15	27	21	33
*	*	57	51	*	21	54	9	48	00	54	6	*	42	30	12	51	39	42	30	33	21	39	27
42	*	*	*	24	*	12	*	6	54	00	48	6	18	30	51	36	*	27	39	21	33	15	27
*	*	*	57	*	27	*	15	54	6	48	00	48	36	24	6	*	45	48	36	39	27	33	21
*	*	*	*	30	*	18	*	12	*	6	48	00	12	24	45	*	*	30	*	21	33	15	27
*	*	*	*	*	*	30	*	24	42	18	36	12	00	12	33	27	*	18	30	9	21	3	15
*	*	*	*	*	*	*	39	36	30	30	24	24	12	00	21	*	33	30	24	21	15	15	3
*	*	*	*	*	33	*	21	*	12	51	6	45	33	21	00	*	*	*	42	42	33	36	24
18	6	24	48	12	33	24	42	30	51	36	*	*	27	*	*	00	12	9	21	18	30	24	36
30	18	18	42	24	21	36	30	42	39	*	45	*	*	33	*	12	00	21	9	30	18	36	30
27	15	33	57	21	42	15	33	21	42	27	48	30	18	30	*	9	21	00	12	9	21	15	27
39	27	27	51	33	30	27	21	33	30	39	36	*	30	24	42	21	9	12	00	21	9	27	21
36	24	42	*	30	51	21	42	15	33	21	39	21	9	21	42	18	30	9	21	00	12	6	18
*	36	36	60	42	39	33	30	27	21	33	27	33	21	15	33	30	18	21	9	12	00	18	12
*	30	*	*	36	*	27	48	21	39	15	33	15	3	15	36	24	*	15	27	6	18	00	12
*	*	48	*	*	48	39	36	33	27	27	21	27	15	3	24	*	30	27	21	18	12	12	00

Figure 4a.iv This is the power 4 matrix, A^4.

00	12	30	54	18	51	30	60	36	*	42	*	48	*	*	*	18	30	27	39	36	48	42	*
12	00	18	42	18	39	30	48	36	57	42	*	*	33	*	*	6	18	15	27	24	36	30	42
30	18	00	24	36	39	48	48	54	57	*	63	*	*	51	*	24	18	33	27	42	36	48	48
54	42	24	00	60	30	72	42	*	51	*	57	*	*	*	63	48	42	57	51	66	60	*	72
18	18	36	60	00	45	12	54	18	63	24	*	30	39	*	*	12	24	21	33	30	42	36	48
51	39	39	30	45	00	57	12	63	21	*	27	*	*	51	33	33	21	42	30	51	39	57	48
30	30	48	72	12	57	00	48	6	54	12	60	18	30	42	*	24	36	15	27	21	33	27	39
60	48	48	42	54	12	48	00	54	9	60	15	*	51	39	21	42	30	33	21	42	30	48	36
36	36	54	*	18	63	6	54	00	48	6	54	12	24	36	57	30	42	21	33	15	27	21	33
*	57	57	51	63	21	54	9	48	00	54	6	54	42	30	12	51	39	42	30	33	21	39	27
42	42	*	*	24	*	12	60	6	54	00	48	6	18	30	51	36	48	27	39	21	33	15	27
*	*	63	57	*	27	60	15	54	6	48	00	48	36	24	6	57	45	48	36	39	27	33	21
48	*	*	*	30	*	18	*	12	54	6	48	00	12	24	45	39	*	30	*	21	33	15	27
*	33	*	*	39	*	30	51	24	42	18	36	12	00	12	33	27	39	18	30	9	21	3	15
*	*	51	*	*	51	42	39	36	30	30	24	24	12	00	21	39	33	30	24	21	15	15	3
*	*	*	63	*	33	*	21	57	12	51	6	45	33	21	00	*	51	51	42	42	33	36	24
18	6	24	48	12	33	24	42	30	51	36	57	39	27	39	*	00	12	9	21	18	30	24	36
30	18	18	42	24	21	36	30	42	39	48	45	*	39	33	51	12	00	21	9	30	18	36	30
27	15	33	57	21	42	15	33	21	42	27	48	30	18	30	51	9	21	00	12	9	21	15	27
39	27	27	51	33	30	27	21	33	30	39	36	42	30	24	42	21	9	12	00	21	9	27	21
36	24	42	66	30	51	21	42	15	33	21	39	21	9	21	42	18	30	9	21	00	12	6	18
48	36	36	60	42	39	33	30	27	21	33	27	33	21	15	33	30	18	21	9	12	00	18	12
42	30	48	*	36	57	27	48	21	39	15	33	15	3	15	36	24	36	15	27	6	18	00	12
*	42	48	72	48	48	39	36	33	27	27	21	27	15	3	24	36	30	27	21	18	12	12	00

Figure 4a.v This is the power 5 matrix, A^5.

00	12	30	54	18	51	30	60	36	69	42	*	48	45	*	*	18	30	27	39	36	48	42	54
12	00	18	42	18	39	30	48	36	57	42	63	45	33	45	*	6	18	15	27	24	36	30	42
30	18	00	24	36	39	48	48	54	57	60	63	*	51	51	69	24	18	33	27	42	36	48	48
54	42	24	00	60	30	72	42	78	51	*	57	*	*	75	63	48	42	57	51	66	60	72	72
18	18	36	60	00	45	12	54	18	63	24	69	30	39	51	*	12	24	21	33	30	42	36	48
51	39	39	30	45	00	57	12	63	21	69	27	*	60	51	33	33	21	42	30	51	39	57	48
30	30	48	72	12	57	00	48	6	54	12	60	18	30	42	63	24	36	15	27	21	33	27	39
60	48	48	42	54	12	48	00	54	9	60	15	63	51	39	21	42	30	33	21	42	30	48	36
36	36	54	78	18	63	6	54	00	48	6	54	12	24	36	57	30	42	21	33	15	27	21	33
69	57	57	51	63	21	54	9	48	00	54	6	54	42	30	12	51	39	42	30	33	21	39	27
42	42	60	*	24	69	12	60	6	54	00	48	6	18	30	51	36	48	27	39	21	33	15	27
*	63	63	57	69	27	60	15	54	6	48	00	48	36	24	6	57	45	48	36	39	27	33	21
48	45	*	*	30	*	18	63	12	54	6	48	00	12	24	45	39	51	30	42	21	33	15	27
45	33	51	*	39	60	30	51	24	42	18	36	12	00	12	33	27	39	18	30	9	21	3	15
*	45	51	75	51	51	42	39	36	30	30	24	24	12	00	21	39	33	30	24	21	15	15	3
*	*	69	63	*	33	63	21	57	12	51	6	45	33	21	00	60	51	51	42	42	33	36	24
18	6	24	48	12	33	24	42	30	51	36	57	39	27	39	60	00	12	9	21	18	30	24	36
30	18	18	42	24	21	36	30	42	39	48	45	51	39	33	51	12	00	21	9	30	18	36	30
27	15	33	57	21	42	15	33	21	42	27	48	30	18	30	51	9	21	00	12	9	21	15	27
39	27	27	51	33	30	27	21	33	30	39	36	42	30	24	42	21	9	12	00	21	9	27	21
36	24	42	66	30	51	21	42	15	33	21	39	21	9	21	42	18	30	9	21	00	12	6	18
48	36	36	60	42	39	33	30	27	21	33	27	33	21	15	33	30	18	21	9	12	00	18	12
42	30	48	72	36	57	27	48	21	39	15	33	15	3	15	36	24	36	15	27	6	18	00	12
54	42	48	72	48	48	39	36	33	27	27	21	27	15	3	24	36	30	27	21	18	12	12	00

Figure 4a.vi This is the power 6 matrix, A^6.

00	12	30	54	18	51	30	60	36	69	42	75	48	45	57	*	18	30	27	39	36	48	42	54
12	00	18	42	18	39	30	48	36	57	42	63	45	33	45	66	6	18	15	27	24	36	30	42
30	18	00	24	36	39	48	48	54	57	60	63	63	51	51	69	24	18	33	27	42	36	48	48
54	42	24	00	60	30	72	42	78	51	84	57	*	75	75	63	48	42	57	51	66	60	72	72
18	18	36	60	00	45	12	54	18	63	24	69	30	39	51	72	12	24	21	33	30	42	36	48
51	39	39	30	45	00	57	12	63	21	69	27	72	60	51	33	33	21	42	30	51	39	57	48
30	30	48	72	12	57	00	48	6	54	12	60	18	30	42	63	24	36	15	27	21	33	27	39
60	48	48	42	54	12	48	00	54	9	60	15	63	51	39	21	42	30	33	21	42	30	48	36
36	36	54	78	18	63	6	54	00	48	6	54	12	24	36	57	30	42	21	33	15	27	21	33
69	57	57	51	63	21	54	9	48	00	54	6	54	42	30	12	51	39	42	30	33	21	39	27
42	42	60	84	24	69	12	60	6	54	00	48	6	18	30	51	36	48	27	39	21	33	15	27
75	63	63	57	69	27	60	15	54	6	48	00	48	36	24	6	57	45	48	36	39	27	33	21
48	45	63	*	30	72	18	63	12	54	6	48	00	12	24	45	39	51	30	42	21	33	15	27
45	33	51	75	39	60	30	51	24	42	18	36	12	00	12	33	27	39	18	30	9	21	3	15
57	45	51	75	51	51	42	39	36	30	30	24	24	12	00	21	39	33	30	24	21	15	15	3
*	66	69	63	72	33	63	21	57	12	51	6	45	33	21	00	60	51	51	42	42	33	36	24
18	6	24	48	12	33	24	42	30	51	36	57	39	27	39	60	00	12	9	21	18	30	24	36
30	18	18	42	24	21	36	30	42	39	48	45	51	39	33	51	12	00	21	9	30	18	36	30
27	15	33	57	21	42	15	33	21	42	27	48	30	18	30	51	9	21	00	12	9	21	15	27
39	27	27	51	33	30	27	21	33	30	39	36	42	30	24	42	21	9	12	00	21	9	27	21
36	24	42	66	30	51	21	42	15	33	21	39	21	9	21	42	18	30	9	21	00	12	6	18
48	36	36	60	42	39	33	30	27	21	33	27	33	21	15	33	30	18	21	9	12	00	18	12
42	30	48	72	36	57	27	48	21	39	15	33	15	3	15	36	24	36	15	27	6	18	00	12
54	42	48	72	48	48	39	36	33	27	27	21	27	15	3	24	36	30	27	21	18	12	12	00

Figure 4a.vii This is the power 7 matrix, A^7.

00	12	30	54	18	51	30	60	36	69	42	75	48	45	57	78	18	30	27	39	36	48	42	54
12	00	18	42	18	39	30	48	36	57	42	63	45	33	45	66	6	18	15	27	24	36	30	42
30	18	00	24	36	39	48	48	54	57	60	63	63	51	51	69	24	18	33	27	42	36	48	48
54	42	24	00	60	30	72	42	78	51	84	57	87	75	75	63	48	42	57	51	66	60	72	72
18	18	36	60	00	45	12	54	18	63	24	69	30	39	51	72	12	24	21	33	30	42	36	48
51	39	39	30	45	00	57	12	63	21	69	27	72	60	51	33	33	21	42	30	51	39	57	48
30	30	48	72	12	57	00	48	6	54	12	60	18	30	42	63	24	36	15	27	21	33	27	39
60	48	48	42	54	12	48	00	54	9	60	15	63	51	39	21	42	30	33	21	42	30	48	36
36	36	54	78	18	63	6	54	00	48	6	54	12	24	36	57	30	42	21	33	15	27	21	33
69	57	57	51	63	21	54	9	48	00	54	6	54	42	30	12	51	39	42	30	33	21	39	27
42	42	60	84	24	69	12	60	6	54	00	48	6	18	30	51	36	48	27	39	21	33	15	27
75	63	63	57	69	27	60	15	54	6	48	00	48	36	24	6	57	45	48	36	39	27	33	21
48	45	63	87	30	72	18	63	12	54	6	48	00	12	24	45	39	51	30	42	21	33	15	27
45	33	51	75	39	60	30	51	24	42	18	36	12	00	12	33	27	39	18	30	9	21	3	15
57	45	51	75	51	51	42	39	36	30	30	24	24	12	00	21	39	33	30	24	21	15	15	3
78	66	69	63	72	33	63	21	57	12	51	6	45	33	21	00	60	51	51	42	42	33	36	24
18	6	24	48	12	33	24	42	30	51	36	57	39	27	39	60	00	12	9	21	18	30	24	36
30	18	18	42	24	21	36	30	42	39	48	45	51	39	33	51	12	00	21	9	30	18	36	30
27	15	33	57	21	42	15	33	21	42	27	48	30	18	30	51	9	21	00	12	9	21	15	27
39	27	27	51	33	30	27	21	33	30	39	36	42	30	24	42	21	9	12	00	21	9	27	21
36	24	42	66	30	51	21	42	15	33	21	39	21	9	21	42	18	30	9	21	00	12	6	18
48	36	36	60	42	39	33	30	27	21	33	27	33	21	15	33	30	18	21	9	12	00	18	12
42	30	48	72	36	57	27	48	21	39	15	33	15	3	15	36	24	36	15	27	6	18	00	12
54	42	48	72	48	48	39	36	33	27	27	21	27	15	3	24	36	30	27	21	18	12	12	00

Figure 4a.viii This is the power 8 matrix, A^8.

00	12	30	54	18	51	30	60	36	69	42	75	48	45	57	78	18	30	27	39	36	48	42	54
12	00	18	42	18	39	30	48	36	57	42	63	45	33	45	66	6	18	15	27	24	36	30	42
30	18	00	24	36	39	48	48	54	57	60	63	63	51	51	69	24	18	33	27	42	36	48	48
54	42	24	00	60	30	72	42	78	51	84	57	87	75	75	63	48	42	57	51	66	60	72	72
18	18	36	60	00	45	12	54	18	63	24	69	30	39	51	72	12	24	21	33	30	42	36	48
51	39	39	30	45	00	57	12	63	21	69	27	72	60	51	33	33	21	42	30	51	39	57	48
30	30	48	72	12	57	00	48	6	54	12	60	18	30	42	63	24	36	15	27	21	33	27	39
60	48	48	42	54	12	48	00	54	9	60	15	63	51	39	21	42	30	33	21	42	30	48	36
36	36	54	78	18	63	6	54	00	48	6	54	12	24	36	57	30	42	21	33	15	27	21	33
69	57	57	51	63	21	54	9	48	00	54	6	54	42	30	12	51	39	42	30	33	21	39	27
42	42	60	84	24	69	12	60	6	54	00	48	6	18	30	51	36	48	27	39	21	33	15	27
75	63	63	57	69	27	60	15	54	6	48	00	48	36	24	6	57	45	48	36	39	27	33	21
48	45	63	87	30	72	18	63	12	54	6	48	00	12	24	45	39	51	30	42	21	33	15	27
45	33	51	75	39	60	30	51	24	42	18	36	12	00	12	33	27	39	18	30	9	21	3	15
57	45	51	75	51	51	42	39	36	30	30	24	24	12	00	21	39	33	30	24	21	15	15	3
78	66	69	63	72	33	63	21	57	12	51	6	45	33	21	00	60	51	51	42	42	33	36	24
18	6	24	48	12	33	24	42	30	51	36	57	39	27	39	60	00	12	9	21	18	30	24	36
30	18	18	42	24	21	36	30	42	39	48	45	51	39	33	51	12	00	21	9	30	18	36	30
27	15	33	57	21	42	15	33	21	42	27	48	30	18	30	51	9	21	00	12	9	21	15	27
39	27	27	51	33	30	27	21	33	30	39	36	42	30	24	42	21	9	12	00	21	9	27	21
36	24	42	66	30	51	21	42	15	33	21	39	21	9	21	42	18	30	9	21	00	12	6	18
48	36	36	60	42	39	33	30	27	21	33	27	33	21	15	33	30	18	21	9	12	00	18	12
42	30	48	72	36	57	27	48	21	39	15	33	15	3	15	36	24	36	15	27	6	18	00	12
54	42	48	72	48	48	39	36	33	27	27	21	27	15	3	24	36	30	27	21	18	12	12	00

Figure 4a.ix This is the power 9 matrix, A^9. It is identical to the matrix in Figure 4a.viii and so the algorithm terminates.

