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PROGRESS REPORT
ON
BASIC RESEARCH IN MATHEMATICS
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and
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PROGRESS REPORT

ON

BASIC RESEARCH IN MATHEMATICS

INTRODUCTION

The preceding progress report, dated July 2, 1951, covered research from September 1, 1950, to June 1, 1951. The present report covers the research carried on from June 1, 1951, to June 15, 1952.

On September 16, 1951, the contract N8 ONR 71400 (University of Michigan project M801) was terminated and the new contract Nonr-330(00) (ONR Project NR 045 087, University of Michigan Project M960) went into force as a continuation of the previous one. Thus the present report describes research under both contracts.

A list of the personnel follows:

June 1, 1951, to September 1, 1951

Professor S. B. Myers, working on Banach spaces.

Dr. H. C. Davis, working on quantum mechanics.

Dr. A. J. Lohwater, working on complex variables.

Dr. Sze-Tsen Hu, working on topological groups.

Dr. D. J. Lewis, working on algebra.

Mr. R. Barrar, working on partial differential equations under the direction of Professor E. H. Rothe.

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Mr. J. G. Hocking, working on topology under the direction of Professor G. S. Young.

Mr. W. E. Jenner, working on algebra under the direction of Professor R. Brauer.

Mr. R. MacDowell, working on Banach spaces under the direction of Professor S. B. Myers.

Mr. W. Smith, working on Banach spaces under the direction of Professor S. B. Myers.

September 1, 1951, to June 15, 1952

Mr. E. Crisler, working on harmonic functions under the direction of Professor W. Kaplan.

Mr. T. W. Hildebrandt, working on theory of elasticity, under the direction of Professor G. E. Hay. (Left project Feb. 1, 1952.)

Mr. J. Kaiser, working on partial differential equations under the direction of Professor E. Rothe. (Left project Feb. 1, 1952).

Mr. R. G. Kuller, working on Banach spaces under the direction of Professor S. B. Myers.

Mr. J. D. Miller, working on topology under the direction of Professor R. L. Wilder. (Joined project Feb. 1, 1952).

Mr. G. Prins, working on foundations of mathematics under the direction of Professor R. L. Wilder.

Mr. R. Raimi, working on Banach spaces under the direction of Professor S. B. Myers.

Mr. J. Shoenfield, working on foundations of mathematics under the direction of Professor R. L. Wilder.

Mr. D. Wend, working on Complex variables, under the direction of Professor W. Kaplan. (Joined project Dec. 1, 1951).

DESCRIPTION OF INDIVIDUAL PROJECTSProfessor S. B. Myers: Rotations of a Banach Space.

A study was initiated of the group G of rotations (norm-preserving linear transformations) of a Banach space B . Various natural topologies are used on G , and an analysis of the structure of G is begun. B can be shown to be a Hilbert space if and only if for every pair of points on the unit sphere of B there is a member of G which interchanges the pair. If B is finite-dimensional, G is a Lie group; transitivity of G over E is probably sufficient to make B a Hilbert space. An important problem is to determine for which infinite-dimensional B one can say that every compact subgroup of G is a Lie group. This relates to an unsolved problem concerning compact groups of homeomorphisms of a manifold.

Dr. H. C. Davis: Second Quantization

Dr. Davis began by examining, according to plan, certain apparent difficulties in the theory of second quantization. It quickly appeared that these were no more than imprecisions in statement, which could be made rigorous in a comparatively straightforward way. A more substantial problem remained in the fact that Fermi-Dirac quantum statistics has never been brought under the general quantum theory of fields, but is based on a purely ad hoc construction. Dr. Davis spent most of the summer looking for a formulation of the one-electron problem from which the correct second-quantized theory could be obtained by the general procedure used for other particles. This seemed to be feasible by the method used, but no conclusive results have yet been found.

Dr. A. J. Lohwater: Boundary Values of Meromorphic and Harmonic Functions

The following theorems were established:

Let $w = f(z)$ be meromorphic in $|z| < 1$ and let $|f(re^{i\theta})|$ have limit l , as $r \rightarrow 1$, for almost all θ of an arc A of $|z| = 1$. If P is a singular point of $f(z)$ on A , then for every real λ either $e^{i\lambda}$ is in the range $R(P)$ of $f(z)$ at P , or else there exists a Jordan arc L in $|z| < 1$ terminating at a point ζ of $|z| = 1$ arbitrarily close to P , such that as $z \rightarrow \zeta$ along L , $f(z) \rightarrow e^{i\lambda}$ (i.e., $e^{i\lambda}$ is an asymptotic value of $f(z)$ "near P "). If, furthermore, $f(z)$ has a finite number of zeros and poles in a neighborhood of the singular point P , then the union of $R(P)$ and

the set of asymptotic values of $f(z)$ "near P " is precisely one of the sets $|w| \leq 1$, $|w| \geq 1$, or the extended plane. These theorems extend results of Carathéodory [Comm. Math. Helv. 19 (1947) p. 266], Nevanlinna [Ann. Acad. Sci. Fennicae (A) 32 (1929) No. 7, p. 28] and Seidel [Trans. Amer. Math. Soc. 36(1), (1934), p. 208].

