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MAGNETICALLY SENSITIVE ELECTRICAL RESISTOR MATERIAL

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ABSTRACT

In this final report a summary is given of theoretical and experimental progress made in the field of the galvanomagnetic effects during the entire period of about three years of this project. For details the reader is referred to previous reports. Part of the present and the previous report has constituted the greater part of the doctoral dissertation of L. P. Kao. The unfinished character of the work is partly due to the very great complexity of the subject, and also partly to the fact that the research is still being carried on.

The scientific motivation of this research is based primarily on two facts which became clear very early. Figure 1 shows that galvanomagnetic effects under very symmetric conditions have a detailed structure. According to the state of the electron theory three years ago, one could conjecture that this structure contains interesting information about the electronic energy surfaces and affords possibly the most direct way to learn all the details of these surfaces. It is of fundamental importance to have a way of getting experimentally at the details of the energy surfaces. In order to develop this way, the theoretical work of this project was undertaken and is being continued. The experimental work was undertaken in order to parallel the theory and to explore the galvanomagnetic effects of certain special systems. It soon became clear that meaningful measurements were hardly possible until the theory had made considerable progress.

In order to enable us to describe the galvanomagnetic effects properly and completely, the phenomenological theory had to be developed first (see Section B). This theory, moreover, is a prerequisite for taking meaningful measurements. It is based on the validity of Ohm's law. The dependence of the components of the conductivity tensor on the magnetic field is represented by a three-dimensional Taylor series in powers of the magnetic field components. The coefficients of the various powers are the galvanomagnetic constants (brackets).

Onsager's relations imply certain parity relations for the galvanomagnetic effects, given in Section B-2. The relations between the brackets due to any type of crystal symmetry have been developed previously and are sketched in Section B-3 for the case of bismuth. Further progress of the phenomenological theory, applied to special cases, is reported in Section B-4, 5, and 6.

Section C describes the methods used in preparing single crystals of bismuth and in measuring its zeroth-, first-, and second-order galvanomagnetic

constants (brackets). Results of measurements are given for temperatures between room temperature and -150°C (see Tables III, V, and VII). The relative smallness of the brackets $[200]_{23}$ and $[011]_{11}$ can be understood qualitatively by noting the similarity between the bismuth lattice and a hexagonal structure. Considering bismuth as a distorted face-centered structure, however, does not clarify any of the relative magnitudes of the brackets.

In Section D an attempt is made at generalizing the electronic theory of the galvanomagnetic effects. By eliminating undue restrictions of the customary approach, general expressions for all the brackets are obtained in terms of the relaxation function $\tau(\underline{k})$ and the energy function $E(\underline{k})$. Under these very general conditions a set of lemmas is derived which governs necessary and sufficient conditions for the vanishing of certain galvanomagnetic effects. This analysis sheds light on, among other things, the question of why previous theories often have led to predictions of zero longitudinal magneto resistance, contrary to experiment, and it is now clear how such discrepancies can be avoided. It is believed that the analysis of the conditions which make for small galvanomagnetic effects is a first step in the direction of finding large effects.

The appendix contains proofs of various theorems and a list of corrections for previous reports.

OBJECTIVE

This project aimed at developing the understanding of the magneto resistance effect (change of electrical resistivity in a magnetic field) by theoretical and experimental research, with the ultimate aim of developing materials with more favorable magneto resistance properties than are available at present.

A. INTRODUCTION

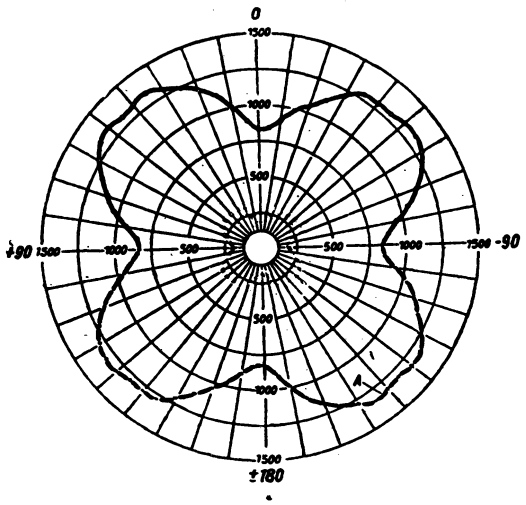
The present report is intended to be self-contained regarding general ideas, while for details the reader will be referred to previous reports.

The purpose of this research has been restricted, up to the present, to gaining more insight into the quite complex directional properties of the galvanomagnetic effects. Soon after the beginning of the research in 1953 it became evident from the literature that the galvanomagnetic effects, that is, the effects of a magnetic field on the electrical conductivity of a single crystal, are very complex. The magneto resistance effect, the Hall effect, the Corbino effect, and several other names represent merely special cases of the galvanomagnetic effect. The complexity of the magneto resistance effect is illustrated by Fig. 1, which represents the variation of the electrical resistance of a single cubic gold crystal as a function of the azimuth angle at which the magnetic field is applied (after Justi[1]). The other galvanomagnetic effects are necessarily of similar complexity.

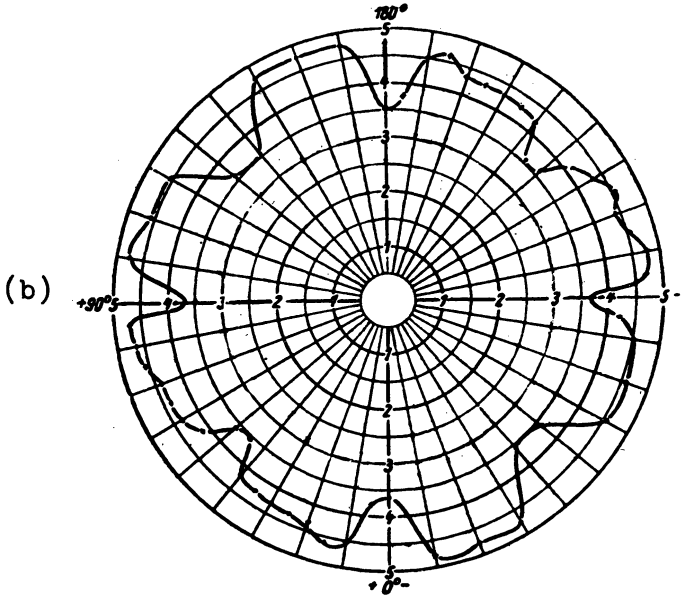
Before any basic understanding, pure or applied, of the information concealed in this articulate effect can be developed, it is first necessary to develop a phenomenological theory. The purpose of a phenomenological theory is to describe measurements systematically, to permit comparison of different measurements, and to guide the experimenter in measuring basic data. It also provides a frame with which future electron theories must comply.

No systematic phenomenological theory for the highly anisotropic galvanomagnetic effects was available and consequently many misconceptions could be found in the literature. We have developed the phenomenological theory, an outline of which is given in Section B of this report, while details are found in previous reports. This theory defines the mode of describing the effects in the following way. The components of the electrical-conductivity tensor are expanded in a power series in terms of the powers of the three components of the magnetic field $\underline{B}$, along suitably chosen coordinates. The coefficients of this series expansion are the material constants by means of which the galvanomagnetic effects are completely described. These coefficients are called "brackets." Measurements must aim at determining them.

Older approaches had terminated such power series after the first or second power of B, but Fig. 1 clearly implies that higher powers are by no means negligible. The coefficients of such power series are not all independent. The symmetry of the crystal, and of the measuring arrangement with respect to it, defines relations between the coefficients which the phenomenol-



(a)



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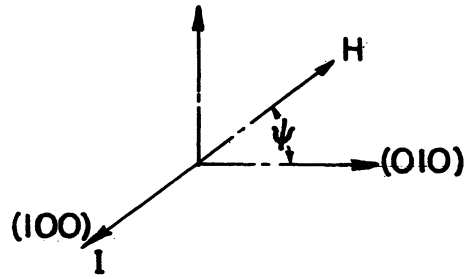

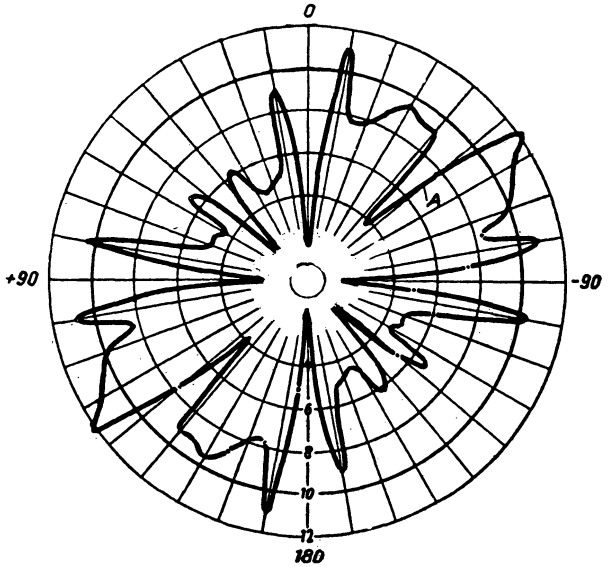

(c)

Fig. 1

Magneto resistance effect $\Delta\rho/\rho_o$ in transverse magnetic field for cubic single crystals with current along (100), as a function of the angle between magnetic field and (010) direction, after E. Justi.

Curve a	Tungsten,	T	=	14.0°K,	H	= 18600 gauss
Curve b	Gold,	T	=	20.4°K,	H	= 35000 gauss
Curve c	Gold,	T	=	4.2°K,	H	= 18600 gauss

ogical theory gives. These relations must be known for any given material of whatever symmetry, before one can plan measurements which will permit the determination of the brackets. In addition, much of the complexity of Fig. 1 will turn out to become qualitatively understandable as a result of the symmetry relations.

We chose bismuth as a first substance for measurement. It is known to have the largest galvanomagnetic effects among the elements. Its anisotropy is pronounced. Meanwhile, a number of investigators have published measurements of some of the brackets of bismuth (or data from which these can be derived), so some checks are possible. The present report gives the results of our own bracket measurements of bismuth, between room temperature and -150°C. The method of preparing oriented single crystals of bismuth and the method of measurement, a discussion of certain sources of error which the new phenomenological theory helped us to avoid, and the results of the measured brackets are given in Section C.

The work is clearly open in three directions. First, brackets of higher order than given presently should be measured for bismuth. Second, other materials, selected for possibly obtaining larger effects than with bismuth, or other desirable features, should be prepared and measured. Third, the electron theory of the galvanomagnetic effects, of which only a very unsatisfactory beginning is presently in existence, should be developed further. Mutual support of work in these three directions can be expected.

Preliminary steps in these directions have been reported previously and some additions are given in Section D of the present report. Work along these directions is being continued.

Because of the relatively large amount of groundwork that was found to be necessary, the present report, although final in name, does not answer the question that initiated this work in anywhere nearly final form. This question was, in what direction to search for materials with more desirable magneto resistance properties. The conclusions are: For pure bismuth the magneto resistance depends on the orientation and the temperature. The effects of temperature are almost independent of the orientation. The most sensitive orientation is obtained when the sample length (direction of current flow) is midway between two binary axes and the magnetic field is along the third binary axis, transverse to the sample. The influence of alloying agents studied so far decreases the magneto resistance, but several avenues have not yet been explored.

B. PHENOMENOLOGICAL THEORY

1. DEFINITIONS

In an isothermal, nonferromagnetic, single crystal of arbitrary symmetry, placed in a homogeneous magnetic field $\underline{B}$, a constant current density $\underline{J}$ is maintained by means of a suitable electric field $\underline{F}$. Evidently,

$$\underline{F} = \underline{F}(\underline{J},\underline{B}) \quad . \tag{1}$$

The dependence of $\underline{F}$ on $\underline{B}$ represents the galvanomagnetic effect. The component F^d of $\underline{F}$ along an arbitrary direction $\underline{d}$ can be measured by means of potential probes. If $\underline{d}$ is parallel to $\underline{J}$, the resulting dependence of $F_{//}(\underline{J},\underline{B})$ is called the magneto resistance effect; if $\underline{d}$ is normal to $\underline{J}$, then $F_{\perp}(\underline{J},\underline{B})$ is called a Hall effect. Both are special cases of the galvanomagnetic ("g.m.") effect.

For weak fields $\underline{B}$, the Hall effect is known to be proportional to B; the magneto resistance effect is proportional to B^2. For stronger fields, and especially at lower temperatures, the dependences are much more complicated. In 1905 Voigt laid the foundation for an appropriate description of the anisotropy of the g.m. effects, but a general treatment of the anisotropy was not available up to the present.

The nature of the problem demands that two coordinate systems be used in the phenomenological theory. The results of g.m. measurements are best described in terms of "laboratory coordinates," which are defined by the geometry of the electrodes applied to the crystal. On the other hand, the effects of the crystal symmetry are best described in terms of "symmetry coordinates," which are defined by the orientation of the crystal axes. Thus the effects of crystal symmetry on measured quantities must be found in two steps:

a. The effects of anisotropy as described in the symmetry coordinates.
b. A transformation of the results from symmetry to laboratory coordinates.

In this way a completely general formalism, covering all crystal symmetries and all geometries of measurement, as well as arbitrary orientation of the magnetic field, has been obtained explicitly.

The laboratory coordinates x^{α} ($\alpha = 1,2,3$) are defined as follows:

x^1 is taken along the current density $\underline{J}$,
x^2 is the plane of $\underline{J}$ and $\underline{d}$, and
x^3 accordingly.

These definitions imply

$$J^2 = J^3 = 0 \quad . \tag{2}$$

In the case of magneto resistance, $\underline{d}$ lies along $\underline{J}$, allowing one degree of freedom for x^2 and x^3 in the plane normal to x^1. Vector and tensor components with respect to the laboratory coordinates will carry Greek superscripts. The convention of summation over repeated indices will be followed.

2. PARITY RELATIONS

Merely on the basis of a couple of very general assumptions a number of far-reaching parity relations can be derived. We assume Ohm's law:

$$F^\alpha = \rho^{\alpha 1} J^1 \quad . \tag{3}$$

The g.m. resistivity components $\rho^{\alpha 1}$ depend on $\underline{B}$. We also assume the validity of Onsager's relations:

$$\rho^{\alpha\beta}(\underline{B}) = \rho^{\beta\alpha}(-\underline{B}) \quad . \tag{4}$$

Taking $\alpha = \beta = 1$ in (3) and (4), it is seen that the magneto resistance is an even function of $\underline{B}$ without exception. Taking $\alpha \neq 1$ in (3) and (4), it is seen that the Hall effect is in general neither an even nor an odd function of $\underline{B}$. However, in a number of special configurations the crystal symmetry may impose an even or an odd parity on the Hall effect. The complete list of such configurations was shown to be as follows.

Consider the crystallographic point group, obtained from that of the crystal by augmenting it with an inversion center.* Then one can easily prove with respect to the rotation axes of this augmented group:

a. The Hall effect is odd if either
   1. $\underline{B}$ lies along a rotation axis of order higher than 2 and either $\underline{J}$ or $\underline{d}$ is normal to $\underline{B}$, or if
   2. $\underline{B}$ is normal to an axis of even order and either $\underline{J}$ or $\underline{d}$ is along that axis.

b. The Hall effect is even if either
   1. $\underline{B}$ lies along any rotation axis and is coplanar with $\underline{J}$ and $\underline{d}$, or if

*Because $\underline{B}$ is an axial vector, it is invariant under inversion, as is $\rho^{\alpha\beta}$. Hence the proper group to be considered is the augmented group.

2. $\underline{B}$, $\underline{J}$, and $\underline{d}$ are normal to the same axis of even order.

c. The Hall effect vanishes if $\underline{B}$ and either $\underline{J}$ or $\underline{d}$ lie along one rotation axis (of any order).

The "new" galvanomagnetic effect recently reported by Goldberg and Davis[2] illustrates cases b.2 and c (see Fig. 2).