Figures containing tables

00	12	*	*	18	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
12	00	18	*	*	*	*	*	*	*	*	*	*	*	*	*	6	*	*	*	*	*	*	*
*	18	00	24	*	*	*	*	*	*	*	*	*	*	*	*	*	18	*	*	*	*	*	*
*	*	24	00	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*
18	*	*	*	00	*	12	*	*	*	*	*	*	*	*	*	12	*	*	*	*	*	*	*
*	*	*	*	*	00	*	12	*	*	*	*	*	*	*	*	*	21	*	*	*	*	*	*
*	*	*	*	12	*	00	*	6	*	*	*	*	*	*	*	*	*	15	*	*	*	*	*
*	*	*	*	*	12	*	00	*	9	*	*	*	*	*	*	*	*	*	21	*	*	*	*
*	*	*	*	*	*	6	*	00	*	6	*	*	*	*	*	*	*	*	*	15	*	*	*
*	*	*	*	*	*	*	9	*	00	*	6	*	*	*	*	*	*	*	*	*	21	*	*
*	*	*	*	*	*	*	*	6	*	00	*	6	*	*	*	*	*	*	*	*	*	15	*
*	*	*	*	*	*	*	*	*	6	*	00	*	*	*	6	*	*	*	*	*	*	*	21
*	*	*	*	*	*	*	*	*	*	6	*	00	12	*	*	*	*	*	*	*	*	*	*
*	*	*	*	*	*	*	*	*	*	*	*	12	00	12	*	*	*	*	*	*	*	3	*
*	*	*	*	*	*	*	*	*	*	*	*	*	12	00	21	*	*	*	*	*	*	*	3
*	*	*	*	*	*	*	*	*	*	*	6	*	*	21	00	*	*	*	*	*	*	*	*
*	6	*	*	12	*	*	*	*	*	*	*	*	*	*	*	00	12	9	*	*	*	*	*
*	*	18	*	*	21	*	*	*	*	*	*	*	*	*	*	12	00	*	9	*	*	*	*
*	*	*	*	*	*	15	*	*	*	*	*	*	*	*	*	9	*	00	12	9	*	*	*
*	*	*	*	*	*	*	21	*	*	*	*	*	*	*	*	*	9	12	00	*	9	*	*
*	*	*	*	*	*	*	*	15	*	*	*	*	*	*	*	*	*	9	*	00	12	6	*
*	*	*	*	*	*	*	*	*	21	*	*	*	*	*	*	*	*	*	9	12	00	*	12
*	*	*	*	*	*	*	*	*	*	15	*	*	3	*	*	*	*	*	*	6	*	00	12
*	*	*	*	*	*	*	*	*	*	*	21	*	*	3	*	*	*	*	*	*	12	12	00

Figure 4b.i This is the initial matrix, B. Figure 4b contains a set of nine tables (i to ix) illustrating the use of Hasse's algorithm on the LA freeway system following the earthquake of January 17, 1994. Travel times are in one-quarter minutes. An asterisk indicates that the travel time between locations is too large to enter the matrix. A double-zero indicates an entry of 0.

00	12	30	*	18	*	30	*	*	*	*	*	*	*	*	*	18	*	*	*	*	*	*	*
12	00	18	42	18	*	*	*	*	*	*	*	*	*	*	*	6	18	15	*	*	*	*	*
30	18	00	24	*	39	*	*	*	*	*	*	*	*	*	*	24	18	*	27	*	*	*	*
*	42	24	00	*	*	*	*	*	*	*	*	*	*	*	*	*	42	*	*	*	*	*	*
18	18	*	*	00	*	12	*	18	*	*	*	*	*	*	*	12	24	21	*	*	*	*	*
*	*	39	*	*	00	*	12	*	21	*	*	*	*	*	*	33	21	*	30	*	*	*	*
30	*	*	*	12	*	00	*	6	*	12	*	*	*	*	*	24	*	15	27	21	*	*	*
*	*	*	*	*	12	*	00	*	9	*	15	*	*	*	*	*	30	33	21	*	30	*	*
*	*	*	*	18	*	6	*	00	*	6	*	12	*	*	*	*	*	21	*	15	27	21	*
*	*	*	*	*	21	*	9	*	00	*	6	*	*	*	12	*	*	*	30	33	21	*	27
*	*	*	*	*	*	12	*	6	*	00	*	6	18	*	*	*	*	*	*	21	*	15	27
*	*	*	*	*	*	*	15	*	6	*	00	*	*	24	6	*	*	*	*	*	27	33	21
*	*	*	*	*	*	*	*	12	*	6	*	00	12	24	*	*	*	*	*	*	*	15	*
*	*	*	*	*	*	*	*	*	*	18	*	12	00	12	33	*	*	*	*	9	*	3	15
*	*	*	*	*	*	*	*	*	*	*	24	24	12	00	21	*	*	*	*	*	15	15	3
*	*	*	*	*	*	*	*	*	12	*	6	*	33	21	00	*	*	*	*	*	*	*	24
18	6	24	*	12	33	24	*	*	*	*	*	*	*	*	*	00	12	9	21	18	*	*	*
*	18	18	42	24	21	*	30	*	*	*	*	*	*	*	*	12	00	21	9	*	18	*	*
*	15	*	*	21	*	15	33	21	*	*	*	*	*	*	*	9	21	00	12	9	21	15	*
*	*	27	*	*	30	27	21	*	30	*	*	*	*	*	*	21	9	12	00	21	9	*	21
*	*	*	*	*	*	21	*	15	33	21	*	*	9	*	*	18	*	9	21	00	12	6	18
*	*	*	*	*	*	*	30	27	21	*	27	*	*	15	*	*	18	21	9	12	00	18	12
*	*	*	*	*	*	*	*	21	*	15	33	15	3	15	*	*	*	15	*	6	18	00	12
*	*	*	*	*	*	*	*	*	27	27	21	*	15	3	24	*	*	*	21	18	12	12	00

Figure 4b.ii. This is the power 2 matrix, B^2.

00	12	30	54	18	*	30	*	36	*	*	*	*	*	*	*	18	30	27	*	*	*	*	*
12	00	18	42	18	39	30	*	*	*	*	*	*	*	*	*	6	18	15	27	24	*	*	*
30	18	00	24	36	39	*	48	*	*	*	*	*	*	*	*	24	18	33	27	*	36	*	*
54	42	24	00	*	63	*	*	*	*	*	*	*	*	*	*	48	42	*	51	*	*	*	*
18	18	36	*	00	45	12	*	18	*	24	*	*	*	*	*	12	24	21	33	30	*	*	*
*	39	39	63	45	00	*	12	*	21	*	27	*	*	*	*	33	21	42	30	*	39	*	*
30	30	*	*	12	*	00	48	6	*	12	*	18	*	*	*	24	36	15	27	21	33	27	*
*	*	48	*	*	12	48	00	*	9	*	15	*	*	*	21	42	30	33	21	42	30	*	36
36	*	*	*	18	*	6	*	00	48	6	*	12	24	*	*	30	*	21	33	15	27	21	33
*	*	*	*	*	21	*	9	48	00	*	6	*	*	30	12	*	39	42	30	33	21	39	27
*	*	*	*	24	*	12	*	6	*	00	48	6	18	30	*	*	*	27	*	21	33	15	27
*	*	*	*	*	27	*	15	*	6	48	00	*	36	24	6	*	*	*	36	39	27	33	21
*	*	*	*	*	*	18	*	12	*	6	*	00	12	24	45	*	*	*	*	21	*	15	27
*	*	*	*	*	*	*	*	24	*	18	36	12	00	12	33	*	*	18	*	9	21	3	15
*	*	*	*	*	*	*	*	*	30	30	24	24	12	00	21	*	*	*	24	21	15	15	3
*	*	*	*	*	*	*	21	*	12	*	6	45	33	21	00	*	*	*	*	*	33	36	24
18	6	24	48	12	33	24	42	30	*	*	*	*	*	*	*	00	12	9	21	18	30	24	*
30	18	18	42	24	21	36	30	*	39	*	*	*	*	*	*	12	00	21	9	30	18	*	30
27	15	33	*	21	42	15	33	21	42	27	*	*	18	*	*	9	21	00	12	9	21	15	27
*	27	27	51	33	30	27	21	33	30	*	36	*	*	24	*	21	9	12	00	21	9	27	21
*	24	*	*	30	*	21	42	15	33	21	39	21	9	21	*	18	30	9	21	00	12	6	18
*	*	36	*	*	39	33	30	27	21	*	27	*	21	15	33	30	18	21	9	12	00	18	12
*	*	*	*	*	*	27	*	21	39	15	33	15	3	15	36	24	*	15	27	6	18	00	12
*	*	*	*	*	*	*	36	33	27	27	21	27	15	3	24	*	30	27	21	18	12	12	00

Figure 4b.iii This is the power 3 matrix, B^3.

00	12	30	54	18	51	30	*	36	*	42	*	*	*	*	*	18	30	27	39	36	*	*	*
12	00	18	42	18	39	30	48	36	*	*	*	*	*	*	*	6	18	15	27	24	36	30	*
30	18	00	24	36	39	48	48	*	57	*	*	*	*	*	*	24	18	33	27	42	36	*	48
54	42	24	00	60	63	*	72	*	*	*	*	*	*	*	*	48	42	57	51	*	60	*	*
18	18	36	60	00	45	12	54	18	*	24	*	30	*	*	*	12	24	21	33	30	42	36	*
51	39	39	63	45	00	57	12	*	21	*	27	*	*	*	33	33	21	42	30	51	39	*	48
30	30	48	*	12	57	00	48	6	54	12	*	18	30	*	*	24	36	15	27	21	33	27	39
*	48	48	72	54	12	48	00	54	9	*	15	*	*	39	21	42	30	33	21	42	30	48	36
36	36	*	*	18	*	6	54	00	48	6	54	12	24	36	*	30	42	21	33	15	27	21	33
*	*	57	*	*	21	54	9	48	00	54	6	*	42	30	12	51	39	42	30	33	21	39	27
42	*	*	*	24	*	12	*	6	54	00	48	6	18	30	51	36	*	27	39	21	33	15	27
*	*	*	*	*	27	*	15	54	6	48	00	48	36	24	6	*	45	48	36	39	27	33	21
*	*	*	*	30	*	18	*	12	*	6	48	00	12	24	45	*	*	30	*	21	33	15	27
*	*	*	*	*	*	30	*	24	42	18	36	12	00	12	33	27	*	18	30	9	21	3	15
*	*	*	*	*	*	*	39	36	30	30	24	24	12	00	21	*	33	30	24	21	15	15	3
*	*	*	*	*	33	*	21	*	12	51	6	45	33	21	00	*	*	*	42	42	33	36	24
18	6	24	48	12	33	24	42	30	51	36	*	*	27	*	*	00	12	9	21	18	30	24	36
30	18	18	42	24	21	36	30	42	39	*	45	*	*	33	*	12	00	21	9	30	18	36	30
27	15	33	57	21	42	15	33	21	42	27	48	30	18	30	*	9	21	00	12	9	21	15	27
39	27	27	51	33	30	27	21	33	30	39	36	*	30	24	42	21	9	12	00	21	9	27	21
36	24	42	*	30	51	21	42	15	33	21	39	21	9	21	42	18	30	9	21	00	12	6	18
*	36	36	60	42	39	33	30	27	21	33	27	33	21	15	33	30	18	21	9	12	00	18	12
*	30	*	*	36	*	27	48	21	39	15	33	15	3	15	36	24	*	15	27	6	18	00	12
*	*	48	*	*	48	39	36	33	27	27	21	27	15	3	24	*	30	27	21	18	12	12	00

Figure 4b.iv This is the power 4 matrix, B^4.

00	12	30	54	18	51	30	60	36	*	42	*	48	*	*	*	18	30	27	39	36	48	42	*
12	00	18	42	18	39	30	48	36	57	42	*	*	33	*	*	6	18	15	27	24	36	30	42
30	18	00	24	36	39	48	48	54	57	*	63	*	*	51	*	24	18	33	27	42	36	48	48
54	42	24	00	60	63	72	72	*	81	*	*	*	*	*	*	48	42	57	51	66	60	*	72
18	18	36	60	00	45	12	54	18	63	24	*	30	39	*	*	12	24	21	33	30	42	36	48
51	39	39	63	45	00	57	12	63	21	*	27	*	*	51	33	33	21	42	30	51	39	57	48
30	30	48	72	12	57	00	48	6	54	12	60	18	30	42	*	24	36	15	27	21	33	27	39
60	48	48	72	54	12	48	00	54	9	60	15	*	51	39	21	42	30	33	21	42	30	48	36
36	36	54	*	18	63	6	54	00	48	6	54	12	24	36	57	30	42	21	33	15	27	21	33
*	57	57	81	63	21	54	9	48	00	54	6	54	42	30	12	51	39	42	30	33	21	39	27
42	42	*	*	24	*	12	60	6	54	00	48	6	18	30	51	36	48	27	39	21	33	15	27
*	*	63	*	*	27	60	15	54	6	48	00	48	36	24	6	57	45	48	36	39	27	33	21
48	*	*	*	30	*	18	*	12	54	6	48	00	12	24	45	39	*	30	*	21	33	15	27
*	33	*	*	39	*	30	51	24	42	18	36	12	00	12	33	27	39	18	30	9	21	3	15
*	*	51	*	*	51	42	39	36	30	30	24	24	12	00	21	39	33	30	24	21	15	15	3
*	*	*	*	*	33	*	21	57	12	51	6	45	33	21	00	*	51	51	42	42	33	36	24
18	6	24	48	12	33	24	42	30	51	36	57	39	27	39	*	00	12	9	21	18	30	24	36
30	18	18	42	24	21	36	30	42	39	48	45	*	39	33	51	12	00	21	9	30	18	36	30
27	15	33	57	21	42	15	33	21	42	27	48	30	18	30	51	9	21	00	12	9	21	15	27
39	27	27	51	33	30	27	21	33	30	39	36	42	30	24	42	21	9	12	00	21	9	27	21
36	24	42	66	30	51	21	42	15	33	21	39	21	9	21	42	18	30	9	21	00	12	6	18
48	36	36	60	42	39	33	30	27	21	33	27	33	21	15	33	30	18	21	9	12	00	18	12
42	30	48	*	36	57	27	48	21	39	15	33	15	3	15	36	24	36	15	27	6	18	00	12
*	42	48	72	48	48	39	36	33	27	27	21	27	15	3	24	36	30	27	21	18	12	12	00

Figure 4b.v This is the power 5 matrix, B^5.

00	12	30	54	18	51	30	60	36	69	42	*	48	45	*	*	18	30	27	39	36	48	42	54
12	00	18	42	18	39	30	48	36	57	42	63	45	33	45	*	6	18	15	27	24	36	30	42
30	18	00	24	36	39	48	48	54	57	60	63	*	51	51	69	24	18	33	27	42	36	48	48
54	42	24	00	60	63	72	72	78	81	*	87	*	*	75	*	48	42	57	51	66	60	72	72
18	18	36	60	00	45	12	54	18	63	24	69	30	39	51	*	12	24	21	33	30	42	36	48
51	39	39	63	45	00	57	12	63	21	69	27	*	60	51	33	33	21	42	30	51	39	57	48
30	30	48	72	12	57	00	48	6	54	12	60	18	30	42	63	24	36	15	27	21	33	27	39
60	48	48	72	54	12	48	00	54	9	60	15	63	51	39	21	42	30	33	21	42	30	48	36
36	36	54	78	18	63	6	54	00	48	6	54	12	24	36	57	30	42	21	33	15	27	21	33
69	57	57	81	63	21	54	9	48	00	54	6	54	42	30	12	51	39	42	30	33	21	39	27
42	42	60	*	24	69	12	60	6	54	00	48	6	18	30	51	36	48	27	39	21	33	15	27
*	63	63	87	69	27	60	15	54	6	48	00	48	36	24	6	57	45	48	36	39	27	33	21
48	45	*	*	30	*	18	63	12	54	6	48	00	12	24	45	39	51	30	42	21	33	15	27
45	33	51	*	39	60	30	51	24	42	18	36	12	00	12	33	27	39	18	30	9	21	3	15
*	45	51	75	51	51	42	39	36	30	30	24	24	12	00	21	39	33	30	24	21	15	15	3
*	*	69	*	*	33	63	21	57	12	51	6	45	33	21	00	60	51	51	42	42	33	36	24
18	6	24	48	12	33	24	42	30	51	36	57	39	27	39	60	00	12	9	21	18	30	24	36
30	18	18	42	24	21	36	30	42	39	48	45	51	39	33	51	12	00	21	9	30	18	36	30
27	15	33	57	21	42	15	33	21	42	27	48	30	18	30	51	9	21	00	12	9	21	15	27
39	27	27	51	33	30	27	21	33	30	39	36	42	30	24	42	21	9	12	00	21	9	27	21
36	24	42	66	30	51	21	42	15	33	21	39	21	9	21	42	18	30	9	21	00	12	6	18
48	36	36	60	42	39	33	30	27	21	33	27	33	21	15	33	30	18	21	9	12	00	18	12
42	30	48	72	36	57	27	48	21	39	15	33	15	3	15	36	24	36	15	27	6	18	00	12
54	42	48	72	48	48	39	36	33	27	27	21	27	15	3	24	36	30	27	21	18	12	12	00

Figure 4b.vi This is the power 6 matrix, B^6.

