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SCALAR DIFFRACTION BY AN ELLIPTIC CYLINDER

by

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Abstract. The method given in Reference 1 is applied to the case of scalar scattering by a perfectly reflecting elliptic cylinder illuminated by waves from a line source parallel to the axis of the cylinder. The surface distribution in the shadow zone is calculated, and the "creeping wave" representation for the scattered field in the shadow zone is derived. It is shown that the results are applicable if and only if $R_0 \omega \gg 1$, where R_0 is the smallest radius of curvature on the cylinder and ω is the wave number.

1. In a recent paper[1] we have developed a theory of scalar diffraction for bodies whose boundary surfaces are level surfaces in coordinate systems in which the scalar wave equation is separable. We applied the theory to the case of diffraction by a prolate spheroid and calculated the surface distribution. In this paper we make a similar application to the case of an elliptic cylinder; but, in addition, we consider the off-angle case and derive the "creeping wave" representation for the scattered field in the shadow zone. The surface distribution is discussed in § 6, and the scattered field is described in § 7.

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[1] N. D. Kazarinoff and R. K. Ritt, On the Theory of Scalar Diffraction and its Application to the Prolate Spheroid, Annals of Phys. 6, 277-299 (1959).

The asymptotic theory which we use to obtain our results gives us more terms of the series, in descending powers of ω, for the exponents of the "creeping wave" terms than does the elegant geometric theory of Keller[2]. These extra terms show that the condition $R_o\omega \gg 1$ is essential for our asymptotic theory to give a meaningful result for the diffracted field. It is reasonable to believe that the same restriction also applies to Keller's geometric theory. More specifically, we show that it is the coefficient C of the attenuation exponent $C \int R^{-2/3} ds$ which behaves in an unknown fashion when $R_o \to 0$; the evaluations of C which have been made by Keller and the authors are performed under the hypothesis that $R_o \omega \gg 1$. In an appendix, we compare the magnitude of the attenuation term which he obtains with the next term of the asymptotic series for the exponent in the case of two prolate spheroids. We also note that Levy[3] has applied the geometric theory to the case of an elliptic cylinder and has given a mathematical derivation of the results thus obtained which is based upon the use of the Watson transform.

Generally, the analysis below closely follows that in Reference 1. Where this is true we only elucidate the principal points in the argument and omit most computations. Elsewhere we give a more complete discussion.

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[2] J. B. Keller, Diffraction by a Convex Cylinder, Trans. I. R. E., AP-4, 312-321 (1956).

[3] B. Levy, Diffraction by an Elliptic Cylinder, New York University, Institute of Mathematical Sciences, Report EM-121, December 1958.

2. We consider an elliptic cylinder with semifocal distance c, eccentricity $e = \operatorname{sech} \xi_o$, and semi-axes a and b. We introduce the (ξ, η, z) coordinate system defined by

$$\begin{aligned} x &= c \cosh \xi \cos \eta \\ y &= c \sinh \xi \sin \eta \\ z &= z \end{aligned} \tag{2.1}$$

On the surface of the cylinder, $\xi = \xi_o$. For this cylinder, the operators L_ξ and L_η appearing in Reference 1, Equation (3.2) are defined by the formulas

$$-L_\xi u = \frac{d^2u}{d\xi^2} + \gamma^2 \sinh^2 \xi u \qquad (\xi \geq \xi_o)$$

and

$$-L_\eta u = \frac{d^2u}{d\eta^2} + \gamma^2 \sin^2 \eta u \qquad (-\pi \leq \eta \leq \pi).$$

The constant γ appearing above is $c(\omega - is)$, where s is a small positive number. The boundary conditions are

$$\left.\frac{du}{d\xi}\right|_{\xi_o} = 0 \quad \text{and} \quad u(\xi, \eta) = u(\xi, \eta + 2\pi). \tag{2.2}$$

Both the operators L_ξ and L_η are of the type considered in Reference 1, Section 3; and for them, respectively,

$$\mathcal{J} q \equiv \mathcal{J}(\gamma^2 \sinh^2 \xi) \geqslant 2\omega s c^2 \sinh^2 \xi_o > 0$$

and

$$\mathcal{J} q \equiv \mathcal{J}(\gamma^2 \sin^2 \eta) \geqslant 0.$$

The radial operator L_ξ is to be considered on the interval $\left[\xi_o, \infty\right)$, $\xi_o > 0$. For L_ξ, $p \equiv 1$, and hence $p(\xi_o) \neq 0$. In order to construct the resolvent Green's function for L_ξ, we consider the homogeneous equation

$$L_\xi\, y - \lambda\, y = 0, \tag{2.3}$$

where

$$\mathcal{J} \lambda < 2\omega s c^2 \sinh^2 \xi_o . \tag{2.4}$$

It has linearly independent solutions w_j, $j = 1, 2$, with the asymptotic forms

$$w_j(\xi) = (\sinh \xi)^{-1/2} e^{\pm i \gamma \cosh \xi} \left\{ 1 + \frac{\mathcal{O}(1)}{\xi} \right\},$$