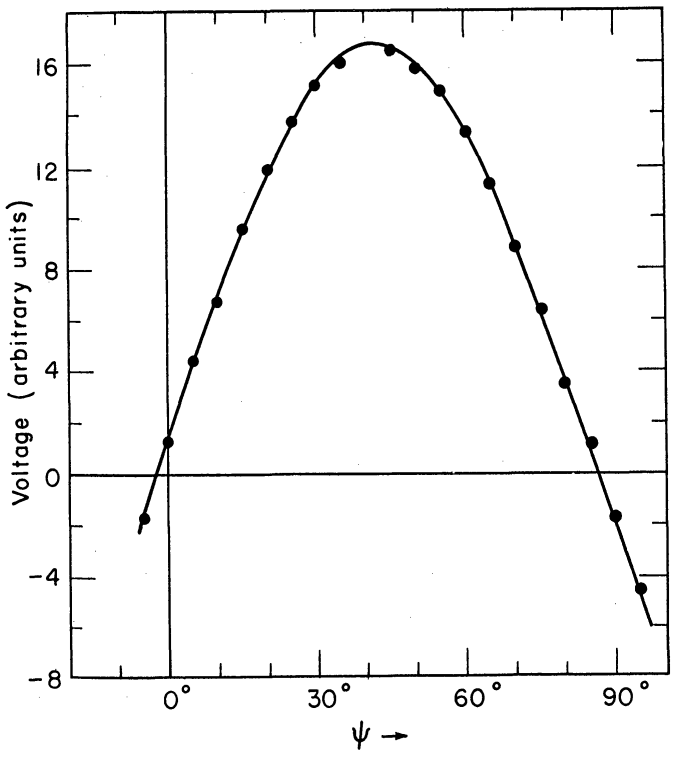

Fig. 2. Dependence of the "planar Hall voltage" on the angle ψ between current and magnetic field for a Ge sample, after C. Goldberg and R. E. Davis.[2]

3. THE EFFECTS OF CRYSTAL SYMMETRY

For the purpose of describing the effects of crystal symmetry it is convenient to refer to "symmetry coordinates" k_i $(i = 1,2,3)$. These are adapted to the crystallographic point group of the crystal augmented again by an inversion center, as follows:

k_3 is taken along the rotation axis of highest order, except for T_h, where it is taken along a twofold axis,
k_1 is taken along a twofold axis normal to k_3 if there be one, and
k_2 accordingly.

Vector and tensor components with respect to the symmetry coordinates will carry Latin subscripts. The components $\sigma_{ij}(\underline{B})$ of the conductivity tensor and $\rho_{ij}(\underline{B})$ of the resistivity tensor are functions of $\underline{B}$, characteristic

of the material at any given temperature, and independent of the electrode geometry of galvanomagnetic measurements.

Most g.m. measurements suggest that $\sigma_{ij}(\underline{B})$ can be expanded in a three-dimensional Taylor series in powers of the components B_1, B_2, B_3, thus:

$$\sigma_{ij}(\underline{B}) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{p=0}^{m} [m-p,\ p,\ n-m]_{ij}\ B_1^{m-p}\ B_2^{p}\ B_3^{n-m} \qquad (5)*$$

$$= \sum_{n=0}^{\infty} B^n \sum_{m=0}^{n} \sum_{p=0}^{m} [m-p,\ p,\ n-m]_{ij}\ \gamma_1^{m-p}\ \gamma_2^{p}\ \gamma_3^{n-m}\ . \qquad (5a)$$

Here γ_i are the direction cosines of $\underline{B}$ with respect to the symmetry coordinates. The coefficients, the "brackets," are independent of $\underline{B}$. They are the true phenomenological g.m. constants, dependent only on the kind of material and the temperature. It was shown that the occurrence of terms cos $n\phi$ in the Fourier analysis of measurements, such as represented in Fig. 1, requires that terms up to the n-th power be retained in (5). It is evident that many terms will often be required, so only a phenomenological theory for all n is satisfactory.

The effects of crystal symmetry on the measured functions $\rho^{\alpha_1}(\underline{B})$ are obtainable in two steps:

a. The effects of anisotropy on the brackets.
b. The dependence of $\rho^{\alpha_1}(\underline{B})$ on $\sigma_{ij}(\underline{B})$.

The effects of crystal symmetry on the brackets are derived from the requirement of invariance under a symmetry operation of the augmented group of the crystal. There are eleven such groups and the resulting symmetry relations for all of these have been tabulated in previous reports, the trigonal and hexagonal tables giving all relations up to and including the sixth power of $\underline{B}$, while the tables of the other groups give the relations for all powers of $\underline{B}$. Also, formulas for the number of independent brackets for each of the eleven classes were tabulated.

For bismuth (augmented group D_{3i}) the first few relations are reproduced in Table I. The interpretation of this table is as follows. The place of a bracket in the table is important. If a place is occupied by its own bracket, as indicated by the first row and column, this bracket is chosen independently. If it is occupied by another expression, the bracket of that

*The measurement of ρ^{α_1} would seem to suggest a power-series expansion of $\rho_{ij}(\underline{B})$ instead of $\sigma_{ij}(\underline{B})$. The latter is preferred because its coefficients admit of a simpler electron theoretical interpretation. However, the symmetry properties of the ρ and σ brackets are exactly the same. The results to be reported for σ brackets can, therefore, be applied directly to ρ brackets if desired.

TABLE I

BRACKET RELATIONS FOR BISMUTH

	ij	11	22	33	
n=0	[000]	$[000]_{11}$	$[000]_{11}$	$[000]_{33}$	
	ij	23	31	12	
n=1	[100]	$[100]_{23}$	0	0	
	[010]	0	$[100]_{23}$	0	
	[001]	0	0	$[001]_{12}$	
	ij	11	22	33	23
n=2	[200]	$[200]_{11}$	$[200]_{22}$	$[200]_{33}$	$[200]_{23}$
	[020]	$[200]_{22}$	$[200]_{11}$	$[200]_{33}$	$-[200]_{23}$
	[002]	$[002]_{11}$	$[002]_{11}$	$[002]_{33}$	0
	[011]	$[011]_{11}$	$-[011]_{11}$	0	$[011]_{23}$
	ij	23	31	12	
	[011]	$[011]_{23}$	0	0	
	[101]	0	$[011]_{23}$	$[011]_{11}$	
	[110]	0	$2[200]_{23}$	$[200]_{11} - [200]_{22}$	

place is dependent and equal to the expression found at that place. For example:

$$[110]_{12} = [200]_{11} - [200]_{22} \quad .$$

Because of the Onsager relations, the ij values 32,13,21 do not require explicit tabulation. Indeed,

$$[m-p,\ p,\ n-m]_{ij} = (-)^n\ [m-p,\ p,\ n-m]_{ji} \quad . \tag{6}$$

Any brackets with $n \leq 2$, not listed in Table I and not reducible to it by (6), vanishes.

The number of independent brackets for bismuth, it was shown, is $1/2\ n^2 + 2n + 2$ when n is even and $1/4\ n^2 + n + 3/4$ when n is odd. Similar formulas for other groups were given previously. The explicit expressions for $\rho^{\alpha 1}$, which result from Table I after transformation to laboratory coordinates, have also been tabulated in previous reports, up to n = 2. The expression for bismuth is:

$$\rho^{\alpha 1}(\underline{B}) = \left\{[000]_{11}^{-1}\,(\ell_1{}^1\ell_1{}^\alpha + \ell_2{}^1\ell_2{}^\alpha) + [000]_{33}^{-1}\,\ell_3{}^1\ell_3{}^\alpha\right\} +$$

$$+ B\left\{[100]_{23}\,[000]_{11}^{-1}\,[000]_{33}^{-1}\,(\gamma_1\ell_1{}^\beta + \gamma_2\ell_2{}^\beta) + [001]_{12}\,[000]_{11}^{-2}\,\gamma_3\ell_3{}^\beta\right\} -$$

$$- B^2([000]_{11}^{-1}\,[000]_{33}^{-2})\left\{\gamma_1{}^2\Big[[200]_{23}\,[000]_{11}\,(\ell_2{}^\alpha\ell_3{}^1 + \ell_3{}^\alpha\ell_2{}^1) +\right.$$

$$([200]_{22}\,[000]_{33} + [100]_{23}^2)\,\ell_2{}^\alpha\ell_2{}^1 + [200]_{11}\,[000]_{33}\,\ell_1{}^\alpha\ell_1{}^1 +$$

$$\left.([200]_{33}\,[000]_{11} + [100]_{23}^2)\,\ell_3{}^\alpha\ell_3{}^1\,[000]_{11}\,[000]_{33}^{-1}\Big] +\right.$$

$$\gamma_2{}^2\Big[[200]_{23}\,[000]_{11}\,(-\ell_2{}^\alpha\ell_3{}^1 - \ell_3{}^\alpha\ell_2{}^1) +$$

$$([200]_{22}\,[000]_{33} + [100]_{23}^2)\,\ell_1{}^\alpha\ell_1{}^1 + [200]_{11}\,[000]_{33}\,\ell_2{}^\alpha\ell_2{}^1 +$$

$$([200]_{33}\,[000]_{11} + [100]_{23}^2)\,\ell_3{}^\alpha\ell_3{}^1\,[000]_{11}\,[000]_{33}^{-1}\Big] +$$

$$\gamma_3{}^2\Big[[002]_{11}\,[000]_{33} + [001]_{12}^2\,[000]_{33}\,[000]_{11}^{-1}\,(\ell_1{}^\alpha\ell_1{}^1 + \ell_2{}^\alpha\ell_2{}^1) +$$

$$+ [002]_{33}\,[000]_{11}^2\,[000]_{33}^{-1}\,\ell_3{}^\alpha\ell_3{}^1] +$$

$$\gamma_2\gamma_3\Big[[011]_{11}\,[000]_{33}\,(\ell_1{}^\alpha\ell_1{}^1 - \ell_2{}^\alpha\ell_2{}^1) +$$

$$+ ([011]_{23}\,[000]_{11} - [100]_{23}\,[001]_{12})\,(\ell_2{}^\alpha\ell_3{}^1 + \ell_3{}^\alpha\ell_2{}^1)\Big] +$$

$$\gamma_3\gamma_1\Big[[011]_{11}\,[000]_{33}\,(\ell_1{}^\alpha\ell_2{}^1 + \ell_2{}^\alpha\ell_1{}^1) +$$

$$+ ([011]_{23}\,[000]_{11} - [100]_{23}\,[001]_{12})\,(\ell_3{}^1\ell_1{}^1 + \ell_1{}^\alpha\ell_3{}^1)\Big] +$$

$$\gamma_1\gamma_2\Big[([000]_{33}\,[200]_{11} - [000]_{33}\,[200]_{22} - [100]_{23}^2)\,(\ell_1{}^\alpha\ell_2{}^1 + \ell_2{}^\alpha\ell_1{}^1) +$$

$$+ 2[200]_{23}\,[000]_{11}\,(\ell_3{}^\alpha\ell_1{}^1 + \ell_1{}^\alpha\ell_3{}^1)\Big]\Big\} +$$

$$+ B^3\Big\{\ldots \qquad (7)$$

Here ℓ_i^α is the direction cosine between the i-th symmetry coordinate axis and the α-th laboratory coordinate axis, and ℓ_k^β vanishes if $\alpha = 1$, and equals the direction cosine between the k-th symmetry axis and the β-th laboratory axis ($1 \neq \beta \neq \alpha$) if $\alpha \neq 1$. Thus (7) expresses the result of measurements in terms of the direction cosines γ_i and the magnitude B of the magnetic field, the direction cosines ℓ fixing the geometry of the electrodes with respect to the

crystal axes, and the material constants (brackets).

It was shown in a previous report how these results can be extended to higher powers of B. The explicit expressions become soon formidable, however, to the point where they are hardly manageable. Thus it is useful to have general rules which cover the behavior of the series. During the period covered by the present report some progress has been made in this direction for a number of frequently occurring special cases. These will be reported in the next section.

4. FURTHER PROGRESS OF THE PHENOMENOLOGICAL THEORY

Some aspects of the phenomenological theory that have presently been worked out are described.

It can easily be shown that

$$\rho^{\alpha 1}(\underline{B}) = \rho_{ji} \ell_j^{\alpha} \ell_i^{1} = \Delta_{ij} \ell_i^{1} \ell_j^{\alpha} / \Delta \quad , \tag{8}$$

where Δ_{ij} is the cofactor of σ_{ij} in $\Delta = \det \sigma_{ij}$, and ℓ_i^{α} is the direction cosine of the laboratory coordinates axis α with respect to the symmetry coordinate i. For $\alpha = 1$ the equation represents the magneto resistance, for $\alpha \neq 1$ the Hall effect, and this is true for all that follows. Substituting the expansions of all σ_{ij} into (8) we obtain

$$\rho^{\alpha 1}(\underline{B}) = \left(\sum_{\eta=0} P_{2\eta}^{\alpha 1} B^{2\eta} + \sum_{\eta=0} Q_{2\eta+1}^{\alpha 1} B^{2\eta+1}\right) / \sum_{\eta=0} M_{2\eta} B^{2\eta} \quad . \tag{9}$$

The explicit expressions of $P_{2\eta}^{\alpha 1}$, $Q_{2\eta+1}^{\alpha 1}$, and $M_{2\eta}$ in terms of the brackets and the two sets of direction cosines can be readily obtained from (8) and (9) since M is the expansion of Δ, and P and Q are expansions of $(\Delta_{ij} \ell_i^{\alpha} \ell_j^{1})$. However, as stated above, these expansions soon require undue labor when one goes beyond the terms quadratic in B. Thus one can ask whether, for some special cases at least, the expansions simplify sufficiently to be manageable. Three classes of special cases were found in which (9) assumes practical forms. The first class, which is valid for all crystal symmetries, all measuring geometries, and arbitrary orientation of the magnetic field, was worked out previously. Here the series (9) is simply broken off after the quadratic terms. Equation (7) belongs to this class. The second class is valid for some crystal symmetries and pertains to special measuring geometry. The third class is valid for isotropic substances only, and permits arbitrary measuring geometry.

a. Galvanomagnetic Tensor for Special Orientations*.—This class of special cases applies to all crystal symmetries except S_2 and C_{2h}. The 3-axis is taken either along a rotational axis of three-, four-, or sixfold symmetry or along a twofold axis if it is accompanied by another twofold axis normal to it. This class is divided into two subclasses according to the relative orientations of the magnetic field and the current.

Case 1. Transverse Field

Assume

(1) Laboratory coordinates are along symmetry coordinates, i.e.,

$$\ell_1^1 = \ell_2^2 = \ell_3^3 = 1 \ .$$

(2) The magnetic field is along the k_3-axis, i.e., $\gamma_3 = 1$.

(3) The current flows along the k_1-axis.

According to the results of previous reports under condition (2) there are only two kinds of nonvanishing brackets, $[o\alpha\omega]_{12}$, $[ooe]_{ii}$, which will be abbreviated by $[\omega]_{12}$, $[e]_{ii}$, $i = 1,2,3$. Here ω and e represent arbitrary odd and even numbers. Then it can be shown that

$$\rho^{11}(\underline{B}) - \rho^{11}(\underline{B} = 0) = - \sum_{\eta=1} B^{2\eta} U_{2\eta} \Big/ [o]_{11} \sum_{\eta=0} B^{2\eta} N_{2\eta} \tag{10}$$

$$\rho^{21}(\underline{B}) = - \sum_{\eta=0} B^{2\eta+1} [2\eta + 1]_{12} \Big/ [o]_{11} \sum_{\eta=0} B^{2\eta} N_{2\eta} \tag{11}$$

$$\rho^{31}(\underline{B}) = 0 \ ; \tag{12}$$

where

$$\sum_{\eta=0} B^{2\eta} N_{2\eta} = \sum_{\eta=0} B^{2\eta} [o]_{11} [2\eta]_{22} + \sum_{\eta=1} B^{2\eta} U_{2\eta} \tag{13}$$

$$U_{2\eta} = \sum_{q=0}^{\eta-1} \left\{ [2(\eta-q)]_{11} [2q]_{22} + [2q+1]_{12} [2\eta - 2q - 1]_{12} \right\} \ . \tag{14}$$

*In most papers on the theory of galvanomagnetic effects the formulas for this class up to the second power of B are used instead of equation (9).

