THE UNIVERSITY OF MICHIGAN

INDUSTRY PROGRAM OF THE COLLEGE OF ENGINEERING

DYNAMICALLY LOADED JOURNAL BEARINGS OF FINITE LENGTH WITH AXIAL FEED

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy in the
University of Michigan
Department of Engineering Mechanics
1964

May, 1964

IP-671

ACKNOWLEDGMENTS

The author wishes to express his gratitude to Professor Jesse Ormondroyd, Chairman of the committee, who suggested the problem and under whose continued guidance the work was performed.

Thanks are also due to Professor George L. West Jr. for his assistance in the initial formulation of the problem and in arranging trips to the Newport News Shipbuilding Company, Newport News, Virginia and the U.S. Naval Engineering Experiment Station, Annapolis, Maryland to secure background data. These trips were sponsored by the University of Michigan Research Institute and the Engineering Mechanics Department, University of Michigan respectively.

The author is also indebted to the other members of his committee for their many helpful suggestions. The use of the I.B.M. 7090 computer at the University Computing Center is gratefully acknowledged.

Finally the author wishes to express his deepest appreciation to his wife, Mary Joan, for her constant patience and understanding, without which this work would not have been possible.

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NOMENCLATURE

Unless otherwise specified, the following symbols are used:

$A_m (m=1,..,\infty)$	functions of z
$B_m (m=1,..,\infty)$	functions of z
$C_m (m=1,..,\infty)$	integration constants
$D^2 = r^2 \frac{d^2}{dz^2}$	differential operator
$\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}$	substantial derivative
$F_i (i=1,2,etc.), F, \overline{F}$	auxillary symbols or functions
$G^{\alpha\beta} (\alpha,\beta=1,2,3)$	dimensionless metric tensor
$H_i (i=1,2)$	ratio of the amplitudes of the first and second harmonic components of the dynamic load to the static load
K	modified Sommerfeld number
L	a characteristic length
$\overline{M}$	mass of the journal plus the mass of the propeller
$N_m (m=1,..,\infty)$	integration constants
O	center of the bearing
O'	center of the journal
$\mathcal{O}$	function used for order of magnitude analysis
R	general radial coordinate
Re	a Reynolds number
$(Re)_c$	critical Reynolds number
S	Sommerfeld number
T	Taylor's parameter

U	velocity of a point on the journal surface tangential to the bearing wall in the circumferential direction
$U' = U/L$	dimensionless form of the velocity U
V	velocity of a point on the journal surface normal to the bearing wall
$V' = V/L$	dimensionless form of the velocity V
$\overline{V}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$	Laplacian operator
W_S	static load on the bearing
W_D	dynamic load on the bearing
W_{D1}, W_{D2}	amplitudes of the first and second harmonic components of the dynamic load
$X^i(i=1,2,3)$	body forces per unit mass corresponding respectively to the general coordinates ξ^i
X, Y, Z	body forces per unit mass corresponding respectively to the x,y,z Cartesian coordinates
a	function of n, $a = n/[1 + (1-n^2)^{1/2}]$
$\vec{a}_{o'}$	acceleration of the journal center
b	number of propeller blades
c	bearing clearance
d	diameter of the journal
e	journal eccentricity
$\vec{e}_N, \vec{e}_t$	unit vectors normal and tangential to the bearing wall
$f_i(i=1,2,etc.), f, \overline{f}$	auxillary symbols or functions
$g_{\alpha\beta}(\alpha,\beta=1,2,3)$	metric tensor
g	determinate of the metric tensor, $g=\lvert g_{\alpha\beta}\rvert$

h	film thickness at any point
h_o	minimum film thickness
ℓ	length of the bearing
$m(m=1,..,\infty)$	series indici
n	eccentricity ratio, $n = e/c$
p	pressure at any point in the bearing
p_o	supply pressure of the lubricating fluid
r	radius of the journal
r_1	radius of the bearing
t	time coordinate
u	velocity corresponding to the x Cartesian coordinate
$u^i(i=1,2,3)$	velocity components corresponding to the general coordinates ξ^i
$u^i_*(i=1,2,3)$	dimensionless form of the general velocity components
v	velocity corresponding to the y Cartesian coordinate
$\vec{v}_B$	velocity of any point on the journal surface
$\vec{v}_{O'}$	velocity of the journal center
$\vec{v}_{B/O'}$	velocity of any point on the journal surface relative to the journal center
w	velocity corresponding to the z Cartesian coordinate
$y^i(i=1,2,3)$	auxillary Cartesian coordinates
$\bar{y}$	function of n, $\bar{y} = n/(1-n^2)^{1/2}$
x,y,z	Cartesian coordinates
$\alpha,\bar{\alpha}$	functions of n

β	angular location of the dynamic load with respect to the static load
$\Gamma^i_{\alpha\beta}(\alpha,\beta,i=1,2,3)$	Euclidean Christoffel symbols
γ	function of n
$\delta^j_i(i,j=1,2,3)$	kronecker delta
ϵ	dimensionless small parameter for an order analysis, $\epsilon = h_o/L$
ζ	function of n
$\xi^i(i=1,2,3)$	general coordinates
Θ	angular location around the journal with respect to the attitude angle
Λ	phase angle of the second harmonic component of the dynamic load
μ	dynamic viscosity
ν	kinematic viscosity
π	dimensionless pressure
ρ	mass density of the lubricant
ϕ	attitude angle of the journal
ψ	function of n
ω	angular velocity of the journal

I. INTRODUCTION

A. Statement of the Problem

The problem being considered here is that of finding a solution for the pressure distribution and simultaneous shaft loci of a 360° journal bearing subjected to dynamic loading. The bearing is lubricated by a circumferential source at one end of the bearing. The lubricating fluid flows out the other end of the bearing. The bearing is considered to be finite in length. The bearing lubricant is water which is supplied at a constant rate and pressure. It is assumed that the bearing and journal surfaces will always remain parallel and that both are completely rigid. The surfaces are further assumed to be perfectly smooth.

The initial motivation for this problem stems from the stern tube bearing of ships, within which the tailshaft seemingly undergoes cavitation damage due to the dynamic loading of the propeller. It has been observed in several ships that the tailshaft is eroded at several definite positions around its periphery and within the confines of the stern bearing. (See Figure 1.1 for the location of the stern bearing and Figure 1.2 for the nature of the tailshaft damage involved.) The number of locations of damage and their positions around the periphery of the journal vary directly as the number of blades which the propeller has. If the number of blades is odd, either three or five, then there will be either three or five locations respectively of tailshaft damage and their positions will be directly in line with the propeller blades. If the number of blades is four, then there will be four locations of damage which are exactly 45° offset from the line of the propeller blades. The

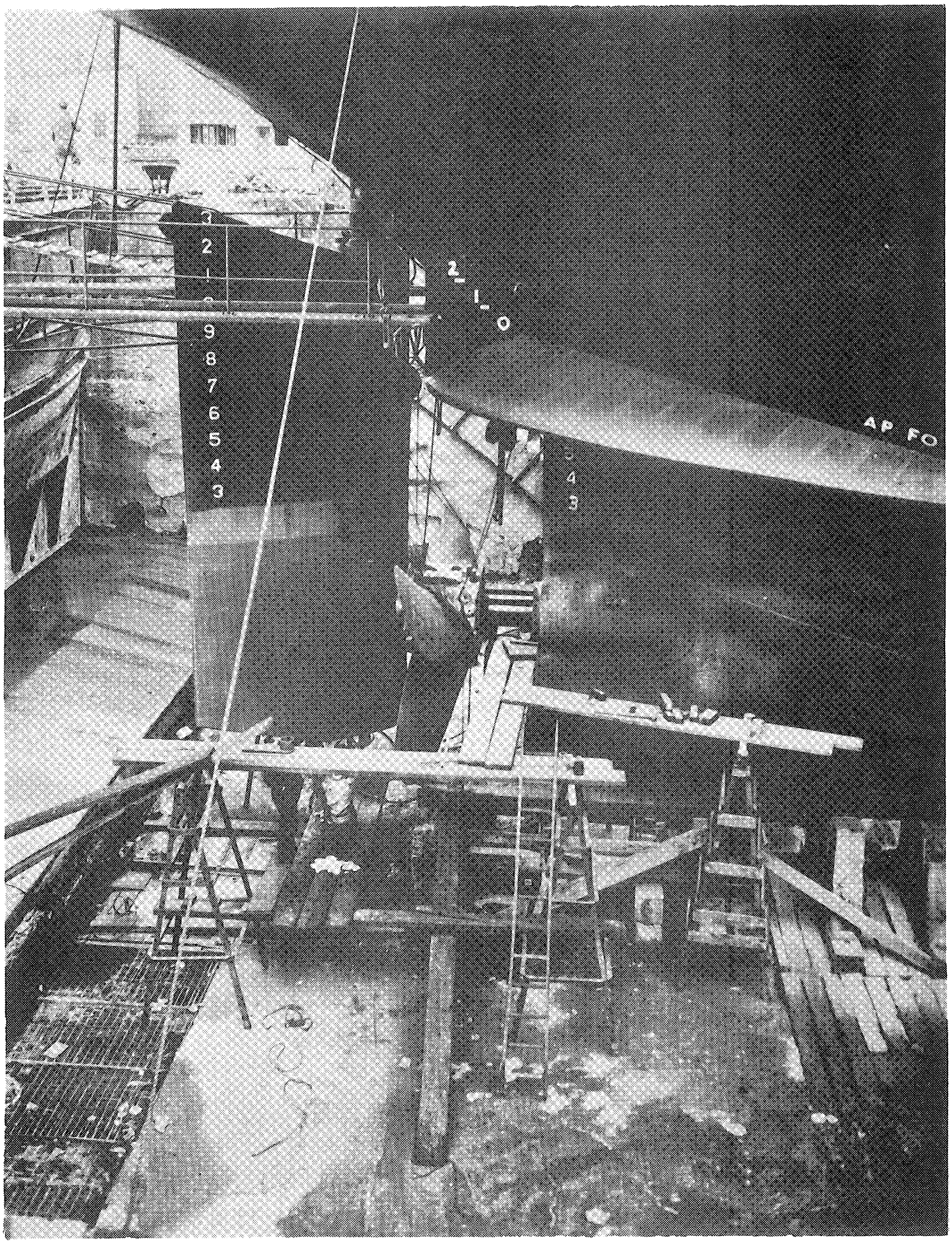

Figure 1.1. Stern Bearing Location.

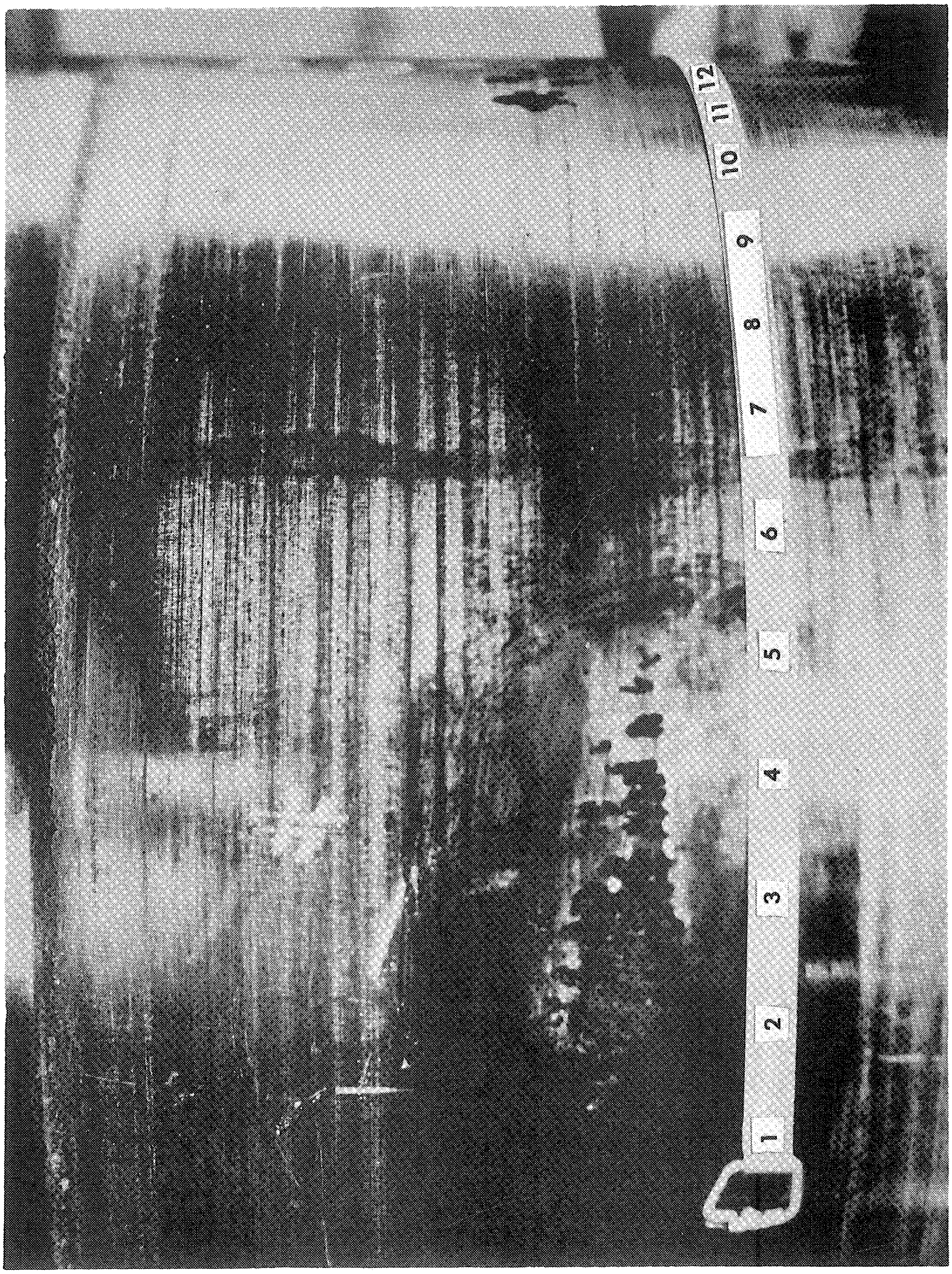

Figure 1.2. Nature of Tailshaft Damage.

location along the length of the bearing where this damage usually occurs is slightly forward of the quarter way point of the bearing as measured from the propeller end of the bearing.

From a survey made by the Bethlehem Steel Company, Shipbuilding Division,[1] possible explanations for an attack of this nature were (1) galvanic attack, (2) stray current leakage, (3) contact erosion with shaft idle and (4) cavitation erosion. The first of these was eliminated by direct electrical measurements which showed no current flow. The second was eliminated by the fact that if grounded D.C. leakage should attempt to leave the hull by jumping across to the shaft, then the damage should occur at the jumping off place, which would be the bearing and not the shaft. The third was eliminated by the multiple locations of damage. This left the fourth, cavitation erosion, as the probable explanation.

While the problem studied in this effort differs from the actual physical problem of stern tube bearings in that is was necessary to make certain assumptions to surpass some mathematical complexities, it is felt that it represents a reasonable initial model of the actual problem. It should be noted in passing that there are innumerable physical situations which correspond very closely to the problem studied here.

The three most important assumptions departing from the actual physical situation of stern tube bearings are (1) the shaft and bearing are to remain parallel at all times, (2) the bearing surface is completely smooth and (3) the propeller loading can be represented by at most the first two harmonic components.

The first of these assumptions has the effect of making the film thickness a function of only angular displacement around the journal. It is believed that the two major consequences of this assumption are a change in the intensity of the pressures obtained and a slight shift along the length of the bearing of the region of minimum pressures developed in the lubricating film. It is not felt however that the general pressure profile would be substantially altered by this assumption. These conjectures will be further explored in the presentation of the results.

The second assumption definitely violates the actual bearing which is composed of staves (See Figure 1.3) spaced in the order of one-half inch around the bearing periphery. The mathematical complexity of incorporating these effects however makes it necessary to assume a smooth bearing surface. Considering the third assumption, although the actual propeller loading is certainly composed of many harmonic components, the first two of these are known to represent the major portion of the propeller loading.

B. Brief History of Hydrodynamic Lubrication

Although the field of Hydrodynamic Lubrication is a relatively old one, it was not until the latter part of the 1950's that solutions for finite length journal bearings considering only static loading became available. Suprisingly few papers on dynamically loaded journal bearings of either finite or infinite length appear in the literature.