in which $\mathcal{O}(1)$ denotes a function which is bounded for $\xi > N$, $|\gamma| > N$, and $|\lambda| < N$. In this and in succeeding formulas, the upper sign is to be used when $j = 1$, the lower one when $j = 2$. Since $\mathcal{R}(i \gamma) > 0$, the only solutions of (2.3) in $L^2(\xi_o, \infty)$ are multiples of w_2; therefore, L_ξ falls into Case I of Reference 1. We next single out the solution ϕ_1 of (2.3) which satisfies the boundary condition (2.2) and a solution ϕ_2 in $L^2(\xi_o, \infty)$:

$$\phi_1(\xi, \lambda) = w_1(\xi, \lambda)\, w_2'(\xi_o, \lambda) - w_2(\xi, \lambda)\, w_1'(\xi_o, \lambda)$$

$$\phi_2(\xi, \lambda) = w_2(\xi, \lambda).$$

At ξ_o, $\phi_1 = -2i\gamma$. Thus, the resolvent Green's function is

$$G(\xi, \xi', \lambda) = \frac{-1}{2i\gamma\, w_2'(\xi_o, \lambda)} \begin{cases} \phi_1(\xi)\phi_2(\xi') & (\xi < \xi') \\ \phi_2(\xi)\phi_1(\xi') & (\xi' < \xi) \end{cases} .$$

The operator L_η is to be considered on $[-\pi, \pi]$. To construct its resolvent Green's function, we consider the homogeneous equation

$$L_\eta y - (-\lambda y) = 0,$$

where λ satisfies the condition (2.4). Using the notation of Meixner and Schäfke[4], we let $y_I(\eta, -\lambda)$ and $y_{II}(\eta, -\lambda)$ be the solutions of the homogeneous equation for which

$$y_I(0) = 1, \qquad y_I'(0) = 0,$$

$$y_{II}(0) = 0, \qquad y_{II}'(0) = 1.$$

It is then a routine computation to show that the resolvent Green's function for the periodic problem is

$$\tilde{G}(\eta, \tau, -\lambda) = \frac{1}{2}\left\{ \frac{y_I(\tau)y_I(\pi-\eta)}{y_I'(\pi)} + \frac{y_{II}(\tau)\, y_{II}(\pi-\eta)}{y_{II}(\pi)} \right\} \qquad (2.5)$$

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[4] J. Meixner and F. Schäfke, Mathieusche Funktionen und Spheroidfunktionen, pp. 98-100, Springer-Verlag, Berlin, (1954).

for $\tau < \eta$. The relation

$$\tilde{G}(\tau, \eta, -\lambda) = \tilde{G}(\eta, \tau, -\lambda) \tag{2.6}$$

then serves to define $\tilde{G}$ for $\eta < \tau$.

3. We are now in a position to write down the contour integral representation, guaranteed by the theory in Reference 1 for the solution $v(\xi, \eta)$ of

$$\left[\nabla^2 + (\omega - is)^2\right] v = \rho(\Xi, \tau) \qquad (s > 0)$$

which we seek. The function $\rho(\Xi, \tau)$ is a distribution corresponding to a line source at (Ξ, τ). The representation is

$$v(\xi, \eta, \Xi, \tau) = \frac{1}{2\pi i} \int_{\Gamma} \tilde{G}(\eta, \tau, -\lambda)\, G(\xi, \Xi, \lambda)\, d\lambda,$$

where Γ is a path in the λ-plane defined by the conditions

$$\lambda = \ell + i\,\delta, \qquad 0 < \delta < 2\omega\, sc^2 \sinh^2 \xi_0,$$

in which ℓ and δ are real. The integration path Γ is oriented in the direction of increasing ℓ. When $s \rightarrow 0^+$, $v(\xi, \eta, \Xi, \tau)$ reduces to the Green's function for the elliptic cylinder relative to the line source at (Ξ, τ).

We shall consider the distribution on the surface of the cylinder and off the surface separately. On the cylinder,

$$v(\xi_0, \eta, \Xi, \tau) = \frac{1}{2\pi i} \int_{\Gamma} \frac{w_2(\Xi, \lambda)\tilde{G}(\eta, \tau, -\lambda)}{w_2'(\xi_0, \lambda)} \, d\lambda . \tag{3.1}$$

As in Reference 1, we shall evaluate this integral by residues, the residues contributed by the zeros of $w_2'(\xi_0, \lambda)$. If $\Xi > \xi > \xi_0$,

$$v(\xi, \eta, \Xi, \tau) = \frac{1}{4\pi\gamma} \int_{\Gamma} \frac{\left[w_1(\xi, \lambda)w_2'(\xi_0, \lambda) - w_2(\xi, \lambda)w_1'(\xi_0, \lambda)\right] w_2(\Xi, \lambda)}{w_2'(\xi_0, \lambda)} \tilde{G}(\eta, \tau, -\lambda) \, d\lambda. \tag{3.2}$$