Case 2. Longitudinal Field

Assume

(1) Laboratory coordinates are along symmetry coordinates, i.e.,

$$\ell_1^1 = \ell_2^2 = \ell_3^3 = 1 \ .$$

(2) The magnetic field is along the k_3-axis, i.e., $\gamma_3 = 1$.

(3) The current flows along the k_3-axis.

Then

$$\rho^{33}(\underline{B}) - \rho^{33}(\underline{B}=0) = - \sum_{\eta=1} B^{2\eta} [2\eta]_{33} \,/\, [o]_{33} \sum_{\eta=0} B^{2\eta} [2\eta]_{33} \qquad (15)$$

$$\rho^{13}(\underline{B}) = \rho^{23}(\underline{B}) = 0 \ . \qquad (16)$$

Note that Equations (10-16) have the same parity as predicted in Section B-2, from which one can in fact write down equations (12) and (16) directly.

b. Explicit Expression of $\rho^{\lambda\mu}(\underline{B})$ for Isotropic Substances.—Upon the introduction of a magnetic field, an isotropic sample behaves anisotropically, as if it possessed a rotation axis of infinite-fold order along the direction of the magnetic field. In this case one can always take the 3-axis along the magnetic-field direction; thus only brackets of the form $[oon]_{ij}$, or $[n]_{ij}$, are involved. There are only three kinds of nonvanishing brackets: $[ooe]_{11}$, $[ooe]_{33}$, and $[oo\omega]_{12}$. Therefore, remembering that $[ooo]_{11} = [ooo]_{33}$, the explicit forms of $P_{2\eta}^{\lambda\mu}$, $Q_{2\eta+1}^{\lambda\mu}$, and $M_{2\eta}$ contained in Equation (9) can be readily written out as follows:

$$P_{2\eta}^{\lambda\mu} = \delta_{\lambda\mu} \sum_{q=0}^{\eta} [2(\eta-q)]_{11} [2q]_{33} + \ell_3^{\lambda} \ell_3^{\mu} \left\{ \sum_{q=0}^{\eta} [2(\eta-q)]_{11} ([2q]_{11} - [2q]_{33}) + \right.$$

$$\left. + \sum_{q=0}^{\eta-1} [2q+1]_{12} [2(\eta-q)+1]_{12} \right\} \qquad (17)$$

$$Q_{2\eta+1} = - \ell_3^{\nu} \sum_{q=0} [2(\eta-q)]_{33} [2q+1]_{12} \qquad (18)$$

$$M_{2\eta} = \sum_{r=0}^{\eta} [2(\eta-r)]_{33} \left\{ \sum_{q=0}^{r} [2(r-q)]_{11} [2q]_{11} + \sum_{q=1}^{r} [2(r-q)+1]_{12} [2q-1]_{12} \right\}, \qquad (19)$$

where ℓ_k^ν has been used as an abbreviation defined by

$$\ell_k^\nu \equiv \ell_i^\lambda \ell_j^\mu - \ell_j^\lambda \ell_i^\mu \quad .$$

Thus, if $\lambda = \mu$, ℓ_k^ν automatically vanished in agreement with the evenness of the magneto resistance, while if $\lambda \neq \mu$, the subscripts ijk must be a permutation of 123 of the same parity as the superscripts $\lambda\mu\nu$. In that case ℓ_k^ν is the direction cosine between the symmetry axis k and the laboratory axis ν. Note that if λ-, μ-, and 3-axis are orthogonal to each other, then $\rho^{\lambda\mu}(\underline{B})$ is a purely odd function of $\underline{B}$. This result conforms with the prediction of Section B-2.

5. PHENOMENOLOGICAL THEORY OF THE CORBINO EFFECT*[3]

a. Galvanomagnetic Effects as Boundary-Value Problems.—In the previous section the phenomenological theory of the isothermal galvanomagnetic effects for single crystals was developed quite generally insofar as Ohm's law is valid, i.e.,

$$\underline{F}^\alpha = \rho^{\alpha\beta}\, \underline{J}^\beta \quad (\alpha,\beta = 1,2,3) \quad . \tag{20}$$

In order to obtain $\rho^{\alpha\beta}$ the general practice is to employ samples of convenient geometry such that the solutions of the equations

$$\text{Curl } \underline{F}\ (\underline{B}) = 0 \tag{21}$$

$$\text{Div } \underline{J}\ (\underline{B}) = 0 \tag{22}$$

satisfy certain boundary conditions. Note that the vectors $\underline{J}$ and $\underline{F}$ are now functions of the constant external magnetic field $\underline{B}$.

In Section B, we used the boundary conditions

$$J^2 = J^3 = 0 \quad . \tag{23}$$

The physical realization of (23) limits us to samples of special geometry. Boundary conditions other than (23) have also been used, related to the various sample geometries employed (described in Table II).

Volterra,[5] in 1915, made a rather extensive study of the galvanomagnetic boundary-value problems for isotropic samples. For single crystals the

*The material presented in this section can be regarded as an immediate extension of the late Prof. W. W. Sleator's work.[4] Prof. Sleator's paper was kindly brought to our attention by Prof. O. Laporte.

TABLE II

VARIOUS GEOMETRIES EMPLOYED*

Geometry	Author
Rectangular plate	Hall, 1879
Split rectangular plate Cruciform	Righi, 1883
Semicircular plate Circular plate with a radial saw cut	Ettingshausen and Nernst, 1886
Circular plate with circular electrodes	Corbino, 1911
Circular plate with point electrodes	Alimenti, 1915; Bordonavo, 1915
Square plate with two perpendicular sets of arms	Heaps, 1918
Circular cylinder with planar radial field	Poppelbaum, 1953

*See References 3, 5, and 6.

problem in general becomes much more complicated. Since the final aim of studying the galvanomagnetic effects is to obtain the material constants, or brackets, it does not seem efficient to adopt an awkward geometry involving inconvenient boundary conditions. Nevertheless, a simple case will be investigated in this chapter for historical and practical reasons. The case involves a circular disk bounded by two concentric circular electrodes. Such an arrangement is known as a Corbino disk. It was employed by Corbine[3] for the first time in 1911. In the literature, however, the "Corbino effect" has been used quite loosely to include galvanomagnetic measurements pertaining to cases where a current (rather than a voltage) transverse to the magnetic field is measured irrespective of the geometry of the sample.*

In the next section it will be shown that a simple relation exists between the Corbino effect and the other galvanomagnetic effects, for isotropic samples or properly cut single-crystal samples with a suitably oriented magnetic field.

b. Corbino Effect for Single Crystals.—Consider a circular disk cut from a single crystal such that the normal is along a three-, four-, or sixfold axis of rotation. Suppose a constant radial current is maintained at constant temperature through two concentric circular electrodes by means of an electric field. When a constant magnetic field is applied normally to the disk, either

*It is clear that the earlier controversy (see Reference 7, p. 127) regarding the relative significance of the Hall and Corbino effects becomes meaningless. They, as well as the magneto resistance effect, are special cases of the general galvanomagnetic effect.

of the following pairs of equations can be used as boundary conditions:

$$F^{\phi} = J^{z} = 0 \tag{24}$$

$$F^{\phi} = F^{z} = 0 \ , \tag{25}$$

with ϕ, r, z denoting a cylindrical laboratory-coordinate system such that the z-axis is along the 3-axis and along the normal. Either (24) or (25) lead to

$$J^{z} = F^{z} = 0 \tag{26}$$

$$J^{r} = \frac{F^{r}\rho_{11}}{\rho_{11}^{2} + \rho_{12}^{2}} = F^{r}\,\sigma_{11} \tag{27}$$

$$J^{\phi} = \frac{F^{r}\rho_{12}}{\rho_{11}^{2} + \rho_{12}^{2}} = -F^{r}\,\sigma_{12} \ , \tag{28}$$

or in terms of the brackets directly:

$$J^{r} = F^{r} \sum_{\eta=0} B^{2\eta}\,[002\eta]_{11} \tag{29}$$

$$J^{\phi} = -F^{r} \sum_{\eta=0} B^{2\eta+1}\,[00(2\eta+1)]_{12} \ . \tag{30}$$

Note that the symmetry of the problem permits one to take any two perpendicular axes in the plane of the disk as the 1- and 2-axes; thus the first halves of Equations (27) and (28) reveal general relations between the Corbino measurements J^{r}, J^{ϕ}, F^{r}, the magneto resistance ρ_{11}, and the Hall effect ρ_{12}. From these relations or Equations (29) and (30) one can immediately draw the following conclusions for a Corbino disk:

1. The radial current J^{r} is always an even function of the normal magnetic field B.

2. The circular current J^{ϕ} (which is usually known as the "Corbino current") is always an odd function of the magnetic field B.

These conclusions are also valid for isotropic samples. Experimental confirmation can be found in References 3, 5, and 6.

The simplicity of Equations (29) and (30) is noteworthy. They afford direct measurements of the brackets, which can then be used to test the microscopic theories much more readily than the usual magneto resistance and the Hall measurements. However, they involve only a limited number of brackets,

and they cannot be used to determine a complete set of brackets.

One might think that another set of boundary conditions such as

$$F^z = J^z = 0 \tag{31}$$

would be as effective as (24) or (25). But this is not the case, because of the way Equations (20) are coupled.

6. APPLICATIONS AND PROGRAM FOR FURTHER WORK ON THE PHENOMENOLOGICAL THEORY

Complete tables of the relations between the brackets for all values of n, the exponent of B, are given in the previous report for all crystal classes except the trigonal and hexagonal, for which the relations have only been tabulated up to n = 6, inclusive. It is desirable to extend the latter tables to include all values of n.

Partial or complete isotropy should be studied in detail, as well as the question of what effects can be expected of a polycrystalline sample of many randomly oriented anisotropic crystallites. The question is in what manner the higher-order brackets average out in a given measurement setup.

The question of what measurements should be taken to determine the galvanomagnetic constants, how this can best be arranged, what checking relations have to be satisfied by the measurements, is of direct experimental importance. The previous report gives the answers for the class D_{3i} of bismuth for n up to two. For the sake of completeness these results are summarized in the next section (C) of the present report. A similar analysis for higher n as well as for the other crystal classes remains to be done.

The results of measurements show that sometimes brackets with the same n-value differ by several orders of magnitude. The occurrence of exceptionally small bracket values among brackets of the same n can be attributed to two causes. In the first place, the atomic arrangement may deviate only slightly from one for which the bracket would vanish identically. In the second place, the details of the electron energy-level structure may be responsible. Analyses of the first kind of causes are required, for any given case, before causes of the second type should be invoked.

The relations between the Fourier coefficients of polar diagrams, as in Fig. 1, and the brackets remain to be worked out. However, one can state in general that the occurrence of a component cos Nϕ leads to a contribution in

$$\frac{\Delta\rho}{\rho} = \sum_{n=0}^{\infty} A_n \cos^n \phi$$

to the coefficients A_n with n = N, N-2, N-4, ... etc. Any particular A_n is the result of brackets of order $\geq$ n. Consequently, the occurrence of the component cos Nϕ requires a measurable influence of brackets of at least the order N. A superficial inspection of Fig. 1c shows, apart from a slight misalignment of the crystal which is responsible for the lack of fourfold symmetry, that values of N up to 24 certainly occur.

C. MEASUREMENTS AND RESULTS

The experimental task of measuring the galvanomagnetic effects consists of several steps. First, single crystals of given purity and dimensions must be prepared, and their crystallographic axes must be determined. Next, measuring equipment has to be made. It turns out that special equipment is required for the absolute determination of the zero-order brackets near room temperature, while relative measurements of all brackets as a function of the temperature are carried out in another setup. The measurements are then made, the galvanomagnetic material constants (brackets) are extracted from them, and their significance is discussed.

All measurements to be reported are for bismuth (group D_{3i}). The independent brackets to be determined are, for n $\leq$ 2,

$$\begin{aligned} n = 0 \quad & [000]_{11}\ [000]_{33} \\ n = 1 \quad & [100]_{23}\ [001]_{12} \\ n = 2 \quad & [200]_{11}\ [200]_{22}\ [200]_{33}\ [002]_{11} \\ & [011]_{11}\ [011]_{23}\ [200]_{23}\ [002]_{33} \ . \end{aligned}$$

The meaning of $[100]_{23}$, for example, it is recalled, is the coefficient of B_1 in the conductivity σ_{23}, i.e., the coefficient of the electric field component along x_3 when the current flows along x_2.

1. PREPARATION OF SINGLE CRYSTALS

The purpose is to grow single crystals of pure Bi with known orientation of crystal axes, in a shape suitable for electrical measurements. The raw material used was bismuth from the American Smelting Co. of New Jersey, and was said to be 99.99+% pure.

The first step is the preparation of Bi rods 2-5 mm in diameter. To obtain this, a 50-ml crucible containing about 50 gm of Bi is placed in a vacuum bell jar. Several glass tubes about 35 cm long with one end closed and an inner diameter of from 2-5 mm are placed vertically into the crucible with the

open ends down. The bell jar is then rinsed three times with He and evacuated to about 10^{-3} mm Hg. (The He rinse is done to prevent the formation of an oxide film on the Bi when heated.) By means of a heater coil around the crucible the Bi is melted and forms a seal around the lower end of the glass tubes. The glass tubes are also heated over their entire length by a vertical oven. Then He is let into the bell jar and pushes the Bi up into the glass tubes. The bell jar is then removed and the tubes are then drawn up at the rate of about 15 cm/hr out of the vertical oven past an air jet so that the Bi solidifies slowly from the upper end. This method prevents the breaking of the glass upon solidifying (Bi expands when solidifying). This method sometimes gives single crystals; the trigonal axis is then usually normal to the length of the tube. The material has a clean, shiny appearance.

The second step is the preparation of single crystals with suitably oriented axes for galvanomagnetic measurements. A Bi rod, still in glass tubing ($\approx$ 0.07-inch inner diameter), is held vertically. A heating coil is moved downward by a clock motor at the rate of about 1 inch/hr. The melting zone in the operation is about 3/8 inch long. The operation is done in about 10^{-2} mm vacuum. (At lower pressures the Bi melt would develop enough vapor pressure to kick itself out of the tube.) To start the operation the upper end of the Bi is heated to melt, and a seed crystal, mounted such that the trigonal axis of the seed is at the desired angle with the rod to be seeded, is moved down to touch the melt. After 10 min the connection is found to be in equilibrium, and the coil is moved downward as far as the desired length of the newly grown single crystal. This method of growing is very satisfactory for the trigonal axis away from the rod and is $\approx$30% successful for the trigonal axis along the rod. Vibration in the room is found to be the disturbing factor. For best results, the seed crystal should be slightly larger than the rod.

After removing the glass rod from the growing apparatus the glass around the Bi is taken off by HF in a lucite container (lucite appears to be resistant to HF) and the Bi surface cleaned by a bath of about 0.05N nitric acid, then rinsed and dried.

Crystals obtained in this way were particularly suitable for measurements of brackets with $n > 0$. For the measurement of the absolute ($n = 0$) conductivities $[000]_{11}$ and $[000]_{33}$, special samples were prepared according to a modified method, as follows. A winding of widening pitch makes it possible to apply a thermal gradient to a steel boat inside a vycor tube. Bismuth is placed in the boat, with thin sheets of mica between Bi and steel. (The purpose of the mica is to prevent any diffusion of steel into Bi; however, it was found that no observable difference results from omitting the mica as far as conductivity measurements are concerned.) After the Bi is inserted, the tube is evacuated and flushed with helium three times. After the last flush the tube is again evacuated to a pressure of approximately 10^{-3} mm of mercury.