00	12	30	54	18	51	30	60	36	69	42	75	48	45	57	*	18	30	27	39	36	48	42	54
12	00	18	42	18	39	30	48	36	57	42	63	45	33	45	66	6	18	15	27	24	36	30	42
30	18	00	24	36	39	48	48	54	57	60	63	63	51	51	69	24	18	33	27	42	36	48	48
54	42	24	00	60	63	72	72	78	81	84	87	*	75	75	93	48	42	57	51	66	60	72	72
18	18	36	60	00	45	12	54	18	63	24	69	30	39	51	72	12	24	21	33	30	42	36	48
51	39	39	63	45	00	57	12	63	21	69	27	72	60	51	33	33	21	42	30	51	39	57	48
30	30	48	72	12	57	00	48	6	54	12	60	18	30	42	63	24	36	15	27	21	33	27	39
60	48	48	72	54	12	48	00	54	9	60	15	63	51	39	21	42	30	33	21	42	30	48	36
36	36	54	78	18	63	6	54	00	48	6	54	12	24	36	57	30	42	21	33	15	27	21	33
69	57	57	81	63	21	54	9	48	00	54	6	54	42	30	12	51	39	42	30	33	21	39	27
42	42	60	84	24	69	12	60	6	54	00	48	6	18	30	51	36	48	27	39	21	33	15	27
75	63	63	87	69	27	60	15	54	6	48	00	48	36	24	6	57	45	48	36	39	27	33	21
48	45	63	*	30	72	18	63	12	54	6	48	00	12	24	45	39	51	30	42	21	33	15	27
45	33	51	75	39	60	30	51	24	42	18	36	12	00	12	33	27	39	18	30	9	21	3	15
57	45	51	75	51	51	42	39	36	30	30	24	24	12	00	21	39	33	30	24	21	15	15	3
*	66	69	93	72	33	63	21	57	12	51	6	45	33	21	00	60	51	51	42	42	33	36	24
18	6	24	48	12	33	24	42	30	51	36	57	39	27	39	60	00	12	9	21	18	30	24	36
30	18	18	42	24	21	36	30	42	39	48	45	51	39	33	51	12	00	21	9	30	18	36	30
27	15	33	57	21	42	15	33	21	42	27	48	30	18	30	51	9	21	00	12	9	21	15	27
39	27	27	51	33	30	27	21	33	30	39	36	42	30	24	42	21	9	12	00	21	9	27	21
36	24	42	66	30	51	21	42	15	33	21	39	21	9	21	42	18	30	9	21	00	12	6	18
48	36	36	60	42	39	33	30	27	21	33	27	33	21	15	33	30	18	21	9	12	00	18	12
42	30	48	72	36	57	27	48	21	39	15	33	15	3	15	36	24	36	15	27	6	18	00	12
54	42	48	72	48	48	39	36	33	27	27	21	27	15	3	24	36	30	27	21	18	12	12	00

Figure 4b.vii This is the power 7 matrix, B^7.

00	12	30	54	18	51	30	60	36	69	42	75	48	45	57	78	18	30	27	39	36	48	42	54
12	00	18	42	18	39	30	48	36	57	42	63	45	33	45	66	6	18	15	27	24	36	30	42
30	18	00	24	36	39	48	48	54	57	60	63	63	51	51	69	24	18	33	27	42	36	48	48
54	42	24	00	60	63	72	72	78	81	84	87	87	75	75	93	48	42	57	51	66	60	72	72
18	18	36	60	00	45	12	54	18	63	24	69	30	39	51	72	12	24	21	33	30	42	36	48
51	39	39	63	45	00	57	12	63	21	69	27	72	60	51	33	33	21	42	30	51	39	57	48
30	30	48	72	12	57	00	48	6	54	12	60	18	30	42	63	24	36	15	27	21	33	27	39
60	48	48	72	54	12	48	00	54	9	60	15	63	51	39	21	42	30	33	21	42	30	48	36
36	36	54	78	18	63	6	54	00	48	6	54	12	24	36	57	30	42	21	33	15	27	21	33
69	57	57	81	63	21	54	9	48	00	54	6	54	42	30	12	51	39	42	30	33	21	39	27
42	42	60	84	24	69	12	60	6	54	00	48	6	18	30	51	36	48	27	39	21	33	15	27
75	63	63	87	69	27	60	15	54	6	48	00	48	36	24	6	57	45	48	36	39	27	33	21
48	45	63	87	30	72	18	63	12	54	6	48	00	12	24	45	39	51	30	42	21	33	15	27
45	33	51	75	39	60	30	51	24	42	18	36	12	00	12	33	27	39	18	30	9	21	3	15
57	45	51	75	51	51	42	39	36	30	30	24	24	12	00	21	39	33	30	24	21	15	15	3
78	66	69	93	72	33	63	21	57	12	51	6	45	33	21	00	60	51	51	42	42	33	36	24
18	6	24	48	12	33	24	42	30	51	36	57	39	27	39	60	00	12	9	21	18	30	24	36
30	18	18	42	24	21	36	30	42	39	48	45	51	39	33	51	12	00	21	9	30	18	36	30
27	15	33	57	21	42	15	33	21	42	27	48	30	18	30	51	9	21	00	12	9	21	15	27
39	27	27	51	33	30	27	21	33	30	39	36	42	30	24	42	21	9	12	00	21	9	27	21
36	24	42	66	30	51	21	42	15	33	21	39	21	9	21	42	18	30	9	21	00	12	6	18
48	36	36	60	42	39	33	30	27	21	33	27	33	21	15	33	30	18	21	9	12	00	18	12
42	30	48	72	36	57	27	48	21	39	15	33	15	3	15	36	24	36	15	27	6	18	00	12
54	42	48	72	48	48	39	36	33	27	27	21	27	15	3	24	36	30	27	21	18	12	12	00

Figure 4b.viii This is the power 8 matrix, B^8.

00	12	30	54	18	51	30	60	36	69	42	75	48	45	57	78	18	30	27	39	36	48	42	54
12	00	18	42	18	39	30	48	36	57	42	63	45	33	45	66	6	18	15	27	24	36	30	42
30	18	00	24	36	39	48	48	54	57	60	63	63	51	51	69	24	18	33	27	42	36	48	48
54	42	24	00	60	63	72	72	78	81	84	87	87	75	75	93	48	42	57	51	66	60	72	72
18	18	36	60	00	45	12	54	18	63	24	69	30	39	51	72	12	24	21	33	30	42	36	48
51	39	39	63	45	00	57	12	63	21	69	27	72	60	51	33	33	21	42	30	51	39	57	48
30	30	48	72	12	57	00	48	6	54	12	60	18	30	42	63	24	36	15	27	21	33	27	39
60	48	48	72	54	12	48	00	54	9	60	15	63	51	39	21	42	30	33	21	42	30	48	36
36	36	54	78	18	63	6	54	00	48	6	54	12	24	36	57	30	42	21	33	15	27	21	33
69	57	57	81	63	21	54	9	48	00	54	6	54	42	30	12	51	39	42	30	33	21	39	27
42	42	60	84	24	69	12	60	6	54	00	48	6	18	30	51	36	48	27	39	21	33	15	27
75	63	63	87	69	27	60	15	54	6	48	00	48	36	24	6	57	45	48	36	39	27	33	21
48	45	63	87	30	72	18	63	12	54	6	48	00	12	24	45	39	51	30	42	21	33	15	27
45	33	51	75	39	60	30	51	24	42	18	36	12	00	12	33	27	39	18	30	9	21	3	15
57	45	51	75	51	51	42	39	36	30	30	24	24	12	00	21	39	33	30	24	21	15	15	3
78	66	69	93	72	33	63	21	57	12	51	6	45	33	21	00	60	51	51	42	42	33	36	24
18	6	24	48	12	33	24	42	30	51	36	57	39	27	39	60	00	12	9	21	18	30	24	36
30	18	18	42	24	21	36	30	42	39	48	45	51	39	33	51	12	00	21	9	30	18	36	30
27	15	33	57	21	42	15	33	21	42	27	48	30	18	30	51	9	21	00	12	9	21	15	27
39	27	27	51	33	30	27	21	33	30	39	36	42	30	24	42	21	9	12	00	21	9	27	21
36	24	42	66	30	51	21	42	15	33	21	39	21	9	21	42	18	30	9	21	00	12	6	18
48	36	36	60	42	39	33	30	27	21	33	27	33	21	15	33	30	18	21	9	12	00	18	12
42	30	48	72	36	57	27	48	21	39	15	33	15	3	15	36	24	36	15	27	6	18	00	12
54	42	48	72	48	48	39	36	33	27	27	21	27	15	3	24	36	30	27	21	18	12	12	00

Figure 4b.ix This is the power 9 matrix, B^9. It is identical to the matrix in Figure 4b.viii and so the algorithm terminates.

Figure for Hasse algorithm in Pascal

```
program hasse(input,output);
const max=999999;
                n=24;
type hed=array[1..n,1..n] of integer;
var a:array[1..n] of hed;
        done:boolean;
        i,j,k,num:integer;
procedure print(matrix:hed);
begin
        for i:=i to n do
            begin
            for j:=1 to n do
                if matrix[i,j]=max then write (' *')
                    else write(matrix[i,j]:4);
            writeln
            end
end;
procedure hedsum(power, init:hed;var next:hed;var flag:boolean);
var row,col,min,middle,temp:integer;
begin
        flag:=true;
        for row:=1 to n do
        for col:=1 to n do
        begin
            min:=power[row,col];
            for middle:=1 to n do
            begin
                temp:=power[row,middle]+init[middle,col];
                if temp¡min the min:=temp
            end;
            next[row,col]:=min;
            if next[row,col]¡¿power[row,col] then flag:=false;
        end
end;
{main program}
begin
        for i:=1 to n do for j:=1 to n do a[1][i,j]:=max;
```

```
        for i:=1 to n do a[1][i,i]:=0;
        repeat
              readln(i,j,num);
              a[1][i,j]:=num;
              a[1][j,i]:=num;
        until eof;
        page; writeln('this is the initial matrix');writeln;
        print(a[1]);
        k:=0;
        repeat
              k:=k+1;
              hedsum(a[k],a[1],a[k+1],done);
              page; writeln('this is power',k+1:5); writeln;
              print(a[k+1]);
        until (done) or (k=n-1);
        writeln;
        writeln('the number of steps was', k:5)
end.
```

Figure 5. Computer program, written in Pascal, of W. C. Arlinghaus; originally presented on a poster by Arlinghaus, Arlinghaus, and Nystuen, "Elements of Geometric Routing Theory–II" Association of American Geographers, National Meetings, Toronto, Ontario, April 1990.

Figures containing tables

00	12	*	*	18	*	*	*	*	*	*	*	*	*	*	*
12	00	18	*	*	*	*	*	*	*	*	*	*	33	*	*
*	18	00	24	*	*	*	*	*	*	*	*	*	*	45	*
*	*	24	00	*	30	*	*	*	*	*	*	*	*	*	*
18	*	*	*	00	42	12	*	*	*	*	*	*	*	*	*
*	*	*	30	42	00	*	12	*	*	*	*	*	*	*	*
*	*	*	*	12	*	00	45	6	*	*	*	*	*	*	*
*	*	*	*	*	12	45	00	*	9	*	*	*	*	*	*
*	*	*	*	*	*	6	*	00	45	6	*	*	*	*	*
*	*	*	*	*	*	*	9	45	00	*	6	*	*	*	*
*	*	*	*	*	*	*	*	6	*	00	45	6	*	*	*
*	*	*	*	*	*	*	*	*	6	45	00	*	*	*	6
*	*	*	*	*	*	*	*	*	*	6	*	00	12	*	*
*	33	*	*	*	*	*	*	*	*	*	*	12	00	12	*
*	*	45	*	*	*	*	*	*	*	*	*	*	12	00	21
*	*	*	*	*	*	*	*	*	*	*	6	*	*	21	00

Figure 7a.i This is the initial matrix, C. Figure 7a contains a set of seven tables (i to vii) illustrating the use of Hasse's algorithm on the LA freeway system and the limited access surface route network (Figure 6) prior to the earthquake of January 17, 1994. Travel times are in one-quarter minutes. An asterisk indicates that the travel time between locations is too large to enter the matrix. A double-zero indicates an entry of 0.

00	12	30	*	18	60	30	*	*	*	*	*	*	45	*	*
12	00	18	42	30	*	*	*	*	*	*	*	45	33	45	*
30	18	00	24	*	54	*	*	*	*	*	*	*	51	45	66
*	42	24	00	72	30	*	36	*	*	*	*	*	*	69	*
18	30	*	72	00	42	12	48	18	*	*	*	*	*	*	*
60	*	54	30	42	00	51	6	*	15	*	*	*	*	*	*
30	*	*	*	12	51	00	45	6	51	12	*	*	*	*	*
*	*	*	36	48	6	45	00	51	9	*	15	*	*	*	*
*	*	*	*	18	*	6	51	00	45	6	51	12	*	*	*
*	*	*	*	*	15	51	9	45	00	51	6	*	*	*	12
*	*	*	*	*	*	12	*	6	51	00	45	6	18	*	51
*	*	*	*	*	*	*	15	51	6	45	00	51	*	24	6
*	45	*	*	*	*	*	*	12	*	6	51	00	12	24	*
45	33	51	*	*	*	*	*	*	*	18	*	12	00	12	33
*	45	45	69	*	*	*	*	*	*	*	27	24	12	00	21
*	*	66	*	*	*	*	*	*	12	51	6	*	33	21	00

Figure 7a.ii. This is the power 2 matrix, C^2.

00	12	30	54	18	60	30	66	36	*	*	*	57	45	57	*
12	00	18	42	30	72	42	*	*	*	51	*	45	33	45	66
30	18	00	24	48	54	*	60	*	*	*	72	63	51	45	66
54	42	24	00	72	30	81	36	*	45	*	*	*	75	69	90
18	30	48	72	00	42	12	48	18	57	24	*	*	63	*	*
60	72	54	30	42	00	51	6	57	15	*	21	*	*	99	*
30	42	*	81	12	51	00	45	6	51	12	57	18	*	*	*
66	*	60	36	48	6	45	00	51	9	57	15	*	*	*	21
36	*	*	*	18	57	15	51	00	45	6	51	12	24	*	57
*	*	*	45	57	15	51	9	45	00	51	6	57	*	33	12
*	51	*	*	24	*	12	57	6	51	00	45	6	18	30	51
*	*	72	*	*	21	57	15	51	6	45	00	51	39	27	6
57	45	63	*	*	*	18	*	12	57	6	51	00	12	24	45
45	33	51	75	63	*	*	*	24	*	18	39	12	00	12	33
57	45	45	69	*	99	*	*	*	33	30	27	24	12	00	21
*	66	66	90	*	*	*	21	57	12	51	6	45	33	21	00

Figure 7a.iii This is the power 3 matrix, C^3.

00	12	30	54	18	60	30	66	36	75	42	*	57	45	57	78
12	00	18	42	30	72	42	78	48	*	51	72	45	33	45	66
30	18	00	24	48	54	60	60	*	69	69	72	63	51	45	66
54	42	24	00	72	30	81	36	87	45	*	51	87	75	69	90
18	30	48	72	00	42	12	48	18	57	24	63	30	63	75	*
60	72	54	30	42	00	51	6	57	15	63	21	*	10599		27
30	42	60	81	12	51	00	45	6	51	12	57	18	30	*	63
66	78	60	36	48	6	45	00	51	9	57	15	63	*	42	21
36	48	*	87	18	57	15	51	00	45	6	51	12	24	36	57
75	*	69	45	57	15	51	9	45	00	51	6	57	45	33	12
42	51	69	*	24	63	12	57	6	51	00	45	6	18	30	51
*	72	72	51	63	21	57	15	51	6	45	00	51	39	27	6
57	45	63	87	30	*	18	63	12	57	6	51	00	12	24	45
45	33	51	75	63	10530		*	24	45	18	39	12	00	12	33
57	45	45	69	75	99	*	42	36	33	30	27	24	12	00	21
78	66	66	90	*	27	63	21	57	12	51	6	45	33	21	00

Figure 7a.iv This is the power 4 matrix, C^4.

00	12	30	54	18	60	30	66	36	75	42	81	48	45	57	78
12	00	18	42	30	72	42	78	48	78	51	72	45	33	45	66
30	18	00	24	48	54	60	60	66	69	69	72	63	51	45	66
54	42	24	00	72	30	81	36	87	45	93	51	87	75	69	57
18	30	48	72	00	42	12	48	18	57	24	63	30	63	75	69
60	72	54	30	42	00	51	6	57	15	63	21	69	105	48	27
30	42	60	81	12	51	00	45	6	51	12	57	18	30	42	63
66	78	60	36	48	6	45	00	51	9	57	15	63	54	42	21
36	48	66	87	18	57	15	51	00	45	6	51	12	24	36	57
75	78	69	45	57	15	51	9	45	00	51	6	57	45	33	12
42	51	69	93	24	63	12	57	6	51	00	45	6	18	30	51
81	72	72	51	63	21	57	15	51	6	45	00	51	39	27	6
48	45	63	87	30	69	18	63	12	57	6	51	00	12	24	45
45	33	51	75	63	105	30	54	24	45	18	39	12	00	12	33
57	45	45	69	75	48	42	42	36	33	30	27	24	12	00	21
78	66	66	57	69	27	63	21	57	12	51	6	45	33	21	00

Figure 7a.v This is the power 5 matrix, C^5.

00	12	30	54	18	60	30	66	36	75	42	81	48	45	57	78
12	00	18	42	30	72	42	78	48	78	51	72	45	33	45	66
30	18	00	24	48	54	60	60	66	69	69	72	63	51	45	66
54	42	24	00	72	30	81	36	87	45	93	51	87	75	69	57
18	30	48	72	00	42	12	48	18	57	24	63	30	42	54	69
60	72	54	30	42	00	51	6	57	15	63	21	69	60	48	27
30	42	60	81	12	51	00	45	6	51	12	57	18	30	42	63
66	78	60	36	48	6	45	00	51	9	57	15	63	54	42	21
36	48	66	87	18	57	6	51	00	45	6	51	12	24	36	57
75	78	69	45	57	15	51	9	45	00	51	6	57	45	33	12
42	51	69	93	24	63	12	57	6	51	00	45	6	18	30	51
81	72	72	51	63	21	57	15	51	6	45	00	51	39	27	6
48	45	63	87	30	69	18	63	12	57	6	51	00	12	24	45
45	33	51	75	42	60	30	54	24	45	18	39	12	00	12	33
57	45	45	69	54	48	42	42	36	33	30	27	24	12	00	21
78	66	66	57	69	27	63	21	57	12	51	6	45	33	21	00

Figure 7a.vi This is the power 6 matrix, C^6.