The initiation of Hydrodynamic Lubrication dates back to Tower,[(2)] who in 1883 seemingly by accident discovered this phenomena. He was engaged in an investigation of the friction characteristics of 157° partial

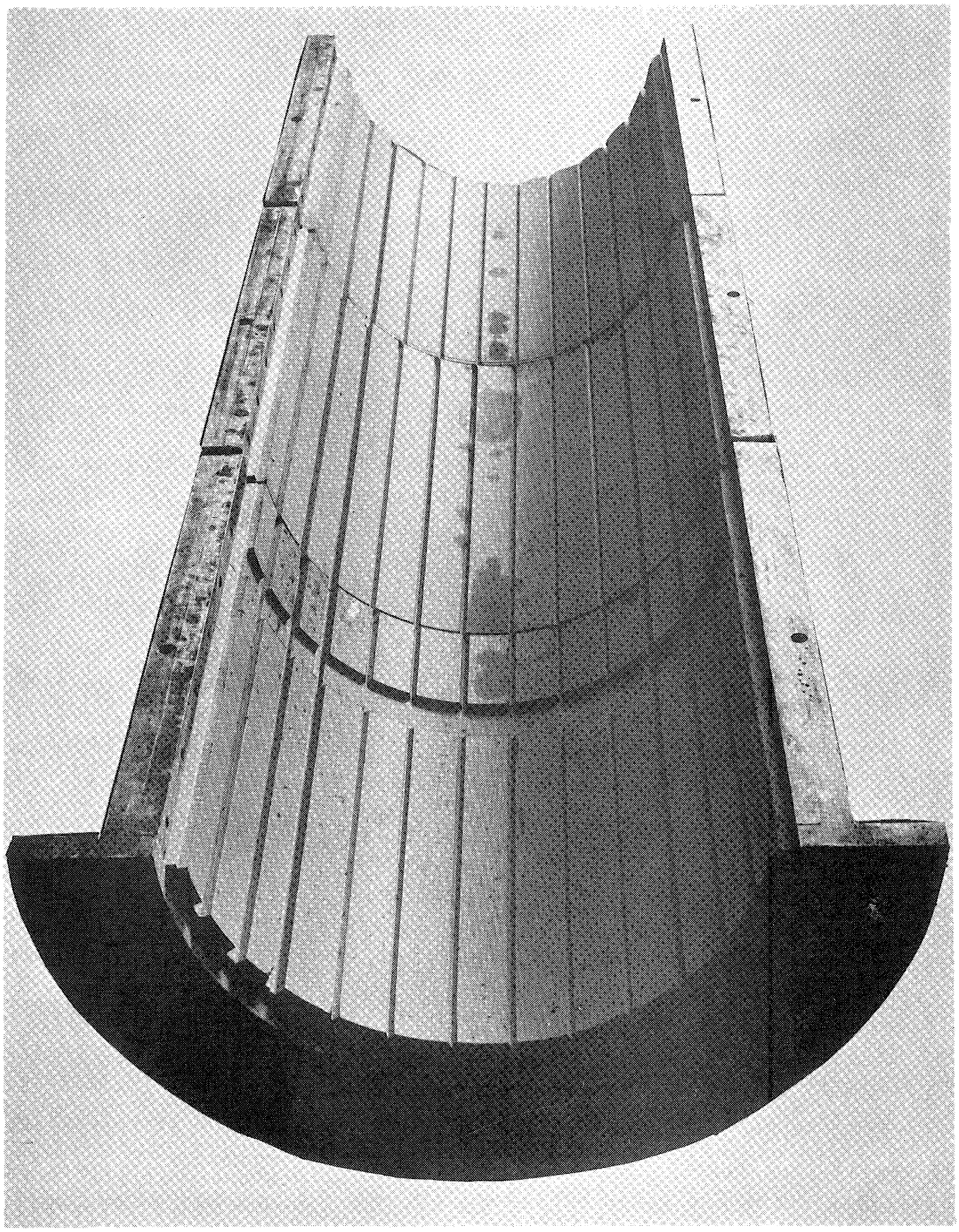

Figure 1.3. Actual Stern Tube Bearing.

railroad car journal bearings. In the course of his experiments it was necessary to drill a hole in the loaded region of the bearing. Plugging the hole with a cork, he noted that in subsequent experiments the cork was continuously forced out of the hole. Upon connecting a pressure gauge he found pressures in excess of 200 psi. while the unit bearing load was only 100 psi. Connecting more pressure gauges around the periphery of the bearing he found a very definite pressure profile. This of course confirmed his growing suspicion that a pressurized fluid film was being developed between the journal and bearing which actually supported the journal.

In 1886 Reynolds(3) put Hydrodynamic Lubrication upon a sound mathematical basis. His work is noted to such a degree for its clarity, scope and understanding, that at present his analysis is basically still followed. Some of his integrations have been improved or extended, but his basic theory is still intact. Considering the case of steady loading and infinite length, Reynolds was able to obtain an approximate solution to his basic equation for a journal bearing in the form of a Fourier series. His results however, were limited to lightly loaded bearing; that is eccentricity ratios of less than 0.5.

Sommerfeld,(4) in 1904, through a series of clever mathematical substitutions succeeded in finding an exact solution to Reynolds equation for steady loading of infinite length journal bearings. The major shortcomings of his results were unrealistic attitude angles and negative pressures of a magnitude that a fluid would be incapable of withstanding. These results were a direct consequence of his assuming a complete oil film around the bearing.

Following Sommerfeld, the next most notable works were that of Harrison;[5] who in 1919 first treated the problem of dynamic loading and Swift,[6] who in 1937 presented a rather extensive treatment of dynamically loaded journal bearings of infinite length. Although the major portion of Swift's effort was devoted to stability of steady loads and alternating loads with no journal rotation, he did consider the case of a sinusoidal load on a rotating journal. His results indicated that at a frequency ratio of one-half (forcing frequency/journal frequency) the load capacity of a dynamically loaded journal bearing is zero. Below one-half the load capacity is less than an equivalent statically loaded bearing and above one-half is greater than an equivalent statically loaded bearing. The general journal orbits were approximately elliptical in shape becoming flatter with increasing frequency ratio. For frequency ratios less than one-half the major axis of the orbit was perpendicular to the load and above one-half parallel to the load. His results however were limited to frequency ratios of one or less and no pressure profiles were obtained.

Burwell,[7] in 1947 extended Swift's results for the case of square wave loading. His results indicated that for equal load amplitudes, the sinusoidal loading gave greater load capacity to the bearing.

A major contribution was made by Tao,[8] who in 1959 succeeded in finding an exact and complete solution of Reynolds equation for statically loaded journal bearings of finite length. His method of approach was a multiple separation of variables technique which led to a form of Heun's equation.

Fedor,[9] in 1960, by adding a sine series to the known solution for a statically loaded journal bearing of infinite length was able to find a solution for a statically loaded journal bearing of finite length with circumferential feed. The analysis presented below will be based on Fedor's method.

In 1961 Hays[10] presented a solution for a finite length journal bearing subjected to a sinusoidal loading considering only the squeeze film effect; that is without journal rotation. His method of approach was the assumption of a double sine-cosine series solution for the pressure function and numerical evaluation of the coefficients on a digital computer. To the best of the author's knowledge he also presented the first pressure profile for a dynamically loaded journal bearing. This distribution, as would be expected for a symmetrically loaded journal bearing, was in the form of a paraboloid; being symmetrical with respect to the length of the bearing and with respect to angular displacement around the journal.

C. General Plan of Attack

The initial phase of the problem deals with the formulation of Reynolds equation governing dynamic loading of finite length journal bearings. Although this equation is well known, the method chosen for its derivation is a relatively new one. Following the recent work of Elrod,[11] who considered the case of statically load journal bearings of finite length, the equation is formulated by a small parameter approach. The reason for this choice is that it eliminates the necessity of making the usual assumptions that the convective inertial terms of the Navier-Stokes equations are negligible, the fluid film curvature is negligible and the pressure is constant across the film thickness.

The solution of the Reynolds equation is based on the recent work of Fedor,[9] who as mentioned above considered the case of a statically loaded finite length journal bearing with circumferential feed. To his solution an assumed series term is added to account for the dynamic loading, which when made to satisfy Reynolds equation and the given boundary conditions will yield the equation for the pressure distribution.

This equation however will contain time indirectly in the form of two unknown velocity components. These velocity components are the translational and rotational velocities of the journal center in its orbit about a steady state position.

These two unknown velocity components must then be solved for from the two scalar equations of motion of the journal mass center. These equations of motion will yield two simultaneous, non-linear, ordinary differential equations which must be evaluated numerically; in this case on a digital computer. In addition to yielding the two unknown velocity components, the path of the journal center will now be known. Thus for a given position in the journal orbit and the corresponding velocity components of the journal center, the pressure profile around and along the length of the bearing may then be evaluated.

II. MATHEMATICAL DESCRIPTION

A. Formulation of Reynolds Lubrication Equation

If the assumptions of constant density and viscosity of the bearing lubricant are made, the Navier-Stokes equations and equation of continuity may be written respectively as

$$\rho \frac{Du}{Dt} = - \frac{\partial p}{\partial x} + \rho X + \mu \nabla^2 u \ , \tag{2.1}$$

$$\rho \frac{Dv}{Dt} = - \frac{\partial p}{\partial y} + \rho Y + \mu \nabla^2 v \ , \tag{2.2}$$

$$\rho \frac{Dw}{Dt} = - \frac{\partial p}{\partial z} + \rho Z + \mu \nabla^2 w \ , \tag{2.3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \ , \tag{2.4}$$

in which x,y,z are Cartesian co-ordinates, u,v,w are the corresponding velocity components, ρ is the density, p is the pressure, μ is the dynamic viscosity, X,Y,Z are the body forces per unit mass,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad \text{and} \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \ .$$

If the further assumptions are made that the flow is everywhere laminar and the inertial terms $\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}, \frac{\partial w}{\partial t}$ and the body forces X,Y,Z are neglible, then it can be shown (See Appendix A for the derivation.) by following the small parameter approach of Elrod[11] that Equations (2.1) and (2.3) may be integrated directly for the velocity components u and w. Equation (2.2) yields $\frac{\partial p}{\partial y} = 0$ and thus the pressure is constant across the film thickness. If the values of u and w are substituted into Equation (2.4) and then integrated with respect to y

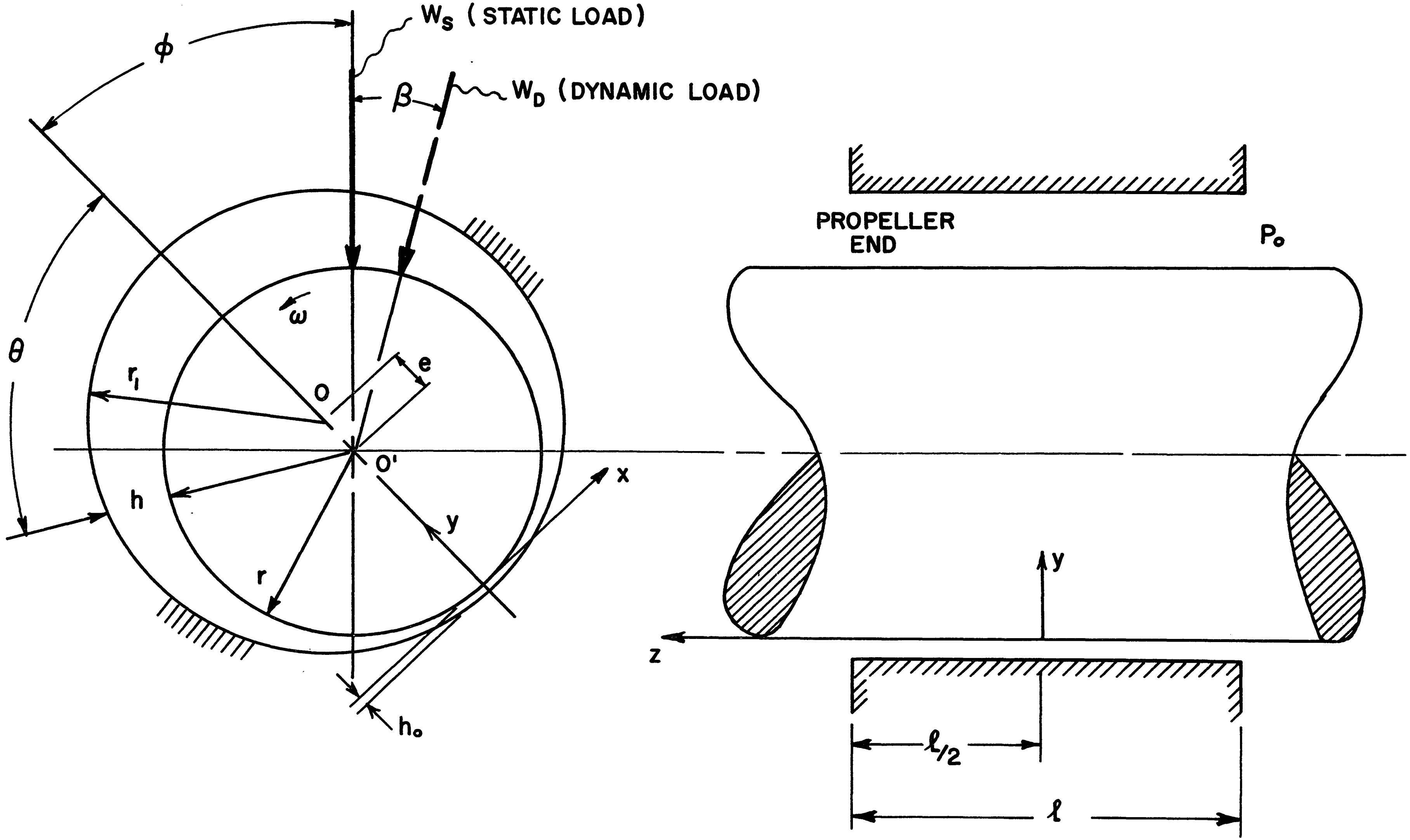

Figure 2.1. Physical Configuration.

across the film thickness h, the Reynolds equation, Equation (2.5), with first order correction terms may be written as

$$\frac{\partial}{\partial x}[h^3(1 - \frac{h}{d}) \frac{\partial p}{\partial x}] + \frac{\partial}{\partial z}[h^3(1 + \frac{h}{d}) \frac{\partial p}{\partial z}]$$

$$= -6\mu U \frac{\partial}{\partial x}[h(1 - \frac{h}{3d})] + 6\mu h(1 - \frac{h}{3d}) \frac{\partial U}{\partial x} + 12\mu V , \qquad (2.5)$$

where h is the film thickness at any point, d is the diameter of the journal and V and U are the velocity components of any point on the journal surface normal and tangential respectively to the bearing.

Justification of the five assumptions necessary for the derivation of Equation (2.5), namely, (1) constant density, (2) constant viscosity, (3) the flow is laminar, (4) $\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = \frac{\partial w}{\partial t} = 0$ and (5) $X = Y = Z = 0$ is now in order. The first of these (1) is very reasonable when it is noted that the compressibility of lubricating oils and water is in the order of one part in two thousand or less. Considering (2), the viscosity of lubricating oils or water is known to decrease with temperature. However, when it is noted that the bearing is supplied by forced feed lubrication and that the hull of the ship, which is directly adjacent to the stern tube bearing, is immersed in the lake water, the temperature rise is very small.

Considering assumption (3) and assuming for a moment that the journal and bearing are concentric, then the critical Reynolds number $(Re)_c$ for Couette flow between circular cylinders may be calculated from the Taylor parameter[12] T as follows,

$$T = -2.0(\frac{r_1}{r} - 1)^4 (Re)_c^2 (\frac{-r^2}{r_1^2 - r^2}) > 0 .$$

For numerical evaluation of the problem considered in this study the data used will be that of the S.S. John G. Munson of the Bradley Transportational Line. This is a single screw ship with a four bladed propeller. From this data $r_1 = 9.2935$ inches and $r = 9.25$ inches. This gives $(Re)_c = 1.810 \times 10^5$. The actual Reynolds number is given by

$$(Re) = \frac{r^2\omega}{\nu} = 4.669 \times 10^5$$

where ν is the kinematic viscosity of water ($\nu = 1.4 \times 10^{-5}$ ft^2/sec) and $\omega = 11$ rad/sec. This value however may be corrected[13] by a factor of $(1.0 + .89n^2)$ where $n = e/c$, e being the eccentricity of the journal and c the bearing clearance. Based on the minimum eccentricity encountered here the correction factor is 1.7. This gives a new value for Reynolds number $(Re) = 2.746 \times 10^5$. This value of (Re) is of course larger than $(Re)_c$ but not significantly and does not necessarily mean that the flow is turbulent. At $(Re)_c$ for Couette flow between circular cylinders a secondary flow is induced which is also a laminar flow. An actual bound on (Re) before turbulent motion occurs has not yet been established but it is known that this secondary flow is actually more stable than the initial laminar flow. It is therefore not unreasonable to assume laminar flow. For purposes of mathematical analysis this is a mandatory assumption.

The validity of assumption (4) can be shown by a comparison of the magnitudes of the inertial terms versus the viscous terms, that is $\frac{\partial u}{\partial t} << \nu \frac{\partial^2 u}{\partial x^2}$. This is equivalent to $\frac{1}{t} << \frac{\nu}{x^2}$. The element of time can be taken to be one half of the period corresponding to the forcing frequency ($\omega = 44$ rad/sec) and that of displacement ($x = 0.0012$ in.).

This corresponds to the maximum journal movement in this period of time. Considering these values

$$\frac{44}{\pi} \ll \frac{1.4 \times 10^{-5} \times 144}{(0.0012)^2} \quad \text{or} \quad 1 \ll 100 \ .$$

Assumption (5) is obviously good. Considering the diameter of the journal, approximately two feet, the hydrostatic head will be in the order of 1.0 psi. This is less than 0.5% of the total pressure.