The heater coil is then heated, allowing the Bi to melt from one end.

After the Bi is completely melted, the heating current is gradually decreased, allowing the steel boat to cool slowly but always maintaining a good thermal gradient. The Bi melt then starts to solidify along this thermal gradient and a single crystal is produced. In this process no precautions need to be taken to insure that each melt starts crystallizing from one spot in the melt. It just happens. There seems to be no preferred direction of growth. Crystals of all orientations have so far been obtained, randomly.

In order to obtain any desired axis orientation, the same method has been employed with a slight variation, using seed crystals. A seed crystal and a Bi rod are introduced into the boat. The Bi rod and part of the seed are melted to insure a good connection. The melt is then cooled from the seed as before. Single-crystal Bi rods of about 12 cm by 7 mm^2 obtained in this way were used for the measurement of the conductivity.

The seed can be gotten either from a previously grown crystal or from a method devised for this purpose as follows. A metal strip in the shape of a quarter circle, one inch wide and 1/2 inch thick, has one continuous V-groove cut along six consecutive sides of a regular 24-sided polygon, so that each straight section is about one inch long and makes an angle of 15° with adjacent sections. A previously grown crystal of arbitrary orientation is bent to lie in the V-groove, preferably with its trigonal axis in the plane of the polygon. The metal strip is then heated at one end and the Bi is melted along a temperature gradient. However, the melting is stopped short of the end of the Bi rod. This allows the end to remain solid and act as its own seed. Cooling is then started toward the opposite end. When finished there remains a single crystal of Bi in the shape of six connected straight sections. It is then cleaved at each bend, thereby producing six given seed crystals all differing by 15° from each preceeding one. Any crystal can be used as a starting point. In this way seed crystals of any desired axis orientation can be obtained.

2. DETERMINATION OF THE CRYSTALLOGRAPHIC AXES

A single crystal of Bi can be cleaved in four directions, three containing the binary axes (imperfect planes) and one normal to the trigonal axis (perfect plane). The planes can be distinguished visually. All crystal axes are determined from the perfect plane.

The trigonal axis is determined accurately by means of a shadowgraphic method. The crystal is mounted in a light beam and a 5 to 10 times enlarged image is observed. The crystal is rotated around its own axis normal to the light path, until the focused shadow of the perfect plane is a straight line. This allows for the measurement of the angle by an ordinary protractor to be accurate within 1/2°or better.

In order to determine the binary axes, the sample is mounted with its perfect plane horizontal under a vertical microscope. On the perfect plane one

can then see three sets of parallel lines intersecting at 60°. These lines are parallel to the binary axes.[8] The determination is better than 1/2°.

3. EQUIPMENT FOR THE ABSOLUTE MEASUREMENT OF ZERO-ORDER BRACKETS NEAR ROOM TEMPERATURE

Many crystals of varied orientations were grown by the procedure described in Section C-1.

A Leeds and Northrup Kelvin bridge was used to determine the resistance of the crystals with an accuracy of two parts in the fourth significant digit. Direct current was used and balancing was accomplished with a sensitive galvanometer.

The sample holder consisted of two knife edges for potential leads and two flat contacts for current leads, and could take samples from three to twenty inches in length. The base of the sample holder was made of lucite with a long V-groove cut into it. Some measurements were taken at ambient temperature in air, whereas another part of the measurements was taken with the samples and holder immersed in a water bath, thermostatically controlled to within 0.1°F. A traveling microscope and a balance were used for the measurement of the distance between the potential probes and for the mass, these quantities being required to make the results absolute.

4. EQUIPMENT FOR THE RELATIVE MEASUREMENT OF BRACKETS AS A FUNCTION OF THE TEMPERATURE

The sample holder and sample to be described below were immersed in a bath of isopentane (mp < -150°C) for temperature control. The liquid was contained in a Dewar and could be moved into and out of the magnetic field of a magnet. The isopentane in the Dewar is cooled by a controlled flow of liquid air through a copper coil submerged in the isopentane until freezing. Then the bath is warmed up slowly after the flow of liquid air is stopped. The measurement can thus be done continuously at all temperatures upward. The temperature is measured by a copper-constantan thermocouple (accuracy ± 1°C).

The magnet consists of a permanent magnet (Alnico), the strength of which can be set by an energizing coil, and then remains constant during many measurement runs. The air gap is 3-1/2 inches wide and the field in the gap was found to be constant to within one-half percent in a central region of 2-1/2-inch diameter. For most measurements the value of the field was 345 ± 3 gauss, measured both by fluxmeter and by a proton resonator. This low value was chosen for studying the lower-order brackets in order to minimize the influence of terms of higher order than the second in B.

The voltage of the potential probes is balanced for zero magnetic

field by means of a simple potentiometer bridge circuit. The change when the sample is moved into the magnetic field is amplified and recorded on a Leeds and Northrup recorder. A sensitivity of 10 μv for full-scale deflection is thus obtained, corresponding to 0.06% of the total sample resistance under average conditions of our measurement.

The sample holder (Fig. 3) is an important part of the equipment. The Dewar flask which must fit between the magnet poles limits its size. The sample holder must permit changing of the orientation of the sample with respect to the magnetic field in all possible ways. For this purpose it is fitted with an assembly of gears and graduated scales which permit the necessary changes to be made without removing the sample from its place in the thermostat or in the magnetic field. The sample holder has gone through various stages of improvement, and while further improvements are still being planned, its latest form will now be described briefly. It is made to allow three rotational degrees of freedom for the Bi sample. The Bi rod, approximately 1 inch long and 2 mm in diameter, is held, by two potential-probe springs, in a V-shaped groove cut in a rectangular lucite piece (1-1/4" x 5/8" x 3/16"). The lucite piece is mounted in a rectangular cut in a circular brass disk. The brass disk is mounted on two brass prongs such that the line joining the two prongs and the Bi rod are mutual perpendicular bisectors.

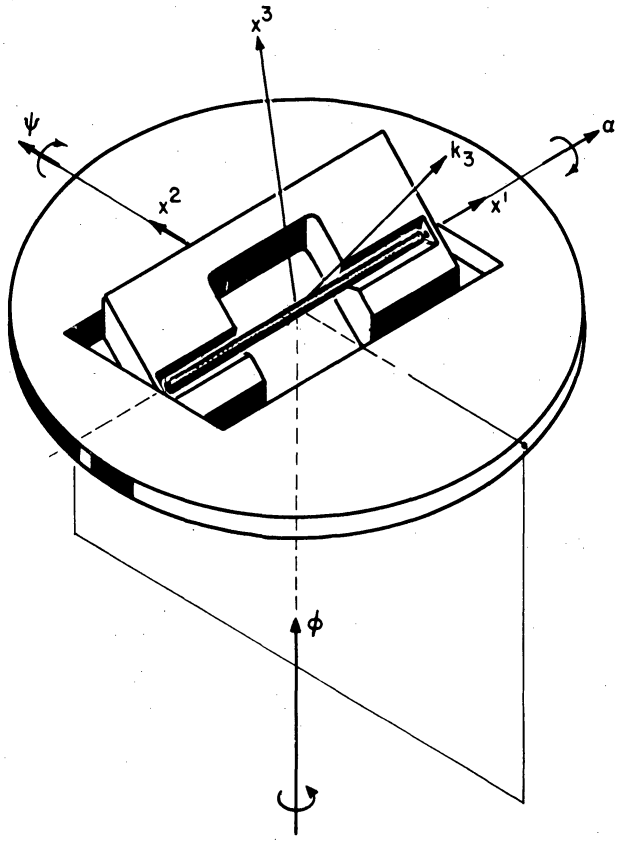

Fig. 3. Schematic diagram of sample holder, showing the three rotation axes.

The axis of rotation of the lucite piece will be denoted as the α-axis, while the axis of rotation of the brass disk is the ψ-axis. Thus α- and ψ-axes are perpendicular. The ψ-axis, in turn, can be rotated about a vertical axis, which will be denoted as ϕ-axis. The current leads are soldered to the ends of the sample. The MR probes are two sharp pieces of bronze pushing the rod against the V-groove with only slight spring pressure. The Hall probes are two bronze straight wires normal to the brass disk, pushing laterally against the sides of the rod, such that the contact points form a line parallel to the ψ-axis. The laboratory coordinates x^α are defined in Section B-1.

With the sample holder described above, the following notation will be used so as to define the orientation uniquely. First, the "positive" direction of the trigonal axis has to be chosen, arbitrarily, as either upward or downward normal to the perfect plane. It was decided to call the direction of k_3 positive if, when the crystal is placed on the sample holder while the brass disk is normal to the ϕ-axis, $\underline{k_3} \cdot \underline{\phi} > 0$. Simultaneously the positive direction of $\underline{x^3}$ is chosen: $\underline{k_3} \cdot \underline{\phi} > 0$, $\underline{x^3} \cdot \underline{\phi} = +1$. Then $\underline{x^1}$ is also chosen as the positive direction if $\underline{x^1} \cdot \underline{k_3} > 0$. Automatically then $\underline{x^2}$ is fixed. The positive rotation axis $\underline{\psi}$ is chosen along $\underline{x^2}$. The symbol α is defined as the angle around $\underline{x^1}$, and is zero when $\underline{k_3} \cdot \underline{x^2} = 0$; ψ is defined as the angle around $\underline{x^2}$, and is zero when $\underline{x^1} \cdot \underline{\phi} = +1$; ϕ is defined as the angle around the downward vertical (Fig. 3 is really drawn upside down to illustrate other features), and $\phi = 0$ when $\underline{x^2} \cdot \underline{B} = +1$; κ is defined as the angle around $\underline{k_3}$ between $\underline{x^2}$ and $\underline{k_1}$ which is along one of the binary axes, measured only when $\alpha = 0$.

5. ABSOLUTE MEASUREMENTS OF ZERO-ORDER BRACKETS (PRINCIPAL CONDUCTIVITIES) AT ROOM TEMPERATURE

In order to measure the zero-order brackets according to the expression

$$\rho = \frac{V}{I}\,\frac{m}{d\ell^2} = \left\{\frac{1}{[000]_{11}} + \cos^2\theta\left(\frac{1}{[000]_{33}} - \frac{1}{[000]_{11}}\right)\right\}, \qquad (32)$$

it is necessary to measure the resistance V/I, the mass per unit length m/ℓ, the length ℓ, and the density d for several samples with different angles θ between the trigonal axis and the rod. The resistivity measured at room temperature near 20°C is plotted for different samples vs $\cos^2\theta$ in Fig. 4 after correcting all data to 20°C, using 0.004 per °C as the thermal coefficient of resistivity. From the slope and intercept of the resulting straight line, the zero-order brackets (principal conductivities) are determined for Bi at 20°C:

$$[000]_{11} = 9.06 \times 10^3 \text{ ohm}^{-1}\text{ cm}^{-1}$$

$$[000]_{33} = 7.21 \times 10^3 \text{ ohm}^{-1}\text{ cm}^{-1} \quad .$$

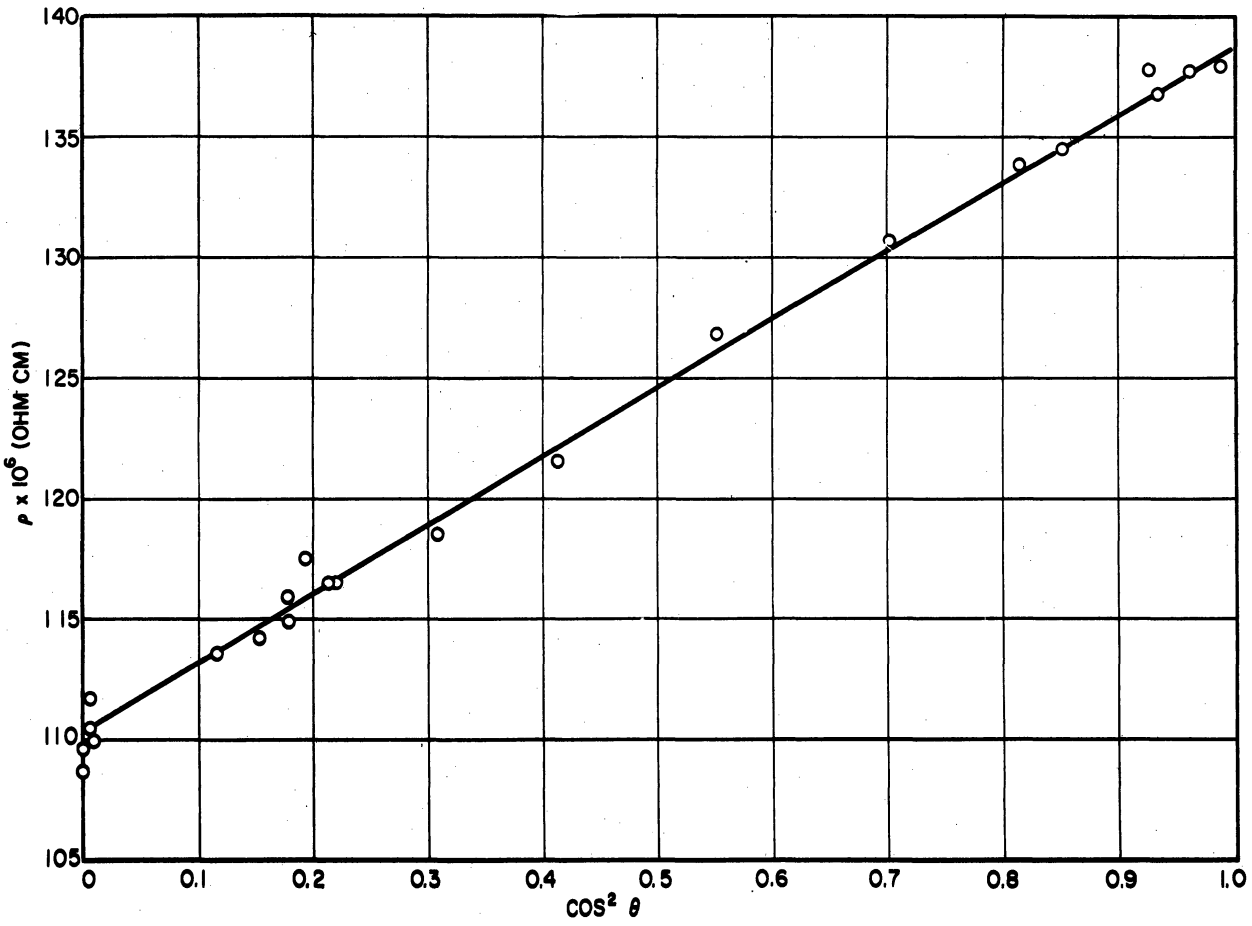

Fig. 4. Resistivity of Bi as a function of the angle Θ between wire and trigonal axis.

The error in the above results is estimated to be less than 0.7%, but a few points deviate more than this amount for reasons that are not understood presently. It is expected that improved control of purity and freedom of strain will improve these results further.

6. MEASUREMENTS OF ZERO-ORDER BRACKETS AT VARIOUS TEMPERATURES

Conductances were measured as a function of the temperature for two crystals whose trigonal axes were accurately parallel and normal to the current, by means of the equipment described in Section C-4. Conductance values were converted to conductivities by means of a conversion factor obtained by comparison with the results of the absolute conductivity measurements at room temperature as described in Section C-5. The results are given in Table III and in Fig. 5. A comparison with the results of other authors is given in the same figure and in Table IV. The agreement is not bad but the deviations are larger than the errors of measurement and must probably be ascribed to slightly different qualities of bismuth used. The ratio of the principal conductivities, which gives the ratio of the overall principal effective masses, varies from 1.18 at low temperatures to 1.25 at room temperature.