In the case where it is practical to evaluate this integral by the residues contributed by the zeros of $w_2'(\xi_0, \lambda)$, the representation (3.2) reduces to

$$v(\xi, \eta, \Xi, \tau) = \frac{-1}{4\pi\gamma} \int_{\Gamma} \frac{w_2(\xi, \lambda)w_1'(\xi_0, \lambda)w_2(\Xi, \lambda)}{w_2'(\xi_0, \lambda)} \tilde{G}(\eta, \tau, -\lambda) \, d\lambda \quad . \tag{3.3}$$

4. Our first objective is to determine the zeros of $w_2'(\xi_0, \lambda)$. We need only sketch the analysis in view of its similarity to that in Reference 1, Sections 6 and 7. The differential equation satisfied by the w_j is

$$\frac{d^2 y}{d\xi^2} + (\gamma^2 \sinh^2 \xi + \lambda) y = 0 .$$

If we let

$$\lambda = -\gamma^2 \sinh^2 \xi_1 , \tag{4.1}$$

then this equation takes the form

$$\frac{d^2 y}{d\xi^2} + \gamma^2 (\sinh^2 \xi - \sinh^2 \xi_1) y = 0 .$$

We define

$$\phi^2(\xi, \xi_1) = \sinh^2 \xi - \sinh^2 \xi_1 ,$$

$$\Phi(\xi, \xi_1) = \int_{\xi_1}^{\xi} \phi(t, \xi_1) \, dt, \qquad \zeta(\xi, \xi_1, \gamma) = \gamma \quad \Phi(\xi, \xi_1) ,$$

and

$$\Psi(\xi, \xi_1) = \Phi^{1/6}(\xi, \xi_1)\, \phi^{-1/2}(\xi, \xi_1) \qquad (\xi \neq \xi_1),$$

with

$$\Psi(\xi_1, \xi_1) = \lim_{\xi \to \xi_1} \Psi(\xi, \xi_1) .$$

In terms of the above notation, the solutions w_j have the asymptotic forms

$$w_j = \gamma^{1/6} e^{\mp i\gamma f(\xi_1)} \left\{ V^{(j)}(\xi) + \frac{B(\xi, \gamma)}{\gamma} \right\}$$

$$w_j' = \gamma^{1/6} e^{\mp i\gamma f(\xi_1)} \left\{ V^{(j)'}(\xi) + B(\xi, \gamma) \right\}$$

when $|\zeta| \leq N$, and the forms

$$w_j = \phi^{-1/2}(\xi, \xi_1)\, e^{\pm i\gamma \left[\zeta(\xi) - f(\xi_1)\right]} \left[1 + B(\zeta^{-1})\right]$$

when $|\zeta| > N$. In these formulas B is used generically for a function which is uniformly bounded for the range of ζ in question and for $|\gamma| > N$,

$$f(\xi_1) = - \int_0^{\pi/2} \sqrt{\sinh^2 \xi_1 + \sin^2\theta}\; d\theta ,$$

and

$$V^{(j)}(\xi) = \left(\frac{\pi}{2}\right)^{1/2} e^{\pm 5\pi i/12}\, \Psi(\xi)\, \zeta^{1/3} H^{(j)}_{1/3}(\zeta) ,$$

where $H^{(j)}_{1/3}$ is a Hankel function.

When $|\lambda| \ll |\gamma|^2$, it can be seen from these formulas that $w_2'(\xi_0, \lambda)$ has no zeros. Provided (ξ_0, η) is not too close to the shadow boundary, the zeros of $w_2'(\xi_0, \lambda)$ corresponding to values of λ with $|\lambda| \gg |\gamma|^2$ have large imaginary parts, and the terms which they contribute to the residue series may be neglected (see, for example, Franz[5] and Levy[3]). When $|\lambda|$ is comparable to $|\gamma|^2$, $w_2'(\xi,\lambda)$ vanishes only if

$$\frac{d}{d\zeta}\left\{ \zeta^{1/3} H^{(2)}_{1/3}(\zeta) \right\} + \mathcal{O}(\gamma^{-1}) = 0 .$$

If $\xi = \xi_o$, ξ_o fixed, and λ is considered as variable, the value ζ_r of ζ which corresponds to the r^{th} zero of $w_2'(\xi_o, \lambda)$ may be thought of as the value of

$$\gamma \int_{\xi_1(\lambda,\gamma)}^{\xi_o} \phi(t, \xi_1(\lambda,\gamma))\ dt$$

which is attained when $\lambda = \lambda_r$, since ξ_1 and λ are related by (4.1). Thus

$$\zeta_r = \zeta(\xi_o, \xi_1(\lambda_r, \gamma)) .$$

Because the zeros h_r of

$$\frac{d}{dt}\left[t^{1/3} H^{(2)}_{1/3}(t) \right]$$

are simple and because this function is analytic in a neighborhood of each of its zeros,

$$\zeta_r = h_r + \mathcal{O}(\gamma^{-1})$$

The zeros h_r and the values of related functions such as $t\, H^{(2)}_{1/3}(t)$ at these zeros are known[6].

The relation

$$\int_{\xi_1(\lambda_r,\gamma)}^{\xi_o} \phi(t, \xi_1(\lambda_r,\gamma))\ dt = h_r\, \gamma^{-1} + \mathcal{O}(\gamma^{-2})$$

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[5] W. Franz, Ueber die Greenschen Funktionen des Zylinders und der Kugel, Z. Naturforsh, 9a, 705-716 (1954).