For the bearing considered here, $h/d = 0.0017$. In all other known journal bearing uses, h/d is even less than this. Therefore, $h/d \ll 1.0$ and these terms may be neglected in Equation (2.5) which then can be written as

$$\frac{\partial}{\partial x}[h^3 \frac{\partial p}{\partial x}] + \frac{\partial}{\partial z}[h^3 \frac{\partial p}{\partial z}] = -6\mu U \frac{\partial h}{\partial x} + 6\mu h \frac{\partial U}{\partial x} + 12\mu V \ . \qquad (2.6)$$

B. Boundary Value Velocities and Film Thickness

Referring to Figure 2.2, the absolute velocity of point B may be written as

$$\vec{v}_B = \vec{v}_{O'} + \vec{v}_{B/O'} \ .$$

In terms of components parallel to the unit vectors $\vec{e}_N$ and $\vec{e}_\Theta$

$$\vec{v}_{O'} = \frac{d}{dt}[e \cos\Theta \ \vec{e}_N + e \sin\Theta \ \vec{e}_\Theta] + \frac{d\vec{\phi}}{dt} x[e \cos\Theta \ \vec{e}_N + e \sin\Theta \ \vec{e}_\Theta],$$

or

$$\vec{v}_{O'} = [\frac{de}{dt} \cos\Theta + e \sin\Theta \frac{d\phi}{dt}] \ \vec{e}_N + [\frac{de}{dt} \sin\Theta - e \cos\Theta \frac{d\phi}{dt}] \ \vec{e}_\Theta \ .$$

Similarly

$$\vec{v}_{O'/O} = -r\omega \sin\delta \ \vec{e}_N + r\omega \cos\delta \ \vec{e}_\Theta \ .$$

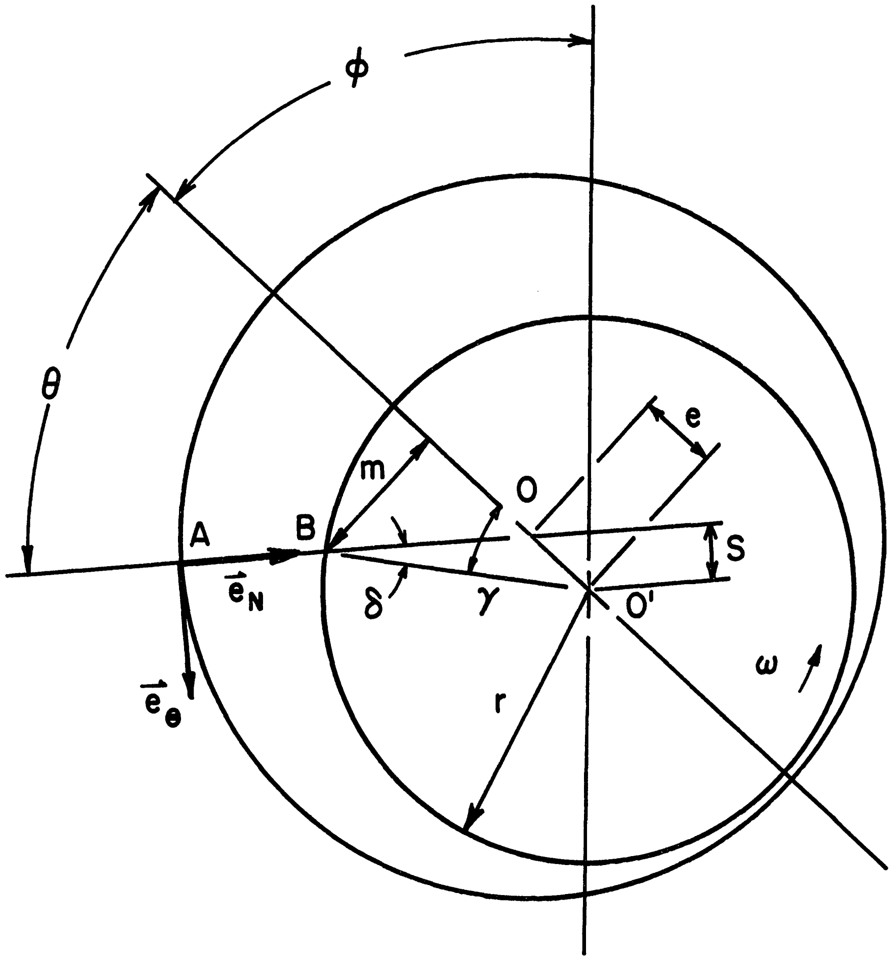

Figure 2.2. Boundary Value Velocities and Film Thickness Configuration.

Therefore

$$\vec{v}_B = [-r\omega \sin\delta + \frac{de}{dt}\cos\theta + e \sin\theta \frac{d\phi}{dt}]\,\vec{e}_N$$

$$+ \;[r\omega \cos\delta + \frac{de}{dt}\sin\theta - e \cos\theta \frac{d\phi}{dt}]\,\vec{e}_\theta \;.$$

Separating $\vec{v}_B$ into it scalar components and noting δ is a small angle, therefore

$$\cos\delta \cong 1.0 \quad \text{and} \quad \sin\delta \cong -\frac{\partial h}{\partial x} ,$$

the components of velocity U and V of a point on the journal tangential and normal respectively to the bearing may be written

$$U = r\omega + \frac{de}{dt}\sin\theta - e\cos\theta\frac{d\phi}{dt} , \qquad (2.7)$$

$$V = r\omega\frac{\partial h}{\partial x} + \frac{de}{dt}\cos\theta + e\sin\theta\frac{d\phi}{dt} . \qquad (2.8)$$

Referring to Figure 2.2 the film thickness h at any point B around the journal may be written as

$$h = \overline{AB} = \overline{OA} - \overline{OB} = (r+c) - \overline{OB} .$$

It is necessary at this point to assume that the journal and bearing always remain parallel such that the film thickness is a function of only angular displacement around the journal.

Now

$$S = e\sin\theta \quad \text{and} \quad S = r\sin\delta ,$$

therefore

$$e\sin\theta = r\sin\delta .$$

Further

$$m = r \sin\delta \quad \text{and} \quad m = \overline{OB} \sin\Theta ,$$

thus

$$\overline{OB} = r \sin\gamma / \sin\Theta .$$

Noting that $\gamma = \Theta - \delta$,

$$\overline{OB} = \frac{r}{\sin\Theta} \sin(\Theta - \delta) = \frac{r}{\sin\Theta} [\sin\Theta \sin\delta - \cos\Theta \cos\delta] .$$

However

$$\sin\delta = \frac{e}{r} \sin\Theta ,$$

and thus

$$\cos\delta = (1 - \frac{e^2}{r^2} \sin^2\Theta)^{1/2} .$$

Therefore

$$\overline{OB} = \frac{r}{\sin\Theta} [\sin\Theta(1 - \frac{e^2}{r^2} \sin^2\Theta)^{1/2} - \frac{e}{r} \sin\Theta \cos\Theta] ,$$

or

$$\overline{OB} = (r^2 - e^2 \sin^2\Theta)^{1/2} - e \cos\Theta .$$

The film thickness h may thus be written as

$$h = r + c - (r^2 - e^2 \sin^2\Theta)^{1/2} + e \cos\Theta .$$

Noting that $e^2 \sin^2\Theta \ll r^2$,

$$h = c + e \cos\Theta = c(1 + n \cos\Theta) . \tag{2.9}$$

Substituting Equations (2.7) and (2.8) into Equation (2.6) and noting $x = r\Theta$, therefore $\frac{\partial}{\partial x} = \frac{1}{r}\frac{\partial}{\partial\Theta}$,

$$\frac{1}{6\mu}[\frac{1}{r^2}\frac{\partial}{\partial\Theta}(h^3 \frac{\partial p}{\partial\Theta}) + \frac{\partial}{\partial z}(h^3 \frac{\partial p}{\partial z})] =$$

$$- \omega \frac{\partial h}{\partial\Theta} + \frac{e}{r}\frac{d\phi}{dt} \cos\Theta \frac{\partial h}{\partial\Theta} - \frac{1}{r}\frac{de}{dt} \sin\Theta \frac{\partial h}{\partial\Theta} \tag{2.10}$$

$$+ \frac{eh}{r}\frac{d\phi}{dt}\sin\Theta + \frac{h}{r}\frac{de}{dt}\cos\Theta$$

$$+ 2\omega\frac{\partial h}{\partial\Theta} + 2e\frac{d\phi}{dt}\sin\Theta + 2\frac{de}{dt}\cos\Theta \ . \qquad \text{(2.10 cont'd)}$$

From Equation (2.9)

$$\frac{\partial h}{\partial\Theta} = - cn \sin\Theta \ .$$

Substituting this result and Equation (2.9) into the right hand side of Equation (2.10) with the exception of the first term

$$\frac{1}{6\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial\Theta}\left(h^3\frac{\partial p}{\partial\Theta}\right) + \frac{\partial}{\partial z}\left(h^3\frac{\partial p}{\partial z}\right)\right] =$$

$$\omega\frac{\partial h}{\partial\Theta} + \frac{d\phi}{dt}\,e\sin\Theta\left(\frac{c}{r} + 2\right)$$

$$+ \frac{de}{dt}\left[\frac{cn}{r} + \frac{c}{r}\cos\Theta + 2\cos\Theta\right] \ . \qquad (2.11)$$

Finally noting

$$\frac{c}{r} \ll 2, \quad \frac{e}{r} \ll 2\cos\Theta, \quad \frac{cn}{r} = \frac{e}{r}$$

and

$$2cn\frac{d\phi}{dt}\sin\Theta = - 2\frac{d\phi}{dt}\frac{\partial h}{\partial\Theta} \ ,$$

Equation (2.11) may we written

$$\frac{1}{6\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial\Theta}\left(h^3\frac{\partial p}{\partial\Theta}\right) + \frac{\partial}{\partial z}\left(h^3\frac{\partial p}{\partial z}\right)\right] =$$

$$\left(\omega - 2\frac{d\phi}{dt}\right)\frac{\partial h}{\partial\Theta} + 2c\frac{dn}{dt}\cos\Theta \ . \qquad (2.12)$$

The solution of Equation (2.12) together with the appropriate boundary conditions will yield the solution for the pressure distribution around the bearing.

C. Boundary Conditions

One of the major difficulties encountered in solving the Reynolds equation is adequately defining the boundary conditions. There are three general approaches adapted in all of the bearing literature. The first of these is the classical Sommerfeld approach, namely

$$p(\Theta) = p(\Theta + 2\pi) \quad \text{and} \quad \frac{\partial p(\Theta)}{\partial\Theta} = \frac{\partial p(\Theta + 2\pi)}{\partial\Theta} .$$

This approach in general yields negative pressures over a considerable portion of the bearing and unrealistic attitude angles. The second method adopted is the same as the first except the pressure is set equal to zero for the entire negative region. The third approach is to let $p = 0$ for all $\Theta > \Theta_1$ and $\frac{\partial p}{\partial\Theta} = 0$ for $\Theta = \Theta_1$. This condition of course requires both the pressure and pressure gradient to be zero at the beginning of the negative region. This last method in general yields results that compare more favorably with practice, in particular with respect to the attitude angle.

The approach used in this study is that of Sommerfeld for the specific reason that the major questions in this study are (1) do negative pressures occur, (2) if so, are they of sufficient magnitude to allow cavitation of the lubricating fluid and (3) if cavitation is possible what is the location of the areas of possible cavitation around and along the length of the journal. Accordingly the boundary conditions adapted are

$$p(0,z) = p(2\pi,z) , \tag{2.13}$$

$$\frac{\partial p}{\partial\Theta}(0,z) = \frac{\partial p}{\partial\Theta}(2\pi,z) , \tag{2.14}$$

$$p(\Theta, \frac{\ell}{2}) = 0 \ , \tag{2.15}$$

$$p(\Theta, -\frac{\ell}{2}) = p_o \ . \tag{2.16}$$

III. THE PRESSURE EQUATION

A. Solution of the Reynolds Equation

In order to determine a solution to Equation (2.12) first consider the infinite bearing; thus $\frac{\partial}{\partial z}(\ ) = 0$. Equation (2.12) may then be written

$$\frac{\partial}{\partial \Theta}(h^3 \frac{\partial p}{\partial \Theta}) = 6\mu r^2(\omega - 2 \frac{d\phi}{dt}) \frac{\partial h}{\partial \Theta} + 12\mu r^2 c \frac{dn}{dt} \cos\Theta . \qquad (3.1)$$

The solution to Equation (3.1) satisfying the first two boundary conditions, Equations (2.13) and (2.14), is a well known solution (See Appendix B.) which may be written as

$$p = \frac{6\mu r^2}{c^2} (\omega - 2 \frac{d\phi}{dt})[\frac{n(2+n \cos\Theta) \sin\Theta}{(2+n^2)(1+n \cos\Theta)^2}]$$

$$+ \frac{6\mu r^2}{c^2 n} \frac{dn}{dt} [\frac{1}{(1+n \cos\Theta)^2}] + \text{arbitrary constant.} \qquad (3.2)$$

The arbitrary constant may be absorbed into the infinite series accounting for the finite length of the bearing. For the total pressure function, in view of Equation (3.2) let

$$p(\Theta,z) = p_o(\frac{1}{2} - \frac{z}{\ell}) + \frac{6\mu r^2}{c^2} (\omega - 2 \frac{d\phi}{dt}) [\frac{n(2+n \cos\Theta) \sin\Theta}{(2+n^2)(1+n \cos\Theta)^2}]$$

$$+ \frac{6\mu r^2}{c^2 n} \frac{dn}{dt} [\frac{1}{(1+n \cos\Theta)^2}]$$

$$- \sum_{m=1}^{\infty} A_m(z) \sin m\Theta - \sum_{m=1}^{\infty} B_m(z) \cos m\Theta . \qquad (3.3)$$

Equation (3.3) satisfies the first two boundary conditions, Equation (2.13) and (2.14). The first term represents the circumferential source function. The next two terms represent the infinite length bearing and the last two terms represent a correction to account for the finite length of the bearing. The first three terms satisfy Equation (3.3) as may readily be shown by substitution. The last two terms can be made to satisfy Equation (3.3) by substituting them into the equation and equating the coefficients of $\sin m\Theta$ and $\cos m\Theta$ equal to zero. If this is done, and for convenience let

$$A_m(z) = A_m \qquad \text{and} \qquad B_m(z) = B_m ,$$

the following recurrence relations are obtained.

For $m = 1$

$$2(D^2-1)\ A_1 + n(D^2+2)\ A_2 = 0 , \tag{3.4}$$

$$2(D^2-1)\ B_1 + n(D^2+2)\ B_2 = 0 , \tag{3.5}$$

and for $m > 1$

$$2(D^2-m^2)A_m + n(D^2-m^2-m+2)A_{m-1} + n(D^2-m^2+m+2)A_{m+1} = 0 , \tag{3.6}$$

$$2(D^2-m^2)B_m + n(D^2-m^2-m+2)B_{m-1} + n(D^2-m^2+m+2)B_{m+1} = 0 , \tag{3.7}$$

where

$$D^2 = r^2 \frac{d^2}{dz^2} .$$

Considering Equations (3.6) and (3.7), not only are they three term recurrence relations, but they are also second order differential equations. However following the work of Fedor(9) assume

$$A_2 = -\gamma A_1 \tag{3.8}$$

and

$$B_2 = -\psi B_1 \tag{3.9}$$

where γ and ψ are positive quantities to be determined. Substituting

Equation (3.8) into Equation (3.4), it becomes

$$[(2-\gamma n)D^2 - 2(1+\gamma n)]A_1 = 0 .$$

Letting

$$\alpha^2 = \frac{2(1+\gamma n)}{(2-\gamma n)} , \tag{3.10}$$

then

$$D^2A_1 - \alpha^2 A_1 = 0 . \tag{3.11}$$

Thus we have a second order, linear, homogeneous equation with constant coefficients. In view of the third boundary condition, Equation (2.15), the solution of Equation (3.11) must be even in z. Accordingly the solution of Equation (3.11) is

$$A_1 = C_1 \cosh \frac{\alpha z}{r} \tag{3.12}$$

where C_1 is a constant of integration. In view of Equation (3.8)

$$A_2 = -\gamma C_1 \cosh \frac{\alpha z}{r} . \tag{3.13}$$

Substituting Equation (3.9) into Equation (3.5), it becomes

$$[(2-\psi n)D^2 - 2(1+\psi n)]B_1 = 0 .$$

Letting

$$\zeta^2 = \frac{2(1+\psi n)}{(2-\psi n)} , \tag{3.14}$$

then

$$D^2B_1 - \zeta^2 B_1 = 0 . \tag{3.15}$$

Again in view of the third boundary condition, Equation (2.15), the solution of (3.15) must be even in z. Accordingly the solution of (3.15) is

$$B_1 = N_1 \cosh \frac{\zeta z}{r} \tag{3.16}$$

where N_1 is a constant of integration. In view of Equation (3.9)

$$B_2 = -\psi N_1 \cosh \frac{\zeta z}{r} \quad . \tag{3.17}$$

Knowing A_1, A_2, B_1, and B_2, all of the A_m and B_m may be evaluated from the general recurrence relations, Equations (3.6) and (3.7) respectively. This is done for the first eight terms of each series in Appendix C. A bound for this particular number of terms will be established later on.