TABLE III

EXPERIMENTAL VALUES OF THE ZERO-ORDER BRACKETS AS A FUNCTION OF TEMPERATURE

T (°C)	$[000]_{11}$	$[000]_{33}$
20	9.06×10^3 $(\Omega\ cm)^{-1}$	7.21×10^3 $(\Omega\ cm)^{-1}$
10	9.36	7.44
0	9.65	7.73
-10	9.97	8.02
-20	10.33	8.36
-30	10.71	8.71
-40	11.10	9.08
-50	11.55	9.52
-60	12.02	9.92
-70	12.53	10.43
-80	13.10	10.95
-90	13.70	11.53
-100	14.40	12.16
-110	15.11	12.85
-120	15.92	13.70
-130	16.83	14.41
-140	17.80	15.28
-150	18.99	16.20

TABLE IV

COMPARISON OF ZERO-ORDER BRACKETS AMONG DIFFERENT WORKERS

Brackets	T (°C)	$\Omega^{-1}\ cm^{-1}$		
		Ours	Okada[9]	Abeles and Meiboom[10]
$[000]_{11}$	+45	--	7.94×10^3	--
	+27	8.90×10^3	--	8.53×10^3
	0	9.65×10^3	9.48×10^3	--
	-70	12.53×10^3	13.16×10^3	--
	-160	20.3×10^3	22.5×10^3	--
	-197	25.9×10^3	--	27.7×10^3
$[000]_{33}$	+45	--	6.26×10^3	--
	+27	7.02×10^3	--	6.65×10^3
	0	7.73×10^3	7.41×10^3	--
	-70	10.43×10^3	10.05×10^3	--
	-160	17.3×10^3	19.2×10^3	--
	-197	22.2×10^3	--	26.6×10^3

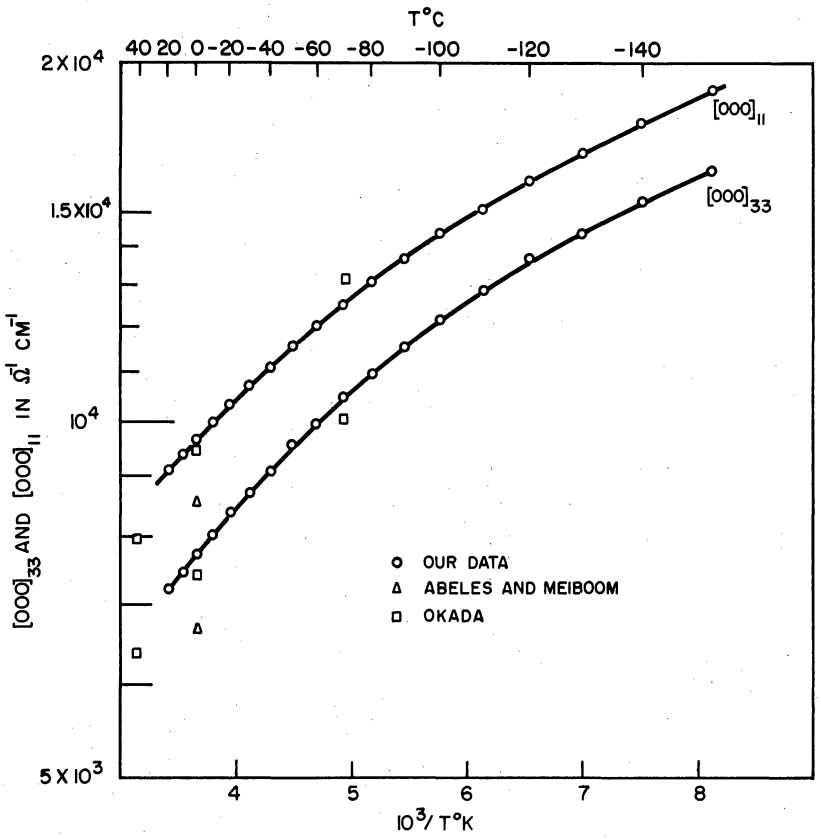

Fig. 5. The principal conductivities of Bi as a function of the temperature.

7. MEASUREMENTS OF FIRST-ORDER BRACKETS (PRINCIPAL HALL CONSTANTS) AT VARIOUS TEMPERATURES

There are two first-order brackets for bismuth, $[100]_{23}$ and $[001]_{12}$. These are measured by means of the Hall probes. The general relation between the transverse voltage V^2 across a cylindrical uniform rod carrying a current I^1 is lengthy. It simplifies considerably for two special settings of the angles involved.

Setting 1.

The sample is placed with the trigonal axis vertical (along the ϕ-axis, i.e., $\psi - \theta = 0°$. Then, using the same notation as in formula (32) one has, up to first order in B,

$$(\pi m/4d\ell)^{1/2} \left\{ \left(\frac{V^2}{I^1}\right)_H - \left(\frac{V^2}{I^1}\right)_0 \right\} = B \left\{ \frac{[100]_{23}}{[000]_{11}\,[000]_{33}} \right\} \cos\theta \sin\phi \,. \quad (33)$$

The linearity with B was checked for B up to 1400 gauss and found to hold with-

out any trace of deviation. The measurements were taken at 345 gauss, setting ϕ to 90° and to 270°. Calculations show that the errors caused by slight misalignment of the sample are negligible. Thus the bracket $[100]_{23}$ is obtained. The results are tabulated in Table V and Fig. 6. Near room temperature the accuracy is 1%; at -150° it is estimated from runs with various samples to be 5%.

Setting 2.

The sample is placed with the trigonal axis horizontal, normal to the ϕ-axis, i.e., $\psi - \theta = -90°$. If α is exactly zero, then the Hall effect is an odd function of B and one has, up to second order in B,

$$(\pi m/4d\ell)^{1/2}\left\{\left(\frac{V^2}{I^1}\right)_H - \left(\frac{V^2}{I^1}\right)_0\right\} = B\left\{\frac{[001]_{12}}{[000]_{11}^2}\right\} \sin\theta \sin\phi \ . \qquad (34)$$

The linearity with B was checked beyond 750 gauss with no detectable deviation. The measurements were again taken at 345 gauss, with ϕ at 90° and at 270°. The values so obtained for the bracket $[001]_{12}$ are two orders of magnitude smaller than $[100]_{23}$ and scattered widely. While for such small signals the noise is about 10% of the signal, the spread of the data was considerably larger. Analysis shows that the errors, due to a slight misalignment of α, are very large in this case. The reason is that in all other orientations one measures a linear combination of $[001]_{12}$ and $[100]_{23}$, and the latter, being so much larger, soon swamps the former. Special runs had to be made, seeking that particular position of α for which the signal was an extreme, in order to find reliable values for $[001]_{12}$. The results presented in the previous report for this bracket are void because we were not yet aware of this difficulty. The results given in Table V and Fig. 6 are believed to be accurate to within 10%, higher values being possible, but not lower ones. Further work is in progress to determine the $[001]_{12}$ bracket more accurately. This is necessary for later determinations of higher-order brackets. Table VI compares our data with those of other authors. This table should replace the corresponding part of the Table IV in the previous report, in which the data of other authors were not correctly converted, in addition to the above-mentioned fact that the values reported for $[001]_{12}$ were off.

8. MEASUREMENTS OF SECOND-ORDER BRACKETS AT VARIOUS TEMPERATURES

There are eight second-order brackets for bismuth: $[200]_{11}$ and $[002]_{33}$ describe the principal longitudinal magneto resistances; $[200]_{22}$, $[200]_{33}$, and $[002]_{11}$ describe the principal transverse magneto resistances; $[011]_{11}$ describes a mixed transverse magneto resistance; $[200]_{23}$ describes the principal quadratic part of the Hall effect; and $[011]_{23}$ describes a planar or mixed quadratic part of the Hall effect.

TABLE V

EXPERIMENTAL VALUES OF THE FIRST-ORDER BRACKETS AS A FUNCTION OF TEMPERATURE

T (°C)	$+[100]_{23}$	$-[001]_{12}$
20	0.94 (Ω cm gauss)$^{-1}$	4.2×10^{-2} (Ω cm gauss)$^{-1}$
10	1.07	
0	1.27	
-10	1.41	
-20	1.68	6.1
-30	1.97	6.9
-40	2.34	7.9
-50	2.73	9.0
-60	3.20	10
-70	3.86	12
-80	4.58	13
-90	5.44	15
-100	6.53	17
-110	7.89	
-120	9.60	
-130	11.5	
-140	14.1	
-150	17.6	

TABLE VI

COMPARISON OF FIRST-ORDER BRACKETS AMONG DIFFERENT WORKERS

Brackets	T (°C)	Gauss^{-1} Ω^{-1} cm^{-1}		
		Ours	Okada[9]	Abeles and Meiboom[10]
$+[100]_{23}$	+45	--	0.60	--
	+27	0.9	--	0.76
	0	1.27	1.19	1.25
	-70	3.86	3.62	3.67
	-160	22.5	25.5	--
	-197	--	--	61.7
$-[001]_{12}$ x 10^2	+45	--	3.8	--
	+27	3.9	--	3.3
	0	5.1	5.4	--
	-70	11.7	15.6	--
	-160	--	86	--
	-197	--	--	207

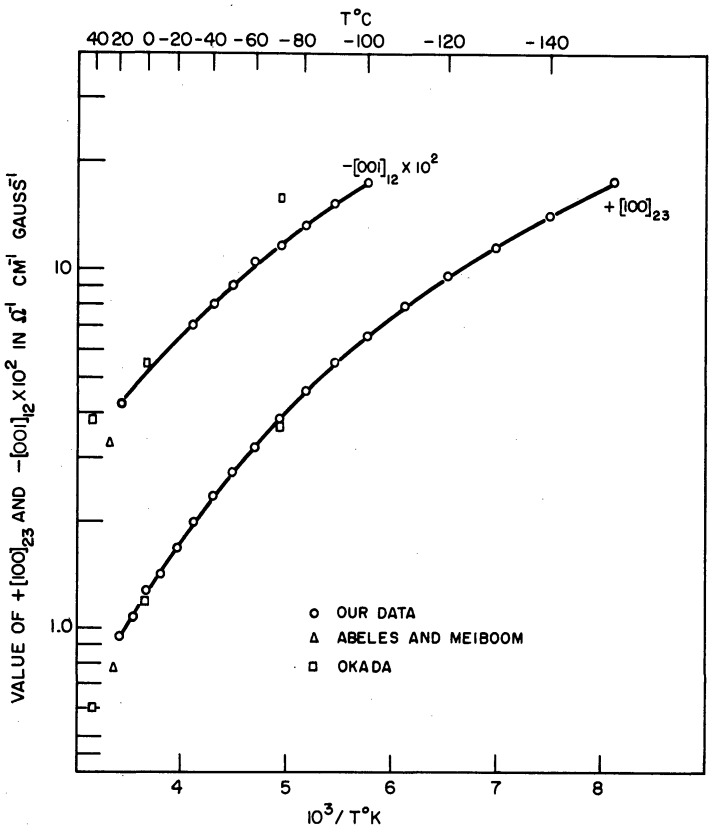

Fig. 6. The principal Hall constants of Bi as a function of the temperature.

The experimental determination of the brackets is carried out by varying ϕ with constant ψ; κ and Θ are constant for each sample, but may differ from one sample to another; α is set accurately equal to zero, according to the measurement of $[001]_{12}$ (see Section C-7).

By varying ϕ only, one can determine the three constants C_0, C_2, and ϕ_2 in the expression

$$[\Delta\rho/\rho_0 = C_0 + C_2 \cos 2(\phi - \phi_2)]_{\psi = \text{const}} , \qquad (35)$$

which describes the magneto resistance up to the second order in B. The three constants depend on the setting of the constant value of ψ and on the value of Θ of the particular sample. In order to determine the various brackets, the three constants C_0, C_2, ϕ_2 must be determined for several settings of ψ and for several samples with different Θ.

It was shown in a previous report that, by using several settings of ψ of one sample only, the following four combinations of second-order brackets can be obtained:

$$[200]_{11} - [200]_{22} , [200]_{23} , [011]_{11} , \text{and } [011]_{23} .$$

The settings of ψ used to obtain these brackets are:

Setting 1: as described in Section C-7 ($\psi - \theta = 0$), yielding

$$\Delta\rho/\rho_o = {}^1C_o + {}^1C_2 \cos 2 (\phi - {}^1\phi_2) \quad . \tag{36}$$

Setting 3: with $\psi - \theta = 45°$.

Setting 4: with $\psi - \theta = -45°$.

Only the difference between Settings 3 and 4 is used, yielding

$$(\rho_{45} - \rho_{-45})/\rho_o = {}^\Delta C_o - {}^\Delta C_2 \cos 2 (\phi - {}^\Delta\phi_2) \quad . \tag{37}$$

The identity

$$\cos 2\,{}^\Delta\phi_2 = {}^\Delta C_o/{}^\Delta C_2 \quad , \tag{38}$$

which is a direct consequence of the relations developed in previous reports, must check.

Using the abbreviations

$$P_o = [000]_{11} \left\{[000]_{33} + \cos^2\theta\,([000]_{11} - [000]_{33})\right\} \tag{39}$$

$$\Delta = -\sin\theta \sin 2\theta \sin 3\kappa \quad , \tag{40}$$

one finds, in terms of the measured quantities ${}^1C_2$, ${}^1\phi_2$, ${}^\Delta C_o$, ${}^\Delta C_2$, ${}^\Delta\phi_2$, and the constants θ and κ of the sample,

$$[200]_{23}\,[000]_{11} = \frac{P_o\,{}^1C_2}{B^2\Delta} \sin\theta \sin 2\,{}^1\phi_2 \tag{41}$$

$$([200]_{11} - [200]_{22})\,[000]_{33} - [100]^2_{23} = \frac{-4P_o\,{}^1C_2}{B^2\Delta} \cos\theta \sin(2\,{}^1\phi_2 + 3\kappa) \tag{42}$$

$$[011]_{11}\,[000]_{33} = \frac{2P_o}{B^2\Delta} \cos\theta\,{}^\Delta C_2 \sqrt{2} \sin 2\,{}^\Delta\phi_2 \tag{43}$$

$$[011]_{23}\,[000]_{11} - [100]_{23}\,[001]_{12} = \frac{2P_o}{B^2\Delta} \left({}^\Delta C_o \sin 3\kappa - \frac{1}{2}\,{}^\Delta C_2 \sqrt{2} \sin 2\,{}^\Delta\phi_2 \cos 3\kappa\right) \tag{44}$$

It was also shown in the previous report that, by using several (at least two) samples with different values of θ, the following four combinations of second-order brackets can be obtained:

$$[200]_{11} + [200]_{22} \;,\; [200]_{33} \;,\; [002]_{11} \text{ , and } [002]_{33} \quad .$$

The settings of ψ used to obtain these are:

Setting 1: as described in Section C-7 ($\psi - \theta = 0$), yielding (36)

Setting 2: as described in Section C-7 ($\psi - \theta = -90°$), yielding

$$\Delta\rho/\rho = {}^2C_o + {}^2C_2 \cos 2 (\phi - {}^2\phi_2) \quad . \tag{45}$$

The identities

$${}^1C_o + {}^1C_2 \cos 2\,{}^1\phi_2 = {}^2C_o + {}^2C_2 \cos 2\,{}^2\phi_2 \tag{46}$$

and

$${}^2C_2 \sin 2\,{}^2\phi_2 = -\frac{1}{2}\,{}^\Delta C_2 \sqrt{2} \sin 2\,{}^\Delta\phi_2 \quad , \tag{47}$$

which are a direct consequence of the relations developed in the previous report, must check.