[6] British Association Mathematical Tables, "The Airy Integral", Cambridge University Press, London and New York, 1946.

may now be used to compute $\xi_r = \xi_1(\lambda_r, \lambda)$ by expanding the integrand on the left hand side in powers of $(t - \xi_1)$ or $(t - \xi_o)$. It is vital to note that both of these expansions are slowly convergent as $\xi_o \longrightarrow 0$. Therefore, the approximation for ξ_r which we obtain by neglecting all but the first two terms is not useful when $\xi_o \longrightarrow 0$ and ω is fixed. We henceforward assume that ξ_o is bounded away from zero. We find, under this hypothesis, that

$$\xi_r - \xi_o = \frac{e^{-\frac{\pi i}{3}} 3^{2/3}}{2(\sinh \xi_o \cosh \xi_o)^{1/3}} \left(\frac{hr}{\gamma}\right)^{2/3} \left\{ 1 - \frac{7e^{-\frac{\pi i}{3}} 3^{2/3} (\sinh^2 \xi_o + \cosh^2 \xi_o)}{60(\sinh \xi_o \cosh \xi_o)^{4/3}} \left(\frac{hr}{\gamma}\right)^{2/3} \right\}$$

$$+ \mathcal{O}(\gamma^{-5/3}) . \qquad (4.2)$$

The specific value of λ_r will not be needed.

Computation of

$$\frac{\partial w_2^1(\xi_o, \lambda)}{\partial \xi_1} \left. \frac{\partial \xi_1}{\partial \lambda} \right|_{\lambda_r}$$

now leads to an approximation for the residue contribution of w_2^1 at λ_r:

$$\frac{\partial}{\partial \lambda} w_2^1(\xi_o, \lambda) \Bigg|_{\lambda_r} = \left\{ \frac{3\pi}{8\gamma \sinh \xi_o \cosh \xi_o} \right\}^{1/2} e^{i\gamma f(\xi_r) + \frac{5\pi i}{4}} h_r H_{1/3}^{(2)}(h_r) \left[1 + \mathcal{O}(\gamma^{-1/3}) \right] . \qquad (4.3)$$

In subsequent work, we shall also need an approximation for $w_1^1(\xi_o, \lambda_r)$. An easy calculation yields the formula

$$w_1^1(\xi_o, \lambda_r) = \left(\frac{\pi}{2}\right)^{1/2} (3\gamma^5 \sinh \xi_o \cosh \xi_o)^{1/6} e^{-i\gamma f(\xi_r) + \frac{3\pi i}{4}}$$

$$\cdot \ h_r^{1/3} \left[\zeta^{1/3} H_{1/3}^{(1)}(\zeta) \right]'_{\zeta = h_r} \left\{ 1 + \mathcal{O}(\gamma^{-2/3}) \right\} .$$

5. We next approximate $\widetilde{G}(\eta, \tau - \lambda_r)$. For $\lambda_r = -\gamma^2 \sinh^2 \xi_r$, the Liouville asymptotic representations for y_I and y_{II} are

$$y_I \sim \left[\frac{K_r(0)}{K_r(\eta)}\right]^{1/2} \cos\left[\gamma \int_0^{\eta} K_r(t)\, dt\right] \tag{5.1}$$

and

$$y_{II} \sim \frac{\sin(\gamma \int_0^{\eta} K_r(t)\, dt)}{\gamma \left[K_r(0)\, K_r(\eta)\right]^{1/2}} ,$$

where

$$K_r(t) = \left[\sin^2 t + \sinh^2 \xi_r\right]^{1/2} . \tag{5.2}$$

These results and the relations (2.5), (2.6), and (5.1) combine to yield the formula

$$\widetilde{G}(\eta, \tau, -\lambda) \sim \frac{-\cos\left(\gamma\left[\int_0^{\pi} K_r(t)\, dt - \left|\int_{\tau}^{\eta} K_r(t)\, dt\right|\right]\right)}{2\gamma \sin(\gamma \int_0^{\pi} K_r(t)\, dt) \left[K_r(\tau)\, K_r(\eta)\right]^{1/2}} . \tag{5.3}$$

6. In this section we consider the surface distribution, and we derive the residue series for the integral in (3.1). The residue series we seek is a sum of terms of the form

$$\frac{w_2(\Xi, \lambda_r)\, \widetilde{G}(\eta, \tau, -\lambda_r)}{\frac{\partial}{\partial \lambda}\left[w_2'(\xi_0, \lambda)\right]_{\lambda = \lambda_r}} .$$

These can be approximated by using the results of §§4 and 5. And, in fact, we need such approximations in order to investigate the convergence of the residue series and to see if the boundary of the region of convergence coincides with the geometric shadow boundary. We perform further approximations by expanding in powers of $(\xi_r - \xi_o)$ and neglecting the terms which cannot be specifically computed using the estimate (4.2) for $\xi_r - \xi_o$. Henceforward, we shall assume that the parameter s involved in γ is zero.