Now applying the third boundary condition, Equation (2.15) $p(\Theta, \frac{\ell}{2}) = 0$, to Equation (3.3)

$$\frac{6\mu r^2}{c^2}(\omega - 2\,\frac{d\phi}{dt})\left[\frac{n(2+n\cos\Theta)\sin\Theta}{(2+n^2)(1+n\cos\Theta)^2}\right] + \frac{6\mu r^2}{c^2 n}\,\frac{dn}{dt}\left[\frac{1}{(1+n\cos\Theta)^2}\right] \tag{3.18}$$

$$- \sum_{m=1}^{\infty} A_m(\tfrac{\ell}{2}) \sin m\Theta - \sum_{m=1}^{\infty} B_m(\tfrac{\ell}{2}) \cos m\Theta = 0 \ .$$

This requires that the first two terms of Equation (3.18) be expanded into Fourier series. Considering the first term

$$\frac{6\mu r^2}{c^2}(\omega - 2\,\frac{d\phi}{dt})\left[\frac{n(2+n\cos\Theta)\sin\Theta}{(2+n^2)(1+n\cos\Theta)^2}\right]$$

$$= \frac{6\mu r^2}{c^2}(\omega - 2\,\frac{d\phi}{dt})\,\frac{1}{(2+n^2)}\left[\frac{n\sin\Theta}{(1+n\cos\Theta)} + \frac{n\sin\Theta}{(1+n\cos\Theta)^2}\right] \ . \tag{3.19}$$

Now

$$\frac{n\sin\Theta}{(1+n\cos\Theta)} = \frac{2a\sin\Theta}{(1+2a\cos\Theta + a^2)} \ ,$$

where

$$a = \frac{n}{1 + (1-n^2)^{1/2}} \quad . \tag{3.20}$$

It is known that[14]

$$\frac{\sin\Theta}{(1+2a\cos\Theta + a^2)} = \sum_{m=1}^{\infty} (-a)^{m-1} \sin m\Theta$$

which is uniformly convergent for $0 < a < 1$. Accordingly

$$\frac{n \sin\theta}{(1+n \cos\theta)} = 2 \sum_{m=1}^{\infty} (-1)^{m-1} a^m \sin m\theta \; . \tag{3.21}$$

Now

$$\frac{n \sin\theta}{(1+n \cos\theta)^2} = n \frac{\partial}{\partial n}\left[\frac{n \sin\theta}{(1+n \cos\theta)}\right] ,$$

therefore from Equation (3.21)

$$\frac{n \sin\theta}{(1+n \cos\theta)^2} = n \frac{\partial}{\partial a}\left[2 \sum_{m=1}^{\infty} (-1)^{m-1} a^m \sin m\theta\right] \frac{\partial a}{\partial n} ,$$

or

$$\frac{n \sin\theta}{(1+n \cos\theta)^2} = 2 \sum_{m=1}^{\infty} \frac{(-1)^{m-1} m\, a^m \sin m\theta}{(1-n^2)^{1/2}} \; . \tag{3.22}$$

Substituting Equations (3.21) and (3.22) into Equation (3.19)

$$\frac{6\mu r^2}{c^2}\left(\omega-2 \frac{d\phi}{dt}\right)\left[\frac{n(2+n \cos\theta) \sin\theta}{(2+n^2)(1+n \cos\theta)^2}\right]$$

$$= \frac{12\mu r^2}{c^2}\left(\omega-2 \frac{d\phi}{dt}\right)\left(\frac{1}{2+n^2}\right) \sum_{m=1}^{\infty} (-1)^{m-1} a^m \left(1 + \frac{m}{(1-n^2)^{1/2}}\right) \sin m\theta \; . \tag{3.23}$$

Considering the second term of Equation (3.18) it is noted

$$n^2 \frac{\partial}{\partial\theta}\left[\frac{1}{(1+n \cos\theta)^2}\right] = \frac{2n^3 \sin\theta}{(1+n \cos\theta)^3} \; . \tag{3.24}$$

Now

$$n \frac{\partial}{\partial n}\left[\frac{n \sin\theta}{(1+n \cos\theta)^2}\right] = - \frac{(-n \sin\theta + n^2 \sin\theta \cos\theta)}{(1+n \cos\theta)^3} \tag{3.25}$$

and

$$\frac{\partial^2}{\partial\theta^2}\left[\frac{n \sin\theta}{(1+n \cos\theta)}\right] = \frac{2n^3 \sin\theta}{(1+n \cos\theta)^3} + \frac{(-n \sin\theta + n^2 \sin\theta \cos\theta)}{(1+n \cos\theta)^3} \; . \tag{3.26}$$

By adding and substracting a term in Equation (3.24) it may be written as

$$n^2 \frac{\partial}{\partial\Theta}[\frac{1}{(1+n\cos\Theta)^2}] = \frac{2n^3 \sin\Theta}{(1+n\cos\Theta)^3} - \frac{(-n\sin\Theta + n^2\sin\Theta\cos\Theta)}{(1+n\cos\Theta)^3}$$

$$+ \frac{(-n\sin\Theta + n^2\sin\Theta\cos\Theta)}{(1+n\cos\Theta)^3} . \quad (3.27)$$

In view of Equations (3.25) and (3.26), Equation (3.27) may be written

$$n^2 \frac{\partial}{\partial\Theta}[\frac{1}{(1+n\cos\Theta)^2}] = \frac{\partial^2}{\partial\Theta^2}[\frac{n\sin\Theta}{(1+n\cos\Theta)}] + n\frac{\partial}{\partial n}[\frac{n\sin\Theta}{(1+n\cos\Theta)^2}] . \quad (3.28)$$

From Equation (3.22)

$$n\frac{\partial}{\partial n}[\frac{n\sin\Theta}{(1+n\cos\Theta)^2}] = 2\sum_{m=1}^{\infty}[\frac{m^2}{(1-n^2)} + \frac{mn^2}{(1-n^2)^{3/2}}](-1)^{m-1}a^m \sin m\Theta . \quad (3.29)$$

Considering Equation (3.21)

$$\frac{\partial}{\partial\Theta}[\frac{n\sin\Theta}{(1+n\cos\Theta)}] = 2\sum_{m=1}^{\infty}(-1)^{m-1}m\, a^m \cos m\Theta \quad (3.30)$$

and therefore

$$\frac{\partial^2}{\partial\Theta^2}[\frac{n\sin\Theta}{(1+n\cos\Theta)}] = -2\sum_{m=1}^{\infty}(-1)^{m-1}m^2 a^m \sin m\Theta . \quad (3.31)$$

Substituting Equations (3.29) and (3.31) into Equation (3.28)

$$n^2\frac{\partial}{\partial\Theta}[\frac{1}{(1+n\cos\Theta)^2}] = -2\sum_{m=1}^{\infty}(-1)^{m-1}m\, a^m[m - \frac{m}{(1-n^2)} - \frac{n^2}{(1-n^2)^{3/2}}]\sin m\Theta. \quad (3.32)$$

Integrating Equation (3.22) with respect to Θ

$$n^2[\frac{1}{(1+n\cos\Theta)^2}] = 2\sum_{m=1}^{\infty}(-1)^{m-1}a^m[m - \frac{m}{(1-n^2)} - \frac{n^2}{(1-n^2)^{3/2}}]\cos m\Theta. \quad (3.33)$$

From Equation (3.33) the second term of Equation (3.18) may thus be written as

$$\frac{6\mu r^2}{nc^2}\frac{dn}{dt}\left[\frac{1}{(1+n\cos\theta)^2}\right]$$

$$= \frac{6\mu r^2}{n^3c^2}\frac{dn}{dt}\left\{2\sum_{m=1}^{\infty}(-1)^{m-1}a^m\left[m - \frac{m}{(1-n^2)} - \frac{n^2}{(1-n^2)^{3/2}}\right]\cos m\theta\right\} . \qquad (3.34)$$

Although termwise integration of a Fourier series is quite legitimate, in this case Equation (3.32), the termwise differentiation of a Fourier series requires more caution. The necessary criteria is that the resultant series must converge uniformly in the interval in question. It is thus necessary to establish the uniform convergence of the series in Equations (3.22), (3.29), (3.30), and (3.31). Considering the series in Equation (3.22)

$$2\sum_{m=1}^{\infty}\frac{(-1)^{m-1}ma^m}{(1-n^2)^{1/2}}\sin m\theta ,$$

this series is uniformly convergent on the interval $0 < a \leqq r$ if $0 < r < 1$.

$$u_m(a) = ma^m \quad \text{and} \quad M_m = mr^m ,$$

noting that $\sin m\theta$ is bounded by one. Then on the stated interval $|a| \leqq r$ and so $|u_m(a)| \leqq M_m$. Since the series $\sum mr^m$ is convergent, the uniform convergence follows by virtue of the Weierstrauss M-test.

The series in Equation (3.30)

$$2\sum_{m=1}^{\infty}(-1)^{m-1}ma^m\cos m\theta$$

is uniformly convergent on the interval $0 < a \leqq r$ if $0 < r < 1$. Noting that $\cos m\theta$ is bounded by one the proof is identical to that for Equation (3.22) directly above.

Considering the series in Equation (3.31)

$$2 \sum_{m=1}^{\infty} (-1)^{m-1} m^2 a^m \sin m\Theta ,$$

this series is uniformly convergent on the interval $0 < a \leqq r$ if $0 < r < 1$. For let

$$u_m(a) = m^2 a^m \quad \text{and} \quad M_m = m^2 r^m ,$$

noting that $\sin m\Theta$ is bounded by one. Then on the stated interval $|a| \leqq r$ and so $|u_m(a)| \leqq M_m$. Since the series $\sum m^2 r^m$ is convergent, the uniform convergence again follows by virtue of the Weierstrauss M-test.

The series in Equation (3.29)

$$2 \sum_{m=1}^{\infty} [\frac{m^2}{(1-n^2)} + \frac{mn^2}{(1-n^2)^{3/2}}] (-1)^{m-1} a^m \sin m\Theta$$

is uniformly convergent on the interval $0 < a \leqq r$ if $0 < r < 1$. Noting that $\sin m\Theta$ is bounded by one and taking advantage of the fact that the sum of two uniformly convergent series is itself uniformly convergent, the proof of the uniform convergence of the individual terms of Equation (3.29) is identical respectively to that of Equations (3.31) and (3.22) above.

Substituting the results of Equations (3.23) and (3.34) into Equation (3.18)

$$\frac{12\mu r^2}{c^2}(\omega-2 \frac{d\phi}{dt})(\frac{1}{2+n^2}) \sum_{m=1}^{\infty} (-1)^{m-1} a^m [1 + \frac{m}{(1-n^2)^{1/2}}] \sin m\Theta$$

$$+ \frac{12\mu r^2}{n^3 c^2} \frac{dn}{dt} \sum_{m=1}^{\infty} (-1)^{m-1} a^m [m - \frac{m}{(1-n^2)} - \frac{n^2}{(1-n^2)^{3/2}}] \cos m\Theta$$

$$- \sum_{m=1}^{\infty} A_m(\frac{\ell}{2}) \sin m\Theta - \sum_{m=1}^{\infty} B_m(\frac{\ell}{2}) \cos m\Theta = 0 . \qquad (3.35)$$

Equating coefficients of $\sin m\Theta$ and $\cos m\Theta$ in Equation (3.35) equal to zero

$$A_m(\frac{\ell}{2}) = \frac{12\mu r^2}{c^2}(\omega - 2\frac{d\phi}{dt})(\frac{1}{2+n^2})(-1)^{m-1}a^m[1 + \frac{m}{(1-n^2)^{1/2}}] \qquad (3.36)$$

and

$$B_m(\frac{\ell}{2}) = \frac{12\mu r^2}{n^3c^2}\frac{dn}{dt}(-1)^{m-1}a^m[m - \frac{m}{(1-n^2)} - \frac{n^2}{(1-n^2)^{3/2}}] \; . \qquad (3.37)$$

Equations (3.36) and (3.37) in addition to yielding sufficient boundary conditions to evaluate the constants of integration for all of the $A_m(z)$ and $B_m(z)$ also permit the evaluation of γ and ψ in Equations (3.8) and (3.9) respectively.

Considering Equations (3.12) and (3.36) for $A_1(\frac{\ell}{2})$

$$C_1 \cosh\frac{\alpha\ell}{2r} = \frac{12\mu r^2}{c^2}(\omega - 2\frac{d\phi}{dt})(\frac{1}{2+n^2})\, a[1 + \frac{1}{(1-n^2)^{1/2}}] \; , \qquad (3.38)$$

and from Equations (3.13) and (3.36) for $A_2(\frac{\ell}{2})$

$$-\gamma C_1 \cosh\frac{\alpha\ell}{2r} = -\frac{12\mu r^2}{c^2}(\omega - 2\frac{d\phi}{dt})(\frac{1}{2+n^2})\, a^2[1 + \frac{2}{(1-n^2)^{1/2}}] \; . \qquad (3.39)$$

Dividing Equation (3.39) by (3.38) and substituting Equation (3.20) for a

$$\gamma = \frac{n[2 + (1-n^2)^{1/2}]}{[1 + (1-n^2)^{1/2}]^2} \; . \qquad (3.40)$$

From Equations (3.16) and (3.37) for $B_1(\frac{\ell}{2})$

$$N_1 \cosh\frac{\zeta\ell}{2r} = \frac{12\mu r^2}{n^3c^2}\frac{dn}{dt}\, a[1 - \frac{1}{(1-n^2)} - \frac{n^2}{(1-n^2)^{3/2}}] \; , \qquad (3.41)$$

and from Equations (3.17) and (3.37) for $B_2(\frac{\ell}{2})$

$$-\psi N_1 \cosh\frac{\zeta\ell}{2r} = -\frac{12\mu r^2}{n^3c^2}\frac{dn}{dt}\, a^2[2 - \frac{2}{(1-n^2)} - \frac{n^2}{(1-n^2)^{3/2}}] \; . \qquad (3.42)$$

Dividing Equation (3.42) by (3.41) and again substituting Equation (3.20) for a

$$\psi = \frac{n[1 + 2(1-n^2)^{1/2}]}{[1 + (1-n^2)^{1/2}]^2} \quad . \tag{3.43}$$

It is to be noted finally that the last boundary condition, Equation (2.16), is satisfied by the chosen form of $p(\Theta,z)$. Now since Equation (3.3), which is repeated for convenience,

$$p(\Theta,z) = p_o(\frac{1}{2} - \frac{z}{\ell}) + \frac{6\mu r^2}{c^2}(\omega - 2\frac{d\phi}{dt})[\frac{n(2+n\cos\Theta)\sin\Theta}{(2+n^2)(1+n\cos\Theta)^2}] \tag{3.3}$$

$$+ \frac{6\mu r^2}{nc^2}\frac{dn}{dt}[\frac{1}{(1+n\cos\Theta)^2}] - \sum_{m=1}^{\infty} A_m(z)\sin m\Theta - \sum_{m=1}^{\infty} B_m(z)\cos m\Theta$$

satisfies Reynolds Equation (3.1) and meets all prescribed boundary conditions, it is a solution of the problem under consideration.

B. Convergence of the Series

Now the question of convergence comes up for the series

$$\sum_{m=1}^{\infty} A_m(z)\sin m\Theta$$

and

$$\sum_{m=1}^{\infty} B_m(z)\cos m\Theta \ .$$

The first of these has been shown(9) to be convergent in the following manner. Making the approximations

$$(m-1)(m+2) \cong (m-1)^2$$

and

$$(m+1)(m-2) \cong (m+1)^2$$

for large m, then the recurrence relation for $A_m(z)$, Equation (3.6), may be written as

$$2[D^2-m^2]A_m(z) + n[D^2-(m-1)^2]A_{m-1}(z) + n[D^2-(m+1)^2]A_{m+1}(z) = 0. \quad (3.44)$$

Letting

$$(D^2-m^2)A_m(z) = F_m \quad (3.45)$$

where

$$F_m = F_m(m,n,z)$$

and substituting into Equation (3.44) yields the following second order difference equation with constant coefficients

$$2F_m + nF_{m+1} + nF_{m-1} = 0 . \quad (3.46)$$

Letting $F_m = U^x$, then from Equation (3.46)

$$U^{x-1}[U^2 + \frac{2}{n} U + 1] = 0 \quad (3.47)$$

or

$$U_{1,2} = \frac{-1 \pm (1-n^2)^{1/2}}{n} .$$

Now in order to insure F_m is finite at $n = 0$ it is noted that

$$U_{1,2} = \frac{n}{-1 \pm (1-n^2)^{1/2}}$$

also satisfies Equation (3.47).