One finds, in terms of the measured quantities ${}^1C_o$, ${}^2C_o$, ${}^2C_2$, ${}^2\phi_2$ and the constants θ of the samples (κ does not occur in these relations),

$$\left\{([200]_{11} + [200]_{22})\,[000]_{33} + [100]_{23}^2\right\}(1 - \cos^2\theta) +$$

$$2\left\{[200]_{33}\,[000]_{11} + [100]_{23}^2\right\}[000]_{11} \cos^2\theta/[000]_{33} = \frac{-2P_o\,{}^1C_o}{B^2} \tag{48}$$

$$([002]_{11} + [001]_{12}^2/[000]_{11})\,[000]_{33}\,(1 - \cos^2\theta) +$$

$$[002]_{33}\,[000]_{11}^2\,[000]_{33}^{-1} \cos^2\theta = -\frac{P_o}{B^2}\,({}^2C_o - {}^2C_2 \cos 2\,{}^2\phi_2) \quad . \tag{49}$$

By measuring several samples with different θ, and plotting the right-hand side vs $\cos^2\theta$, a straight line should result, permitting the determination of all four quantities; or, by using samples with θ equal to 0° or 90°, the solution is very simple.

The results for all eight brackets are given in Table VII, and in Fig. 7 the logs are plotted against the reciprocal temperature together with data from other authors. In Table VIII the room-temperature data of various authors are compared.

The brackets fall into two groups according to their magnitude. We consider the large brackets accurate to about 10% since they are derived from several samples in good agreement with each other. The great similarity in their temperature dependence also suggests a relatively small margin of error.

TABLE VII

EXPERIMENTAL VALUES OF THE SECOND-ORDER BRACKETS x 10^5 AS A FUNCTION OF TEMPERATURE, IN GAUSS^{-2} Ω^{-1} cm^{-1}

T(°C)	$-[200]_{11}$	$-[200]_{22}$	$-[200]_{33}$	$-[002]_{11}$	$-[002]_{33}$	$\pm[200]_{23}$*	$\pm[011]_{11}$*	$+[011]_{23}$
20	8.38	24.5	20.0	2.4	1.15	1.3	2.6	3
10	10.5	31.0	25	2.9	1.5			
0	13.2	41.0	34	3.67	2.1			
-10	16.8	50.0	43	4.75	2.7			
-20	21.7	65.6	57	6.0	3.4			
-30	27.9	85.0	74	7.5	4.25			
-40	36.9	112	98	9.3	5.3			
-50	50.0	137.5	127	11.9	7.0			
-60	67.8	194	168	15.3	9.2			
-70	90.5	263	228	20.5	12.3			
-80	121	346	303	27.3	16.3			
-90	161	463	405	36.1	22.3			

*The ambiguity in the sign of $[200]_{23}$ and $[011]_{11}$ is essential. It is due to the two possible choices of a positive direction along a given binary axis.

TABLE VIII

COMPARISON OF SECOND-ORDER BRACKETS AT 27°C AMONG DIFFERENT WORKERS

Brackets	Values x 10^5 x (-1) in Gauss^{-2} Ω^{-1} cm^{-1}		
	Ours*	Okada's	Abeles and Meiboom's
$[200]_{11}$	7.3	7.0	6.5
$[200]_{22}$	22.5	20.0	19.3
$[200]_{33}$	19.2	18.5	15.2
$[002]_{11}$	2.1	4.4	2.25
$[002]_{33}$	1.1	1.5	0.7
$[200]_{23}$	1.3	0.5	--
$[011]_{11}$	2.6	1.8	--
$[011]_{23}$	- 3.0	- 0.5	--

*Our results for $[200]_{23}$, $[011]_{11}$, and $[011]_{23}$ are for T≈25°C; Okada's values are obtained from interpolation between 0° and 45°C. Abeles and Meiboom's values are given at 27°C.

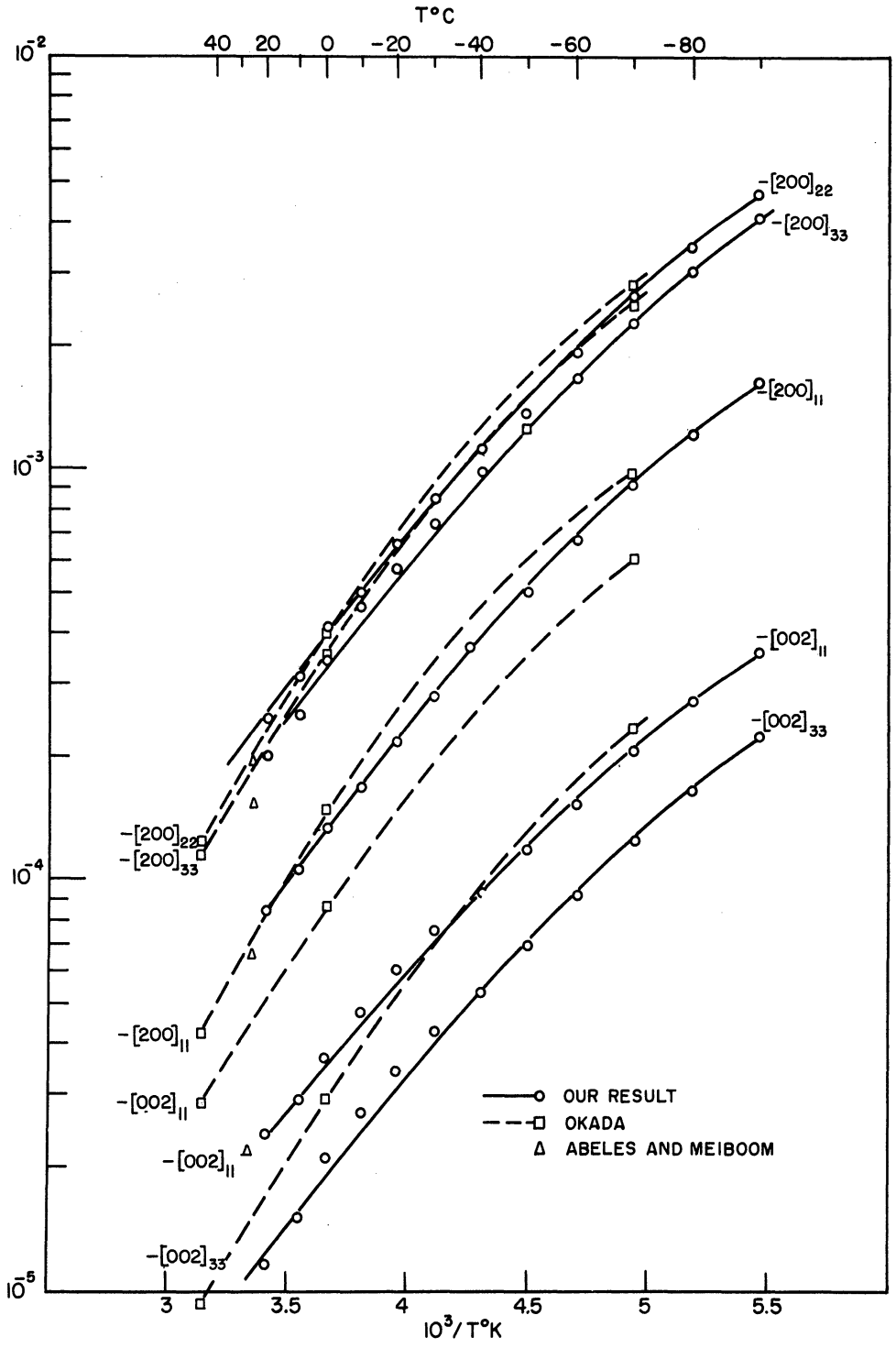

Fig. 7. Results of measurements of second-order brackets for Bi.

The small brackets, on the other hand, especially the last three, are particularly sensitive to slight errors in κ, and because of the manner in which Θ occurs, the mixed brackets are only insensitive to errors in Θ when the latter lies near 45°. This condition was satisfied by using samples grown especially for this purpose, as described. The net result of the various sources of error is an uncertainty of about 1 or 2 units x 10^{-5} for all second-order brackets at room temperature and proportionally at lower temperatures.

The data presented in the previous report for second-order brackets were only preliminary, as stated, and are superseded by the present figures.

9. DISCUSSION

At this point a purely phenomenological discussion will be given. The bismuth structure can be considered as almost belonging to the hexagonal class D_{6h}. Another way to describe the structure is to consider it as a slightly distorted face-centered cubic structure whose body diagonal forms the trigonal axis. It is worth-while to check how far a qualitative explanation of the relative magnitudes of the various brackets can be attributed to such slight distortions.

For the first-order brackets the similarity with D_{6h} gives no information. The similarity with O_h suggests equality of the two first-order brackets, contrary to observation. Thus, the difference of the first-order brackets can only be explained with more detailed electron theory.

For the second-order brackets the similarity with D_{6h} gives two relations:

$$[200]_{23} \approx 0$$

$$[011]_{11} \approx 0 \quad .$$

The similarity with O_h gives five relations:

$$2[200]_{23} \approx [011]_{11}$$

$$[200]_{33} \approx [002]_{11}$$

$$2([200]_{22} - [200]_{33}) - ([200]_{11} - [200]_{22}) + [011]_{23} \approx 0$$

$$2([200]_{11} - [200]_{22}) - ([002]_{11} - [002]_{33}) - [011]_{23} \approx 0$$

$$([200]_{22} - [200]_{33}) + \frac{1}{\sqrt{2}}[200]_{23} \approx 0 \quad .$$

The first two relations indicate that the smallness of $[200]_{23}$ and $[011]_{11}$ may be due to the similarity of the bismuth structure to D_{6h}. The first and last

relations of comparison with O_h are reasonably satisfied, but the other three relations are so much off that one may well conclude that very little influence is exerted by the similarity with the cubic symmetry and that electron theory must be called upon to explain the relative magnitudes of these brackets.

D. ELECTRON THEORY

1. INTRODUCTION

Various authors[11,12] have developed an electron theory for galvanomagnetic effects. The purpose of the present section is twofold. In the first place, existing electron theories introduce at an early stage drastic simplifying assumptions and restrictions. The present approach tries to keep the treatment more general, thus giving a starting platform for further development in the future. In the second place, a few theorems were discovered in the framework of the present more general treatment. These theorems refer to the (necessary and sufficient) conditions under which certain parts of the galvanomagnetic effects vanish. They are of interest, for example, in giving a general understanding of why earlier simplified theories usually led to the predicting of zero longitudinal magneto resistance; a simple criterion is established which enables one at the outset to judge any approximate theory in this respect. Furthermore, the theorems are of interest as a possible starting point for understanding why the galvanomagnetic constants of some materials are particularly small. It is reasonable to expect that progress in obtaining large galvanomagnetic effects is furthered by understanding the factors which tend to make the effects small. However, further work will be necessary to clarify all the issues connected with these theorems.

2. SURVEY OF THE RESTRICTIONS OF EXISTING THEORIES

All the existing microscopic theories[11,12] can be conveniently classified into two categories: the single-band model and the multiband model.* Practically all models make the following fundamental assumptions.

(A) Ohm's law is valid, i.e.,

$$J_i = \sigma_{ij}(\underline{B})\, F_j \qquad (i,j = 1,2,3) \quad . \tag{50}$$

For the multiband model with q noninteracting bands the conductivity tensor components $\sigma_{ij}(\underline{B})$ are composed additively of the corresponding components of all bands:

*The two-band model and the many-valley model encountered in the literature are special cases of the multiband model to be worked out presently. The same is true for the single-band model.

$$\sigma_{ij}(\underline{B}) = \sum_{\lambda=1}^{q} \sigma_{ij}^{(\lambda)}(\underline{B}) \quad . \tag{51}$$

(B) According to the Bloch scheme, the charge carriers within each of the q noninteracting energy bands are associated with a group velocity

$$\underline{V}^{(\lambda)} = \nabla_{\underline{k}} E^{(\lambda)}/\hbar \ , \qquad \lambda = 1,2,\ldots,q \ , \tag{52}$$

where $\underline{k}$ is the quasi-momentum and $\hbar$ is Dirac's constant. When $\lambda = 1$, the superscript will be omitted for the single-band model here and in all that follows.

(C) The number of electrons or holes per unit volume in the λ-th band whose wave vector $\underline{k}$ lies within the range $\underline{k}$ to $\underline{k} + d\underline{k}$ is

$$\frac{1}{4\pi^3} f^{(\lambda)}(\underline{k},t)\, d\underline{k} \quad . \tag{53}$$

The distribution function $f^{(\lambda)}(\underline{k},t)$ becomes $f_o^{(\lambda)}(E)$ when $\underline{B} = \underline{F} = 0$. The latter function is defined by the Fermi-Dirac distribution and the distribution of energy levels.

(D) In the presence of an electric field $\underline{F}$ and a magnetic field $\underline{B}$, $f^{(\lambda)}$ obeys the Boltzmann transport equation:

$$-\frac{e}{\hbar}\left[\underline{F} + \frac{1}{c}\underline{V}^{(\lambda)} \times \underline{B}\right] \cdot \nabla_{\underline{k}} f^{(\lambda)}(\underline{k},t) + \left[f^{(\lambda)}(\underline{k},t) - f_o^{(\lambda)}(E)\right]/\tau^{(\lambda)}(\underline{k}) = 0 \ , \tag{54}$$

-e and c being the electron charge and the speed of light. The expression $\tau^{(\lambda)}(\underline{k})$ is the relaxation time pertaining to the λ-th band whose existence is assumed. The noninteraction of the bands is expressed by the fact that no coupling exists between the equations for different values of λ (different bands).*

These assumptions are used in practically all theories of electron

*To be more specific, let us consider the case of a rather empty s-band that overlaps nearly full p-band having slow holes and fast holes. By assumption, the electrons in the s-band are the first kind of carriers, the slow and the fast holes in the p-band are the second and the third kinds of carriers, respectively; these carriers are moving in the first, the second, and the third bands, respectively, independent of each other's motion. Furthermore, no transition of electrons to the p-band and no conversion from the slow holes to the fast ones will take place, and vice versa.

conduction in solids. For their validity, see References 11-16.

The single-band model ($\lambda = q = 1$) was used by Jones and Zener[17] in 1934 and improved by Davis[18] in 1939 and by Seitz[19] in 1950. It can be expected to be a reasonable approximation only for alkaline metals and for low-concentration n-type or p-type semiconductors. Even for p-type Ge and Si its validity is limited because of the degeneracy of the valence bands. Davis' and Seitz's treatments of the single-band model are restricted as follows.

Restriction 1: Validity is limited to special orientations of the magnetic field and current relative to each other and to the crystallographic axes (Davis).

Restriction 2: Validity is limited to low values of the magnetic field (Seitz).

Restriction 3: Validity is limited to special classes of crystal symmetry (Seitz and Davis).

Although the idea can be traced back to Riecke[20] in 1898, the two-band model ($q = 2$) was first introduced by Jones[21] in 1936. Later, developments by Sondheimer and Wilson,[22] by Chambers,[23] and many others[24] followed. These studies are restricted as follows.

Restriction 4: The energy is no more general than a quadratic function in reciprocal space.

Restriction 5: The parameters used are (magnetic) field dependent, rendering it impossible to study the effect of crystal symmetry in general.

Another drawback of the two-band model in the hands of Wilson et al. is that, because of the restrictions used, it always gives zero longitudinal magneto resistance contrary to all known experiments.