We first consider $w_2(\Xi, \lambda_r)$. Since Ξ is large,

$$w_2(\Xi,\lambda_r) = \exp\left\{-i\gamma\left[\int_{\xi_r}^{\Xi} H_r(t)\,dt + \int_0^{\pi/2} K_r(\theta)\,d\theta\right]\right\} \frac{\left[1 + \mathcal{O}(\gamma^{-1} e^{-\Xi})\right]}{\left[H_r(\Xi)\right]^{1/2}} ,$$

where

$$H_r(t) = (\sinh^2 t - \sinh^2 \xi_r)^{1/2} . \tag{6.1}$$

Performing an integration by parts, we find that

$$w_2(\Xi,\lambda_r) = \exp\left\{-i\gamma\left[\coth\Xi\, H_r(\Xi) + \int_0^{\ell_r(\Xi)} K_r(t)\,dt\right]\right\} \frac{\left[1 + \mathcal{O}(\gamma^{-1} e^{-\Xi})\right]}{\left[H_r(\Xi)\right]^{1/2}} , \tag{6.2}$$

with

$$\ell_r(t) = \sin^{-1}\left\{\frac{\sinh \xi_r}{\sinh t}\right\} . \tag{6.3}$$

The exponential factors in the remainder of the r^{th} residue term are

$$e^{-i\gamma\int_0^{2\pi} K_r(t)\,dt} \; \frac{\cos\left[\gamma\left\{\int_0^{\pi} K_r(t)\,dt - \left|\int_{\tau}^{\eta} K_r(t)\,dt\right|\right\}\right]}{\sin\left[\gamma\int_0^{\pi} K_r(t)\,dt\right]} ,$$

or

$$\frac{i\left\{e^{-i\gamma\left\{\left|\int_{\tau}^{\eta} K_r(t)\,dt\right| - \int_0^{\pi/2} K_r(t)\,dt\right\}} + e^{-i\gamma\left\{3\int_0^{\pi/2} K_r(t)\,dt - \left|\int_{\tau}^{\eta} K(t)\,dt\right|\right\}}\right.}{1 - e^{-4i\gamma\int_0^{\pi/2} K_r(t)\,dt}} . \tag{6.4}$$

It is important to note that because

$$\int_0^{\pi/2} K_r(t)\,dt = \int_0^{\pi/2} K_o(t)\,dt + (\xi_r - \xi_o)\int_0^{\pi/2} \frac{\sinh\xi_o \cosh\xi_o dt}{K_o(t)} + \ldots,$$

and because

$$\mathcal{J}(\xi_r - \xi_o) < 0,$$

$$\mathcal{R}\left(i\gamma \int_0^{\pi/2} K_r(t)\,dt\right) > 0 . \tag{6.5}$$

Since $\mathcal{R}(\gamma)$ is large, it follows that the dominant term in (6.4) is

$$\exp\left\{-i\gamma\left[\left|\int_{\tau}^{\eta} K_r(t)\,dt\right| - \int_0^{\pi/2} K_r(t)\,dt\right]\right\}.$$

Therefore, by the relations (3.1), (6.2), (5.3), (4.3), and (6.4), the surface distribution

$$v(\xi_o, \eta, \Xi, \tau) \sim \frac{e^{3\pi i/4}}{2\gamma}\left\{\frac{2\gamma \sinh\xi_o \cosh\xi_o}{3\pi}\right\}^{1/2} \cdot$$

$$\cdot \sum_r \frac{\exp\left\{-i\gamma\left[\coth\Xi\, H_r(\Xi) + \int_0^{\ell_r(\Xi)} K_r(t)\,dt + \left|\int_{\tau}^{\eta} K_r(t)\,dt\right| - \int_0^{\pi/2} K_r(t)\,dt\right]\right\}}{h_r H_{1/3}^{(2)}(h_r)\left[K_r(\eta)\, K_r(\tau)\, H_r(\Xi)\right]^{1/2}}, \tag{6.6}$$

where only the dominant part of the first creeping wave term has been included. In this formula the functions H_r, K_r and ℓ_r are defined by the relations (6.1), (5.2), and (6.3), respectively.

Let us assume that $\eta > \tau$. Then if we expand K_r in powers of $(\xi_r - \xi_o)$, we find that the exponential terms in the above summation take the form

$$\exp\left(-i\gamma\left\{\left(\int_\tau^\eta + \int_0^{\ell_0(\Xi)} - \int_0^{\pi/2}\right) K_0(t)\,dt + \coth\Xi\; H_0(\Xi) \right.\right.$$

$$+ (\sinh\xi_0 \cosh\xi_0)(\xi_r - \xi_0)\left(\int_\tau^\eta + \int_0^{\ell_0(\Xi)} - \int_0^{\pi/2}\right)\frac{dt}{K_0(t)}$$

$$+ \frac{(\xi_r - \xi_0)^2}{2}\left(\int_\tau^\eta + \int_0^{\ell_0(\Xi)} - \int_0^{\pi/2}\right) \cdot$$

$$\left.\left. \cdot \left(\frac{\sinh^4\xi_0 + (\cosh^2\xi_0 + \sinh^2\xi_0)\sin^2 t}{K_0^3(t)} + \frac{\tanh\Xi\,\cosh\xi_0}{H_0(\Xi)}\right) dt \right\}\right).$$