Accordingly

$$F_m = T_1[\frac{n}{-1 + (1-n^2)^{1/2}}]^m - T_2[\frac{n}{1 + (1-n^2)^{1/2}}]^m .$$

In order to insure $F_\infty = 0$ at $n = 0$ it follows that $T_1 = 0$. Letting $F_0 = f(z)$, then

$$F_m = (-1)^m f(z)[\frac{n}{1 + (1-n^2)^{1/2}}]^m = (-1)^m a^m f(z). \quad (3.48)$$

Substituting Equation (3.48) into Equation (3.45)

$$(D^2-m^2)A_m(z) = (-1)^m a^m f(z) \ . \qquad (3.49)$$

Since $A_m(z)$ is an even function the complementary solution is

$$A_m(z) = C_m \cosh \frac{mz}{r} \ .$$

A particular solution by variation of parameters is

$$A_m(z) = (-1)^m a^m (\tfrac{r}{m}) \int f(z') \sinh[\tfrac{m}{r}(z-z')] \, dz'$$

or replacing the integral by its maximum value M, then the solution to Equation (3.49) may be written

$$A_m(z) = C_m \cosh \frac{mz}{r} - \frac{(-1)^m a^m M}{m^2} \ . \qquad (3.50)$$

Evaluating Equation (3.50) at $z = \frac{\ell}{2}$ it follows that

$$C_m = \frac{A_m(\frac{\ell}{2}) + \frac{(-1)^m a^m M}{m^2}}{\cosh \frac{m\ell}{2r}} \ .$$

Equation (3.50) may then be written as

$$A_m(z) = \frac{A_m(\frac{\ell}{2})\cosh \frac{mz}{r}}{\cosh \frac{m\ell}{2r}} + \frac{(-1)^m a^m M}{m^2}\left[\frac{\cosh \frac{mz}{r}}{\cosh \frac{m\ell}{2r}} - 1\right] \ .$$

Therefore the inequality exists

$$|A_m(z)| \leqq \left|\frac{A_m(\frac{\ell}{2})\cosh \frac{mz}{r}}{\cosh \frac{m\ell}{2r}}\right| + \left|\frac{a^m M}{m^2}\right| \left|\frac{\cosh \frac{mz}{r}}{\cosh \frac{m\ell}{2r}} - 1\right|$$

or more strongly

$$|A_m(z)| \leqq |A_m(\tfrac{\ell}{2})| + |\frac{a^m M}{m^2}| \ .$$

Letting

$$R_m = |A_m(\tfrac{\ell}{2})| + |\frac{a^m M}{m^2}|$$

then it follows from the Cauchy ratio test that $\sum_{m=1}^{\infty} R_m$ is absolutely convergent since $0 < a < 1$ for $0 < n < 1$ and

$$\lim_{m \to \infty} \left| \frac{A_{m+1}(\frac{\ell}{2})}{A_m(\frac{\ell}{2})} \right| = a$$

and

$$\lim_{m \to \infty} \left| \frac{a^{m+1}M}{(m+1)^2} \Big/ \frac{a^m M}{m^2} \right| = a \ .$$

Now $|\sin m\Theta| \leqq 1$ and $|A_m(z)| \leqq R_m$ for large m and therefore by the comparison test the series

$$\sum_{m=1}^{\infty} A_m(z) \sin m\Theta$$

absolutely converges for

$$-\frac{\ell}{2} \leqq z \leqq \frac{\ell}{2} ,$$

$$0 \leqq \Theta \leqq 2\pi ,$$

$$0 < n < 1 .$$

The proof for

$$\sum_{m=1}^{\infty} B_m(z) \cos m\Theta$$

is identical since the recurrence relation is identical and $|\cos m\Theta| \leqq 1$.

C. Estimate of the Series Error

An upper bound on the error involved in a finite number of terms of the series

$$\sum_{m=1}^{\infty} A_m(z) \sin m\Theta$$

can be determined in the following manner. A little reflection on this series will reveal that the $A_m(z)$ are attenuated at $z = 0$ and reach a maximum at $z = \frac{\ell}{2}$ where the axial flow in the bearing will attain its greatest importance. This is further substantiated by both the mathematical form (see Appendix C) and the numerical evaluation of the $A_m(z)$. Accordingly the following inequality may be written that

$$\sum_{m=1}^{\infty} |A_m(z)||\sin m\Theta| \leqq \sum_{m=1}^{\infty} |A_m(\tfrac{\ell}{2})||\sin m\Theta|$$

or

$$\sum_{m=1}^{\infty} |A_m(z)| \leqq \sum_{m=1}^{\infty} |A_m(\tfrac{\ell}{2})| \ .$$

Considering Equation (3.36) which is rewritten for convenience

$$A_m(\tfrac{\ell}{2}) = \frac{F_1}{(2+n^2)} (-1)^{m-1} a^m [1 + \frac{m}{(1-n^2)^{1/2}}] \qquad (3.36)$$

where

$$F_1 = \frac{12\mu r^2}{c^2}(\omega - 2 \frac{d\phi}{dt}) ,$$

it is noted that $A_m(\frac{\ell}{2})$ is the combination of the two geometric terms

$$a^m \quad \text{and} \quad ma^m = a \frac{d}{da} (a^m) \ .$$

By virtue of the known sum of the geometric series

$$\sum_{m=0}^{\infty} r^m = \frac{1}{(1-r)}$$

and its derivative

$$\sum_{m=1}^{\infty} mr^{m-1} = \frac{1}{(1-r)^2}$$

it follows that

$$\sum_{m=1}^{\infty} |A_m(\tfrac{\ell}{2})| = \frac{F1}{(2+n^2)}[\frac{1}{(1-a)} + \frac{a}{(1-n^2)^{1/2}(1-a)^2} - 1] . \qquad (3.51)$$

Considering the series

$$\sum_{m=1}^{\infty} B_m(z) \cos m \Theta$$

it is again noted that

$$\sum_{m=1}^{\infty} |B_m(z)||\cos m\Theta| \leqq \sum_{m=1}^{\infty} |B_m(\tfrac{\ell}{2})||\cos m\Theta|$$

or

$$\sum_{m=1}^{\infty} |B_m(z)| \leqq \sum_{m=1}^{\infty} |B_m(\tfrac{\ell}{2})| .$$

Rewriting Equation (3.37) for convenience

$$B_m(\tfrac{\ell}{2}) = \frac{F_2}{n^3}(-1)^{m-1}a^m[m - \frac{m}{(1-n^2)} - \frac{n^2}{(1-n^2)^{3/2}}] \qquad (3.37)$$

where

$$F_2 = \frac{12\mu r^2}{c^2}\frac{dn}{dt} ,$$

then again by virtue of the known sum for the geometric series and its derivative it follows that

$$\sum_{m=1}^{\infty} |B_m(\tfrac{\ell}{2})| = \frac{F_2}{n^3}[\frac{a}{(1-a)^2} - \frac{a}{(1-a)^2(1-n^2)} - \frac{n^2}{(1-a)(1-n^2)^{3/2}} + 1] . \qquad (3.52)$$

It is to be noted that the error involved with the series $B_m(z)$ is approximately twice as great as that for $A_m(z)$ for any given number of terms. A typical maximum error for the series

$$\sum_{m=1}^{\infty} B_m(z) \cos m\Theta$$

considering the first eight terms and corresponding to a steady state position of $n = 0.80$ is 3.5 percent. However in the actual evaluation, the boundary condition for a supply pressure of 30.0 psig was checked with a maximum error of 0.2 percent for eight terms of the series. The error interior to the ends of the bearing would of course be even less than this.

D. Condition on the Inlet Pressure

In order to insure that the oil film is continuous, the condition is imposed that the axial pressure gradient at the end of the bearing is negative. It follows then from Equation (3.3) that

$$\left(\frac{\partial p}{\partial z}\right)_{z = \frac{\ell}{2}} = \left[- \frac{p_o}{\ell} - \sum_{m=1}^{\infty} \frac{\partial A_m}{\partial z} \sin m\Theta - \sum_{m=1}^{\infty} \frac{\partial B_m}{\partial z} \cos m\Theta \right]_{z = \frac{\ell}{2}} < 0. \qquad (3.53)$$

If the first term is taken from each series, an equation can be found that imposes a minimum value for the feed pressure p_o. Greater accuracy on this lower bound may be found by including more terms of the two series. From Equation (3.12)

$$\left(\frac{\partial A_1}{\partial z}\right)_{z = \frac{\ell}{2}} = C_1\left(\frac{\alpha}{r}\right) \sinh \frac{\alpha\ell}{2r} ,$$

where from Equation (3.36)

$$C_1 = \frac{12\mu r^2}{c^2}\left(\omega - 2\,\frac{d\phi}{dt}\right)\left(\frac{1}{2+n^2}\right)\left(1 + \frac{1}{(1-n^2)^{1/2}}\right) \frac{a}{\cosh \frac{\alpha\ell}{2r}} .$$

From Equation (3.16)

$$\left(\frac{\partial B_1}{\partial z}\right)_{z = \frac{\ell}{2}} = N_1\left(\frac{\zeta}{r}\right) \sinh \frac{\zeta\ell}{2r} ,$$

where from Equation (3.37)

$$N_1 = \frac{12\mu r^2}{n^3c^2}\frac{dn}{dt}\left[1-\frac{1}{(1-n^2)}-\frac{n^2}{(1-n^2)^{3/2}}\right]\frac{a}{\cosh\frac{\zeta\ell}{2r}} \;.$$

Substituting these results into Equation (3.53) and noting Equation (3.20) for a

$$p_o > -\frac{12\mu r\ell}{c^2}\left[(\omega-2\frac{d\phi}{dt})(\frac{1}{2+n^2})(\alpha)(\frac{n}{(1-n^2)^{1/2}})(\tanh\frac{\alpha\ell}{2r})(\sin\Theta)\right.$$

$$\left.-(\frac{dn}{dt})(\frac{1}{(1-n^2)^{3/2}})(\zeta)(\tanh\frac{\zeta\ell}{2r})(\cos\Theta)\right] \;. \tag{3.54}$$

Now Θ is the angle at $z = \frac{\ell}{2}$ where the tendency is the greatest for positive gradients to exist. Thus the angle Θ in question is the angle which maximizes the right hand side of Equation (3.54). Differentiating the right hand side of Equation (3.54) with respect to Θ and equating to zero gives

$$\Theta = \tan^{-1}\left[-(\frac{\alpha}{\zeta})\frac{\tanh\frac{\alpha\ell}{2r}}{\tanh\frac{\zeta\ell}{2r}}\;\frac{\omega-2\frac{d\phi}{dt}}{\frac{dn}{dt}}\;\frac{n(1-n^2)}{2+n^2}\right] \;. \tag{3.55}$$

IV. EQUATIONS OF MOTION OF THE JOURNAL

A. Formulation of the Equations of Motion

Referring to Figure 4.1 the displacement of the journal center with respect to a fixed reference point, in this case the center of the bearing, may be written as

$$\overrightarrow{OO'} = e\,\vec{e}_N \,.$$

The velocity of point O' is then

$$\vec{v}_{O'} = \frac{de}{dt}\vec{e}_N + e\,\frac{d\phi}{dt}\vec{e}_\Theta \,.$$

It follows that the acceleration of point O' may be written as

$$\vec{a}_{O'} = \left[\frac{d^2e}{dt^2} - e\left(\frac{d\phi}{dt}\right)^2\right]\vec{e}_N + \left[e\,\frac{d^2\phi}{dt^2} + 2\,\frac{de}{dt}\,\frac{d\phi}{dt}\right]\vec{e}_\Theta \,.$$

Considering the scalar equations of motion

$$\Sigma F_N = ma_N \quad \text{and} \quad \Sigma F_t = ma_t \,,$$

then for the normal direction

$$W_D \cos(\phi + \beta) + \int_{-\ell/2}^{\ell/2}\int_0^{2\pi} p(\Theta,z)\cos\Theta r d\Theta dz + W_S \cos\phi = \bar{M}\left[\frac{d^2e}{dt^2} - e\left(\frac{d\phi}{dt}\right)^2\right] \tag{4.1}$$

and for the tangential direction

$$-W_D \sin(\phi + \beta) + \int_{-\ell/2}^{\ell/2}\int_0^{2\pi} p(\Theta,z)\sin\Theta r d\Theta dz - W_S \sin\phi = \bar{M}\left[e\,\frac{d^2\phi}{dt^2} + 2\,\frac{de}{dt}\,\frac{d\phi}{dt}\right] \tag{4.2}$$

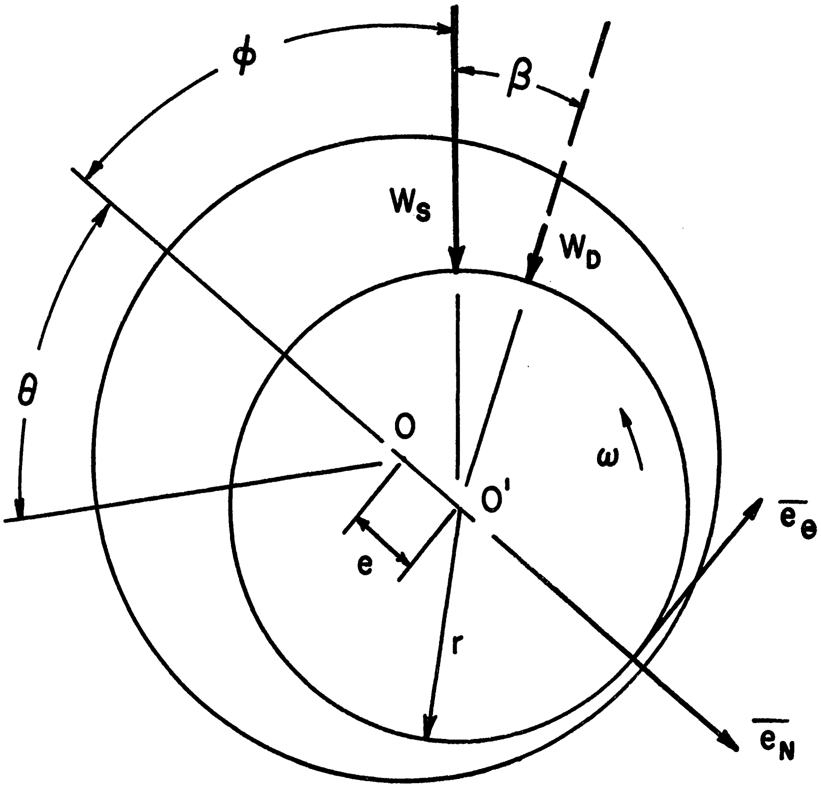

Figure 4.1. Geometry for Journal Equations of Motion.

where $\overline{M}$ represents the mass of the journal plus the mass of the propeller. The pressure function $p(\Theta,z)$ is defined by Equation (3.3) which for convenience is repeated below:

$$p(\Theta,z) = p_o(\frac{1}{2} - \frac{z}{\ell}) + \frac{6\mu r^2}{c^2}(\omega - 2\,\frac{d\phi}{dt})[\frac{n(2+n\cos\Theta)\sin\Theta}{(2+n^2)(1+n\cos\Theta)^2}]$$

$$+ \frac{6\mu r^2}{c^2 n}\,\frac{dn}{dt}\,[\frac{1}{(1+n\cos\Theta)^2}]$$

$$- \sum_{m=1}^{\infty} A_m(z)\sin m\Theta - \sum_{m=1}^{\infty} B_m(z)\cos m\Theta\,. \qquad (4.3)$$

B. Evaluation of the Pressure Integrals

Considering the evaluation of the integrals of the pressure function in Equation (4.1), then for the first term of Equation (4.3)

$$\int_{-\ell/2}^{\ell/2}\int_0^{2\pi} p_o(\frac{1}{2} - \frac{z}{\ell})\cos\Theta r d\Theta dz = p_o r \int_{-\ell/2}^{\ell/2} [(\frac{1}{2} - \frac{z}{\ell})\sin\Theta]_0^{2\pi} dz = 0. \quad (4.4)$$

The integral of the second term of Equation (4.3) may be written as

$$\int_{-\ell/2}^{\ell/2}\int_0^{2\pi} \frac{6\mu r^2}{c^2}(\omega-2\,\frac{d\phi}{dt})[\frac{n(2+n\cos\Theta)\sin\Theta}{(2+n^2)(1+n\cos\Theta)^2}]\cos\Theta r d\Theta dz$$

$$= F\int_{-\ell/2}^{\ell/2}\{\int_0^{\pi}\frac{(2+n\cos\Theta)\sin\Theta\cos\Theta d\Theta}{(1+n\cos\Theta)^2} + \int_{\pi}^{2\pi}\frac{(2+n\cos\Theta)\sin\Theta\cos\Theta d\Theta}{(1+n\cos\Theta)^2}\}\,dz \qquad (4.5)$$

where

$$F = \frac{6\mu r^3}{c^2}(\omega-2\,\frac{d\phi}{dt})\,\frac{n}{(2+n^2)}\,.$$

Letting $\Theta = \beta + 2\pi$ in the second term of the right hand side of Equation (4.5), then the right hand side of Equation (4.5) becomes

$$F\int_{-\ell/2}^{\ell/2}\{\int_0^{\pi}\frac{(2+n\cos\Theta)\sin\Theta\cos\Theta d\Theta}{(1+n\cos\Theta)^2} + \int_{-\pi}^{0}\frac{(2+n\cos\beta)\sin\beta\cos\beta d\beta}{(1+n\cos\beta)^2}\}\,dz\,. \qquad (4.6)$$

Replacing $\beta = -\gamma$, inverting the limits of the second term in (4.6) and substituting this result into Equation (4.5) it follows that

$$\int_{-\ell/2}^{\ell/2}\int_0^{2\pi} \frac{6\mu r^2}{c^2}(\omega - 2\frac{d\phi}{dt})[\frac{n(2+n\cos\Theta)\sin\Theta}{(2+n^2)(1+n\cos\Theta)^2}]\cos\Theta r d\Theta dz$$

$$= F\int_{-\ell/2}^{\ell/2}\{\int_0^{\pi}\frac{(2+n\cos\Theta)\sin\Theta\cos\Theta d\Theta}{(1+n\cos\Theta)^2} - \int_0^{\pi}\frac{(2+n\cos\gamma)\sin\gamma\cos\gamma d\gamma}{(1+n\cos\gamma)^2}\} dz \qquad (4.7)$$

$$= 0 .$$

Considering the third term of Equation (4.3) then

$$\int_{-\ell/2}^{\ell/2}\int_0^{2\pi}\frac{6\mu r^2}{nc^2}\frac{dn}{dt}[\frac{1}{(1+n\cos\Theta)^2}]\cos\Theta r d\Theta dz$$

$$= \frac{6\mu r^3}{nc^2}\frac{dn}{dt}\int_{-\ell/2}^{\ell/2}[\frac{1}{(1-n^2)}\frac{\sin\Theta}{(1+n\cos\Theta)} - \frac{n}{(1-n^2)}\int\frac{d\Theta}{(1+n\cos\Theta)}]_0^{2\pi} dz$$

$$= \frac{6\mu r^3}{nc^2}\frac{dn}{dt}\int_{-\ell/2}^{\ell/2}\{\frac{-2n}{(1-n^2)}[\frac{2}{(1-n^2)^{1/2}}\tan^{-1}(\sqrt{\frac{1-n}{1+n}}\tan\frac{\Theta}{2})]_0^{\pi}\} dz .$$