00	12	30	54	18	60	30	66	36	75	42	81	48	45	57	78
12	00	18	42	30	72	42	78	48	78	51	72	45	33	45	66
30	18	00	24	48	54	60	60	66	69	69	72	63	51	45	66
54	42	24	00	72	30	81	36	87	45	93	51	87	75	69	57
18	30	48	72	00	42	12	48	18	57	24	63	30	42	54	69
60	72	54	30	42	00	51	6	57	15	63	21	69	60	48	27
30	42	60	81	12	51	00	45	6	51	12	57	18	30	42	63
66	78	60	36	48	6	45	00	51	9	57	15	63	54	42	21
36	48	66	87	18	57	6	51	00	45	6	51	12	24	36	57
75	78	69	45	57	15	51	9	45	00	51	6	57	45	33	12
42	51	69	93	24	63	12	57	6	51	00	45	6	18	30	51
81	72	72	51	63	21	57	15	51	6	45	00	51	39	27	6
48	45	63	87	30	69	18	63	12	57	6	51	00	12	24	45
45	33	51	75	42	60	30	54	24	45	18	39	12	00	12	33
57	45	45	69	54	48	42	42	36	33	30	27	24	12	00	21
78	66	66	57	69	27	63	21	57	12	51	6	45	33	21	00

Figure 7a.vii This is the power 7 matrix, C^7. The algorithm terminates in six steps; this matrix is identical to A^6.

Figures containing tables

00	12	*	*	18	*	*	*	*	*	*	*	*	*	*	*
12	00	18	*	*	*	*	*	*	*	*	*	*	33	*	*
*	18	00	24	*	*	*	*	*	*	*	*	*	*	45	*
*	*	24	00	*	*	*	*	*	*	*	*	*	*	*	*
18	*	*	*	00	42	12	*	*	*	*	*	*	*	*	*
*	*	*	*	42	00	*	6	*	*	*	*	*	*	*	*
*	*	*	*	12	*	00	45	6	*	*	*	*	*	*	*
*	*	*	*	*	6	45	00	*	9	*	*	*	*	*	*
*	*	*	*	*	*	6	*	00	45	6	*	*	*	*	*
*	*	*	*	*	*	*	9	45	00	*	6	*	*	*	*
*	*	*	*	*	*	*	*	6	*	00	45	6	*	*	*
*	*	*	*	*	*	*	*	*	6	45	00	*	*	*	6
*	*	*	*	*	*	*	*	*	*	6	*	00	12	*	*
*	33	*	*	*	*	*	*	*	*	*	*	12	00	12	*
*	*	45	*	*	*	*	*	*	*	*	*	*	12	00	21
*	*	*	*	*	*	*	*	*	*	*	6	*	*	21	00

Figure 7b.i This is the initial matrix, *D*. Figure 7b contains a set of seven tables (i to vii) illustrating the use of Hasse's algorithm on the LA freeway system and the limited access surface route network (Figure 6) following the earthquake of January 17, 1994. Travel times are in one-quarter minutes. An asterisk indicates that the travel time between locations is too large to enter the matrix. A double-zero indicates an entry of 0.

00	12	30	*	18	60	30	*	*	*	*	*	*	45	*	*
12	00	18	42	30	*	*	*	*	*	*	*	45	33	45	*
30	18	00	24	*	*	*	*	*	*	*	*	*	51	45	66
*	42	24	00	*	*	*	*	*	*	*	*	*	*	69	*
18	30	*	*	00	42	12	48	18	*	*	*	*	*	*	*
60	*	*	*	42	00	51	6	*	15	*	*	*	*	*	*
30	*	*	*	12	51	00	45	6	51	12	*	*	*	*	*
*	*	*	*	48	6	45	00	51	9	*	15	*	*	*	*
*	*	*	*	18	*	6	51	00	45	6	51	12	*	*	*
*	*	*	*	*	15	51	9	45	00	51	6	*	*	*	12
*	*	*	*	*	*	12	*	6	51	00	45	6	18	*	51
*	*	*	*	*	*	*	15	51	6	45	00	51	*	27	6
*	45	*	*	*	*	*	*	12	*	6	51	00	12	24	*
45	33	51	*	*	*	*	*	*	*	18	*	12	00	12	33
*	45	45	69	*	*	*	*	*	*	*	27	24	12	00	21
*	*	66	*	*	*	*	*	*	12	51	6	*	33	21	00

Figure 7b.ii. This is the power 2 matrix, D^2.

00	12	30	54	18	60	30	66	36	*	*	*	57	45	57	*
12	00	18	42	30	72	42	*	*	*	51	*	45	33	45	66
30	18	00	24	48	*	*	*	*	*	*	72	63	51	45	66
54	42	24	00	*	*	*	*	*	*	*	*	*	75	69	90
18	30	48	*	00	42	12	48	18	57	24	*	*	63	*	*
60	72	*	*	42	00	51	6	57	15	*	21	*	*	*	*
30	42	*	*	12	51	00	45	6	51	12	57	18	*	*	*
66	*	*	*	48	6	45	00	51	9	57	15	*	*	*	21
36	*	*	*	18	57	6	51	00	45	6	51	12	24	*	57
*	*	*	*	57	15	51	9	45	00	51	6	57	*	33	12
*	51	*	*	24	*	12	57	6	51	00	45	6	18	30	51
*	*	72	*	*	21	57	15	51	6	45	00	51	39	27	6
57	45	63	*	*	*	18	*	12	57	6	51	00	12	24	45
45	33	51	75	63	*	*	*	24	*	18	39	12	00	12	33
57	45	45	69	*	*	*	*	*	33	30	27	24	12	00	21
*	66	66	90	*	*	*	21	57	12	51	6	45	33	21	00

Figure 7b.iii This is the power 3 matrix, D^3.

00	12	30	54	18	60	30	66	36	75	42	*	57	45	57	78
12	00	18	42	30	72	42	78	48	*	51	72	45	33	45	66
30	18	00	24	48	90	60	*	*	78	69	72	63	51	45	66
54	42	24	00	72	*	*	*	*	*	*	96	87	75	69	90
18	30	48	72	00	42	12	48	18	57	24	63	30	63	75	*
60	72	90	*	42	00	51	6	57	15	63	21	*	105	*	27
30	42	60	*	12	51	00	45	6	51	12	57	18	30	*	63
66	78	*	*	48	6	45	00	51	9	57	15	63	*	42	21
36	48	*	*	18	57	6	51	00	45	6	51	12	24	36	57
75	*	78	*	57	15	51	9	45	00	51	6	57	45	33	12
42	51	69	*	24	63	12	57	6	51	00	45	6	18	30	51
*	72	72	96	63	21	57	15	51	6	45	00	51	39	27	6
57	45	63	87	30	*	18	63	12	57	6	51	00	12	24	45
45	33	51	75	63	105	30	*	24	45	18	39	12	00	12	33
57	45	45	69	75	*	*	42	36	33	30	27	24	12	00	21
78	66	66	90	*	27	63	21	57	12	51	6	45	33	21	00

Figure 7b.iv This is the power 4 matrix. D^4.

00	12	30	54	18	60	30	66	36	75	42	81	48	45	57	78
12	00	18	42	30	72	42	78	48	78	51	72	45	33	45	66
30	18	00	24	48	90	60	87	66	78	69	72	63	51	45	66
54	42	24	00	72	114	84	*	*	102	93	96	87	75	69	90
18	30	48	72	00	42	12	48	18	57	24	63	30	42	75	69
60	72	90	114	42	00	51	6	57	15	63	21	69	105	48	27
30	42	60	84	12	51	00	45	6	51	12	57	18	30	42	63
66	78	87	*	48	6	45	00	51	9	57	15	63	54	42	21
36	48	66	*	18	57	6	51	00	45	6	51	12	24	36	57
75	78	78	102	57	15	51	9	45	00	51	6	57	45	33	12
42	51	69	93	24	63	12	57	6	51	00	45	6	18	30	51
81	72	72	96	63	21	57	15	51	6	45	00	51	39	27	6
48	45	63	87	30	69	18	63	12	57	6	51	00	12	24	45
45	33	51	75	42	105	30	54	24	45	18	39	12	00	12	33
57	45	45	69	75	48	42	42	36	33	30	27	24	12	00	21
78	66	66	90	69	27	63	21	57	12	51	6	45	33	21	00

Figure 7b.v This is the power 5 matrix, D^5.

00	12	30	54	18	60	30	66	36	75	42	81	48	45	57	78
12	00	18	42	30	72	42	78	48	78	51	72	45	33	45	66
30	18	00	24	48	90	60	87	66	78	69	72	63	51	45	66
54	42	24	00	72	11484		11190		102	93	96	87	75	69	90
18	30	48	72	00	42	12	48	18	57	24	63	30	42	54	69
60	72	90	11442		00	51	6	57	15	63	21	69	60	48	27
30	42	60	84	12	51	00	45	6	51	12	57	18	30	42	63
66	78	87	11148		6	45	00	51	9	57	15	63	54	42	21
36	48	66	90	18	57	6	51	00	45	6	51	12	24	36	57
75	78	78	10257		15	51	9	45	00	51	6	57	45	33	12
42	51	69	93	24	63	12	57	6	51	00	45	6	18	30	51
81	72	72	96	63	21	57	15	51	6	45	00	51	39	27	6
48	45	63	87	30	69	18	63	12	57	6	51	00	12	24	45
45	33	51	75	42	60	30	54	24	45	18	39	12	00	12	33
57	45	45	69	54	48	42	42	36	33	30	27	24	12	00	21
78	66	66	90	69	27	63	21	57	12	51	6	45	33	21	00

Figure 7b.vi This is the power 6 matrix, D^6.

00 12 30 54 18 60 30 66 36 75 42 81 48 45 57 78
12 00 18 42 30 72 42 78 48 78 51 72 45 33 45 66
30 18 00 24 48 90 60 87 66 78 69 72 63 51 45 66
54 42 24 00 72 11484 11190 102 93 96 87 75 69 90
18 30 48 72 00 42 12 48 18 57 24 63 30 42 54 69
60 72 90 11442 00 51 6 57 15 63 21 69 60 48 27
30 42 60 84 12 51 00 45 6 51 12 57 18 30 42 63
66 78 87 11148 6 45 00 51 9 57 15 63 54 42 21
36 48 66 90 18 57 6 51 00 45 6 51 12 24 36 57
75 78 78 10257 15 51 9 45 00 51 6 57 45 33 12
42 51 69 93 24 63 12 57 6 51 00 45 6 18 30 51
81 72 72 96 63 21 57 15 51 6 45 00 51 39 27 6
48 45 63 87 30 69 18 63 12 57 6 51 00 12 24 45
45 33 51 75 42 60 30 54 24 45 18 39 12 00 12 33
57 45 45 69 54 48 42 42 36 33 30 27 24 12 00 21
78 66 66 90 69 27 63 21 57 12 51 6 45 33 21 00

Figure 7b.vii This is the power 7 matrix, D^7. The iteration terminates after 6 steps; this matrix is identical to D^6.

5. SAMPLE OF HOW TO DOWNLOAD THE ELECTRONIC FILE BACK ISSUES OF *SOLSTICE* ON A GOPHER

Solstice is available on a GOPHER from the Department of Mathematics at Arizona State University: PI.LA.ASU.EDU port 70

BACK ISSUES OF *SOLSTICE* AVAILABLE ON FTP

This section shows the exact set of commands that work to download *Solstice* on The University of Michigan's Xerox 9700. Because different universities will have different installations of TeX, this is only a rough guideline which *might* be of use to the reader. (BACK ISSUES AVAILABLE using anonymous ftp to open um.cc.umich.edu, account GCFS; type cd GCFS after entering system; then type ls to get a directory; then type get solstice.190 (for example) and download it or read it according to local constraints.) Back issues will be available on this account; this account is ONLY for back issues; to write Solstice, send e-mail to Solstice@UMICHUM.bitnet or to Solstice@um.cc.umich.edu . Issues from this one forward are available on FTP on account IEVG (substitute IEVG for GCFS above).

First step is to concatenate the files you received via bitnet/internet. Simply piece them together in your computer, one after another, in the order in which they are numbered, starting with the number, "1."

The files you have received are ASCII files; the concatenated file is used to form the .tex file from which the .dvi file (device independent) file is formed. They should run, possibly with a few harmless "vboxes" over or under. ASSUME YOU HAVE SIGNED ON AND ARE AT THE SYSTEM PROMPT, #.

create -t.tex

percent-sign t from pc c:backslash words backslash solstice.tex to mts -t.tex char notab

(this command sends my file, solstice.tex, which I did as a WordStar (subdirectory, "words") ASCII file to the mainframe)

run *tex par=-t.tex

(there may be some underfull (or certain over) boxes that generally cause no problem; there should be no other "error" messages in the typesetting–the files you receive were already tested.)

run *dvixer par=-t.dvi

control *print* onesided

run *pagepr scards=-t.xer, par=paper=plain

SOLSTICE:

AN ELECTRONIC JOURNAL OF GEOGRAPHY AND MATHEMATICS

WINTER, 1994

Volume V, Number 2
Institute of Mathematical Geography
Ann Arbor, Michigan

The purpose of *Solstice* is to promote interaction between geography and mathematics. Articles in which elements of one discipline are used to shed light on the other are particularly sought. Also welcome, are original contributions that are purely geographical or purely mathematical. These may be prefaced (by editor or author) with commentary suggesting directions that might lead toward the desired interaction. Individuals wishing to submit articles, either short or full- length, as well as contributions for regular features, should send them, in triplicate, directly to the Editor-in-Chief. Contributed articles will be refereed by geographers and/or mathematicians. Invited articles will be screened by suitable members of the editorial board. IMaGe is open to having authors suggest, and furnish material for, new regular features.

The opinions expressed are those of the authors, alone, and the authors alone are responsible for the accuracy of the facts in the articles.

Send all correspondence to: Sandra Arlinghaus, Institute of Mathematical Geography, 2790 Briarcliff, Ann Arbor MI 48105. sarhaus@umich.edu

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Concern for manuscript security. Prospective authors sometimes worry, quite reasonably, that an article published in electronic format might easily be altered by a subscriber and then re-sent to others. Of course, the same sorts of concern might arise with conventional publishing, given the availability of machines that seem able to photocopy, cut, paste, and bind, all in one step. The strategy at IMaGe, with *Solstice* is roughly the following: we use an oriental rug as the model. A *Solstice* document is viewed as a weaving of words against the warp and the woof of columns and rows in an electronic matrix. The spacing of the document, though right- justified, is done by hand (rather than by word-processor). In that way, an original pattern of blank spaces is created that is difficult to mimic - an important feature in defeating any significant alteration. Indeed, deliberate spacing "errors" that hopefully do not detract from the overall visual effect, are introduced at the whim of the Editor. Any re-transmitted document that does not match the document originally transmitted from IMaGe (when the two copies are superimposed) is therefore an altered, bogus copy.

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Unless otherwise noted, all regular "features" are written by the Editor–in–Chief.

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7. INDEX to Volumes I (1990), II (1991), III (1992), IV (1993) and V (1994, part 1) of *Solstice*.

8. OTHER PUBLICATIONS OF IMaGe All transmitted as part 9 of 9.

1. WELCOME TO NEW READERS AND THANK YOU

Welcome to new subscribers! We hope you enjoy participating in this means of journal distribution. Instructions for downloading the typesetting have been repeated in this issue, near the end. They are specific to the TeX installation at The University of Michigan, but apparently they have been helpful in suggesting to others the sorts of commands that might be used on their own particular mainframe installation of TeX. New subscribers might wish to note that the electronic files are typeset files—the mathematical notation will print out as typeset notation. For example,

$$\Sigma_{i=1}^{n}$$

when properly downloaded, will print out a typeset summation as i goes from one to n, as a centered display on the page. Complex notation is no barrier to this form of journal production.

Thanks much to subscribers who have offered input. Helpful suggestions are important in trying to keep abreast, at least somewhat, of the constantly changing electronic world. Some suggestions from readers have already been implemented; others are being worked on. Indeed, it is particularly helpful when the reader making the suggestion becomes actively involved in carrying it out. We hope you continue to enjoy *Solstice*.

2. PRESS CLIPPINGS—SUMMARY

Volume 72, Number 4, October 1993 issue of *Papers in Regional Science: The Journal of the Regional Science Association* carried an article by Gunther Maier and Andreas Wildberger entitled "Wide Area Computer Networks and Scholarly Communication in Regional Science." Maier and Wildberger noted that "Only one journal in this directory can be considered to be related to Regional Science, *Solstice: An Electronic Journal of Geography and Mathematics*."

Beyond that, brief write-ups about *Solstice* have appeared in the following publications:

1. **Science**, "Online Journals" Briefings. [by Joseph Palca] 29 November 1991. Vol. 254.

2. **Science News**, "Math for all seasons" by Ivars Peterson, January 25, 1992, Vol. 141, No. 4.

3. **Newsletter of the Association of American Geographers**, June, 1992.

4. **American Mathematical Monthly**, "Telegraphic Reviews" — mentioned as "one of the World's first electronic journals using TeX," September, 1992.

5. **Harvard Technology Window**, 1993.