Each of the above integrals is a real quantity. The only terms with an imaginary part in the above exponent are $-i\gamma$, $(\xi_r - \xi_0)$ and $(\xi_r - \xi_0)^2$. Using the transformation (2.1), we find that $c\int K_0(t)\,dt$ represents an integral $\int ds$ where ds is the differential of arc on the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ and that $(ab)^{2/3} \cdot c^{-1} \cdot \int K_0^{-1}(t)\,dt$ represents an integral $\int R^{-2/3}\,ds$ where R is the local radius of curvature on the ellipse. This tells us that to a first approximation, the attenuation of the "creeping waves" is as predicted by Keller[2]. Now by (4.2),

$$\gamma(\sinh\xi_0 \cosh\xi_0)(\xi_r - \xi_0) \sim \frac{(ab)^{2/3}}{c}\, C_1 ,$$

where

$$C_1 = \frac{(-9\omega h_r^2)^{1/3}}{2}\left[1 - \frac{7e^{-\pi i/3}}{60}\left(\frac{3h_r}{\omega}\right)^{2/3}\left(R_0^{-2/3} + \frac{R_0^{1/3}}{a}\right)\right],$$

$R_o = b^2/a$ is the radius of curvature at the ends of the semi-major axis of the ellipse. Thus the second term in the above exponent becomes

$$+ i\, C_1 \left(\int_{s(\tau)}^{s(\eta)} + \int_{s(0)}^{s(\ell_o(\Xi))} - \int_{s(0)}^{s(\pi/2)} \right) R^{-2/3}\, ds \; .$$

The formula for C_1 reveals an essential limitation upon our theory and perhaps that of Keller; namely, the above expansion of the creeping wave exponents in descending powers of ω is meaningful only if $R_o \omega \gg 1$. In particular, for a fixed ω, R_o cannot be taken too small. Thus, we have derived mathematically the expected physical restriction upon theories of this kind.

The $(\xi_r - \xi_o)^2$ term in the exponent is of less interest. We have carried out its computation only in the case of the prolate spheroid; see the Appendix. As the discussion there would indicate, when $\xi_o \to \infty$ and the cylinder becomes circular the $(\xi_r - \xi_o)$ and $(\xi_r - \xi_o)^2$ terms cancel in such a way as to produce the expected exponent.

It remains to investigate the convergence of the residue series when summed in the "creeping wave" form. The condition (6.5) shows that the convergence will not be rapid unless

$$f(\xi, \eta) \equiv \left(\int_{\tau}^{\eta} + \int_{0}^{\ell_o(\Xi)} - \int_{0}^{\pi/2} \right) K_o^{-1}(t)\, dt > 0 \; .$$

In fact for a given τ, the condition

$$f(\tau, \eta) = 0 \tag{6.7}$$

determines the boundary of the region of convergence. We shall show that this is indeed the optical shadow boundary. To do this, it is convenient to put the elliptic integrals of the first kind involved in $f(\xi,\eta)$ into Legendre form. We then find

$$f(\xi,\eta) = \left(\int_0^{\cos\tau} + \int_0^{-\cos\eta} - \int_0^{\sqrt{1-\frac{\sinh^2\xi_0}{\sinh^2\Xi}}} \right) \frac{e\ dt}{\left[(1-t^2)(1-e^2t^2)\right]^{1/2}} .$$

The addition formula for integrals of the first kind can now be used to write

$$\int_0^{\cos\tau} + \int_0^{-\cos\eta} \quad \text{as} \quad \int_0^{\alpha} ,$$

where

$$\alpha(\tau,\eta) = \frac{\cosh\xi_0\left\{\cos\tau\sin\eta(\cosh^2\xi_0-\cos^2\eta)^{1/2}-\cos\eta\sin\tau(\cosh^2\xi_0-\cosh^2\tau)^{1/2}\right\}}{\cosh^2\xi_0-\cos^2\tau\cos^2\eta} .$$

A tangent to the ξ_0-ellipse drawn from the point (Ξ,τ) touches the ellipse at the point $(\xi_0\ \eta)$ such that

$$\cosh\Xi\ \sinh\xi_0\cos\tau\cos\eta + \sinh\Xi\ \cosh\xi_0\ \sin\tau\sin\eta = \sinh\xi_0\ \cosh\xi_0 . \qquad (6.8)$$

When $\tau = 0$, it is a trivial matter to verify that (6.7) and (6.8) are equivalent. For nonzero τ, the verification is easy but tedious.