Evaluating the limits with respect to Θ and integrating with respect to z

$$\int_{-\ell/2}^{\ell/2}\int_0^{2\pi}\frac{6\mu r^2}{nc^2}\frac{dn}{dt}[\frac{1}{(1+n\cos\Theta)^2}]\cos\Theta r d\Theta dz$$

$$= -\frac{12\mu\pi r^3\ell}{c^2}\frac{1}{(1-n^2)^{3/2}}\frac{dn}{dt} . \qquad (4.8)$$

In view of the orthogonality relationship

$$\int_0^{2\pi}\cos rx \sin sx\, dx = 0 ,$$

where r and s are integers then the integral of the fourth term of Equation (4.3) becomes

$$\int_{-\ell/2}^{\ell/2}\int_{0}^{2\pi}\left[-\sum_{m=1}^{\infty} A_m(z)\sin m\Theta\right]\cos\Theta r d\Theta dz = 0 . \qquad (4.9)$$

From the additional orthogonality relationship that

$$\int_0^{2\pi} \cos rx \cos sx\, dx = \begin{matrix} 0, & r \neq s \\ \pi, & r = s \neq 0 \end{matrix}$$

then the integral of the last term of Equation (4.3) becomes

$$\int_{-\ell/2}^{\ell/2}\int_{0}^{2\pi}\left[-\sum_{m=1}^{\infty} B_m(z)\cos m\Theta\right]\cos\Theta r d\Theta dz = -r\pi\int_{-\ell/2}^{\ell/2} B_1(z)dz . \qquad (4.10)$$

From Equation (3.16)

$$B_1(z) = N_1 \cosh\frac{\zeta z}{r}$$

and thus Equation (4.10) may be written

$$\int_{-\ell/2}^{\ell/2}\int_{0}^{2\pi}\left[-\sum_{m=1}^{\infty} B_m(z)\cos m\Theta\right]\cos\Theta r d\Theta dz = -\frac{2r^2\pi}{\zeta} N_1 \sinh\frac{\zeta\ell}{2r} . \qquad (4.11)$$

However from Equation (3.41)

$$N_1 = \frac{1}{\cosh\frac{\zeta\ell}{2r}}\left\{\frac{12\mu r^2}{n^3c^2}\frac{dn}{dt}\left[\frac{n}{1+(1-n^2)^{1/2}}\right]\left[1-\frac{1}{(1-n^2)}-\frac{n^2}{(1-n^2)^{3/2}}\right]\right\}$$

and Equation (4.11) may be written as

$$\int_{-\ell/2}^{\ell/2}\int_{0}^{2\pi}\left[-\sum_{m=1}^{\infty} B_m(z)\cos m\Theta\right]\cos\Theta r d\Theta dz$$

$$= -\frac{24\mu r^4\pi}{n^2c^2\zeta}\frac{dn}{dt}\left[\frac{1}{1+(1-n^2)^{1/2}}\right]\left[1-\frac{1}{(1-n^2)}-\frac{n^2}{(1-n^2)^{3/2}}\right]\tanh\frac{\zeta\ell}{2r} . \qquad (4.12)$$

Considering next the evaluation of the integrals of the pressure function in Equation (4.2), then for the first term of Equation (4.3)

$$\int_{-\ell/2}^{\ell/2}\int_{0}^{2\pi} p_o\left(\frac{1}{2}-\frac{z}{\ell}\right)\sin\Theta r d\Theta dz = -p_o r \int_{-\ell/2}^{\ell/2}\left[\left(\frac{1}{2}-\frac{z}{\ell}\right)\cos\Theta\right]_0^{2\pi} dz = 0 \ . \qquad (4.13)$$

The integral of the second term of Equation (4.3) may be written as

$$\int_{-\ell/2}^{\ell/2}\int_{0}^{2\pi} \frac{6\mu r^2}{c^2}\left(\omega-2\frac{d\phi}{dt}\right)\left[\frac{n(2+n\cos\Theta)\sin\Theta}{(2+n^2)(1+n\cos\Theta)^2}\right]\sin\Theta r d\Theta dz$$

$$= \frac{6\mu r^3}{c^2}\left(\omega-2\frac{d\phi}{dt}\right)\left(\frac{n}{(2+n^2)}\right)\int_{-\ell/2}^{\ell/2}\int_{0}^{2\pi}\left[\frac{(2+n\cos\Theta)\sin^2\Theta}{(1+n\cos\Theta)^2}\right] d\Theta dz \ . \qquad (4.14)$$

Letting

$$\overline{F} = \int_0^{2\pi}\frac{(2+n\cos\Theta)\sin^2\Theta}{(1+n\cos\Theta)^2}d\Theta = \int_0^{2\pi}\frac{[1+(1+n\cos\Theta)]\sin^2\Theta}{(1+n\cos\Theta)^2}d\Theta$$

$$= \int_0^{2\pi}\frac{\sin^2\Theta}{(1+n\cos\Theta)}d\Theta + \int_0^{2\pi}\frac{\sin^2\Theta}{(1+n\cos\Theta)^2}d\Theta$$

and making the substitution that

$$u = \cos\Theta \quad \text{and} \quad du = -\sin\Theta d\Theta \ ,$$

then

$$\overline{F} = -\int_0^{2\pi}\frac{\sqrt{1-u^2}\,du}{1+nu} - \int_0^{2\pi}\frac{\sqrt{1-u^2}\,du}{(1+nu)^2} \ .$$

Making the further substitution that

$$z = 1 + nu \quad \text{and} \quad dz = ndu \ ,$$

then

$$\overline{F} = -\frac{1}{n^2}\int_0^{2\pi}\frac{\sqrt{(n^2-1)+2z-z^2}}{z}dz - \frac{1}{n^2}\int_0^{2\pi}\frac{\sqrt{(n^2-1)+2z-z^2}}{z^2}dz \ .$$

The solution of these integrals may be written as

$$\overline{F} = -\frac{1}{n^2}\Big[\sqrt{(n^2-1)+2z-z^2} + \int_0^{2\pi}\frac{dz}{\sqrt{(n^2-1)+2z-z^2}}$$

$$+ (n^2-1)\int_0^{2\pi}\frac{dz}{z\sqrt{(n^2-1)+2z-z^2}} - \frac{1}{z}\sqrt{(n^2-1)+2z-z^2}$$

$$+ \int_0^{2\pi}\frac{dz}{z\sqrt{(n^2-1)+2z-z^2}} - \int_0^{2\pi}\frac{dz}{\sqrt{(n^2-1)+2z-z^2}}\Big]$$

or

$$\overline{F} = -\frac{1}{n^2}\Big[(1-\frac{1}{z})\sqrt{(n^2-1)+2z-z^2} + n^2\int_0^{2\pi}\frac{dz}{z\sqrt{(n^2-1)+2z-z^2}}\Big]$$

$$= -\frac{1}{n^2}\Big[(1-\frac{1}{z})\sqrt{(n^2-1)+2z-z^2} + \frac{n^2}{\sqrt{1-n^2}}\sin^{-1}\Big\{\frac{2z+2(n^2-1)}{z\sqrt{4n^2}}\Big\}\Big].$$

However

$$z = (1+n\cos\theta)$$

and therefore

$$\overline{F} = -\frac{2}{n^2}\Big[\frac{n^2\sin\theta\cos\theta}{(1+n\cos\theta)} + \frac{n^2}{(1-n^2)^{1/2}}\sin^{-1}\Big(\frac{n+\cos\theta}{1+n\cos\theta}\Big)\Big]_0^{\pi}$$

$$= \frac{2\pi}{(1-n^2)^{1/2}}\ .$$

Substituting this value of $\overline{F}$ back into Equation (4.14) and integrating with respect to z then

$$\int_{-\ell/2}^{\ell/2}\int_0^{2\pi}\frac{6\mu r^2}{c^2}(\omega-2\frac{d\phi}{dt})\Big[\frac{n(2+n\cos\theta)\sin\theta}{(2+n^2)(1+n\cos\theta)}\Big]\sin\theta r d\theta dz$$

$$= \frac{12\mu r^3\pi\ell}{c^2}(\omega-2\frac{d\phi}{dt})\frac{n}{(2+n^2)(1-n^2)^{1/2}}\ . \qquad (4.15)$$

Considering the integral of the third term in Equation (4.3) then it follows directly that

$$\int_{-\ell/2}^{\ell/2}\int_{0}^{2\pi} \frac{6\mu r^2}{nc^2}\frac{dn}{dt}\left[\frac{1}{(1+n\cos\Theta)^2}\right]\sin\Theta r d\Theta dz$$

$$= \frac{6\mu r^3}{nc^2}\frac{dn}{dt}\int_{-\ell/2}^{\ell/2}\left[\frac{1}{n(1+n\cos\Theta)}\right]_0^{2\pi} dz = 0 \quad (4.16)$$

From the orthogonality relationship

$$\int_0^{2\pi} \sin rx \sin sx\, dx = \begin{cases} 0, & r \neq s \\ \pi, & r = s > 0 \end{cases}$$

then the integral of the fourth term of Equation (4.3) becomes

$$\int_{-\ell/2}^{\ell/2}\int_{0}^{2\pi}\left[-\sum_{m=1}^{\infty} A_m(z)\sin m\Theta\right]\sin\Theta r d\Theta dz = -r\pi\int_{-\ell/2}^{\ell/2} A_1(z)dz . \quad (4.17)$$

From Equation (3.12)

$$A_1 = C_1 \cosh\frac{\alpha z}{r}$$

and thus Equation (4.17) may be written

$$\int_{-\ell/2}^{\ell/2}\int_{0}^{2\pi}\left[-\sum_{m=1}^{\infty} A_m(z)\sin m\Theta\right]\sin\Theta r d\Theta dz = -\frac{2r^2\pi}{\alpha} C_1 \sinh\frac{\alpha\ell}{2r} . \quad (4.18)$$

However from Equation (3.38)

$$C_1 = \frac{1}{\cosh\frac{\alpha\ell}{2r}}\left[\frac{12\mu r^2}{c^2}\left(\omega - 2\frac{d\phi}{dt}\right)\left(\frac{1}{2+n^2}\right)\left(\frac{n}{(1-n^2)^{1/2}}\right)\right]$$

and Equation (4.18) may be written as

$$\int_{-\ell/2}^{\ell/2}\int_{0}^{2\pi}\left[-\sum_{m=1}^{\infty} A_m(z)\sin m\Theta\right]\sin\Theta r d\Theta dz$$

$$= -\frac{24r^4\mu\pi}{\alpha c^2}\left(\omega - 2\frac{d\phi}{dt}\right)\frac{n}{(2+n^2)(1-n^2)^{1/2}}\tanh\frac{\alpha\ell}{2r} . \quad (4.19)$$

In view of the orthogonality relationship that

$$\int_0^{2\pi} \cos rx \sin sx \, dx = 0 ,$$

where r and s are integers, then the integral of the last term in Equation (4.3) becomes

$$\int_{-\ell/2}^{\ell/2} \int_0^{2\pi} [- \sum_{m=1}^{\infty} B_m(z)\cos m\Theta] \sin\Theta r d\Theta dz = 0 . \qquad (4.20)$$

C. Simplification of the Equations of Motion

If it is noted that

$$\frac{1}{n^2} [\frac{1}{1+(1-n^2)^{1/2}}][1 - \frac{1}{(1-n^2)} - \frac{n^2}{(1-n^2)^{3/2}}] = - \frac{1}{(1-n^2)^{3/2}}$$

and the results of Equations (4.4), (4.7), (4.8), (4.9) and (4.12) are substituted into Equation (4.1), then the equation of motion for the normal direction becomes

$$W_D \cos(\phi+\beta) + W_S \cos\phi + \frac{12\mu r^3 \pi\ell}{c^2} [\frac{2r}{\zeta\ell} \tanh \frac{\zeta\ell}{2r} - 1]\frac{1}{(1-n^2)^{3/2}} \frac{dn}{dt} = \bar{M}[\frac{d^2e}{dt^2} - e(\frac{d\phi}{dt})^2] . \qquad (4.21)$$

Similarly, substituting the results of Equations (4.13), (4.15), (4.16), (4.19) and (4.20) into Equation (4.2), then the equation of motion for the tangential direction becomes

$$-W_D \sin(\phi+\beta) - W_S \sin\phi - \frac{12\mu r^3 \pi\ell}{c^2} [\frac{2r}{\alpha\ell} \tanh \frac{\alpha\ell}{2r} - 1](\omega-2 \frac{d\phi}{dt}) \frac{n}{(2+n^2)(1-n^2)^{1/2}}$$

$$= \bar{M}[e \frac{d^2\phi}{dt^2} + 2 \frac{de}{dt} \frac{d\phi}{dt}] . \qquad (4.22)$$

Considering Equations (4.21) and (4.22) the inertial terms can be shown to be negligible. Since the bearing is going to oscillate about a steady state position with a maximum radial travel equal to the minimum film thickness, which is in the order of 0.001 inches, and the circular frequency of the dynamic load is four times the angular velocity of the journal for a four-bladed propeller, which in this case is 11.0 radians/second, then the maximum acceleration expected will be

$$a = r\omega^2 = \frac{0.001}{12}[4 \times 11]^2 = 0.16 .$$

This as a percentage of gravity is

$$\frac{0.16}{32.2} \times 100 = 0.5\% .$$

The inertial forces are therefore in the order of 1/200 of the gravity forces and shall be neglected.

The dynamic load will be assumed to be of the form

$$W_D = W_{D_1} \sin b\omega t + W_{D_2} \sin 2b(\omega t - \Lambda) \tag{4.23}$$

where W_{D_1} and W_{D_2} represent the amplitudes of the first and second harmonic components of the propeller loading, b is the number of blades and Λ a phase angle. It should be noted that these are termed the first and second harmonic components in terms of the blade number and are actually not the first and second harmonics in the usual sense.

Utilizing the fact that the inertial forces are negligible, substituting Equation (4.23) for W_D and introducing the dimensionless constants

$$H_1 = \frac{W_{D1}}{W_S}, \quad H_2 = \frac{W_{D2}}{W_S} \tag{4.24}$$

and the constant

$$K = \frac{12\mu r^3 \pi \ell}{c^2 W_S} \tag{4.25}$$

Equations (4.21) and (4.22) may be written respectively as

$$[\{H_1 \sin b\omega t + H_2 \sin 2b(\omega t - \Lambda)\} \cos\beta + 1] \cos\phi$$

$$-[\{H_1 \sin b\omega t + H_2 \sin 2b(\omega t - \Lambda)\} \sin\beta] \sin\phi$$

$$+ K[\frac{2r}{\zeta\ell} \tanh \frac{\zeta\ell}{2r} - 1] \frac{1}{(1-n^2)^{3/2}} \frac{dn}{dt} = 0 \tag{4.26}$$

and

$$-[\{H_1 \sin b\omega t + H_2 \sin 2b(\omega t - \Lambda)\} \cos\beta + 1] \sin\phi$$

$$-[\{H_1 \sin b\omega t + H_2 \sin 2b(\omega t - \Lambda)\} \sin\beta] \cos\phi$$

$$- K[\frac{2r}{\alpha\ell} \tanh \frac{\alpha\ell}{2r} - 1](\omega - 2 \frac{d\phi}{dt}) \frac{n}{(2+n^2)(1-n^2)^{1/2}} = 0 . \tag{4.27}$$

It is convenient at this point to make the change of variable

$$\bar{y} = \frac{n}{(1-n^2)^{1/2}} \tag{4.28}$$

In terms of the variable $\bar{y}$ it is noted that

$$\frac{1}{(1-n^2)^{3/2}} \frac{dn}{dt} = \frac{d\bar{y}}{dt}$$

and

$$\frac{n}{(2+n^2)(1-n^2)^{1/2}} = \frac{\bar{y}(1+\bar{y}^2)}{(2+3\bar{y}^2)} .$$

Substituting these relations into Equations (4.26) and (4.27) and solving for the time derivatives of the dependent variables $\bar{y}$ and ϕ, then Equations (4.26) and (4.27) become respectively

$$\frac{d\bar{y}}{dt} = \frac{[\{H_1 \sin b\omega t + H_2 \sin 2b(\omega t-\Lambda)\} \cos\beta + 1] \cos\phi}{K[1 - \frac{2r}{\zeta\ell} \tanh \frac{\zeta\ell}{2r}]} \tag{4.29}$$

$$- \frac{[\{H_1 \sin b\omega t + H_2 \sin 2b(\omega t-\Lambda)\} \sin\beta] \sin\phi}{K[1 - \frac{2r}{\zeta\ell} \tanh \frac{\zeta\ell}{2r}]}$$

and

$$\frac{d\phi}{dt} = \frac{\omega}{2} - \frac{[\{H_1 \sin b\omega t + H_2 \sin 2b(\omega t-\Lambda)\} \cos\beta + 1] \sin\phi}{2K[1 - \frac{2r}{\alpha\ell} \tanh\frac{\alpha\ell}{2r}] \frac{\bar{y}(1+\bar{y}^2)}{(2+3\bar{y}^2)}}$$

$$- \frac{[\{H_1 \sin b\omega t + H_2 \sin 2b(\omega t-\Lambda)\} \sin\beta] \cos\phi}{2K[1 - \frac{2r}{\alpha\ell} \tanh\frac{\alpha\ell}{2r}] \frac{\bar{y}(1+\bar{y}^2)}{(2+3\bar{y}^2)}} \quad . \tag{4.30}$$

In terms of the new variable $\bar{y}$ Equation (3.10) for α^2 becomes

$$\alpha^2 = \frac{2[\gamma\bar{y} + (1+\bar{y}^2)^{1/2}]}{2(1+\bar{y}^2)^{1/2} - \gamma\bar{y}} \tag{4.31}$$

where from Equation (3.40)

$$\gamma = \frac{\bar{y}[1 + 2(1+\bar{y}^2)^{1/2}]}{[1 + (1+\bar{y}^2)^{1/2}]^2} \quad . \tag{4.32}$$

Similarly Equation (3.14) for ζ^2 becomes

$$\zeta^2 = \frac{2[\psi\bar{y} + (1+\bar{y}^2)^{1/2}]}{2(1+\bar{y}^2)^{1/2} - \psi\bar{y}} \tag{4.33}$$

where from Equation (3.43)

$$\psi = \frac{\bar{y}[2 + (1+\bar{y}^2)^{1/2}]}{[1 + (1+\bar{y}^2)^{1/2}]^2} \tag{4.34}$$

Substituting Equation (4.32) into Equation (4.31)

$$\alpha = \left[\frac{2[\bar{y}^2\{1+2(1+\bar{y}^2)^{1/2}\} + (1+\bar{y}^2)^{1/2}\{1+(1+\bar{y}^2)^{1/2}\}^2]}{2(1+\bar{y}^2)^{1/2}\{1+(1+\bar{y}^2)^{1/2}\} - \bar{y}^2\{1+2(1+\bar{y}^2)^{1/2}\}} \right]^{1/2} \tag{4.35}$$

and substituting Equation (4.34) into Equation (4.33)

$$\zeta = \left[\frac{2[\bar{y}^2\{2+(1+\bar{y}^2)^{1/2}\} + (1+\bar{y}^2)^{1/2}\{1+(1+\bar{y}^2)^{1/2}\}^2]}{2(1+\bar{y}^2)^{1/2}\{1+(1+\bar{y}^2)^{1/2}\}^2 - \bar{y}^2\{2+(1+\bar{y}^2)^{1/2}\}} \right]^{1/2} . \tag{4.36}$$

Values of $\bar{y}$, γ, α, ψ and ζ verses n are given in Table 4.1.

