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COLLEGE OF LITERATURE, SCIENCE, AND THE ARTS
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Final Report

ASYMPTOTIC SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT

The project has supported the investigation of three mathematical problems: a Peano-like existence proof for solutions of the Goursat problem for quasi-linear equations has been found, a Runge-Kutta process for the numerical solution of this problem has been developed, and the asymptotic behavior of solutions of ordinary linear differential equations having two simple turning points has been determined.

ACTIVITIES

Three research papers have been prepared in connection with the project. One has already been published. The remaining two are part of the doctoral thesis of Robert H. Moore, whose study has been supported by the project, and will be submitted for publication soon. Mr. Moore will receive his Ph.D. degree in June. During 1958, Mr. Charles Grobe received support from the project. He participated in seminar work but his main energies were spent in preparation for his preliminary examinations for the Ph.D., of which he has now passed half.

The director attended and presented a paper to the International Congress of Mathematicians, 1958, with partial support from the project. He also was enabled to attend the 1958 winter meeting of the American Mathematical Society. Mr. Moore presented his results to the April meeting of the Society in New York.

PAPERS PUBLISHED AND IN PREPARATION

1. Moore, R. H., "On approximations of the solution of the Goursat problem for quasi-linear equations," to be submitted for publication.
2. Moore, R. H., "A Runge-Kutta procedure for the solution of the Goursat problem for quasi-linear equations," to be submitted for publication.
3. Kazarinoff, N. D., "Asymptotic theory of second order differential equations with two simple turning points," Archive for Rational Mechanics and Analysis, 2, 129-150 (1958).

SUMMARY OF RESULTS

I. APPROXIMATIONS OF THE SOLUTION OF THE GOURSAT PROBLEM

Consider the hyperbolic partial differential equation

$$u_{xy} = f(x, y, u, u_x, u_y)$$

given together with the initial conditions

$$y(x, 0) = \sigma(x), \quad u(0, y) = \tau(y), \quad \sigma(0) = \tau(0),$$

to be satisfied by u on the rectangle $R: 0 \leq x \leq a; 0 \leq y \leq b$. In comparison, also consider the ordinary differential equation

$$u' = f(x, u), \quad u(0) = \sigma,$$

to be satisfied by u on $I: 0 \leq x \leq a$. These two problems will be referred to, respectively, as 2-HP and 1-Ord.

Many authors have realized the analogy between these initial value problems. Mr. Moore's thesis is divided into two parts, each concerned with exploiting this analogy. The object of Part I is to develop an existence proof for 1-Ord. The important part of this development is that a broad class of "approximate solution functions" are indeed approximations to a solution in the sense that, under a natural ordering, they converge uniformly to a solution. The conclusion of existence of a solution under the given conditions has been known for some time. The object of Part II is to develop for 2-HP an analog of the Runge-Kutta procedure for numerically approximating solutions of 1-Ord. The procedure developed is similar to that given by Kutta both in application and order of accuracy. It is intended to test the procedure by running some examples having known solutions on the computer at the Naval Ordnance Research Laboratory, Silver Spring, Maryland.

Mr. Moore's principal theorem follows. Let

$$\mathcal{X} = [(x, y) \mid 0 \leq x \leq a, 0 \leq y \leq b],$$

and let \mathcal{V} be three-dimensional real space with points (v^0, v^1, v^2) . Let f be a real-valued function defined on $\mathcal{X} \times \mathcal{V}$ such that f is bounded and continuous and satisfies the Lipschitz condition

$$|f(x, y, v^0, v^1, v^2) - f(x, y, v^0, \bar{v}^1, \bar{v}^2)| < L[|v^1 - \bar{v}^1| + |v^2 - \bar{v}^2|]$$

on all of $\mathcal{X} \times \mathcal{V}$. Let σ and τ be continuously differentiable real-valued functions defined on $0 \leq x \leq a$ and $0 \leq y \leq b$. Suppose $\{m^\nu\}$ and $\{m^{*\nu}\}$ are convergent sequences of meshes in \mathcal{X} and $\mathcal{X} \times \mathcal{V}$, respectively, and $\{c^{j\nu}\}$ ($j = 0, 1, 2$) are sequences of real-valued functions defined on $\mathcal{X} \times \mathcal{V}$ such that

$c^{j\nu}$ is constant on each cell of $m^{*\nu}$ and such that $\{c^{j\nu}\}$ converges uniformly to f as $\nu \rightarrow \infty$ (j fixed). Let \mathcal{N}^ν denote the set of nodes of m^ν .