Let $u(r, \theta)$ be a harmonic function with bounded mean modulus for $r < 1$. If $u(r, \theta) \rightarrow 0$ as $r \rightarrow 1$ for almost all θ and if on the remaining set (of measure 0) the radial limit, wherever it may exist, is finite, then $u(r, \theta) \equiv 0$ in $r < 1$; is on the remaining set (of measure 0) the radial limit is infinite only on a denumerable set $\{\theta_k\}$ ($k = 1, 2, \dots$), then there exist real constants C_k with $\sum |C_k| < \infty$ such that, in $r < 1$,

$$u(r, \theta) = \sum_{k=1}^{\infty} C_k \frac{1 - r^2}{1 + r^2 - 2r \cos(\theta - \theta_k)}$$

These results include an unpublished uniqueness theorem of P. C. Rosenbloom and clear up a question raised by Zygmund [Math. Zs. 25 (1926) p. 284]. Some of the results are included in a paper by Dr. Lohwater in the Proceedings of the American Mathematical Society vol. 3 (1952) pp. 478-279.

Dr. Sze-Tsen Hu: Topological Groups

The following are the accomplishments of this project:

- (1) The preparation of a paper entitled "Cohomology Theory in Topological Groups" for publication in the Michigan Mathematical Journal.
- (2) The completion of the major part of a research on finite dimensional compact connected groups. Let G be a finite dimensional compact connected group and G_0 a closed connected subgroup.

The purpose of this work is to reduce the Čech cohomology ring $H(G/G_0)$ of the homogeneous space G/G_0 with real coefficients to that of the homogeneous space of some compact connected Lie group. In fact, it is well known that there is a totally disconnected closed central normal subgroup Z of G such that the Quotient group $M = G/Z$ is a Lie group. Let $h: G \rightarrow M$ denote the natural projection and let $M_0 = h(G_0)$. The main result of this work states that the cohomology rings $H(G/G_0)$ and $H(M/M_0)$ are isomorphic. In particular, if G_0 reduces to the neutral element e , we obtain $H(G) \approx H(M)$. This result enables us to generalize a number of theorems proved for compact connected Lie groups by H. Hopf and his school. A paper entitled "On Finite Dimensional Compact Connected

"Groups" has been prepared and submitted to the Annals of Mathematics.

Dr. D. J. Lewis: Topics in Algebra

Dr. Lewis made a concentrated study of class field theory by attending lectures, reading, and conferring with Professors Artin and Brauer. He reworked his thesis of 1950 on "Cubic Homogeneous Polynomials over p-adic Number Fields" and prepared it for publication. Efforts were again made to classify quartic homogeneous polynomials over finite fields which have only singular solutions; some sharper results were obtained. This problem was reduced to the classification of homogeneous polynomials which assume only quadratic residues modulo a prime. All efforts to resolve the latter question proved fruitless. A number of results concerning the norm groups of algebras were obtained and are being prepared for publication. The principal result is that the norm group of a central simple algebra of dimension nine over an algebraic number field is exactly that field.

Mr. R. Barrar: Parabolic Partial Differential Equations

(Research Direction by Professor E. H. Rothe)

During the project period Mr. Barrar completed the work necessary to apply the method developed by Schauder for elliptic equations (Math. Zs. vol. 38) to linear parabolic partial differential equations. The existence theorems obtained are an improvement over those previously published. A first draft of a doctoral dissertation was prepared on the basis of these results.

Mr. J. Hocking: Topology

(Research Direction by Professor G. S. Young)

Mr. Hocking continued to work in the direction indicated in the progress report of July 2, 1951. He encountered difficulties in the proof of one of the key results and found that a much deeper analysis was required.

Mr. W. E. Jenner: Arithmetic of Rings and Algebras

(Research Direction by Professor R. Brauer)

Mr. Jenner worked on his doctoral dissertation and nearly completed it. He studied the arithmetic of rings and algebras. In particular, he considered representations of ideals as direct intersections. An ideal is called a block ideal if it cannot be written as a direct intersection (except in a trivial manner). Under suitable assumptions,

every ideal has a unique representation as a direct intersection of block ideals. In the case of an algebra A over an algebraic number field K , the block ideal components of a prime ideal \mathfrak{p} of K in A are studied. These block ideals generate an abelian group. The block ideal components of a prime ideal \mathfrak{p} of K are completely determined by the behavior of \mathfrak{p} in the center of K . The results are used to obtain information about multiplicative decomposition of ideals of K in A .

Mr. R. MacDowell: Banach Spaces

(Research Direction by Professor S. B. Myers)

Mr. MacDowell continued work on his dissertation. His characterization of the Banach space of continuous functions over the product of two compact Hausdorff spaces has led him to a study of the space of continuous functions on a fiber bundle.

Mr. W. Smith: Banach Algebras

(Research Direction by Professor S. B. Myers)

Mr. Smith continued his study of Banach algebras of continuous mappings of a compact Hausdorff space into a Banach algebra.