Considering Equations (4.35) and (4.36) for α and ζ, it is seen that these expressions are quite complex for computational work. These may be approximated by polynomials of the form

$$\alpha \cong 0.002204\bar{y}^3 - 0.04980\bar{y}^2 + 0.6518\bar{y} + 0.7843 \tag{4.37}$$

and

$$\zeta \cong 0.008050\bar{y}^3 - 0.1215\bar{y}^2 + 0.6154\bar{y} + 0.8332 \, , \tag{4.38}$$

where for

$$0.6 < n < 1.0 \quad \text{or} \quad 0.75 < \bar{y} < \infty$$

the maximum error involved in either expression is 0.68 per cent.

The solution of the two simultaneous, non-linear, first order differential equations given by Equations (4.29) and (4.30) will give the path of the journal in its orbit about a steady state position and the velocity components corresponding to the translational and rotational motion of the journal center in this orbit.

Thus for a given point (n, ϕ) in this orbit, and the corresponding velocity components $(\frac{dn}{dt}, \frac{d\phi}{dt})$ the pressure may be evaluated around the circumference and along the length of the bearing from Equation (4.3).

TABLE 4.1

$\overline{y}$, γ, α, ψ, ζ vs n

n	$\overline{y}$	γ	α	ψ	ζ
0.00	0.0000	0.0000	1.0000	0.0000	1.0000
0.10	0.1005	0.0753	1.0056	0.0751	1.0056
0.20	0.2041	0.1520	1.0229	0.1510	1.0227
0.30	0.3145	0.2321	1.0527	0.2285	1.0519
0.40	0.4364	0.3176	1.0970	0.3085	1.0942
0.50	0.5574	0.4115	1.1593	0.3923	1.1516
0.60	0.7500	0.5185	1.2460	0.4815	1.2274
0.70	0.9802	0.6466	1.3702	0.5785	1.3273
0.80	1.3333	0.8125	1.5634	0.6875	1.4622
0.85	1.6136	0.9214	1.7120	0.7484	1.5492
0.90	2.0647	1.0633	1.9371	0.8171	1.6566
0.95	3.0424	1.2756	2.3691	0.8962	1.7955
0.96	3.4286	1.3359	2.5223	0.9141	1.8290
0.97	3.9900	1.4080	2.7314	0.9329	1.8652
0.98	4.9247	1.4990	3.0497	0.9530	1.9048
0.99	7.0179	1.6280	3.6676	0.9749	1.9488
1.00	∞	2.0000	∞	1.0000	2.0000

V. RESULTS AND DISCUSSION

A. Physical Data of the Case Considered for Numerical Evaluation

Numerical results for the simultaneous location of the journal center and the corresponding pressure distribution in the bearing will be presented under separate headings as they are actually two distinct steps in the numerical evaluation.

The data considered for numerical results of this problem is from the stern tube bearing of the S. S. John G. Munson of the Bradley Transportational Line. The pertinent characteristics of the bearing are:

$\omega = 11.0$ radians/second,

$r = 9.25$ inches,

$\ell = 74.0$ inches,

$c = 0.0435$ inches,

$b = 4$,

$\mu = 1.4 \times 10^{-7}$ pound-second/inch^2,

$p_o = 30.0$ (pound/inch^2) gage .

The actual static load on the bearing is unknown due to the effect of journal deflection. If the effect of this deflection is ignored the static load can be shown to be approximately 7×10^4 pounds. This static load however would yield a steady state eccentricity of 0.999966 and a minimum film thickness of 1.47×10^{-5} inches. This is considerably below the limit for which a lubricating film would be developed. It is felt that the journal actually rests on the after end of the bearing due to its static deflection and allows a film to be developed in the remainder of the bearing. For this reason the steady state eccentricity of the journal will be assumed and the value of W_S is then fixed.

With respect to the location of the dynamic load and the amplitude ratios of its first and second harmonic components little is known. From the sparse information available it appears that the ranges of β and H_1 are respectively $270 < \beta < 360$ and $0 < H_1 < 1.0$. No information is available on H_2 although it is certainly considerably less than H_1.

B. Journal Orbits

The solutions of the two simultaneous differential equations, Equations (4.29) and (4.30), describing the journal orbits were obtained numerically by the Runge-Kutta fourth order method on an I.B.M. 7090 digital computer. The step size used in the numerical integration was $t = 0.005$ seconds. As a check on the accuracy of the integration a step size of $t = 0.001$ seconds indicated a discrepancy of only one part in 5×10^6 for a total of 0.5 seconds.

The integration was treated as an initial value problem utilizing the static loading position as initial values for n and ϕ. The integration was then continued until the journal orbit repeated itself within an error of 0.5 percent. In all cases considered here this occurred by at least the second time around the orbit. It should be noted that due to the assumption of a complete oil film, the steady state position for ϕ from Equations (4.29) and (4.30) will always be $\phi = 90°$.

Figure 5.1, 5.2 and 5.3 show the orbits obtained considering only the first harmonic component of the dynamic load for three values of β (270°, 300°, 330°) and using a steady state position of $n = 0.85$ and $\phi = 90°$. These figures illustrate that it takes 9.0 cycles in terms of the dynamic loading frequency (44.0 radians/second) or 2.25 cycles in terms of the journal frequency (11.0 radians/second) to complete

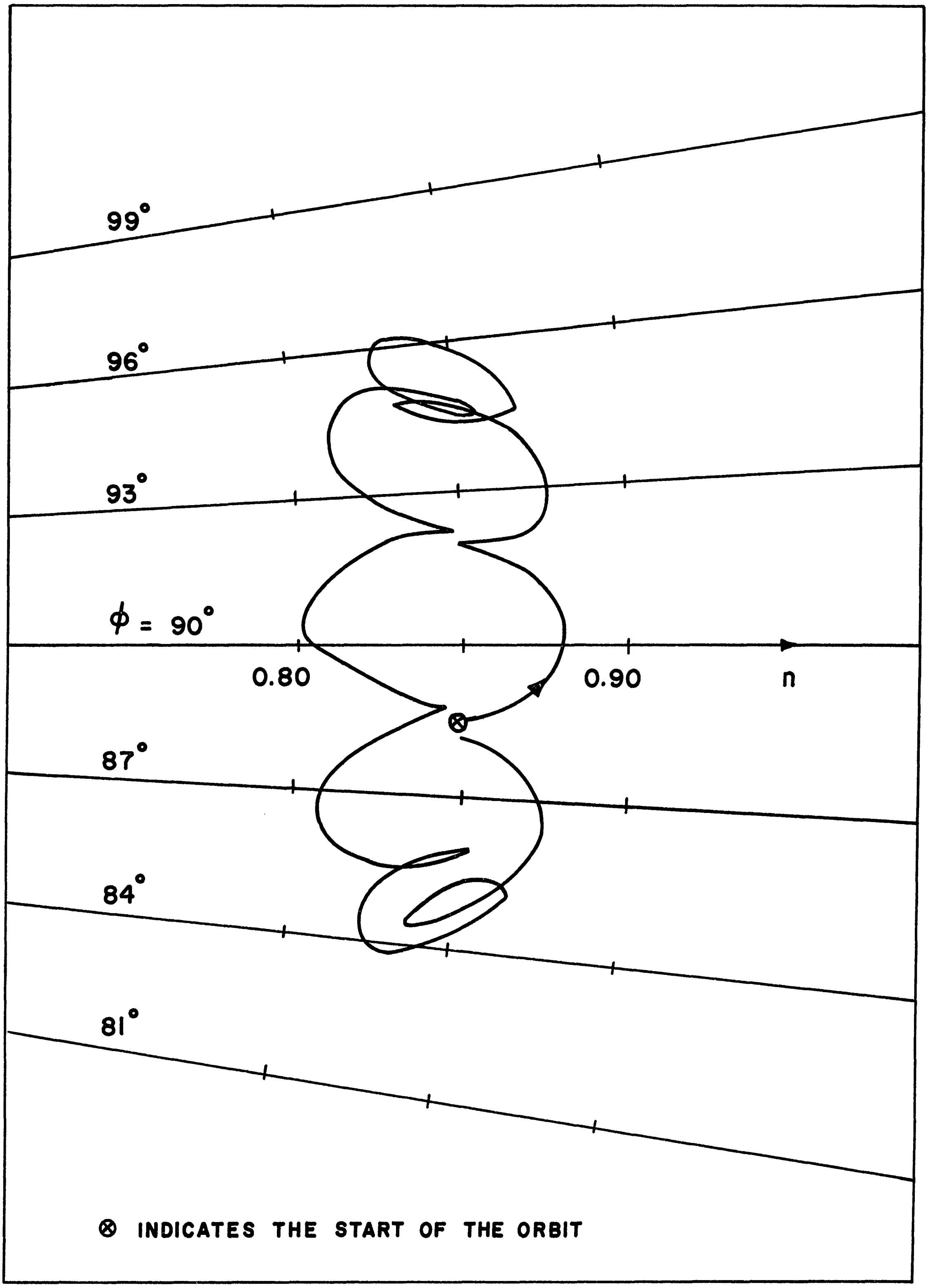

Figure 5.1. Locus of the Journal Center for $H_1 = 0.8$, $H_2 = 0.0$, $b = 4$, $\beta = 270°$ and $K = 0.1796$. Steady State Coordinates are $\phi = 90°$ and $n = 0.85$.

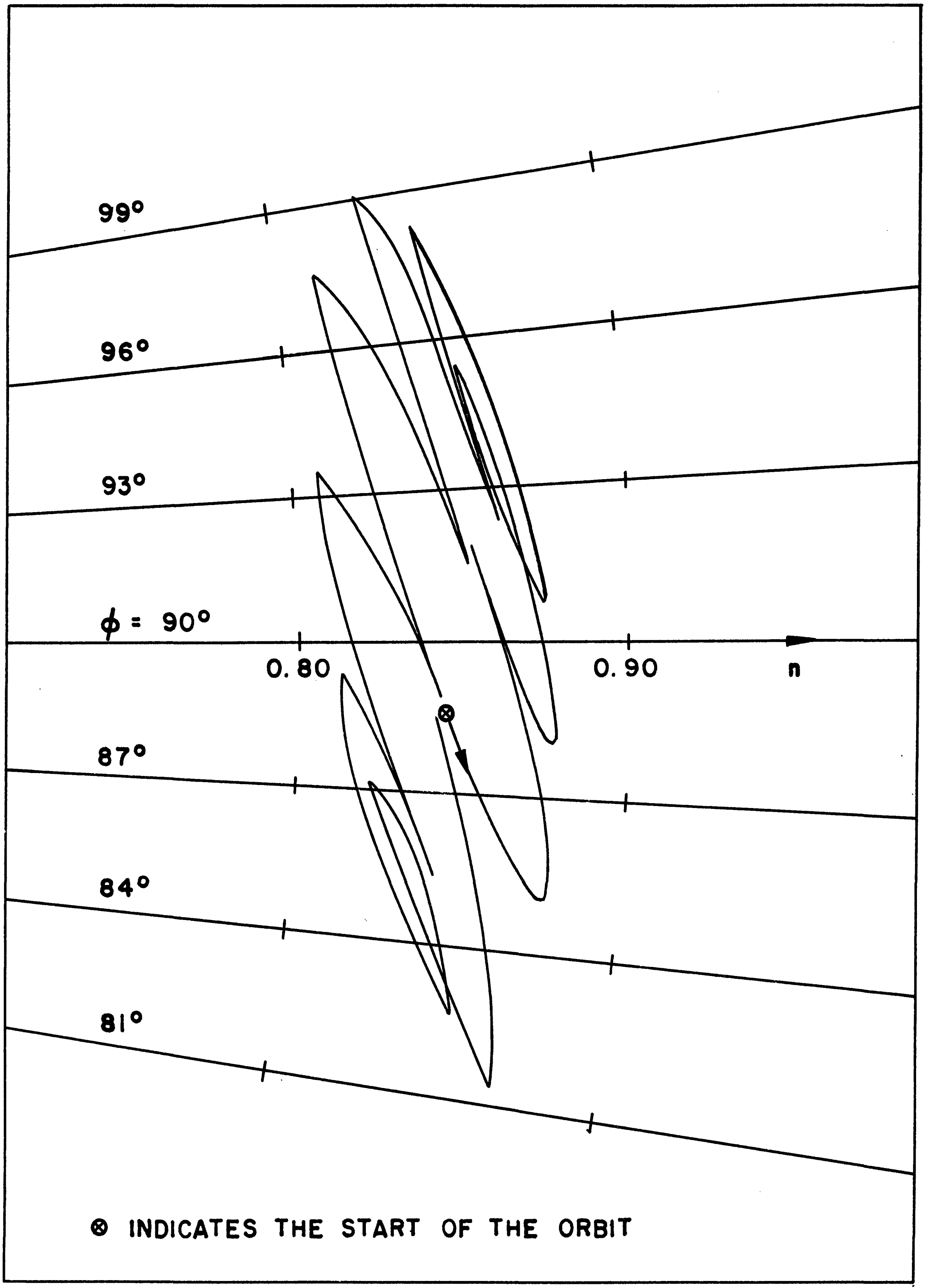

Figure 5.2. Locus of the Journal Center for $H_1 = 0.8$, $H_2 = 0.0$, $b = 4$, $\beta = 300°$ and $K = 0.1796$. Steady State Coordinates are $\phi = 90°$ and $n = 0.85$.