7. Lastly, we discuss the far field in the shadow zone. In particular, we derive the residue series for the integral in (3.3) for large ξ and $\Xi > \xi > \xi_0$, or for large Ξ and $\xi > \Xi > \xi_0$. The r^{th} residue R_r is precisely

$(-2i\gamma)^{-1} w_2(\xi, \lambda_r) w_1'(\xi_0, \lambda_r)$ times the r^{th} residue in the case of the surface distribution. It therefore follows from the relations (6.2), (4.4), and (6.6) that

$$R_r = F_r E_r \quad ,$$

where

$$R_r = \frac{(\sinh \xi_0 \cosh \xi_0)^{2/3} \left[\zeta^{1/3} H^{(1)}_{1/3}(\zeta) \right]'_{\zeta = h_r}}{2 \cdot 3^{1/3} (\gamma h_r)^{2/3} H^{(2)}_{1/3}(h_r) \left[H_r(\Xi) H_r(\xi) K_r(\eta) K_r(\tau) \right]^{1/2}}$$

and

$$E_r = \frac{\cos\left\{ \gamma \left[\int_0^{\pi} K_r(t)\, dt - \left| \int_{\tau}^{\eta} K_r(t)\, dt \right| \right] \right\}}{\sin\left(\gamma \int_0^{\pi} K_r(t)\, dt \right)} \, .$$

$$\cdot \exp\left\{ -i\gamma \left[\coth \xi H_r(\xi) + \coth(\Xi) H_r(\Xi) + \left(\int_0^{\ell_r(\xi)} + \int_0^{\ell_r(\Xi)} - \int_0^{\pi} \right) K_r(t)\, dt \right] \right\} .$$

In these formulas the functions H_r, K_r and ℓ_r are defined by the relations (6.1), (5.2), and (6.3), respectively.

We can also write E_r as a sum of creeping wave terms:

$$E_r = i \sum_{n=0}^{\infty} \left\{ e^{i\gamma \left(\left| \int_{\tau}^{\eta} \right| - (2n-1) \int_0^{\pi} \right) K_r(t)\, dt} + e^{-i\gamma \left[(2n+1) \int_0^{\pi} - \left| \int_{\tau}^{\eta} \right| \right] K_r(t)\, dt} \right\} .$$

$$\cdot \exp\left\{ -i\gamma \left[\coth \xi H_r(\xi) \coth \Xi\, H_r(\Xi) + \left(\int_0^{\ell_r(\xi)} + \int_0^{\ell_r(\Xi)} \right) K_r(t)\, dt \right] \right\} .$$

Let

$$I_r(f) = -i\gamma \left[\left| \int_{\tau}^{\eta} \right| - (2n-1)\left(\int_0^{\pi} + \int_0^{\ell_r(\xi)} + \int_0^{\ell_r(\Xi)} \right) \right] f(t)\, dt \ .$$

Then a typical exponent is

$$-i\gamma \coth\xi\, H_0(\Xi) - i\gamma \coth\Xi\, H_0(\Xi) - i\gamma\, I_0(K_0) + i\, C_1 \frac{(ab)^{2/3}}{c} I_0(K_0^{-1}) + \dots \ .$$

The description of the terms involving I_0 in terms of physical parameters is essentially the same as that given at the end of § 6; and, of course, the remarks made there upon the region of validity of the expansion also apply. It is a considerably more tedious matter to verify that the creeping wave expansion converges in the geometric shadow zone.

APPENDIX

In our paper[1] we developed an expression for the surface distribution induced by a plane wave whose plane is perpendicular to the axis of the spheroid. This expression is in the form of the well-known "creeping wave" representation, namely

$$\sum_{n=0}^{\infty} (-1)^n \sum_r A_r \left\{ e^{i\nu_r\left[d_r(\eta)+nL_r\right]+\frac{\pi i}{4}} + e^{i\nu_r\left[d_r^*(\eta)+nL_r\right]-\frac{\pi i}{4}} \right\} .$$

We should like to point out the restriction upon our result and that given by the geometrical optics theory of Keller[2]. The restriction stems from the terms $(d_r(\eta) + nL_r)$ and $(d^*(\eta) + nL_r)$. We have shown that

$$i\nu_r d_r(\eta) = i\sqrt{a^2 - b^2}\,\omega \int_0^{\eta} \left(\frac{\xi_r^2 - t^2}{1 - t^2}\right)^{1/2} dt ,$$

where a and b are the semi-major and minor axes of the spheroid, ω is the wave number, and ξ_r is related to the r^{th} zero of the Airy function (Ref. 1, Sect. 10). If one expands the integral above in a series of ascending powers of $\xi_r - \xi_o$, where $\xi_o^{-1} = e$ is the eccentricity, one obtains the result (taking into account only the first 3 terms of the expansion)

$$i\nu_r d_r(\eta) \sim i\omega \int_{\text{Arc}\cos\eta}^{\pi/2} ds + i(c_1\omega^{1/3} + c_2\omega^{-1/3}) \int_{\text{Arc}\cos\eta}^{\pi/2} R^{-2/3} ds$$

$$+\, i\, c_3\omega^{-1/3} \int_0^{\eta} \frac{dt}{\left[(1-e^2t^2)^3(1-t^2)\right]^{1/2}} ,$$

where R is the local radius of curvature, s is arc length,

$$c_1 = \frac{e^{-\pi i/3}(3h_r)^{2/3}}{2}$$

$$c_2 = \frac{e^{-2\pi i/3}(3h_r)^{4/3}}{8}\left\{\frac{(8 - 7R_o/a)}{15R_o^{2/3}} + \frac{R_o^{1/3}}{a}\right\}$$

$$c_3 = -\frac{e^{-2\pi i/3}(3h_r)^{4/3}R_o^{2/3}}{8a} ,$$

$R_o = b^2/a$ is the radius of curvature at the tip of the spheroid, and h_r is the r^{th} zero of $\left[t^{1/3} H^{(2)}_{1/3}(t)\right]'$.