Mr. E. Crisler: Harmonic Functions and Integrable Functions

(Research Direction by Professor W. Kaplan)

If $f(\theta)$ is Lebesgue integrable for $0 \leq \theta \leq 2\pi$, then the Poisson integral formula defines a corresponding function $u(r, \theta)$, harmonic for $r < 1$, with radial limit equal to $f(\theta)$ for almost all θ . Results of Kaplan, Boothby, and others provide certain decompositions of the family of level curves of $u(r, \theta)$. From these decompositions one obtains decompositions of the function $f(\theta)$. Mr. Crisler has been studying these decompositions with the purpose of obtaining new characterizations of the class of integrable functions.

Mr. T. W. Hildebrandt: Stresses and Deflections in Preloaded Spherical Shells.

(Research Direction by Professor G. E. Hay)

The problem under consideration is the static behavior of a shallow spherical shell supporting the loads (1) a uniform normal pressure ρ_1 (the preload), and (2) an axial force P uniformly distributed over a small circular area of the shell.

Starting with the general differential equations for a spherical shell as given by Reissner¹, a system of linearized equations is developed, together with specific conditions for linearization. These conditions are as follows:

$$\frac{|P|}{\pi a^2 \xi_0^2} \ll \frac{2E}{B} \left(\frac{h}{a}\right)^2 \quad \text{for } \mu = \frac{|\rho| a^2}{4} \frac{B}{Eh^2} \leq \frac{1}{2}$$

$$\ll \frac{2\mu}{B} \frac{2E}{B} \left(\frac{h}{a}\right)^2 \quad \text{for } \mu \geq \frac{1}{2} ,$$

where a = radius of the shell, h = thickness, ξ_0 = colatitude of edge of area supporting P , E = Young's Modulus, $B = \frac{Eh}{\sqrt{12(1-\nu^2)}}$, and ν = Poisson's ratio. In words, the maximum numerical value of the pressure due to the axial force P is independent of the preload ρ for $\mu \leq 1/2$, but may increase linearly with ρ for $\mu \geq 1/2$. The same linearized system is valid for both cases of the linearization conditions.

The system has a closed solution for all values of μ when the preload is directed outward ($\text{sgn } \rho = +1$), and for $\mu < 1$ when the preload is directed inward ($\text{sgn } \rho = -1$). For the case $\mu < 1$, the solution appears in terms of Bessel functions of complex argument, while for $\mu \geq 1$, $\text{sgn } \rho = +1$, the solution appears in terms of Bessel functions of pure imaginary argument (modified Bessel functions). Thus all functions which appear in the solutions are currently available in tabulated form.

Mr. J. Kaiser: Bessel Functions in Connection with Elliptic Partial Differential Equations in a Space of Constant Curvature

(Research Direction by Professor E. H. Rothe)

The Laplacian operator can be defined in an invariant manner for an arbitrary Riemannian manifold with positive definite metric. This research concerns the equation $\Delta_2 u + \lambda u = 0$ for such a generalized Laplacian Δ_2 . It is assumed that the space is three dimensional and that the curvature is constant. In appropriate coordinates a separation of variables is possible. One of the Sturm-Liouville problems arising through the separation is studied in detail and the solutions are expressed in terms of generalized Bessel functions. The results overlap recent ones of M. U. Olevsky.

¹ E. Reissner, "On Axisymmetrical Deformations of Thin Shells of Revolution". Proc. Symp. on Appl. Math., 3, pp. 27-52, 1950.

Mr. R. G. Kuller: Banach Algebras

(Research Direction by Professor S. B. Myers)

Mr. Kuller has been studying Banach algebras, particularly the algebra of differentiable real-valued functions. He will probably need another year to complete his dissertation.

Mr. J. M. Miller: Topology

(Research Direction by Professor R. L. Wilder)

Mr. Miller has taken an active part in several seminars on an advanced level. This has necessitated his doing a wide range of reading in topology, group theory, topological algebra, and function spaces. He should be ready to start on his doctoral dissertation in the near future.

Mr. G. Prins: Foundations of Mathematics

(Research Direction by Professor R. L. Wilder)

Mr. Prins has concentrated on laying a foundation for his research program by studying the works of Carnap, Church, Gödel, and other writers on symbolic and mathematical logic. In addition he has studied chapters of a forthcoming book by Beth, as well as other notes that have been made available to him. He should be about ready to commence his doctoral dissertation.

Mr. R. Raimi: Banach Spaces

(Research Direction by Professor S. B. Myers)

Mr. Raimi has been working on various questions concerning linear transformations in Banach spaces, with possible applications to ergodic theory. He will probably need another year to complete his dissertation.

Mr. J. Shoenfield: Foundations of Mathematics

(Research Direction by Professor R. L. Wilder)

Mr. Shoenfield has written the first draft of his doctoral dissertation, entitled "Models of Formal Systems". Therein he investigates axiomatic systems and independence and consistency proofs by use of models within formal systems.

Mr. D. Wend: Riemann Surfaces

(Research Direction by Professor W. Kaplan)

By application of the Schwarz reflection principle, one can use a portion of a parabolic Riemann surface to generate, by reflections, a class of parabolic surfaces. Mr. Wend has been studying this geometric process, with special emphasis on Riemann surfaces of inverses of polynomials.