We denote the kl 'th node point of \mathcal{N}^ν by ξ_{kl}^ν and set

$$c_{kl}^{j\nu} = c^{j\nu}(\xi_{kl}^\nu, v_{kl}^\nu)$$

where

$$v_{kl}^\nu = (v_{kl}^{0\nu}, v_{kl}^{1\nu}, v_{kl}^{2\nu}) \quad \text{and} \quad V_{kl}^{j\nu} = v_{kl}^{j\nu}(\xi_{kl}^\nu).$$

Also let $\sigma_k^\nu = \sigma(\xi_{k0}^\nu)$ and $\tau_l^\nu = \tau(\xi_{0l}^\nu)$, and define $\sigma_k^{1\nu}$ and $\tau_l^{1\nu}$ similarly.

The integral equation problem equivalent to 2-HP has the following finite analogue:

$$V_{kl}^{0\nu} = \sigma_k^\nu + \tau_l^\nu - \tau_0 + \sum_{\alpha=0}^{k-1} \sum_{\beta=0}^{l-1} c_{\alpha\beta}^\nu \Delta_\alpha^{1\nu} \Delta_\beta^{2\nu}$$

$$(*) \quad V_{kl}^{1\nu} = \sigma_k^{1\nu} + \sum_{\beta=0}^{l-1} c_{k\beta}^\nu \Delta_\beta^{2\nu}$$

$$V_{kl}^{2\nu} = \tau_l^{1\nu} + \sum_{\alpha=0}^{k-1} c_{\alpha l}^\nu \Delta_\alpha^{1\nu},$$

where by convention $\sum_0^{-1} = 0$.

Theorem. Under the foregoing hypotheses a sequence of functions $V^{\nu}: \mathfrak{X}^{\nu} \rightarrow \mathfrak{V}$ is defined by the equations (*) with the property that a subsequence converges uniformly to a continuous function $V: \mathfrak{X} \rightarrow \mathfrak{V}$. The first component V^0 of V is a solution to 2-HP and the second and third components are its derivatives.

It is through this theorem that the convergence of the Runge-Kutta procedure developed in Part II is established.

II. ASYMPTOTIC SOLUTION OF DIFFERENTIAL EQUATIONS WITH TWO SIMPLE TURNING POINTS

A general asymptotic theory of second-order ordinary differential equations with two simple turning points and containing a numerically large parameter is derived. Differential equations of the form

$$(**) \quad \frac{d^2 y}{ds^2} - \lambda^2 P(s, \lambda)y = 0, \quad P(s, \lambda) = \sum_0^{\infty} p_j(s)\lambda^{-j},$$

are considered for complex λ , $|\lambda| > N$, and s in a closed, simply connected, perhaps unbounded region \mathcal{D}_s of the complex plane in which p_0 has precisely two simple zeros.

If the relation

$$\int_1^z (t^2-1)^{1/2} dt - \int_{\alpha}^s p_0^{1/2}(t)dt = 0$$

defines a schlicht mapping

$$z(s): \mathcal{D}_s \leftrightarrow \mathcal{D}_z$$

with $[-1, 1] \subset \mathcal{D}_z$, then the differential equations considered can be rewritten

as

$$\frac{d^2 u}{dz^2} - \lambda^2 Q(z, \lambda)u = 0, \quad Q(z, \lambda) = \sum_0^{\infty} q_j(z)\lambda^{-j}$$

with $q_0(z) = z^2 = 1$. The solutions of this equation are then approximated by those of the transformed Weber equation

$$\frac{d^2v}{dz^2} - \lambda(\lambda z^2 - \nu)v = 0 ,$$

where ν is a suitably chosen parameter. These yield asymptotic expansions to $n+1$ terms, where n is an arbitrary nonnegative integer, for solutions of (**). The specific nature of these approximations is too complex to be gone into here.

Interest in this problem stems mainly from possible applications for the theory derived. In certain regions, the differential equations for the angular and radial spheroidal functions are of the type (**). This is also true of the Whittaker equation for certain configurations of its parameters. Equations of type (**) also arise in other problems related to wave motion and diffraction.

To apply satisfactorily the theory developed to the discussion of Whittaker functions with both parameters large, one extension should and can be made, namely, the parameter ν above must be allowed to approach zero. This permits a discussion of the limiting behavior of solutions of (**) under confluence of turning points. Mathematically, this is perhaps the most interesting aspect of the two turning point problem.