In the geometrical optics theory of Keller[2] only the terms $\int ds$ and $c_1 \int R^{-2/3} ds$ are present.

Two observations of interest can be made from these formulas. Firstly, we note that in the case of the sphere (e = 0 and b = a) the term

$$\frac{i\, e^{-2\pi i/3}(3h_r)^{4/3}R_o^{1/3}}{\omega^{1/3} a}\int R^{-2/3}\, ds$$

is the negative of the term

$$i\, c_3\, \omega^{-1/3}\int_0^{\eta}\frac{dt}{\left[(1-e^2t^2)^3(1-t^2)\right]^{1/2}} .$$

This is consistent with the known results for the sphere. Secondly, we observe that if $R_o \to 0$, that is $e \to 1$, and if ω is fixed, the c_2 and c_3 terms completely dominate the attenuation. Let us estimate $\dfrac{\mathcal{I}\, c_2}{\omega^{2/3}\mathcal{I}\, c_1}$ in the case n = r = 0, the case of the most significant term in the creeping wave representation. One finds

$$\frac{\mathcal{J}_{c_2}}{\omega^{2/3}\,\mathcal{J}_{c_1}} = \frac{\left|3h_o\right|^{2/3}}{\omega^{2/3}\,4}\left\{\frac{(8-7R_o/a)}{15R_o^{2/3}} + \frac{R_o^{1/3}}{a}\right\},$$

where $\left|h_o\right|^{2/3} \approx \frac{3}{2}(1.0188)$.

If a = 6 in., b = .6 in., and λ = 1.25 in.,

$$\frac{\mathcal{J}_{c_2}}{\omega^{2/3}\mathcal{J}_{c_1}} \approx 3/8 \;;$$

while if a = 1 in., b = .1 in., and λ = 1.25 in.,

$$\frac{\mathcal{J}_{c_2}}{\omega^{2/3}\mathcal{J}_{c_1}} \approx 4 .$$

Therefore, for such spheroids and such a λ it appears that our theory and that of Keller will not give a significant result for the diffracted field. That is to say, the condition $R_o\omega >> 1$ is essential for the expansion of the creeping wave exponents in descending powers of ω to be meaningful. This condition is a consequence of the fact that in order for the expansion of

$$\int_0^{\eta}\left(\frac{\xi_r^2 - t^2}{1 - t^2}\right)^{1/2} dt$$

in powers of $\xi_r - \xi_o$ to be useful, we must know $\xi_r - \xi_o$; whereas $\xi_r - \xi_o$ has been estimated under the hypothesis that ξ_o is bounded away from 1. Further even if $R_o\omega >> 1$, since our theory is only an asymptotic one, we have no à priori way of predicting for a particular choice of parameters whether or not the additional terms in this series, which we have found, give a more or less accurate result than that obtained by consideration of only the first term. The examples a = 6 in., 1 in. have been chosen for discussion because of recent measurements of Olte and Silver[7] on such spheroids.

[7] A. Olte and S. Silver, New Results on Backscattering from Cones and Spheroids, URSI-Toronto Symposium, June 1959.

A CORRECTION TO A PREVIOUS REPORT

In "Studies in Radar Cross Sections **XXX**, The University of Michigan, Radiation Laboratory Report 2591-4-T, (August, 1958)", two errors are made which lead to erroneous conclusions.

Equation (2.5) is incorrect. It should read:

$$w'' + \left\{ p^2 \left[\frac{x_r^2 - x^2}{(x_r^2 - 1)(1 - x^2)} \right] + \frac{1}{(1 - x^2)^2} \right\} w = 0 .$$

In its original form, the equation led to an incorrect evaluation of the creeping wave exponents. This was observed in Reference 1 of the present report, and the corrected exponents appear in the appendix of the present report.

Also, the formula for $\frac{\partial y'}{\partial \lambda}$, on page 39, should now read:

$$\left. \frac{\partial y'}{\partial \lambda} \right|_{\lambda=\lambda_r} \sim (x_o^2 - 1)^{-1} \left\{ \frac{3\pi}{8 \gamma x_o} \right\}^{1/2} h_r H_{1/3}^{(2)}(h_r) \, e^{i \gamma f(x_r) + \frac{5\pi i}{4}} .$$

This results in a corrected formula (8.27):

$$\lim_{s \to 0} R_n = 2 \cdot 3^{-1/2} \, i \left[h_n H_{1/3}^{(2)}(h_n) \right] \left\{ (1 - \eta^2)(1 - \varepsilon^2 \eta^2) \right\}^{-1/4} \cdot$$

$$\cdot \left\{ \frac{e^{i\nu_n d(\eta) + \frac{\pi i}{4}} + e^{i\nu_n d^*(\eta) - \frac{\pi i}{4}}}{1 + e^{i\nu_n L}} \right\} .$$