6. **Graduating Engineering Magazine**, 1993.

7. **Earth Surface Processes and Landforms**, 18(9), 1993, p. 874.

8. **On Internet**, 1994.

If you have read about *Solstice* elsewhere, please let us know the correct citations (and add to those above). Thanks. We are happy to share information with all and are delighted when others share with us, as well.

Publications of the Institute of Mathematical Geography have, in addition, been reviewed or noted in

1. *The Professional Geographer* published by the Association of American Geographers;

2. The *Urban Specialty Group Newsletter* of the Association of American Geographers;

3. *Mathematical Reviews* published by the American Mathematical Society;

4. *The American Mathematical Monthly* published by the Mathematical Association of America;

5. *Zentralblatt* fur Mathematik, Springer-Verlag, Berlin

6. *Mathematics Magazine*, published by the Mathematical Association of America.

7. *Newsletter* of the Association of American Geographer.

8. *Journal of The Regional Science Association.*

9. *Journal of the American Statistical Association.*

3. ARTICLES

The Paris Metro: Is Its Graph Planar?

Sandra L. Arlinghaus, William C. Arlinghaus, and Frank Harary
The University of Michigan,
Lawrence Technological University, and
New Mexico State University.

"Over the river and through the woods,
To Grandmother's house we go.
The horse knows the way to carry the sleigh
Through the white and drifting snow."

Song of unknown origin

To appear in *Structural Models in Geography* by this set of authors.

The reader should read this article with a map of the Paris Metro in hand.

In the Euclidean plane, crossing lines intersect at a point in the plane; the line segment determined by X, Y and X', Y' intersect at a point Z (Figure 1). The graph that includes the four nodes X, Y, X', Y' and the two edges XY' and $X'Y$ does *not* have a fifth node at any other location (Figure 2). To make this viewpoint consistent with our narrow Euclidean mindset, think of stretching the edge $X'Y$ so that there is no visual hint of "intersection" – the horse knows to go over the river even though an aerial view of the wintry landscape sees the river and road as two "intersecting" dark tracings across the white, snowy backdrop.

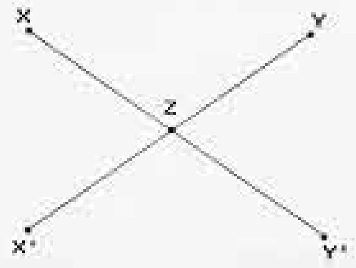

Figure 1. Draw nodes X and Y, left to right, horizontally. Draw nodes X' and Y', left to right, horizontally below the first set. Join X to Y' and join X' to Y using straight segments. Label their intersection as Z.

Planar graphs

To capture this idea more formally, we introduce the concept of embedding; the approach and material in this section follows closely that of Harary (1969, pp. 102-113). A graph is *embedded* in a surface when it is drawn on that surface in such a way that no two edges intersect (geometrically). The graph in Figure 2 has not been embedded in the plane: the edges $X'Y$ and XY' intersect. The graph in Figure 3 has been embedded in the plane. The

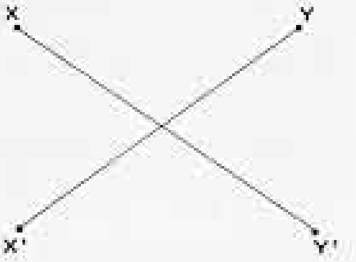

Figure 2. Draw nodes X and Y, left to right, horizontally. Draw nodes X' and Y', left to right, horizontally below the first set. Join X to Y' and join X' to Y using straight segments.

connection pattern of the graphs in Figures 2 and 3 is identical: topologically, they are said to be *homeomorphic*. They are equivalent structural models. Thus, we distinguish between a planar graph and a plane graph. A graph is *planar* if it can be embedded in the plane (as can Figure 2); a graph is *plane* if it has already been embedded in the plane (as has Figure 3). The graph in Figure 2 is planar but not plane; the graph in Figure 3 is both planar and plane.

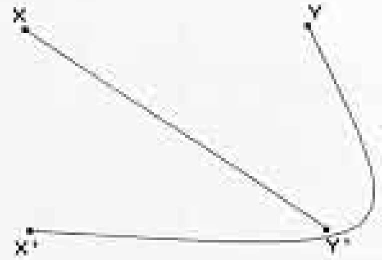

Figure 3. Draw nodes X and Y, left to right, horizontally. Draw nodes X' and Y', left to right, horizontally below the first set. Join X to Y' using a straight segment and join X' to Y using a curved line that does not pass through the segment joining X to Y'.

A graph that cannot be embedded in the plane is called *nonplanar*. There are two nonplanar graphs of particular importance. One is the graph composed of two sets of three nodes: think of one set of three nodes arranged horizontally and of the other set as arranged horizontally below the first set. Edges join each node of the top set to each node of the bottom set: a total of nine edges (Figure 4 shows the detail of labeling). This set is denoted as $K_{3,3}$. The other critical nonplanar graph is denoted as K_5. It is composed of a pentagon and all edges joining the nodes (Figure 5 shows detail).

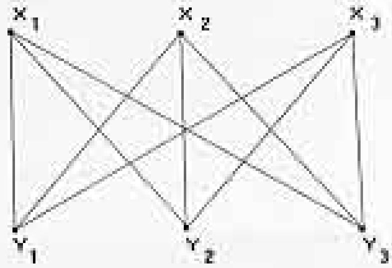

Figure 4. Draw nodes X_1, X_2, X_3 from left to right as one set of nodes arranged horizontally. Draw nodes Y_1, Y_2, Y_3 from left to right as another set of nodes arranged horizontally, below the first set. Draw edges X_1Y_1, X_1Y_2, X_1Y_3; X_2Y_1, X_2Y_2, X_2Y_3; X_3Y_1, X_3Y_2, X_3Y_3 to form the nonplanar $K_{3,3}$ graph.

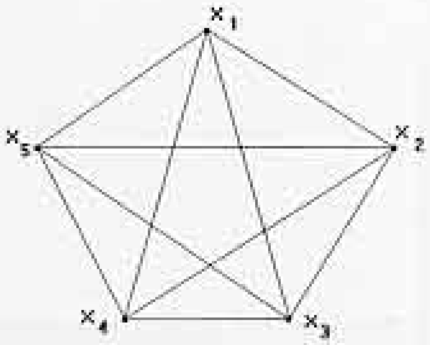

Figure 5. Draw nodes X_1, X_2, X_3, X_4, and X_5 arranged as nodes of a regular pentagon. Join the nodes as a pentagon: along edges X_1X_2, X_2X_3, X_3X_4, X_4X_5, X_5X_1. Join the remaining nodes: along edges X_1X_3, X_1X_4, X_2X_4, X_2X_5, X_3X_1, X_3X_5.

Generally, one might look at geometric intersections to suggest whether or not a given graph is planar: simple-looking geometric intersection patterns can often be unscrambled in the plane to eliminate any geometric intersections (as was Figure 2 in Figure 3). More complicated geometric intersection patterns ($K_{3,3}$, K_5) cannot be undone (Harary, 1969). As with the four color problem, and as is often the case, what is a simple problem to consider is in fact a difficult one to solve. It was not until 1930 that Kuratowski finally solved the long-standing problem of characterizing planar graphs. The statement of the theorem is simple; its proof is not (see Harary, 1969, for proof).

Kuratowski's Theorem

A graph is planar if and only if it has no subgraph homeomorphic to K_5 or to $K_{3,3}$.

The K in the notation honors Kuratowski for his achievement. With this elegant theorem in hand, we now turn to consider planarity in the geographic world.

The Paris Metro

The Paris Metro is a subway system that, for the most part, under the streets of Paris, links the classical "Portes" – City "Gates"– to each other as the many routes criss-cross the Seine in association with the various bridges (Figure 6). The Paris Metro map is a graph; there are numerous nodes representing local stations along a single train route as well as larger stations at which one can transfer from one metro route to another. There are directed arcs, forming a cycle, in the south west of the map leading to the Porte d'Auteuil, and in the northwest leading to Pré St. Gervais. All other arcs represent two-way Metro linkages. The map is complicated in appearance; subway lines often follow surface traffic patterns. Pedestrians need access to subway routes from sidewalks. Indeed, the Paris Metro map reflects the surface pattern of the numerous rotary, star-shaped intersections and tortuous "rues" that add much to Parisian charm. The Metro graph is strongly connected; choose any two metro stops – they are mutually reachable within the entire system, although a transfer might be required. Any well-designed mass transit system should clearly have this style of connectedness, lest passengers be stranded. There are a number of nodes with indegree and outdegree in excess of four. Anyone who has traversed the maze of possible transfers at Montparnasse-Bienvenue, for example, will be aware of how complicated a trip from "here" to "there" can be. Because there are quite a few transfer nodes with a number of incident edges, it is natural to consider whether or not a $K_{3,3}$ or a K_5 might be contained as a subgraph of the Metro graph. David Singmaster has shown that the London Underground is non-planar; is the Metro graph planar?

Planarity and the Metro

Indeed, the Metro is not planar, either; when the map is strictly considered as a digraph, it is an easy matter to choose six nodes and a set of edges to form a $K_{3,3}$. If one wishes, however, to eliminate the possibility of a transfer from one train to the other, in order to have direct geo-graphical adjacency as well as graphical adjacency, it is also possible to find a $K_{3,3}$ under these tighter constraints.

The Metro stops of Etoile ("star") and Nation are joined on the north by a single Metro route arching across the northern part of the city; they are joined on the south by a single arch paralleling the southern perimeter of Paris; and, they are joined across a diametral route, through Châtelet as a "center," by a single Metro route passing under the Champs Elysées, Concord, Palais Royal, Hôtel de Ville and the Bastille. When Montparnasse-Bienvenue and Stalingrad are chosen also, as nodes intermediate on these southern and northern arches, along with Gare de l'Est as a final node, this set of nodes can be joined in a $K_{3,3}$ with

Figure 6. Map of the Paris Metro.

only direct geographic linkage (requiring no transfers) between pairs of nodes along distinct edges. Label the nodes as follows (Figure 7):

1. Etoile
2. Montparnasse-Bienvenue
3. Nation
4. Châtelet
5. Gare de l'Est
6. Stalingrad

Each odd-numbered node is joined to each even-numbered node along distinct edges, as required for a $K_{3,3}$. Thus the Paris Metro, viewed as a structural model, is nonplanar; to travel from Montparnasse-Bienvenue to the Gare de l'Est requires, when represented as a

Figure 7. Metro map with labeled nodes and distinguished edges linking the nodes.

map in the plane, that the edge from node 2 to node 5 cross at least one of the other edges of the $K_{3,3}$. The geographical and social implications of this lack of planarity are significant.

Significance of lack of planarity

One might imagine a subway system to exist in a plane parallel to the plane of surface traffic, some number of feet below the surface. Experience with even simple subway systems defeats this notion; trains run on elevated tracks in regions with high water tables or on landfill; their elevation is altered to cross natural barriers such as rivers. There is considerable topographic relief in most subway systems. Natural difficulties can force a subway system out of a planar environment. Thus, collisions between trains on different routes, in different (intersecting) planes, must be considered; the separation of routes into different layers (planes) offers protection from collision–except where the planes intersect.

In the case of Paris, there are Metro lines at different levels; trains enter selected stations at different depth levels. Passengers trying to switch from one Metro route to another at Montparnasse-Bienvenue may recall running up or down stairs and through connecting tunnels to execute a transfer. The Metro map shows the route north from Montparnasse-Bienvenue, toward Odéon, Châtelet, the Gare de l'Est, and the Porte de Clignancourt to "cross" routes 12 (from Mairie d'Issy to the Porte de la Chapelle) and 10 (from the Gare d'Orleans-Austerlitz to the Porte d'Auteuil). If these crossings were "real," rather than over- or under-passes, there could be serious metro collisions at them. Map evidence suggests that it is the Orleans/Clignancourt route that is at a different level as routes 10 and 12 intersect at nearby Sèvres Babylon station. A lack of planarity can be used to advantage by engineers planning new stations or new routes in a tightly-packed transport system.

References

Harary, F. 1969. *Graph Theory*. Reading, Mass., Addison-Wesley.

Kuratowski, K. 1930. Sur le problème des courbes gauches en topologie. *Fund. Math.*, **15**, 271-283.

Interruption!

Sandra Lach Arlinghaus
The University of Michigan

Interruption, from the Latin–*rumpere* (to break) plus *inter* (between, among), means literally "to break into (between)." The concept of "interruption" can be employed to guide research direction between apparently disparate objects of study; "interruption" is a meta-concept like "symmetry," "duality," and a host of others. We are all familiar with flat maps of the Earth that are interrupted. Indeed, all flat maps of the Earth are interrupted; the one-point compactification of the sphere guarantees that this is so from a topological standpoint. From a more pragmatic standpoint, we know that it is not possible to remove the peel from an orange and place it flatly in the plane – the peel will rip.

Classical interruption in mapping

It is this pragmatic view of mapping the Earth into the plane that conjures up most visual images of an "interrupted" map projection – one in which some cuts have been made (typically in the oceans) in order to preserve some degree of a desirable property, such as conformality or equality of area. Philbrick's (1963) Sinu-Mollweide has the northern hemisphere continuous with slits in the oceans in the southern hemisphere; Goode's Homolosine Equal Area projection (Goode, various years) has interruptions in oceans in both hemispheres. Either of these projections would be viewed, clearly, as an "interrupted" projection.

However, would all who see these as interrupted also view a cylindrical projection (Miller, for example) as "interrupted"? Of course it is, for once the sphere is projected onto the surface of the cylinder, the cylinder must then be "developed" or unrolled into a section of the plane. The development of a surface in the plane is a cut – a form of breaking into the cylinder – an interruption. The difference is that the interruption in a Miller cylindrical projection often determines the boundary of the map in the plane – our eye seeks closure and when the cut coincides with the map boundaries we use for closure, the visual effect is less jarring; the interruption is masked by the boundary.

Abstract variants on interruption and mapping

Going farther abstractly, one might consider rather than a map on a cylinder, a map on a Möbius strip; Tobler (1961) described a scheme in which a pin, poked through a map on a Möbius strip, emerges at its antipodal point. When this procedure is continued a finite number of times, the boundaries of a region and its antipodal region are traced out simultaneously on this one-sided map. This novel approach suggests ways to trace out partial, discrete, boundaries. Spilhaus (1979) suggests that to construct a continuous map of the antipodes one "show which land is opposite other land ... by taking a pair of maps of two hemispheres and putting them back to back with the North Pole covering the South Pole." Neither construction touches on deeper non-Euclidean aspects of this style of construction (Arlinghaus, 1987).

From the viewpoint of interruption, however, what is interesting is the mere idea of considering a map on a Möbius strip. The cylinder and the Möbius strip are both developable surfaces in the plane and they are but two members of a broader class. Because developable surfaces, when interrupted and placed in the plane, are those whose boundaries can easily mask the cuts of interruption, they are a class of particular interest. This broader class of surface may be viewed as composed of two structurally parallel sequences of transformations

– one easily visualized and the other visualized easily only by analogy with the first (Figure 1). (This sort of characterization is common in a variety of books that deal with elementary topology, as for example in Courant and Robbins, 1941.)

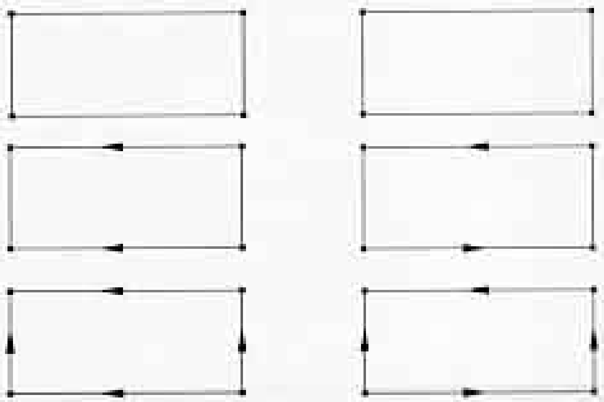

Figure 1. Two sequences: on the left, a rectangle is rolled up into a cylinder, and then the cylinder is joined, end-to-end, to form a torus. On the right, a rectangle, given a half-twist, is rolled up into a Möbius strip, and then joined (with another half twist), end-to-end, to form a Klein bottle.

Visual sequence:

1. A plane rectangle may be rolled into a cylinder by gluing together the upper left to the upper right corners and the lower left to the lower right corners. The result is a cylinder with diameter that of the length of the top of the rectangle.
2. A cylinder may be rolled into a torus by gluing one circular end of the cylinder to the other – the seam along which gluing takes place is the circle that matches the ends of the straight line seam along the length of the cylinder.

Abstract sequence:

1. A plane rectangle may be rolled into a Möbius strip by gluing together the upper left to the lower right corners of the rectangle and the lower left to the upper right corners of the rectangle. The result is a Möbius strip; the gluing action imparts a half-twist to the rectangular strip.
2. A Möbius strip may be rolled into a Klein bottle by gluing one "circular" end of the Möbius strip to the other, as with the torus.

What can be glued can be unglued (in this context); thus, cylinder, torus, Möbius strip, and Klein bottle are developable surfaces in the plane. One can view each of them as a surface on which to map; difficulty in such an approach is encountered only when the need to visualize physical objects is relied upon. Conceptually, from a structural viewpoint, the Möbius strip is no more difficult to consider than is the cylinder; the Klein bottle no more difficult than is the torus.