The "many-valley" model is equivalent to a multiband model with ellipsoidal energy surfaces for each one of the noninteracting bands, arranged to conform with the crystal symmetry of the substance.* The model was used by Blochinzev and Nordheim[25] in 1933 and was recently elaborated by many authors.[10,26-28]

3. THE SINGLE-BAND MODEL

a. The Definition of the Brackets.—If we write

*If this model is interpreted as containing many valleys in a single band, one must justify the assumption that only intravalley transitions can take place and that overlapping ellipsoids contribute independently and additively.

$$f(k,t) = f_o(E) - \Phi \frac{\partial f_o}{\partial E}$$

and substitute into (54), omitting the superscript $\lambda = 1$ and neglecting terms in F^2 or higher, we have

$$\frac{1}{\tau(\underline{k})} \Phi + \frac{e}{\hbar} \underline{F} \cdot \nabla_k \underline{E} - \frac{e}{\hbar^2 c} \underline{B} \cdot \underline{\Omega} \Phi = 0 \quad , \tag{55}$$

where $\underline{\Omega}$ is an operator defined by

$$\underline{\Omega} \equiv - \nabla_k E \times \nabla_k = (\Omega_1, \Omega_2, \Omega_3) \quad . \tag{56}$$

The solution Φ from (55) is well known.* It can be expressed in an ascending power series of $\underline{B}$, as follows:

$$\Phi = - \sum_{n=0}^{\infty} \left(\frac{e}{\hbar^2 c} \underline{B} \cdot \tau \underline{\Omega} \right)^n \left(\frac{e}{\hbar} \tau \nabla_k E \cdot \underline{F} \right) \quad . \tag{57}$$

Consequently, the magneto-electric current density $\underline{J}$ can be written as

$$\underline{J} = (-e/4\pi^3) \iiint \underline{V} f(\underline{k},t) d\underline{k}$$

$$= (e/4\pi^3\hbar) \iiint \nabla_k E \, \Phi \frac{\partial f_o}{\partial E} \, d\underline{k} \quad , \tag{58}$$

or in component form,

$$J_i = \sum_{n=0}^{\infty} B^n \sum_{m=0}^{n} \sum_{p=0}^{m} [m-p,p,n-m]_{ij} \, \gamma_1^{m-p} \, \gamma_2^{p} \, \gamma_3^{n-m} \, F_j \quad , \tag{59}$$

where $i,j = 1,2,3$, and

$$[m-p,p,n-m]_{ij} = ab^n \iiint \left(- \frac{\partial f_o}{\partial E} \right) \frac{\partial E}{\partial k_i} P \left\{ (\tau\Omega_1)^{m-p} (\tau\Omega_2)^{p} (\tau\Omega_3)^{n-m} \right\} \left(\tau \frac{\partial E}{\partial k_j} \right) d\underline{k} \quad , \tag{60}$$

with

$$a = e^2/4\pi^3\hbar^2 \; ; \quad b = e/\hbar^2 c \quad . \tag{61}$$

*See Reference 11, p. 225, or Reference 17. Note that a sign difference in the odd powers of Φ is introduced according to the definition (56).

$P\{\ \}$ means the sum of all permutations of (m-p) operators $\tau\Omega_1$, p operators $\tau\Omega_2$, and (n-m) operators $\tau\Omega_3$, in different order. This sum consists of $n!/\{(m-p)! \cdot p!(n-m)!\}$ terms. Φ has the dimension of energy, $\Omega/\hbar^2$ that of mass^{-1}, and $b\tau\Omega$ that of gauss^{-1}; hence a bracket of order n has the dimension of Ohm^{-1} cm^{-1} gauss^{-n}. By comparing Equations (59) and (50) with (5a) it is obvious that the brackets defined by (60) are identical to those introduced in the phenomenological theory.

It is interesting to observe that the assumptions II (Onsager's relation) and III[power-series expansion of $\sigma_{ij}(\underline{B})$] of the phenomenological theory come out automatically as consequences of Equations (55) to (59). Assumption II is a consequence of the integration by parts of (60, leading to (6), while assumption III is a consequence of the solution of the Boltzmann equation. This bears out the fact that the microscopic model is consistent with the well-established thermodynamical relations (4). Note that (60) is obtained without any restriction on the form of $\tau(\underline{k})$ and $E(\underline{k})$, except that $\tau(\underline{k})$ exists and that $\tau(\underline{k})$ and $E(\underline{k})$ must have the same symmetry as the crystal plus an inversion center. The addition of the inversion center, which is a necessary consequence of the principle of time reversal, was arrived at in the phenomenological theory from the fact that the vector $\underline{B}$ is an axial vector (antisymmetric second-rank tensor). Thus all brackets behave like the components of a tensor of even rank, in the sense that they transform identically into themselves under the operation of an inversion.

Since all isothermal galvanomagnetic effects can be expressed in terms of the brackets according to (9) for arbitrary orientations of $\underline{J}$, $\underline{B}$, and the crystal axes, Equations (9) and (60) form the basis for a general microscopic theory of the isothermal galvanomagnetic effects according to the single-band model, without any of the restrictions mentioned in the previous section. Therefore, this theory contains all previous works as special cases. For crystal classes D_{2h}, D_{4h}, D_{6h}, T_h, O_h, Equations (10), (15), and (11) become Equations (8), (9), and (7) of Davis, respectively, except that all higher-power terms are written out explicitly, without introducing anything new in principle.

Further development of the theory starting from this point depends on what information is available about $\tau(k)$ and $E(k)$. Unfortunately, at the present, no reliable information about these quantities is available. A summary of various representative, simple assumptions about $\tau(k)$ and $E(k)$ is given in Table IX. In an effort to obtain further insight into the theory, the following three mathematical lemmas are useful.

b. Three Mathematical Lemmas.—

A number of general properties of the galvanomagnetic brackets can receive some clarification by means of the following lemmas, proofs of which are given in the appendix.

TABLE IX

TYPICAL ASSUMPTIONS FOR $\tau(\underline{k})$ AND $E(\underline{k})$

$\tau(\underline{k})$	$E(\underline{k})$	Authors	Applications
Constant		Kohler[29]	Kohler diagram
$\tau(E)$	$E = ak^2 + \epsilon$	Jones and Zener[17]	Li
$\tau(E)$	$E = \sum a_i k_i^2$	Jones[21]	Bi
$\tau(E)$	E even in k_1, k_2, k_3	Davis[18]	--
$\tau = a_1k^2 + a_2y_4^*$	$E = b_1k^2 + b_2y_4^*$		
$\tau = a_1k^2 + a_2y_4^*$	$E = ak^2$	Seitz[19]	Cubic crystals
	$E = \sum_{ij} \alpha_{ij}k_ik_j$	Shoenberg[30]	Bi
$\tau = \tau(E)$	Warped surface	Lax and Mavroides[31]	Ge and Si

*y_4 is the cubic harmonic of 4th order.

Lemma I. A Variation of the Schwartz Inequality.

Let $\underline{f}$, $\underline{g}$, $\underline{F}$, $\underline{G}$ be four vector functions of $\underline{k}$, then:

$$(1)\quad I \equiv \int \underline{f}\cdot\underline{f}d\underline{k} \int \underline{g}\cdot\underline{g}d\underline{k} + \int \underline{F}\cdot\underline{F}d\underline{k} \int G\cdot\underline{G}d\underline{k} - 2\int \underline{f}\cdot\underline{G}d\underline{k} \int \underline{F}\cdot\underline{g}d\underline{k} \geq 0 , \tag{62}$$

where each integral sign represents a threefold integration and

$$d\underline{k} \equiv dk_1\, dk_2\, dk_3 .$$

(2) The equality of Equation (62) holds if and only if

$$\underline{f} = c\underline{G}$$
$$\underline{F} = c\underline{g} , \tag{63}$$

where c is an arbitrary constant.

Lemma II. The necessary and sufficient condition that

$$U_{2\eta} = 0 \tag{64}$$

for all η is

$$\Omega_3(\tau E_1) = cE_2 , \tag{65}$$

where c is an arbitrary constant, $E_i = \partial E/\partial k_i$ (i = 1,2,3), and

$$U_{2\eta} \equiv \sum_{q=0}^{\eta-1} \left\{ [2(\eta-q)]_{11} [2q]_{22} + [2q+1]_{12} [2\eta - 2q - 1]_{12} \right\} , \qquad (66)$$

with the abbreviation $[\eta]_{ij} = [0,0,\eta]_{ij}$.

An interesting corollary of this lemma is as follows. If $U_{2\eta} = 0$ for any value of η, then it is zero for all values of η. The proof is left to the reader.

Lemma III. The necessary and sufficient condition that

$$[002\eta]_{33} = 0 \qquad (67)$$

for all $\eta > 0$ is

$$\Omega_3(\tau E_3) = 0 , \qquad (68)$$

provided that τ is finite and positive, which is plausible on account of its physical meaning.

A consequence of this lemma is that if $[002\eta]_{33} = 0$, then $[002\eta']_{33} = 0$ for all $\eta' \geq \eta$.

Note that similar lemmas also hold if both the inner and outer indices 1,2,3 in Equations (64) through (68) are permuted cyclically. Therefore it is expedient to combine Lemmas II and III into one lemma as follows.

Lemma. The necessary and sufficient condition that

$$U_{2\eta}^{ij} = 0 \quad \text{and} \quad [2\eta]_{ii} = 0 \qquad (69)$$

is

$$\Omega_i(\tau E_j) = E_k C_{ijk} \qquad i,j,k = 1,2,3 , \qquad (70)$$

where

$$C_{ijk} = \begin{cases} C_j & \text{if ijk is an even permutation of 123.} \\ -C_j' & \text{if ijk is an odd permutation of 123.} \\ 0 & \text{if any two indices of ijk are equal.} \end{cases} \qquad (71)$$

C_j and C_j' are arbitrary constants. $U_{2\eta}^{31}$ is defined such that Equation (66) stands for $U_{2\eta}^{jj}$; $[2\eta]_{ii}$ stands for a bracket in which all the inner indices are zero except the i-th one, i.e., $[2\eta,0,0]_{11}$, $[0,2\eta,0]_{22}$, and $[0,0,2\eta]_{33}$.

c. Special Conditions Satisfying Equation (70).—By inspection one can find various special conditions under which either Equation (65) or Equation (68) or both, i.e., (70), are satisfied. These conditions are listed in Table X. In using this table, one follows each row and reads: If I and II are given, then III is satisfied.

For example, the second row states: If $\tau(\underline{k})$ is a function of the energy $E(\underline{k})$, which is given by

$$E = \sum_{i=1,2,3} a_i k_i^2 ,$$

and if the temperature is at absolute zero, the Equation (70) is satisfied [of course (68) is also satisfied]. However, if one reads the 4th or 5th row, then only Equation (68) is satisfied.

TABLE X

SPECIAL CONDITIONS SATISFYING EQUATION (70)

I	II	III	
Relaxation Time $\tau(k)$	Energy $E(k)$	Equations Satisfied	
$\tau(\underline{k})$ = constant	$E(\underline{k}) = \sum_{i=1,2,3} a_i k_i^2$	$\Omega_i(\tau E_j) = C_{ijk} E_k$	$\Omega_i(\tau E_i) = 0$
$\tau(\underline{k}) = \tau(E)$, T = 0°K			
$\tau(\underline{k}) = Ck / \frac{dE}{dk}$	$E(\underline{k}) = E(k)$		
$\tau(\underline{k})$ = constant	$E(\underline{k}) = E\left(\sum_n a_n k^n\right)$		
$\tau(\underline{k}) = \tau(E)$	$E(\underline{k}) = E\left(\sum_n \sum_i a_{n,i} k_i^n\right)$		

The verification of Table X is straightforward and is left to the reader.

d. Discussion.—A number of interesting consequences are deduced directly from the three mathematical lemmas. These will be stated for the single-band model, whereas their validity for the multiband model is clear. The physical significance of the mathematical lemma given by Equations (69) and (70) is revealed by the vanishing of the even part of the galvanomagnetic tensor $\rho^{\alpha_1}(\underline{B})$. That is, under certain conditions one can conclude by means of

this lemma that the magneto resistance vanishes, or that the Hall effect is odd, or that terms quadratic in the magnetic field are absent. The results are stated in terms of three corollaries. Corresponding statements about the Corbino effect can be easily deduced from these corollaries and Equations (30) and (31).

Corollary I. A Condition for Zero Magneto Resistance Change.

If: (1) for all crystal symmetries except S_2 and C_{2h}, the 3-axis is taken either along a rotational axis of three-, or four-, or sixfold symmetry, or along a twofold axis if it is accompanied by another twofold axis normal to it;

(2) laboratory coordinates coincide with symmetry coordinates, i.e.,

$$\ell_1^1 = \ell_2^2 = \ell_3^3 = 1 \ ,$$

(3) the magnetic field is along k_3, i.e., $\gamma_3 = 1$; then the necessary and sufficient condition for vanishing isothermal magneto resistance change $\rho^{11}(\underline{B}) - \rho^{11}(B=0) = 0$, is Equation (70).

Proof: The proof is trivial if the mathematical Lemmas II and III are applied to Equations (10) and (15).

Corollary II. A Condition for Odd Hall Effect.

If: (1) for crystal symmetries T_h and O_h, the coordinates are taken as described in Section B-3;

(2) the magnetic field is along the k_i axis;

then the necessary and sufficient condition for $\rho^{\alpha\beta}(\underline{B})$ being an odd function of $\underline{B}$ is Equation (70).

Proof: The proof is lengthy and will be found in the appendix.

Corollary III. A Condition for the Vanishing of Quadratic Terms.

If: (1) $E(\underline{k})$ and $\tau(\underline{k})$ are even functions in k_1, in k_2, and in k_3;

(2) only terms up to B^2 are considered;

then Equation (70) is the necessary and sufficient condition for

$$\rho^{\alpha 1}(\underline{B}) - \rho^{\alpha 1}(\underline{B}=0) = \begin{cases} 0 & \text{if } \alpha = 1 \\ \text{Odd function of } \underline{B}, & \text{if } \alpha \neq 1 \ . \end{cases}$$

Condition (1) limits the validity of the corollary to crystal symmetries of D_{2h}, D_{4h}, D_{6h}, T_h, and O_h. These are general enough to include all elements except Po, Hg, As, Sb, Bi, and the transition metals. Note that for T_h and O_h, $\rho^{\alpha\beta}(\underline{B}=0)$ is zero because of the orthogonality and symmetry conditions.

Proof: See the appendix.

Corollaries I to III, supplemented by Table VIII, might be considered as the microscopic counterparts to the parity statements of Section B. However, the phenomenological statements are perfectly general, while the validity of the corollaries is limited to the microscopic models. On the other hand, the microscopic theory can, at least in principle, be extended so as to describe what happens when the conditions of the corollaries are only approximately met. This is a branch which remains for future development.

Remarks About the Longitudinal Magneto Resistance Change.—Many microscopic models lead to zero longitudinal magneto resistance change, contrary to experiment. What conditions made these calculated values zero? An answer is provided immediately by Lemma III and Table X. For example, it can be shown that, for $\eta \geq 1$, $[002\eta]_{33}$ and its cyclic equivalents are identically zero if

(a) $\tau(\underline{k}) = \tau(E)$ only [or $(\underline{k})$ is a constant],

(b) $E(\underline{k})$ is a function of $\sum_n a_n k^n$ or of $\sum_n \sum_i a_{n,i} k_i^n$.

Now $[002\eta]_{33}$ is directly proportional to the longitudinal magneto resistance change [see Equation (15)] when both the current and the magnetic field are along the 3-axis. This axis is also taken as the principal axis of a crystal class of order higher than C_{2h}. Consequently, zero longitudinal magneto resistance change resulted in the works of Jones for Bi[21] and of Abeles and Meiboom for p-type Ge.[26] Similar results under wider classes of conditions can be predicted from Table X.

It is often claimed that condition (a) can be used for interpreting experiments.* Assumptions (a) and (b) probably constitute good approximations for certain materials under suitable conditions, e.g., Cu and Ag at high temperature and under small magnetic field. Since there are more functions satisfying Equation (68) than Equation (65) (see Table X), this may imply that Equation (67) is approximately true more often than Equation (64). Thus one could speculate that one more often finds the longitudinal magneto resistance change smaller than the transverse one. However, this should not be used to rule out cases where the situation is otherwise.**

*See, for example, References 26 and 29.

**See, for example, Fig. 4 of Pearson and Suhl (Reference 32).

About the Nonnegative Magneto Resistance Change.—It has been found experimentally that, as a rule, $\rho^{\alpha\beta}(\underline{B}) - \rho^{\alpha\beta}(\underline{B}=0)$ is a positive quantity,* for substances involving no transition elements. Davis[18] and others,** using Equation (10) for $\eta = 1$, stated that the Schwartz inequality

$$U_2^{31} = [001]_{12}^2 + [002]_{11}\,[000]_{22} \geq 0 \tag{72}$$

provides the explanation. However, one has to use Equation (9) to claim a general proof. Unfortunately, even for the cases of Equations (10) and (15), no general proof of the positive definite nature is established as yet. Nevertheless, one can make some statements as follows.