The utility of considering various mapping surfaces–GIS

A current maxim of those concerned with the protection of various elements of the environment is "to think globally, act locally." While this may have fine implications for landfill management, it is a dangerous cartographic practice. Globally we should think of a sphere or some other approximation of the Earth's surface that is topologically equivalent (homeomorphic) to the sphere. Locally we tend to think of our immediate part of the Earth as flat; recently, Barmore (1992; 1994) has shown the difficulty in determining geographic centers of various sorts when concerns for curvature are not involved in policy decisions. In earlier times, this sort of lack of tying knowledge of the earth as a sphere to a local plane environment was evident: from Eratosthenes' measurement of the Earth to the great voyages undertaken at the end of the Middle Ages and beginning of the Renaissance in Western Europe.

Most mapping is done from the global/spherical viewpoint to the local/planar viewpoint; it need not be, and when the mapping is from developable surface to plane, or from sphere to object homeomorphic to the sphere, then maps that hide interruption can be constructed. One place where this issue has, for the most part, not been addressed at all, is in the electronic environment of the Geographic Information Systems (GISs). In a recent paper, Tobler (1993) speaks to this issue at some length and notes, in particular, that of the hundreds of GISs available, "The one exception, explicitly designed to consider the spheroidal earth, is the 'Hipparchus' system developed by Hrvoje Lukatela of Calgary, Alberta (Lukatela 1987)." GISs such as this apparently offer a way to make maps directly from spherical data, eliminating the middle step of imitating the traditional drafting processes of the human arm and the planar decisions associated with those. This sort of idea seems quite natural–why should we use the computer to imitate the classical drafting process; why not use it to take advantage of the underlying mathematical characteristics of the real problems of dealing with surfaces?

Another route to this sort of end might be to construct data structures in the environment of the mathematics of the Klein bottle, torus, Möbius strip, or cylinder, and then to develop (as in "unroll") the mathematics to make plane maps. Either way – from sphere to sphere homeomorph, or from developable surface to plane, one might look forward to more elegantly constructed electronic programs for executing mapping – with the usual hoped-for consequence that elegance in theory leads to leaps in practice.

Future directions

What is important to consider for maps is important to consider for other representations of the earth's surface. Cartographic considerations can guide disparate research projects of spatial character.

Structural models (Harary, Norman, and Cartwright, 1965), one form of abstract graphs (Harary 1969), can offer yet another way to map the Earth. These abstract graphs serve as "maps" whenever any discrete set of real-world locations and flows can be captured in channels linking locations: the locations serve as the nodes for the graph and the channels serve as edges linking nodes. Thus, a set of cities and the railroad tracks joining them may be represented visually as a structural model – the cities are nodes and the tracks are edges of the model. Indeed, a set of individuals, at least some of whom share a common belief, may also be represented as a structural model; the individuals are nodes and the belief, if shared, is represented along edges linking appropriate individuals. There are numerous examples one might construct. What is important is that these models, as are maps, are also subject to interruption. Because it is abstractly preferable to avoid or to mask interruption, it is important to know how it arises.

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4. REPRINT

Imperfections in the Uniform Plane

Michael F. Dacey

with Forewords by John D. Nystuen, The University of Michigan

In this section, *Solstice* Board member, John D. Nystuen, selects a paper from the collected papers of the Michigan Inter-University Community of Mathematical Geographers (MICMOG) (of which he is Editor) to reprint here, some 30 years after its initial presentation. In addition to the reprint of work of Michael Dacey, Nystuen's original Foreword, and introduction of Dacey and his work to the assembled MICMOG group, is also reprinted. In addition, a new Foreword by Nystuen takes a look at the Dacey paper in retrospect. The paper is reprinted with permission of Nystuen, on behalf of the Michigan Inter-University Community of Mathematical Geographers.

Foreword, December, 1994

John D. Nystuen

Thirty years ago Michael Dacey contributed to the development of spatial statistics in highly original ways. Many of the ideas he used and introduced to the literature in the 1960s are now part of generally accepted spatial theory. For example, he was one of the first to use the idea of a dimensional transformation to permit evaluations of the spatial association of point and area phenomena. The transformational approach proved useful as a general concept as Keith Clarke has demonstrated in his interesting book (Clarke, 1990). Arthur Getis, a colleague of Dacey's, and Barry Boots used many of Dacey's ideas in their book (Getis and Boots, 1978) about modelling spatial process.

Today, vigorous effort is being expended on incorporating spatial analysis functions into Geographic Information Systems (GIS) software. We are re-issuing one of Dacey's seminal works to bring to the attention of contemporary scholars an important source of many of the concepts now becoming accessible to general uses of GIS technology. Dacey's work now speaks to another generation.

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Clarke, Keith C. 1990. *Analytical and Computer Cartography*, Prentice-Hall, Englewood Cliffs, NJ.

Getis, A. and Boots, B. 1978. *Models of Spatial Processes*, Cambridge University Press, Cambridge.

Foreword, May, 1964

John D. Nystuen

We are pleased to present to our readers a paper by Professor Michael F. Dacey. Many of us are aware, if only vaguely, of his provocative and voluminous writings. Professor Dacey has penetrated deeply into realms where few, if any, have gone before. He travels alone and has left but a thin trail of mimeographed papers as scent. The track is now long and difficult to follow and he does not rest. He has allowed one of his works to become discussion paper #4 of our series. We hope this will expose his activities to a wider audience. Some may be inspired to join him in the new work that he is doing. I hope so. Certainly we must keep in contact with him. Regrettably many of his results depend upon his previous statements now difficult to obtain. I will attempt in this foreword a short review of the pertinent ideas by way of a summary of this paper. I have also added, with his permission, a glossary of symbols at the end of the paper.

Michael Dacey has for several years explored abstract spatial patterns using probabilistic methods. This paper is one of a series of such studies. Most of the work provides empirical examples of the concepts. The contrast in methodologies displayed between discussion paper #3 (W. Bunge, "Patterns of Location") and this one is marked. Professor Bunge turns away from probabilistic formulations (see page 3 of "Patterns of Location") and Professor Dacey rejects deterministic models (see page 1 below). I believe the relative worth of these two broad approaches to abstract geography will receive increasing attention in the literature. There is much precedent for concern over this question in other disciplines. Clearly Dacey accepts the value of a probabilistic approach.

It may aid the reader if the paper is viewed as consisting of six parts.

1. Professor Dacey first describes an abstract model of imperfections in a uniform plane. The characteristics of this model are specified in a general way. I believe that Professor Dacey is the first to suggest models where non-random patterns are disturbed by random variables (see Dacey and Tung, 1962).
2. The point pattern which results from the above mentioned model is to be summarized quantitatively in such a fashion that it can be compared with some actual geographic point pattern. Professor Dacey calls upon his previous extensive investigations of nearest neighbor statistics to do this job [1]. He specifies how measures of the distances to the 1st nearest, 2nd nearest, ... kth nearest neighbors of a sample of points in the point pattern may be used to describe the point pattern by probability distributions of these lengths. The strategy is to then compare the probability distributions of the model with a geographic pattern using a simple χ^2 statistic. Professor Dacey is aware that nearest neighbor methods may be used to compare point-to-area relations as well as point-to-point relations. A point pattern is not simply a set of points. The points occupy a space for which a metric is defined. The metric makes possible distance measures between the points. The fact that there is a space creates the boundary problems mentioned in the text. The original purpose of these statistics was to test if points were more clustered or more even than random. Imagine a study area which is mostly empty but has in one small region an even distribution of points. Measuring distances between points and using the nearest neighbor test would indicate a point pattern more even than random. In one sense, however, they are clustered for they occupy only a small section of the study area. There is a strategy for this situation. Use another point set to represent the area. This may be done by using an even distribution of points in the area or by assigning points to the area at random. The second set of points now represents the study area. The area has been abstracted into a point pattern and the nearest neighbor method may be used. Measures between the two point sets now reveals the original point pattern to be clustered. The decision concerning which method to employ depends upon whether the phenomenon studied has a postulated interaction of point-to-point or point-to-area. The text indicates the procedure for using either method.
3. Theoretical order distances are specified by equations (16) and (17). The probability functions are made more explicit and operational by assuming each lattice point is disturbed by the same two dimensional normal variate. Professor Dacey has ample evidence

that these particular probability distributions are useful for this purpose.[2]

4. Solutions of the equations in the previous section would yield an analytic solution regarding expected order distances for various disturbance models. However, these equations prove very difficult to evaluate. Recourse to a simulated solution is sought. An *almost periodic disturbance model* is postulated. Its parameters are estimated from data on an actual pattern of urban places in Iowa. Using these parameters, a set of points conforming to the structure of the theoretical model is generated with random digits and tables of normal deviates. This artificial pattern is one of many possible representations of the theoretical pattern. It is presumed to display the type of pattern expected from an analytic solution if one could be found.
5. The author now has two patterns: one, a simulated theoretical pattern which conforms to the structure of the model; and the other, an actual urban place pattern in Iowa. He also is able to make the appropriate nearest neighbor measures which characterize each pattern. The frequency distributions are then compared using the χ^2 statistic.
6. In an addendum, the author presents further testing of his model by taking advantage of a computer program which generates the distance measures required. The paper ends.

It must be clear to the reader from the contents of this paper that Michael Dacey has indeed traveled over much ground. He has previously developed many of the results needed in this study. Many of his solutions and applications are ingenious. He exhibits an understanding of the theoretical implications of his work. He has a wide knowledge of the literature on probability and is able to adopt simulation methods and computer technology to his purpose. All he lacks is someone to talk to.

Endnotes

1. Examples of his statements on nearest neighbor measures include: "Analysis of Central Place Patterns by Nearest Neighbor Method," Seattle, May 1959, mimeographed; "Analysis of Central Place and Point Patterns by a Nearest Neighbor Method," *Proc. of IGU Symposium in Urban Geography*, Lund, 1960, pp. 55-75; "Identification of Randomness in Point Patterns," (with Tze-hsiung Tung), Philadelphia, June 1962; mimeographed. (Dacey and Tung is now forthcoming in the *Journal of Regional Science*, v. 4.

2. See references at the end of the paper and also: "Order Neighbor Statistics for a Class of Random Patterns in Multidimensional Space," *Annals*, Association of American Geographers, v. 53 (Dec. 1963): 505-515, "Certain Properties of Edges on a Polygon in a Two Dimensional Aggregate of Polygons Having Randomly Distributed Nuclei," Philadelphia, June 1963, mimeographed.

Imperfections in the Uniform Plane

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See end of article for additional information

A statistical formulation of the spatial properties of central place system is proposed. Currently, the theoretical locations of central places are specified by geometric or algebraic quantities. This type of statement leads to certain rejection of central place models, for it is inconceivable that any observed pattern of central places corresponds exactly to the specified geometry. A probabilistic formulation is preferred for empirical analysis because deviations from the precise locations are contained within the statement of the model.

In the classical theory of Christaller (1933) and Lösch (1939) central places form a honeycomb pattern or hexagonal lattice on the undifferentiated, unbounded plane. A probabilistic statement of this location pattern incorporates deviations from the precise lattice locations, and the deviations are subject to stochastic processes. This initial formulation of a probabilistic central place distribution uses the concept of imperfections in the uniform plane to define these deviations. Imperfections may be combined with the central place geometry in many ways. Here one basic formulation and two closely related models are proposed. The models possess some properties of the Christaller-Lösch system and evidently are not inconsistent with the spirit of central place theory.

This report has two purposes. First, a general model of imperfections in the uniform plane is constructed. Second, the application of a particular model to a map pattern is evaluated.

The map pattern of urban places in Iowa has been selected for an initial examination of the imperfection concept. The empirical test involves interpretation of parameters of the model in terms of phenomena commonly studied by geographers and estimation of these parameters from the Iowa map pattern. Because the formal statement of the model contains equations that are difficult to evaluate analytically, this initial study has used a simulation technique to obtain summary measures on theoretical patterns. Properties of a fabricated pattern are compared with the Iowa map pattern, and the level of agreement is found acceptable to the first approximation.

The Christaller Spatial Model

The theoretical distribution of central places may be expressed in terms of a plane lattice. Let P represent a plane symmetry lattice. Choosing any arbitrary point of this lattice as an origin point O, the location of any other given lattice point can be defined with respect to this origin by a vector T

$$T = ut_1 + vt_2 \tag{1}$$

where u and v are integers. The vector notation implies that the plane is constructed as a linear lattice having a translation period t_1 which is repeated periodically at an interval t_2. The translation periods t_1 and t_2 may be regarded as vectors separated by the angle g. Using K to denote a collection, the lattice points of P are defined by

$$P = KT = K(ut_1 + vt_2). \tag{2}$$

Central place theory conventionally uses a hexagonal lattice for which the translations t_1 and t_2 are of the same unit length and the angle of periodic rotation is $g = \pi/3$.

A more general discussion is obtained by not restricting attention to the hexagonal lattice. In this report P represents any plane lattice which may have a three-, four-, or six-fold axis. In applying the lattice to a particular problem, the translation periods t_1 and t_2 and the angle of rotation g need specification.

Types of Imperfections in the Uniform Plane

Three types of imperfection in the uniform plane are studied in this report. These imperfections are closely related to certain kinds of imperfections found in nearly perfect crystals. An introduction to crystal imperfections is found in Van Bueren (1961, especially Chapters 2-4) and an excellent synthesis of the concept of imperfection in the solid state is given by Seitz (1952). The basic principles of our formulation draw heavily upon concepts used in the study of crystals and the solid state; the mathematical formulation is, however, quite different.

The imperfections under consideration are identified as (i) dislocations or disturbances, (ii) vacant lattice sites and (iii) interstitial points. These three types of imperfections are most easily defined by considering two maps containing point symbols. For the present purposes assume the maps have identical area and number of points. One map represents a finite domain of the lattice P. The other map, called S, may show fabricated locations or the positions of actual objects. Figure 1 is "good" map S overlaid on a square P.

i. The term dislocation is more descriptive of the first imperfection, but it has a definite meaning in crystallography and solid state physics; so we shall call this imperfection a disturbance. A disturbance occurs when the location of a point is not exactly at a theoretical lattice site but is 'sufficiently' close so that with high degree of certainty a disturbed point is correctly associated with its theoretical location.

ii. A vacant lattice site occurs where no point is 'close' to a theoretical lattice site. Where two or more points occur in the vicinity of a lattice site, it is not called a vacant lattice site even though the one point correctly associated with that theoretical location may not be identifiable.

iii. An interstitial imperfection occurs in the uniform plane where a point is not identified with any lattice site. Interstitial locations occur where a point is too distant from a theoretical location to be associated with high degree of certainty with a particular lattice site, or where two or more points are located 'close' to a lattice site and the one point correctly assigned to that theoretical location is not identifiable.

These imperfections are not given precise definitions. In constructing the imperfection model more precise definitions are given.

A Model of the Imperfect Plane

One basic formulation and two modifications are described. All imperfections under consideration are the result of stochastic processes, in the space rather than the more common time dimension. The principal feature of an imperfection model is the imperfection in pattern related to disturbances or shocks from geometrically exact locations (Figure 1). While this single type of imperfection is adequate for many physical systems, it is probably too restrictive to encompass patterns formed by economic, social or cultural systems. To handle complex map patterns two additional types of two dimensional stochastic processes were studied. One type of imperfection generates interstitial points and is defined by a two dimensional, uniform, random variable. The other type of imperfection generates clusters of points and is defined by spatially contiguous probability distributions. Because the pattern of urban places in Iowa is relatively homogeneous and contains no examples of large metropolitan centers, it was not necessary to incorporate a contagious process in a model for the Iowa map pattern. For this reason, only the first two types of imperfections are discussed in this report.

The Disturbance Effect

Each lattice point of P is associated with a stochastic variable ξ. The ξ is the disturbance variable and defines the realized location of a point with respect to its theoretical lattice site. It is convenient to separate ξ into its two polar components: a distance ρ and a rotation angle θ. So, $\xi \equiv (\rho, \theta)$.

The displacement of the point s_{ab} from its equilibrium position $(at_1 + bt_2)$ is given by the random variable ξ_{ab}. So, the disturbed position of this point is

$$s_{ab} = at_1 + bt_2 + \xi_{ab}. \tag{3}$$

It is assumed that the same stochastic variable is associated with each lattice site. Then, if a point is disturbed from each lattice site the collection of randomly disturbed points is

$$S_1 = K(ut_1 + vt_2 + \xi_{ab}), \tag{4}$$

u and v integers. This notation indicates that ξ has translation period t_1 which is repeated periodically at an interval t_2. In this sense the stochastic variable is carried through space and is associated in turn with each lattice site. Accordingly, in point set S_1 each lattice site $(at_1 + bt_2)$ has exactly one corresponding disturbed point s_{ab}.

Vacant Lattice Sites

It is not necessary to apply a disturbance to each lattice site. Instead a lattice site and the variable ξ_{ab} may be taken in conjunction with a binary or on-off operator which nullifies the vectors defining some disturbed points so that the corresponding lattice sites are vacant. As a consequence, there is a sparser network of disturbed points than lattice sites. Because a disturbed point is not associated with each lattice site, the disturbance term is said to be repeated almost periodically. A more precise definition of the almost periodic disturbance is given.