In the case of Equation (15), if η is an odd integer, then by integrating by parts η times it is clear that†

$$-[002\eta]_{33} \geq 0 \quad . \tag{73}$$

If also in the power-series expansion

$$\sum_{\eta=1} B^{2\eta}\,[002\eta]_{33} \tag{74}$$

the magnitude of a term is always greater than that of the subsequent term, then (74) is always nonnegative. We do not know of any convincing evidence that this monotonic condition holds.†† If it does, it means that the longitudinal magneto resistance change is nonnegative for isotropic substances, and for crystals except C_{2h} and S_2 when the measuring current is along the principal axis.

By means of Lemma I it can be shown in general that

$$-U_{2\eta}^{ij} \geq 0 \quad \text{if} \quad \eta = 2\lambda + 1 \tag{75}$$

$$-U_{2\eta}^{ij} \leq 0 \quad \text{if} \quad \eta = 2\lambda \quad . \tag{76}$$

Thus a similar statement can be made about the transverse magneto resistance change. That is, if in Equation (74) and in

$$\sum_{\eta=1} B^{2\eta}\,U_{2\eta} \tag{77}$$

*There are exceptions to this statement; for example, see Reference 33.

**See, for example, Brooks (Reference 12).

†This statement is true for any of the brackets $[002\eta]_{ii}$, $i = 1,2,3$, and those obtained from $[002\eta]_{ii}$ by cyclically permuting the indices 1,2, and 3.

††However, Blom stated (Reference 34, p. 94): "Experimentally we find as a rule that the higher (power) terms are smaller than the lower (power) ones."

the magnitude of each term is larger than that of the subsequent one, then, from Equations (10), (13), and (14), the transverse magneto resistance change is always nonnegative, for cases where the laboratory coordinates coincide with the symmetry coordinates or where an isotropic sample is used. However, even if the monotonic nature of the expression (74) and (77) were granted, one still has to study the most general Equation (9) for the magneto resistance.

Therefore, it should be emphasized that at present no theory predicts the positive magneto resistance change in general.

4. THE MULTIBAND MODEL

The multiband model, as described in Section C-2, is so postulated that all formal derivations of the single-band model can be taken over directly simply by writing

$$[m\text{-}p,p,n\text{-}m]_{ij} = \sum_{\lambda=1,2,\ldots,q} [m\text{-}p,p,n\text{-}m]_{ij}^{(\lambda)} \tag{78}$$

with

$$[m\text{-}p,p,n\text{-}m]_{ij}^{(\lambda)} = ab^n \int\int\int \left(-\frac{\partial f_o^{(\lambda)}}{\partial E^{(\lambda)}}\right) \frac{\partial E^{(\lambda)}}{\partial k_i} \times$$

$$\times P\left\{(\tau^{(\lambda)}\Omega_1(\lambda))^{m-p}(\tau^{(\lambda)}\Omega_2(\lambda))^{p}(\tau^{(\lambda)}\Omega_3(\lambda))^{n-m}\right\}\left(\tau^{(\lambda)} \frac{\partial E^{(\lambda)}}{\partial k_j}\right)d\underline{k} \; . \tag{79}$$

If $\lambda = 1$, then, by dropping the superindex 1, Equation (79) becomes (60). Therefore all the formal developments of Section B can now be interpreted according to the multiband model, without any of the restrictions mentioned in Section C-2.

As an example, we can reformulate the often-used two-band formula as follows. If one limits oneself to terms up to B^2 and to special geometries and symmetries as specified for Equations (10) through (15), and interprets the brackets according to two noninteracting bands, then Equations (10) and (14) are comparable to Equations (8.523.4) and (8.521.9), respectively, of Wilson's book.[11] The additional formula of (15) would be zero if more restrictions such as used by Wilson et al. on E's and τ's were made. The Equations (10) through (15) contain no field-dependent variables, such as the carrier densities n_1 and n_2, so the effect of the crystal symmetry can be studied. Furthermore, by means of tables in previous reports, they can be extended to cases where both geometry and symmetry are arbitrary, and to cases where more noninteracting bands and higher powers of B are desired.

The mathematical lemmas, given by Equations (64) through (71) in terms of the single-band model, can all be re-established for the multiband model. Therefore, the mathematical lemma given by Equations (69) through (71) can be repeated, replacing Equation (70) by

$$\Omega_i^{(\lambda)}[\tau^{(\lambda)}E_j^{(\lambda)}] = C_{ijk}\,E_k^{(\lambda)}\,, \qquad \lambda = 1,2,\ldots q\,. \tag{80}$$

Note that Equation (80) is more stringent than (70) in the sense that the same constant C_{ijk} is required for all values of λ. Consequently, all three corollaries, for the single-band model, will hold also for the multiband model if and only if condition (80) is read instead of condition (70). Then one can repeat all the remarks about the zero, the nonnegative, and the relative magnitudes of the longitudinal and transverse magneto resistance.

All these formal results followed from the assumption that transitions between bands do not occur. However, experiments have shown that recombination of electrons and holes does occur. Studies* allowing transitions between bands based on the present work may prove to be very fruitful.

*One example of such studies was given recently by Rittner.[35]

E. APPENDIX

1. PROOFS OF THE LEMMAS AND COROLLARIES OF SECTION D-3-b

Lemma I.

Proof: Schwartz's inequality is

$$\int \underline{f} \circ \underline{f}\, d\underline{k} \int g \circ g\, d\underline{k} - \left(\int f \circ g\, d\underline{k} \right)^2 \geq 0 \,. \tag{A.1}$$

The equality of Equation (A.1) holds if and only if

$$\underline{f} = c\, \underline{g} \,. \tag{A.2}$$

One can now rewrite the left-hand side of Equation (62) as

$$I \equiv \left\{ \left(\int \underline{f} \circ \underline{f}\, d\underline{k} \int \underline{g} \circ \underline{g}\, d\underline{k} \right)^{1/2} - \left(\int \underline{F} \circ \underline{F}\, d\underline{k} \int \underline{G} \circ \underline{G}\, d\underline{k} \right)^{1/2} \right\}^2 +$$

$$+ 2 \left\{ \left[\left(\int \underline{f} \circ \underline{f}\, d\underline{k} \int \underline{g} \circ \underline{g}\, d\underline{k} \right) \left(\int \underline{F} \circ \underline{F}\, d\underline{k} \int \underline{G} \circ \underline{G}\, d\underline{k} \right) \right]^{1/2} - \right.$$

$$\left. - \int \underline{f} \circ \underline{G}\, d\underline{k} \int \underline{F} \circ \underline{g}\, d\underline{k} \right\} \,.$$

The first term of I is nonnegative. We can rewrite the second term as

$$2 \left\{ \left[\int \underline{f} \circ \underline{G}\, d\underline{k} + \alpha_1 \right]^{1/2} \left[\int \underline{F} \circ \underline{g}\, d\underline{k} + \alpha_2 \right]^{1/2} - \int \underline{f} \circ \underline{G}\, d\underline{k} \int \underline{F} \circ \underline{g}\, d\underline{k} \right\} \,,$$

which is also nonnegative because both α_1 and α_2 are nonnegative constants according to Equation (A.1). This proves the first part of the lemma.

It is obvious that Equation (63) is sufficient for the equality of

(62) to hold. Now we prove the necessity: If the equality of Equation (62) holds, then one has both terms of I vanishing separately. Since the second term of I vanishes, it is required by Equation (A.2) that

$$f = C_1 G$$

$$F = C_2 g ,$$

where C_1, C_2 are two arbitrary constants. Since the first term of I vanishes, it is necessary to have

$$C_1 = C_2 = C .$$

q.e.d.

Lemma II.

Proof: Sufficiency. If (65) holds, then

$$[2q + 2]_{11} = - c^2 [2q]_{22}$$

and

$$[2q + 1]_{12} = - c[2q]_{22} .$$

Substituting these into (66), it can be seen that the terms within $\{\ \}$ vanish for all values of η and q.

Necessity. If (64) is true in general, then

$$U_2 = [2]_{11}[0]_{22} + [1]_{12}^2 = 0 . \qquad (A.3)$$

If the explicit form of the brackets is written out according to Equation (60), the last expression is a Schwartz equality. According to (A.2) the necessary and sufficient condition for this to vanish is that the square root of the integrands of $[2]_{11}$ and $[0]_{22}$ be linearly dependent. This is just Equation (65).

q.e.d.

Lemma III.

Proof: Since the proof for sufficiency is obvious, one only needs to prove the necessity. If Equation (67) holds, then

$$[002]_{33} = ab^2 \iiint - \frac{\partial f_o}{\partial E} E_3 (\Omega_3 \tau)^2 (\tau E_3)\, d\underline{k}$$

$$= +ab^2 \iiint \left(\frac{\partial f_o}{\partial E} \tau \right) \left\{ \Omega_3 (\tau E_3) \right\}^2 d\underline{k} = 0 .$$

Since $-(\partial f_0/\partial E)_\tau$ is nonnegative, (68) is true.

q.e.d.

Corollary II.

Proof: It is sufficient to prove the corollary for the case i = 3; i.e., $\gamma_3 = 1$.

Sufficiency. Using the conditions (1) and (2) with i = 3, it follows that:

$$P_{2\eta}^{\alpha\beta} = \sum_{q=0}^{\eta} \left\{ \ell_1^\alpha \ell_1^\beta [2q]_{22}[2(\eta-q)]_{33} + \ell_2^\alpha \ell_2^\beta [2q]_{11}[2(\eta-q)]_{33} + \ell_3^\alpha \ell_3^\beta [2(\eta-q)]_{11}[2q]_{22} \right\} +$$

$$+ \sum_{q=0}^{\eta-1} \ell_3^\alpha \ell_3^\beta [2q+1]_{12}[2\eta-2q-1]_{12}$$

$$= \ell_1^\alpha \ell_1^\beta [2\eta]_{22}[0]_{33} + \ell_2^\alpha \ell_2^\beta [2\eta]_{11}[0]_{33} + \ell_3^\alpha \ell_3^\beta [0]_{11}[2\eta]_{22} +$$

$$+ \sum_{q=0}^{\eta-1} (\ell_1^\alpha \ell_1^\beta [2q]_{22} + \ell_2^\alpha \ell_2^\beta [2q]_{11})\, [2(\eta-q)]_{33} + U_{2\eta}^{31} , \qquad (A.4)$$

where

$$U_{2\eta}^{31} = \sum_{q=0}^{\eta-1} \left([2(\eta-q)]_{11}[2q]_{22} + [2q+1]_{12}[2\eta-2q-1]_{12} \right) .$$

Because of condition (1) we have

$$[0]_{11} = [0]_{22} = [0]_{33}$$

and

$$[2\eta]_{22} = [2\eta]_{11} .$$

The first term on the right side of Equation (A.4) is zero because of the orthogonality condition, and the last two terms are zero because of the lemma.

Necessity. If $P_{2\eta}^{\alpha\beta} = 0$ for arbitrary ℓ_i^α and ℓ_j^β, then the second and third terms of (A.4) must vanish separately, that is $U_{2\eta}^{31} = 0$. Thus the proof is established by using the lemma.

Corollary III.

Proof: It suffices to prove the corollary for D_{2h} only, because all the other symmetries contain D_{2h} as subgroup. When $\alpha \neq 1$ the conclusion deals with the Hall potential; when $\alpha = 1$ it deals with the magneto resistance change. The proof for both cases is established by showing $R_{ij}^{\alpha 1} = 0$ if and only if Equation

(70) holds. [See Equation (11) for the definition of $R_{ij}^{\alpha 1}$.]

Sufficiency.

$$\underline{i = j}.$$

$R_{ii}^{\alpha 1}$ for D_{2h}, given by Table XV of Report No. 2136-5-P, consists of terms either of the form of (65) or (68). These are all zero, according to the lemma, if Equation (70) is true for all values of i and j.

$$\underline{i \neq j}.$$

The proof will be established by showing that a typical coefficient, for example,

$$R_{23}^{\alpha 1} = [011]_{23}[000]_{11} - [010]_{31}[001]_{12} ,$$

is zero if Equation (70) holds. Then the proofs for the coefficients of $R_{31}^{\alpha 1}$ and $R_{12}^{\alpha 1}$ follow identically if one permutes the indices 1,2,3 cyclically. Now if (70) holds, one has

$$[011]_{23} = C_2C_1[000]_{22}$$

$$[010]_{31} = -C_1[000]_{11}$$

$$[001]_{12} = -C_2[000]_{22} .$$

Consequently

$$[011]_{23}[000]_{11} - [010]_{31}[001]_{12} = 0 .$$

Necessity. If $R_2^{\alpha 1} = 0$ for arbitrary values of γ's, then $R_{ii}^{\alpha 1}$ = and $R_{i<j}^{\alpha 1}$ are zero separately. Since $R_{ii}^{\alpha 1} = 0$ also for arbitrary values of ℓ's, then each coefficient of $\ell_i^{\alpha}\ \ell_i^{\beta}$ in $R_{ii}^{\alpha 1}$ is zero separately. Therefore, Equation (70) is true according to the lemma.

The proof is completed by recalling the second part ($i < j$) of the sufficiency proof.

2. LIST OF CORRECTIONS TO PREVIOUS REPORTS

The reports to which these corrections refer carry at the upper-right corner of the cover page the serial number 2136-n-P, with n = 1,2, ... 6.

Report	Page	Line	Reads	Should Read
n = 1	4	27	n = 1	n = 0
	11	14	13	31
	16	12	q = 0	p = 0
	18	Table		many errors; delete

Report	Page	Line	Reads	Should Read
n = 3	21	21	1	18
	23	12	3	4
	23	21	12	11
	23	30	$[202]_{23}$	$[202]_{33}$
	27	Fig. 1b		H should be parallel to the $\phi = 0$ line
	32	Fig. 4	abscissa, label 10^{5}	10^{-5}
	33	Table VIII		column 6 head should be C x 10^{16}
n = 4	3	31	$(-)^{n}[]_{ij}$	$(-)^{n}[]_{ji}$
	4	26	$\sigma_{ij}(-H)$	$\sigma_{ji}(-H)$
	9			add to $-B_{22}D = \ldots$ a term $-\ell_2 \ell_3 [200]_{23}[000]_{11}$
	11	16	$[022]_{11}$	$[002]_{11}$
	19	Fig. 5	ψ around outer semicircle	should be ϕ
n = 5	2	2	$F_{1,2,3}$	$F^{1,2,3}$
	2	3	F_d	F^d
	27	bottom	VII.	VIII and by replacing e by w and vice versa
	33	6		delete: and Isotropic case
	33	19	$\ell_i^{\alpha}\ell_j^{1}$	$\ell_j^{\alpha}\ell_i^{1}$
	35	1	$\rho^{\alpha 1}_{2\eta}$	$P^{\alpha 1}_{2\eta}$
	38			add footnote: The whole table should be divided by a factor (1-AB).
	40	11	$)\delta^{\alpha 1} + ($	$)\delta^{\alpha 1} - ($
	41	24	c	-c
n = 6	2	5		add $[200]_{23}$ and $[002]_{33}$
	12	3		reflect Fig. 3
	16	14,23	$\frac{F_2}{J_1}$	$\frac{F^2}{J^1}$
	17	5		
	22	Fig. 8	curve labels $[000]_{12}[000]_{23}$	$[001]_{12}$ $[100]_{23}$
	26	Eq. 22	$[000]_{33}$	$[000]^{-1}_{33}$
	27			see the corrected results in present report
	28			
	31	Fig. A.1		omit B^2 in caption
	32	Fig. A.2	The caption in upper corner should read $\frac{\Delta\rho}{\rho}$ = signal in μV x $\frac{1.336 \times 10^{-4}}{\mu V}$	

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