A binary operator to produce vacant lattice sites is defined for $(at_1 + bt_2)$, denoted in symbols by β_{ab}, such that for $0 \le \lambda \le 1$,

$$\beta_{ab} = 1, \quad \text{with probability } \lambda$$

$$\beta_{ab} = 0, \quad \text{with probability } 1 - \lambda. \tag{5}$$

The vectors defining location of the disturbed point s_{ab} are multiplied by β_{ab} so that the disturbed point is realized with probability λ and is not defined with probability $(1 - \lambda)$. In more precise form, the location of the disturbed point having equilibrium position $(at_1 + bt_2)$ is

$$s_{ab} = \beta_{ab}(at_1 + bt_2 + \xi_{ab}) \tag{6}$$

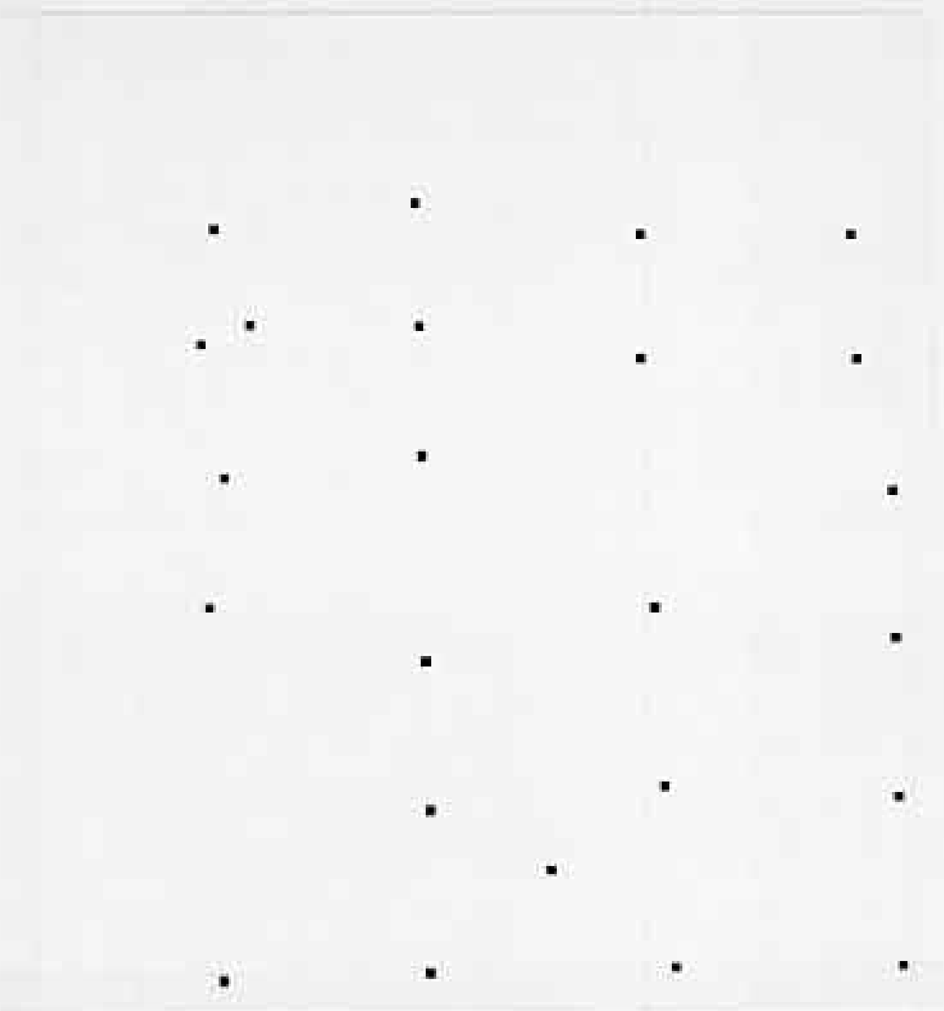

Figure 1. Map of imperfection model. Most symbols show disturbance effect on a square lattice. There are two vacant lattice sites, and two examples of interstitial points. Most map patterns are, of course, not this regular. This figure shows a six by four square lattice which has been altered as suggested.

with the usual convention that $s_{ab} = 0$ does not define a point at the lattice site 0. So, for $\beta_{ab} = 0$ the disturbed point s_{ab} does not exist, while for $\beta_{ab} = 1$ location is found precisely in the manner for the period disturbance.

Each lattice site is associated with the same stochastic variable and with the same binary operator. Accordingly, the relation (6) is carried through space with translation period t_1 repeated periodically at interval t_2. The collection of points generated by the almost periodic disturbance is

$$S_2 = K(\beta_{uv}(ut_1 + vt_2 + \xi_{uv}) \qquad (7)$$

u and v integers. The S_2 is completely identified by the underlying lattice P, the probability λ, and the parameters specifying the components ρ and θ of the stochastic variable ξ. It is summarized by the parameter set $S(t_1, t_2; \lambda, \xi)$.

Uniform Random Disturbance

This collection of points, denoted by R, is a random point set. To make the definition explicit, an arbitrary origin is selected and the lattice point O of P is convenient. The R is specified by the theoretical frequency of points within distance r of the origin. Where the parameter γ is the expectation that a unit area contains a point belonging to R, put

$$p = \pi\gamma r^2 \qquad (8)$$

where $\gamma > 0$. The frequency p describes any arbitrary disk of radius r, so that the distribution ξ is independent of the specified origin. It is a property of R, Feller (1957) that the distribution conforms to a Poisson process. The probability of finding exactly j points of R within any disk of radius r is $p^j e^{-p}/j!$.

Definition of the Basic Model

The model to be considered in this report is defined by the combination of an S and the R point sets; call this model M and

$$M = S \cup R. \qquad (9)$$

This model is summarized by the parameter set $M(t_1, t_2; \lambda, \xi; \mu)$, where $\mu = (\lambda + \gamma)$. For a model containing S and R points only, μ is the mean density of points per unit area.

Several interesting formulations of M are defined by special values of the parameters λ and γ.

The *periodic disturbance model* M_1 is given by $\lambda = 1$, for one disturbed point is associated with each lattice site. A *complete periodic disturbance model* also has $\gamma = 0$, for each point is disturbed from a lattice site and there are no random points from R.

The *almost periodic disturbance model*, called M_2, is given by $0 < \lambda < 1$. The magnitude of γ determines if M_2 has a one-to-one correspondence of points to lattice sites or if M_2 has more or less points than lattice sites. If $\gamma = 1 - \lambda$ the theoretical density of points belonging to S_2 and R equals the density of lattice sites. If $\gamma > 1 - \lambda$ the expected number of points exceeds the number of lattice sites, while the expected number of points is less for $\gamma < 1 - \lambda$.

The point set given for $\lambda = 0$ is a random point pattern. It is of course recognized that R is only one of many point sets that could be combined with S_1 or S_2 disturbed points.

Description of Pattern

The disturbance models are described by the underlying lattice P, the density measures λ and γ and the disturbance process ξ. The combination of these parameters produce disturbed and interstitial points and vacant lattice sites in the uniform plane. In a formal sense a model is completely specified by the lattice parameters and the several probability functions. This specification of a model does not, however, describe or summarize in any useful fashion the point pattern generated by a particular model. But, numerical summary of point pattern M is prerequisite to test of the hypothesis that an observed map pattern is similar to an imperfection pattern.

To measure the level of correspondence between observed and theoretical patterns there is need for (i) measurements on one or more properties of the observed pattern and (ii) theoretical values for the same properties on the pattern defined by the model. In addition, if parameter values for the model are estimated from the observed pattern, the properties for test of similarity between observed and theoretical patterns should be independent of the properties initially used to estimate parameters.

In this report pattern is summarized by two classes of order distance statistics. The methods are described briefly and then their utility as descriptive measures of pattern are indicated.

Point to Point Order Distances

Let i represent any arbitrary point in a point pattern Q. The measured map distance from i to the j nearest point is represented by R_{ij}. J measurements are taken from i and are ordered to satisfy the inequalities

$$R_{i1} < R_{i2} < \cdots < R_{ij} < \cdots < R_{iJ} \tag{10}$$

and the R_{ij} is called the j order distance. For description of a bounded map pattern the j order distance is recorded only if R_{ij} is less than the distance from i to the nearest map boundary. The chance of bias due to the influence of boundaries is reduced by this constraint, but there is loss of information to the pattern description because all distance relations are not utilized.

The R_{ij} measurements reflect the arbitrary map metric. The dimensional constant which eliminates effect of scale is $d^{1/2}$, where d is the density of points in Q. Measurements in Q are reduced to standardized distance by the transformation

$$r_{ij} = d^{1/2} R_{ij}. \tag{11}$$

Standard distances are used in this report to describe all patterns.

Let I denote a collection of points in Q, and $i \in I$. One description of Q uses standard distances from each origin point $i \in I$ to the J nearest points.

Locus to Point Order Distances

A second description of pattern uses distance measurements from coordinate locations to points. Let L define a set of locations in Q and in general a locus $\ell \in L$ is not a point symbol of Q. The measured distance in Q from locus ℓ to the h nearest point is denoted by $R_{\ell h}$. The measurements from ℓ are ordered by distance and put in standard form; in symbols

$$r_{\ell 1} < r_{\ell 2} < \cdots < r_{\ell h} < \cdots < r_{\ell H} \tag{12}$$

$$r_{\ell h} = d^{1/2} R_{\ell h}. \tag{13}$$

The second description of Q uses standard distances from each locus $\ell \in L$ to the H nearest points. The boundary constraint pertains to these distances also.

Sampling Methods

The elements of I may consist of all or a sample of points in Q. For this study a census was taken, largely because of small pattern size.

The loci in L necessarily constitute a sample, and these locations may be designated by random, stratified or uniform sampling methods. The most efficient mesh for plane sampling has been studied by a number of writers, as Zubrzycki (1961) and Dalenius, Hajek, and Zubrzycki (1961), but there are no general conclusions. This study used random sampling, largely because the patterns of interest contain high degree of uniformity in spacing and random sampling is probably less sensitive to this type of spatial bias. However, this topic requires study.

Summary Description of Pattern

A point pattern may be summarized by (i) the lower moments of the j and h order distances or (ii) the frequency distributions of these order distances. The j order point to point distances provide a quantitative summary of the arrangement of points with respect to other points of the pattern, but these distances do not explicitly reflect the arrangement of points with respect to the map space. The complementary h order locus to point distances provide a quantitative summary of the arrangement of points with respect to the loci in L. To the degree the sample mesh of L is a measure of the map space, h order distances also summarize the arrangement of points with respect to the map space. Because these two classes of distances reflect two different aspects of pattern, this type of summary statement captures many of the subtle characteristics composing a point pattern.

Comparison of Map Patterns

The descriptive measures provide a basis for evaluating the degree of similarity between two or more patterns. patterns are called similar if the order distances summarizing each of the patterns have"the same statistical parameters. The standardized distances allow direct comparison of any two point patterns, for the distances represented by the variable r (either r_{ij} or $r_{\ell h}$ are normalized to account for differences in scale, unit measurement and density of points. Using either means or frequency distributions of order distances, the hypothesis that two or more sets of measurements belong to the same statistical population may be tested by standard procedures.

Theoretical Order Distances

This paragraph considers the basic derivation of order distances for imperfection models. The derivations are simplified by studying (i) lattices for which $t_1 = t_2$, (ii) nearest neighbor situations only, and (iii) the stochastic variable ξ defined by the normal law.

Two nearest neighbor lattice sites are separated by the distance t $(= t_1 = t_2)$. Let the random variable X denote the distance between two disturbed points associated with any two nearest neighbor lattice sites. It requires only elementary geometry to show that the distance between points (ρ_1, θ_1) and (ρ_2, θ_2) is

$$x = ((\rho_1 \cos\theta_1 - \rho_2 \cos\theta_2 + t)^2 + ((\rho_1 \sin\theta_1 - \rho_2 \sin\theta_2)^2)^{1/2}. \tag{14}$$

The simplest derivation of order distances is for the complete periodic disturbance model ($\lambda = 1$ and $\gamma = 0$) on the hexagonal lattice. Let m $(= 6)$ denote the number of nearest neighbors to each lattice site. We consider the distances from an arbitrary point i at $(at_1 + bt_2 + \xi_{ab})$. It is assumed that the m nearest points to i are disturbed from nearest neighbor lattice sites only. The x_k is the distance from point i to the k $(= 1, 2, \ldots, m)$ nearest point.

If the disturbance term is identical and independent for each lattice site, the m distances from i may be interpreted as m independent observations in a sample of size m from the population defined by the random variable X. Because the observations are ordered from shortest to longest, x_k is the kth order statistic. It is well known that the distribution function of the kth order statistic is given by

$$\Psi(x_k) = \frac{m!}{(k-1)!(m-k)!} F^{k-1}(\omega) F^{k-1}(\omega)(1 - F(\omega))^{m-k} f(\omega) \qquad (15)$$

where $f(\omega) = dF(\omega)$ and the variable X, after making the probability transformation for a specified $f(\rho)$ and $f(\theta)$, is substituted for ω. The z crude moment of the k order statistic for the complete periodic disturbance model is

$$\mu_z{}'(x_k) = \frac{m!}{(k-1)!(m-k)!} F^{k-1}(\omega) \int_0^{\infty} \omega^z F^{k-1}(\omega)(1 - F(\omega))^{m-k} f(\omega) d\omega. \qquad (16)$$

The derivation is far more complex if the lattice is not hexagonal and undoubtedly requires more advanced concepts than provided by elementary probability methods. Moreover, even in this simplified case, numerical evaluation of (16) is not necessarily possible by elementary procedures.

In the statement of disturbance models the normal law was interpreted in polar coordinates by the folded half-normal distribution; that is, the distribution function for location about a lattice site is

$$F(\xi) = F(\rho, \theta) = \int_0^{\rho} \int_0^{\theta} f(\rho) f(\theta)\, d\rho\, d\theta \qquad (17)$$

where

$$f(\rho) = \sqrt{2}\exp(-\rho^2/2\sigma^2)/(\sigma\sqrt{\pi}) \qquad \rho > 0$$
$$f(\theta) = (2\pi)^{-1} \qquad 0 < \theta < 2\pi.$$

It seems appropriate to accept that $f(\xi)$ is identical for each lattice site so that the parameter σ is constant throughout the lattice space. Using (17) to define (14) and substituting the resulting probability transformation into (16) gives an expression for order statistics that, for me, is totally intractable.

Some simplification is gained by interpreting the normal law by the bivariate or circular normal distribution. In this case the distance variable X has a well known form. It may be shown that the distribution function is

$$F(x) = 1/2\exp(-t^2/2\eta^2) \int_0^{(x/\eta)^2} e^{-x/2} I_0(tx^{1/2}/\eta)\, dx \quad x > 0 \qquad (18)$$

where $\eta = 2\sigma^2$ and $I_0(\bullet)$ is the modified Bessel function of the first kind of zero order. This expression is recognized as the integral of the non-central χ^2 with two degrees of freedom. In a slightly different form it occurs as a basic distribution function in bombing or coverage problems, Germond (1950). By substituting (18) for $F(\omega)$, (16) gives the z crude moment of order statistics from a non-central χ^2 distribution; however, tables of values have not been published.

It is apparent that even the simplest imperfection model yields equations that are difficult to evaluate. Where $\lambda \neq 1$ and/or $\gamma \neq 0$ the equation systems are immensely more complex and numerical evaluation may be considered, for any practical purpose at this time, impossible. In order to circumvent these mathematical problems the imperfection model has been evaluated by simulation of an equation system for a given set of parameter values.

Analysis of the Pattern of Urban Places in Iowa

The imperfection models were designed to produce types of patterns and distributions studied in the social sciences. Moreover, the particular class of patterns motivating the present formulation are formed by map representations of urban places. As a partial evaluation of the adequacy of the imperfection model to replicate town and city patterns, the distribution of urban places in Iowa, 1950, is studied.

Many parameters of the Iowa distribution are already available in Dacey (1963a). These data provide empirical estimates of parameters for application of the imperfection model to the Iowa pattern. Using estimated parameters, the degree of correspondence of M_2 with the observed pattern of urban places is analyzed. Simulation is used to evaluate the theoretical imperfection model.

Almost Periodic Disturbance Model

The almost periodic disturbance model M_2 is specified by three sets of parameters:

t_1, t_2 and g identify the underlying lattice P,

ξ specifies the disturbance term generating the point set S_2 and

λ and γ are the scale densities for the point sets S_2 and R, respectively.

These three sets of parameters are given numerical values by relating the imperfection concept to structural features of the Iowa map pattern. In this construction, each parameter is described in terms of the corresponding property of the Iowa pattern. Since the theoretical pattern is synthetically fabricated, the definitions and interpretations of parameters are biased toward operational statements.

Lattice Parameters

The M_2 is fabricated as a rectangular map space containing the domain of a square lattice. The domain is of dimensions 12 by 18 and contains 96 points. Thus, the parameters are $t_1 = t_2 = 1$, $g = \pi/2$.

The primitive cells of the square lattice have an abstract correspondence to counties, and in this context lattice points represent the geographic center of counties. This lattice has some resemblance to the Iowa map. In gross form Iowa is roughly a rectangle and most counties in Iowa are approximately square. However, the counties do not form a square grid, largely because of surveying adjustments for the earth's curvature. An alternative, and possibly a closer, approximation to the Iowa structure is the diamond lattice.

The lattice has 96 squares while Iowa has 99 counties. There is no formal advantage to using a lattice of approximately the same dimensions as the study area.

For specification of other parameters the following relations are established between M_2 and the Iowa map:

i. square lattice cells of M_2 are equated with Iowa counties,

ii. lattice points of M_2 are equated with geographic centers of counties,

iii. S_2 and R points are equated with urban places.

Using this dictionary (α) the distribution function for distance from lattice site to S_2 point is estimated from the observed distances from geographic center of counties to nearest urban place and (β) the frequency distribution of points in primitive lattice cells is estimated from the observed frequency distribution of urban places in counties. These two properties are evidently independent of the order distances used to summarize observed and theoretical patterns.

Disturbance Variables

In my earlier study of Iowa it was shown that for interior counties containing an urban place the distance from the geographic center to nearest urban place was closely approximated by the folded half-normal distribution, as defined for $f(\rho)$ in (17), with scale parameter $\sigma = 0.2286$. Observed and calculated frequency distributions are compared in Table 1.

The angular component θ of the disturbance term is taken as a uniform random variable, as defined in (17). No evidence is presented for this assumption, so the uniform variable is entered into the model on the theoretical consideration that a completely chance factor occurs in the disturbance process. However, in examining the location of places with respect to geographic centers I found no evidence of directional bias.

On the basis of these estimates, the vector component ρ and the angular component θ of the disturbance variable ξ are defined for M_2 by the folded, uniform bivariate distribution (17).

Scale Variables

The remaining two parameters of M_2 are the density measures λ and γ. Because M_2 contains only S_2 and R points, the density of all points is $\mu = \lambda + \gamma$. For the Iowa map pattern there are 93 places and 99 counties, so the estimated density of total points in M_2 is $(93/99) = \mu$.

The individual densities λ and γ were estimated from the frequency distribution of urban places among Iowa counties, Table 2. A two parameter probability density function that gives a good fit to the observed frequencies has been stated by Dacey (1963b). By assuming that each disturbed point in S_2 is always located in the primitive cell of its theoretical lattice site and that each random point in R has an equal probability of occurring in each primitive cell, the probability that a cell contains x points is

$$f(x;\lambda,\mu) = (\gamma^{x+1}e^{-\gamma}/x!) + (x\lambda\gamma^{x-1}e^{-\gamma}/x!) \tag{19}$$

where $\gamma = \mu - \lambda$ and $x = 0, 1, \ldots$. The parameter λ was estimated by the method of moments from the distribution of urban places among Iowa counties. Table 2 compares observed and expected frequencies for the parameters $\lambda = 0.74$, $\gamma = 0.20$ and $\mu = 0.94 \cong 93/99$.

Comparison of M_2 and Iowa

A synthetic pattern was constructed from the pattern M_2 for the parameters

$$t_1 = t_2 = 1 \quad g = \pi/2$$

$$\sigma = 0.2286 \quad \lambda = 0.7396 \quad \gamma = 0.1979.$$

These parameters were applied to a space containing 96 lattice sites, so that M_2 contained 71 S_2 points and 19 R points. Tables of random digits and standard normal deviates were used to generate a synthetic M_2. Because of the small pattern size, random digits and normal deviates were tested for randomness.

The M_2 and Iowa patterns were described by (i) distances from origin points to the 10 nearest neighbors and (ii) distances from loci to the 10 nearest points. The boundary constraint was applied so that the number of recorded measurements tends to decrease as the order of neighbor increases.

Order mean distances are listed in Table 3 for point to point measurements and in Table 4 for locus to point measurements. The tabulated data on M_2 give mean distances for the 10 lower order neighbors and the number of recorded measurements for each order. Distances obtained from the Iowa map were standardized by multiplying each observed mean order distance by the square root of the density of urban places. The tabulated data on Iowa give the standardized mean distances and approximate miles for the 10 lower order neighbors. Also tabulated are the absolute and percentage differences between the observed and calculated mean order distances. Many other properties of M_2 and Iowa were collected but are not included in this report.

There are many reasons for not conducting an elaborate analysis for goodness-of-fit of the M_2 data to the Iowa data. Important reasons include the small size of the fabricated M_2 and difficulty in transforming frequency distributions into the normal form. These and similar problems could, largely, be handled in a more careful experimental design. More control was not exercised because I wanted a fast, crude evaluation of an imperfection model to determine whether it possessed any empirical reference; and, hence, merited detailed consideration. A fair test of the imperfection approach to urban systems requires a substantially more sophisticated model than M_2

Though recognizing the 'imperfections' in M_2, it seems sufficiently provocative to justify release of this highly preliminary report. While statistical methods were used to evaluate hypotheses of no difference between M_2 and Iowa (which were not rejected by the available data), reports on levels of significance and other statistical findings do not seem particularly critical at this stage of development.

Evaluation

The synthetic pattern M_2 reproduces with considerable fidelity the Iowa map pattern of urban places. The correspondence between M_2 and Iowa is a statistical rather than a cartographic similarity. This criterion of similarity determines the type of conclusions that can be drawn from the present study.

Both patterns were summarized by sets of distance measurements. These distances represent, however, quite different conceptualizations. The Iowa pattern refers to an observed distribution that exists in the real world, and at a point in time a study area has a single pattern of urban places. In contrast, the synthetic pattern represents a probabilistic model that is an abstract construction. This model does not describe one map pattern. Instead, the model defines a set of theoretical values. It is possible to interpret the model and synthetically construct a pattern that is representative of the model; yet, the model generates only one of an infinity of different patterns that correspond precisely to the statement of the model.

In more formal terms, the reduction of the distribution of urban places to order distances in a one-to-one mapping but the reduction of the model to a pattern is a one-to-many mapping. So, for the Iowa distribution only one pattern is formally possible (all representations must be conformal) while the mapping of the model is multi-valued. Consequently, while a single map describes the Iowa pattern, there is no cartographic summary of the pattern contained within the theoretical model.

While we reduce a map to a set of numbers we do not return a corresponding set of numbers to the map form. The cost of reducing the Iowa map pattern to a system of equations describing an imperfection model is the loss of the map description of that pattern. Whether this loss is compensated by the substantially greater analytical utility of a mathematical construction is a question that each student must resolve for himself.

In evaluating these questions the role of simulation should be correctly interpreted. Simulation was used only after all parameters of the model were estimated. This is not general in social science investigations of large, complex systems by means of simulation. Often, the model is simulated many times, each run using a different set of parameter values. The model being simulated is then adjudged successful if some set of parameters provides a good fit to the data at hand. This iterative approach is based upon an a priori acceptance of the model. In this application the simulation is used primarily to study properties of a complex model, but it does not provide any independent means of verifying the model itself. Simulation was not used for this purpose; for the imperfection concept simulation serves as the poor man's (mathematically poor, that is) numerical integration of a completely specified probabilistic model which can not be evaluated by analytic methods.

Table 1

Frequency Distributions of Observed and Calculated Standardized Distances, c_1, from Geographic Center of Interior Counties Containing an Urban Place to Nearest Urban Place

Distance	Freq.	Dist.	Error	$\frac{(f_0-f_c)^2}{f_0}$
c_1/σ	f_0	f_c	$f_0 - f_c$	
0- .243	11	11.72	- .72	0.471
- .486	11	11.04	- .04	0.000
- .729	11	9.82	1.18	0.127
- .972	8	8.23	- .23	0.005
-1.215	6	6.51	- .51	0.237
-1.458	3	4.85	*-1.85*	*0.052*
-1.701	5	3.39	*1.61*	
-1.944	2	2.28	**- .28**	**0.265**
-2.187	2	1.41	**.59**	
-2.430	2	.83	**1.17**	
Over 2.430	0	.92	**- .92**	
Total	61	61		1.157 ($\equiv \chi^2$)
				df=4

$.90 > Pr(\chi^2 = 1.157) > .75$

Iowa data, f_0 from Dacey (1963a). The standard deviation is $\sigma = 0.2286$. The calculated frequency, f_c, is from the unit half-normal distribution.

Table 2

Comparison of Observed Distribution of Urban Places per County in Iowa, 1950, with Expected Distribution of Points per Primitive cell of M_2

Number of Places x	Frequency Distributions Observed $g(x)$	Expected $E(x)$
0	21	21.1
1	64	64.2
2	13	12.4
3	1	1.2
≥ 4	0	.1

Observed values are from Dacey (1963a). Expected values are computed from (20) with $\lambda = .74$ and $\gamma = .2$.

Table 3

Comparison of j Order Distances for M_2 and Iowa Maps

Order	M_2		Iowa		Error	
j	n_j	$\bar{r}_j$	$d_0^{1/2}\bar{R}_j$	Mi.	$\bar{r}_j - d_0^{1/2}\bar{R}_j$	As % of Iowa
1	65	0.63	0.66	16	-.03	4.7
2	58	0.84	0.84	21	.00	
3	56	0.98	0.99	25	-.01	1.4
4	55	1.12	1.12	28	.00	
5	53	1.24	1.24	31	.00	
6	46	1.35	1.36	34	-.01	1.0
7	44	1.46	1.49	37	-.03	2.1
8	41	1.54	1.60	40	-.06	4.0
9	37	1.65	1.68	42	-.03	2.0
10	36	1.74	1.78	44	-.04	2.0

Iowa data are from Dacey (1963a).

Table 4

Comparison of h Order Distances for M_2 and Iowa Maps

Order	M_2		Iowa		Error	
h	n_j	$\bar{r}_h$	$d_0^{1/2}\bar{R}_h$	Mi.	$\bar{r}_h - d_0^{1/2}\bar{R}_h$	As % of Iowa
1	40	0.42	0.41	10	.01	4.7
2	36	0.72	0.72	18	.00	
3	32	0.97	0.93	23	.04	4.2
4	31	1.07	1.13	28	-.06	4.8
5	29	1.21	1.26	31	-.05	4.0
6	28	1.32	1.39	35	-.07	4.8
7	28	1.43	1.45	36	-.02	1.8
8	27	1.55	1.56	39	-.01	0.8
9	22	1.62	1.65	41	-.03	1.6
10	20	1.71	1.74	43	-.03	1.9

Iowa data are from Dacey (1963a).

October 14, 1963 Philadelphia, Pennsylvania

This original paper by Dacey, when printed in the *Papers* of the Michigan Inter - University Community of Mathematical Geographers, was supplemented with an 'Addendum' reflecting computer programs current at the time by Professor Duane F. Marble and Mr. Marvin Tener, and a second examination of the Iowa data by Dacey (December 13, 1963). A Glossary by Nystuen offered expanded explanations of complicated material for readers uncomfortable with notation. The added materials are not reprinted here.

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Readers of *Solstice* might also be interested to note the following additional references to Dacey's work, not noted in the MICMOG publication.

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5. FEATURES

Construction Zone: The Braikenridge-MacLaurin Construction

The projective plane is often thought of as the Euclidean plane with a line of infinity attached. The line at infinity is composed of the infinity of points at infinity, each of which can be viewed as the intersection point for sets of parallel lines. Such generality can offer enlightenment.

The Braikenridge-MacLaurin construction (Coxeter 1974) offers a strategy for constructing a conic through five given points in the projective plane. Imaginary lights suggest how the construction traces out the locus of a conic in the projective plane.

Given five points, A, B, C, A', B' (Figure 1). Represent each of these by a relatively large white light bulb. Join A to B' and A' to B by lighting, one at a time, a series of small white light bulbs from A to B' and from B to A'. Designate the intersection point of these two lines, N, by a white bulb larger than those along the lines, but not quite as large as those representing the five given points. Choose an arbitrary line, z_1, through N; draw it using a sequence of small red lights. Join A' to C by a line of small red lights. Label the intersection M, of $A'C$ and z_1, with a medium-sized red light. Join B' to C by a line of red lights. Label the intersection L of $B'C$ and z_1 with a medium-sized red light. Join A to M by a line of small red lights and join B to L by a line of small red lights. Label the intersection C_1' of AM and BL with a medium red light. The point C_1' lies on the conic.

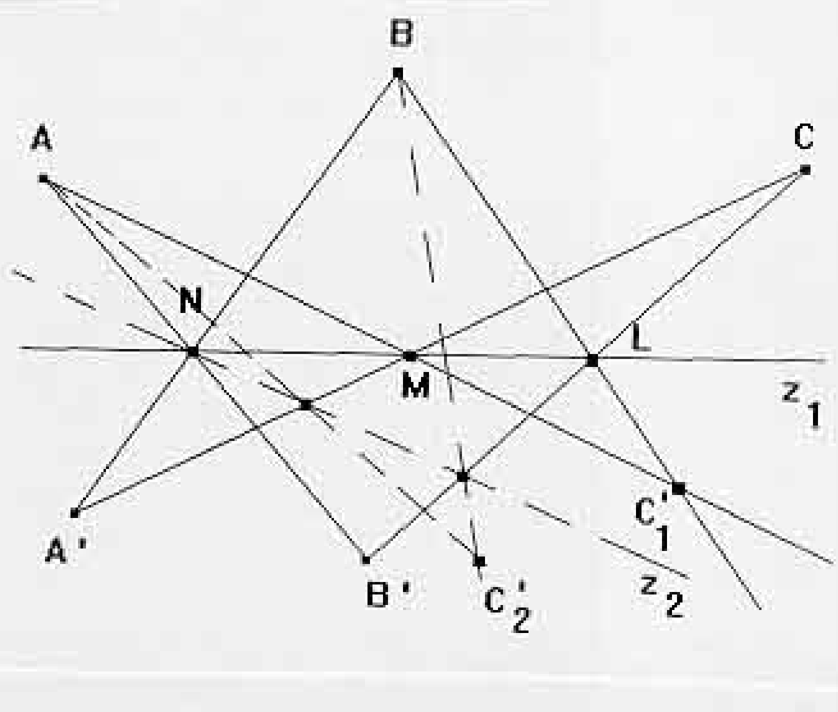

Figure 1. Braikenridge-MacLaurin Construction of a conic through five given points, A, B, C, A', and B' in the projective plane.

Now turn off all red lights except the one representing C'_1. Draw, using a sequence of small green lights, a line z_2 (different from z_1), through N. Repeat this construction, using green lights, producing in the end another point, C'_2, on the conic. Leave the green light representing C'_2 on and turn the others (green ones) off. Repeat this process using enough (three) different colors (a "Four-color Theorem" type of idea) to trace out the locus of the conic in lights!

Happy Holidays!

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Population Environment Dynamics Course and Monograph

Once again, *Solstice* board member William D. Drake invited S. Arlinghaus to co-teach a course in Population Environment dynamics based on Drake's ideas of transition theory. For the third consecutive year their efforts, together with those of the many fine students, have resulted in an interesting monograph, authored almost totally by the students. The student authors and content of *Population - Environment Dynamics: Towards Public Policy Strategies* are as listed below:

Deborah Carr, Stability in Rural Communities: Myth or Reality?

Cheri DeLaRosia, Population-Environment Trends in the Modernization of Thailand;

Rohinton Emmanuele, A City in Transition: Urban Demographic Changes in Detroit and Their Impact on Urban Greenness and Climate;

Noah Hall, Coastal Protection and the Coastal Population-Environment Dynamic;

Timothy Macdonald, NAFTA and the Human Element, A Region in Transition;

Soonae Park, Demographic Transition and Economic Growth in Korea: Comparison between Asian Countries;

Carlos de la Parra, Analysis of Transitions in the U.S.-Mexico Border;

Brent Plater, Population Policy and Environmental Quality;

Shelley Price, A Framework of Pollution Prevention and Life-Cycle Design: Aiding Developing Nations through Transition to Industrialization;

Richard Wallace, Motor Vehicle Transport and Global Climate Change: Policy Scenarios;

Tracy Yoder, An Inquiry into Determinates of Fertility.

6. SAMPLE OF HOW TO DOWNLOAD THE ELECTRONIC FILE BACK ISSUES OF *SOLSTICE* ON A GOPHER

Solstice is available on a GOPHER from the Department of Mathematics at Arizona State University: PI.LA.ASU.EDU port 70

BACK ISSUES OF *SOLSTICE* AVAILABLE ON FTP

This section shows the exact set of commands that work to download *Solstice* on The University of Michigan's Xerox 9700. Because different universities will have different installations of TEX, this is only a rough guideline which *might* be of use to the reader. (BACK ISSUES AVAILABLE using anonymous ftp to open um.cc.umich.edu, account IEVG; type cd IEVG after entering system; then type ls to get a directory; then type get solstice.190 (for example) and download it or read it according to local constraints.) Back issues will be available on this account; this account is ONLY for back issues; to write Solstice, send e-mail to sarhaus@umich.edu.

First step is to concatenate the files you received via bitnet/internet. Simply piece them together in your computer, one after another, in the order in which they are numbered, starting with the number, "1."

The files you have received are ASCII files; the concatenated file is used to form the .tex file from which the .dvi file (device independent) file is formed. They should run, possibly with a few harmless "vboxes" over or under. ASSUME YOU HAVE SIGNED ON AND ARE AT THE SYSTEM PROMPT, #.

create -t.tex

percent-sign t from pc c:backslash words backslash solstice.tex to mts -t.tex char notab

(this command sends my file, solstice.tex, which I did as a WordStar (subdirectory, "words") ASCII file to the mainframe)

run *tex par=-t.tex

(there may be some underfull (or certain over) boxes that generally cause no problem; there should be no other "error" messages in the typesetting–the files you receive were already tested.)

run *dvixer par=-t.dvi

control *print* onesided

run *pagepr scards=-t.xer, par=paper=plain