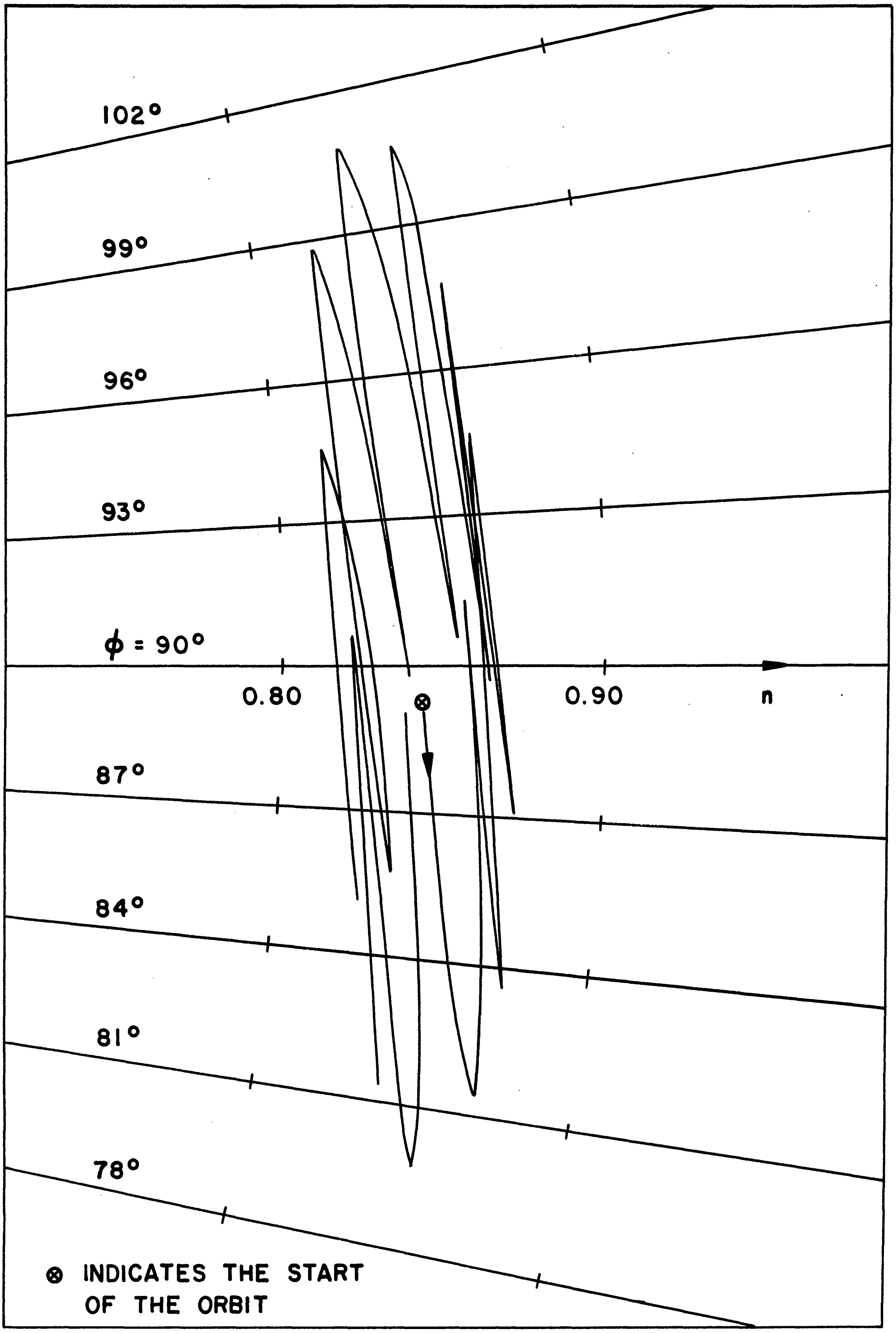

Figure 5.3. Locus of the Journal Center for $H_1 = 0.8$, $H_2 = 0.0$, $b = 4$, $\beta = 330°$ and $K = 0.1796$. Steady State Coordinates are $\phi = 90°$ and $n = 0.85$.

one orbit of the journal center. The orbits are almost symmetrical about a line connecting the mid-point of the first cycle and the end of the fifth cycle. This shall be designated as ϕ_{mean} in the discussion.

It is seen that ϕ_{mean} is very sensitive to a change in the value of β locating the dynamic load, decreasing at a much faster rate than β is increasing. This amounts to a rapid conversion from one velocity $(\frac{dn}{dt})$ to the other velocity $(\frac{d\phi}{dt})$ and as shall be seen below considerably effects the magnitudes of the pressures developed in the bearing.

The principal characteristics of the above orbits along with other orbits investigated for $H_2 = 0.0$ and a steady state position of $n = 0.85$ and $\phi = 90°$ are summarized in Table 5.1. These results show that the size of the orbits vary directly as the amplitude ratio H_1 but than ϕ_{mean} is almost independent of H_1 .

It should be noted that the conditions of $\phi = 90°$ and $\beta = 0°$ are a singular point of Equations (4.29) and (4.30) and represent an unstable condition. Figure 5.4 shows the orbit obtained for these conditions, which although appearing similar in form to the preceding orbits continues to grow in size.

Considering again the case of $H_2 = 0.0$, the effect of increasing the eccentricity of the initial conditions is shown in Figure 5.5. Although the orbits are elongated in the ϕ direction with a corresponding decrease in the n direction the general shape of the orbits is maintained. The former point is to be expected as resistance to motion in the n direction increases with increasing eccentricity.

TABLE 5.1

CHARACTERISTICS OF THE ORBITS OF THE JOURNAL CENTER FOR THE CASE OF H_2 = 0.0, b = 4, K = 0.1796 AND STEADY STATE COORDINATES OF n = 0.85 AND ϕ = 90°

H_1	β	n_{max}	n_{min}	ϕ_{max}	ϕ_{min}	ϕ_{mean}
0.8	0°	-	-	-	-	0.00°
0.8	270°	0.8804	0.8025	96.20°	83.88°	84.51°
0.8	300°	0.8790	0.8054	99.16°	82.29°	21.06°
0.8	330°	0.8750	0.8138	100.76°	79.85°	7.88°
0.4	270°	0.8664	0.8272	93.15°	86.90°	84.73°
0.4	300°	0.8655	0.8284	94.48°	85.85°	21.11°
0.4	330°	0.8630	0.8323	95.47°	84.75°	7.90°

TABLE 5.2

CHARACTERISTICS OF THE ORBITS OF THE JOURNAL CENTER FOR THE CASE OF H_2 = 0.0, b = 4, K = 0.0969 AND STEADY STATE COORDINATES OF n = 0.95 AND ϕ = 90°

H_1	β	n_{max}	n_{min}	ϕ_{max}	ϕ_{min}	ϕ_{mean}
0.8	0°	-	-	-	-	0.00°
0.8	270°	0.9611	0.9309	96.32°	83.79°	65.49°
0.8	300°	0.9606	0.9321	99.28°	82.16°	8.13°
0.8	330°	0.9591	0.9356	101.18°	79.82°	3.35°
0.4	270°	0.9561	0.9409	93.26°	86.80°	65.86°
0.4	300°	0.9557	0.9413	94.59°	85.74°	8.24°
0.4	330°	0.9548	0.9429	95.55°	84.69°	3.51°

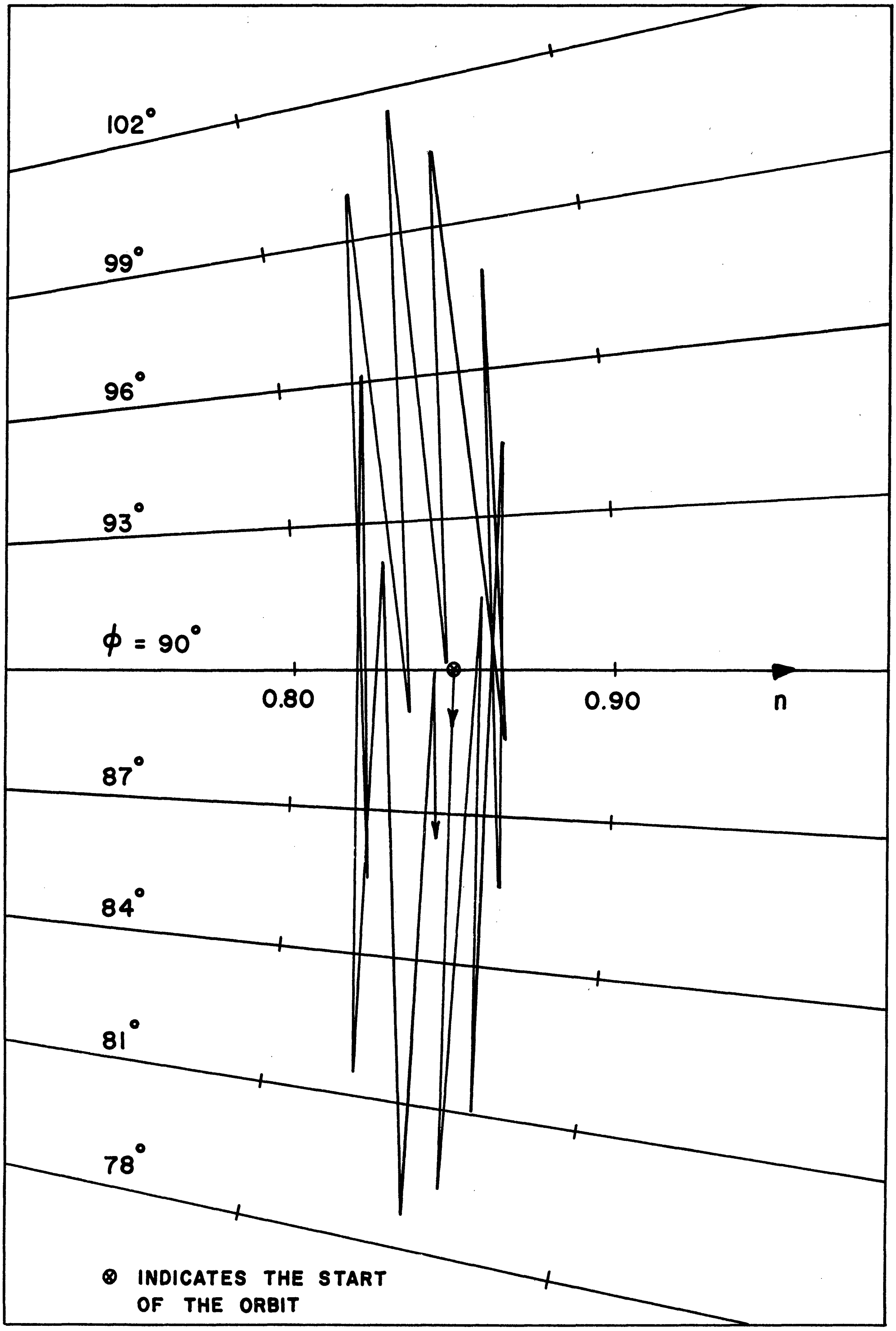

Figure 5.4. Locus of the Journal Center for $H_1 = 0.8$, $H_2 = 0.0$, $b = 4$, $\beta = 0°$ and $K = 0.1796$. Initial Coordinates are $\phi = 90°$ and $n = 0.85$.

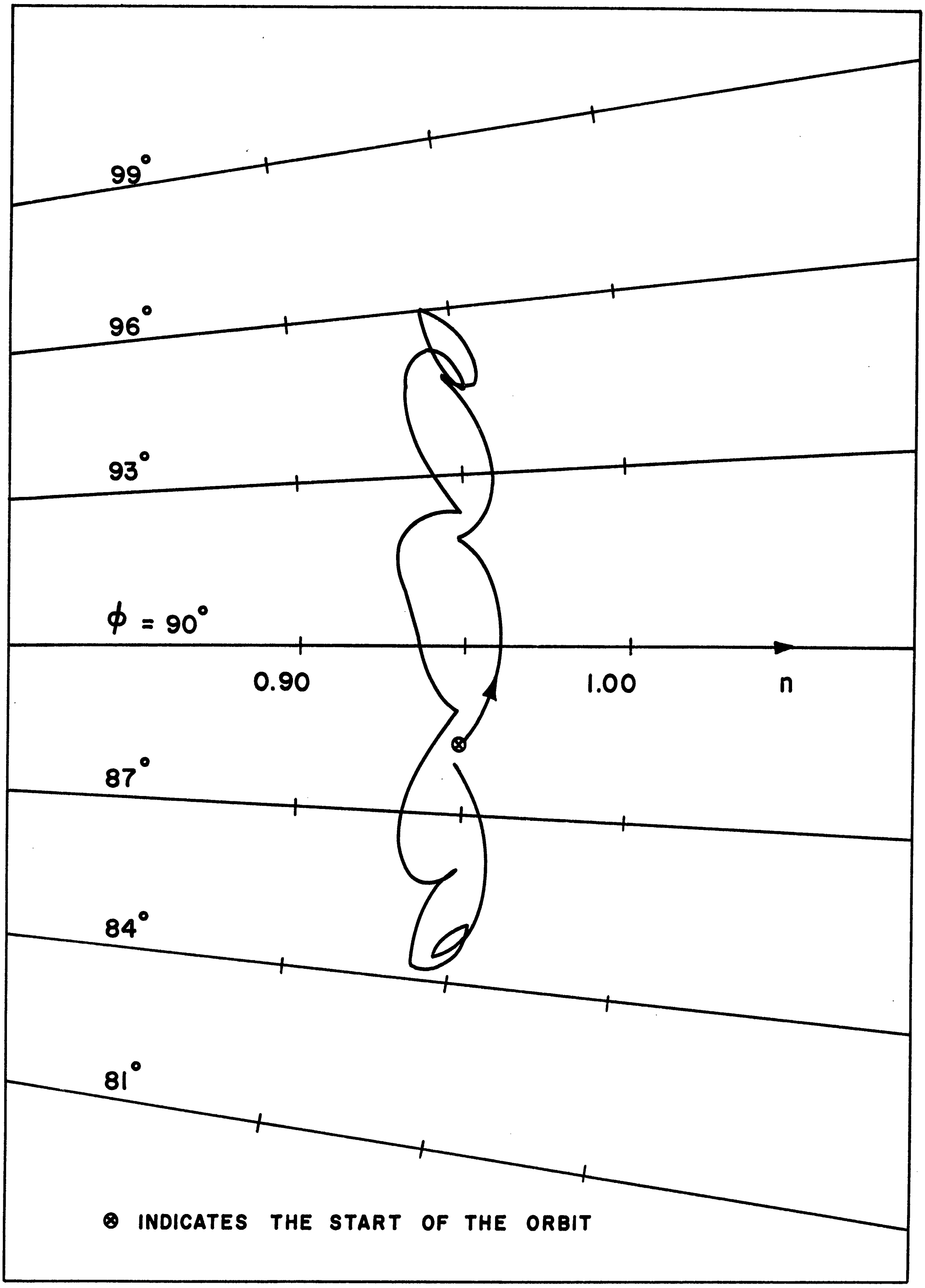

Figure 5.5. Locus of the Journal Center for $H_1 = 0.8$, $H_2 = 0.0$, $b = 4$, $\beta = 270°$ and $K = 0.0969$. Steady State Coordinates are $\phi = 90°$ and $n = 0.95$.

Table 5.2 summarizes the principal characteristics of orbits investigated for steady state conditions of $\phi = 90°$ and $n = 0.95$ with $H_2 = 0.0$. It is again noted that ϕ_{mean} is essentially independent of H_1, depending largely on β. A comparison of Tables 5.1 and 5.2 shows that ϕ_{mean} also depends on the steady state value of n but to a much smaller extent than β.

Including the second harmonic component of the dynamic load at various phase angles Λ produces no substantial changes in the general shape of the journal orbits. Principal characteristics of the orbits investigated are summarized in Tables 5.3 and 5.4. The variations of ϕ_{mean} verses β, the amplitude ratios H_1 and H_2, and for differential initial eccentricities follow the same patterns as those for the case of $H_2 = 0.0$. It is also seen that ϕ_{mean} is essentially independent of Λ.

Figure 5.6 shows a typical journal orbit obtained if the blade number is changed to $b = 3$. In this case it takes only 7.0 cycles in terms of the dynamic loading frequency (33.0 radians/second) or 2.33 rotations of the journal to complete one orbit.

If the blade number is changed to $b = 5$ it now takes 11.0 cycles in terms of the loading frequency (55.0 radians/second) or 2.2 rotations of the journal to complete one orbit. A typical orbit obtained for this case is shown in Figure 5.7.

TABLE 5.3

CHARACTERISTICS OF THE ORBITS OF THE JOURNAL CENTER CONSIDERING BOTH THE FIRST AND SECOND HARMONIC COMPONENTS OF THE DYNAMIC LOAD FOR b = 4, K = 0.1796 AND STEADY STATE COORDINATES OF n = 0.85 AND ϕ = 90°

H_1	H_2	Λ	β	n_{max}	n_{min}	ϕ_{max}	ϕ_{min}	ϕ_{mean}
0.8	0.4	0.00°	270°	0.8805	0.7885	97.74°	82.37°	84.15°
0.8	0.4	3.75°	270°	0.8815	0.7903	97.48°	82.52°	83.59°
0.8	0.4	7.50°	270°	0.8819	0.7940	96.88°	83.01°	83.23°
0.8	0.4	11.25°	270°	0.8817	0.7999	96.10°	83.70°	83.18°
0.8	0.4	11.25°	300°	0.8799	0.8049	99.26°	81.95°	28.09°
0.4	0.2	0.00°	270°	0.8665	0.8216	93.81°	86.26°	84.42°
0.4	0.2	3.75°	270°	0.8671	0.8223	93.70°	86.33°	83.86°
0.4	0.2	7.50°	270°	0.8673	0.8238	93.44°	86.54°	83.45°
0.4	0.2	11.25°	270°	0.8672	0.8262	93.10°	86.85°	83.25°

TABLE 5.4

CHARACTERISTICS OF THE ORBITS OF THE JOURNAL CENTER CONSIDERING BOTH THE FIRST AND SECOND HARMONIC COMPONENTS OF THE DYNAMIC LOAD FOR b = 4, K = 0.0969 AND STEADY STATE COORDINATES OF n = 0.95 AND ϕ = 90°

H_1	H_2	Λ	β	n_{max}	n_{min}	ϕ_{max}	ϕ_{min}	ϕ_{mean}
0.8	0.4	0.00°	270°	0.9612	0.9251	97.86°	82.29°	64.11°
0.8	0.4	3.75°	270°	0.9615	0.9258	97.60°	82.44°	62.88°
0.8	0.4	7.50°	270°	0.9617	0.9274	97.01°	82.93°	62.22°
0.8	0.4	11.25°	270°	0.9616	0.9298	96.22°	83.62°	62.33°
0.8	0.4	11.25°	300°	0.9609	0.9315	99.38°	81.87°	11.09°
0.4	0.2	0.00°	270°	0.9561	0.9387	93.93°	86.17°	64.55°
0.4	0.2	3.75°	270°	0.9563	0.9389	93.82°	86.23°	63.33°
0.4	0.2	7.50°	270°	0.9564	0.9395	93.56°	86.45°	62.69°
0.4	0.2	11.25°	270°	0.9564	0.9404	93.22°	86.75°	62.80°

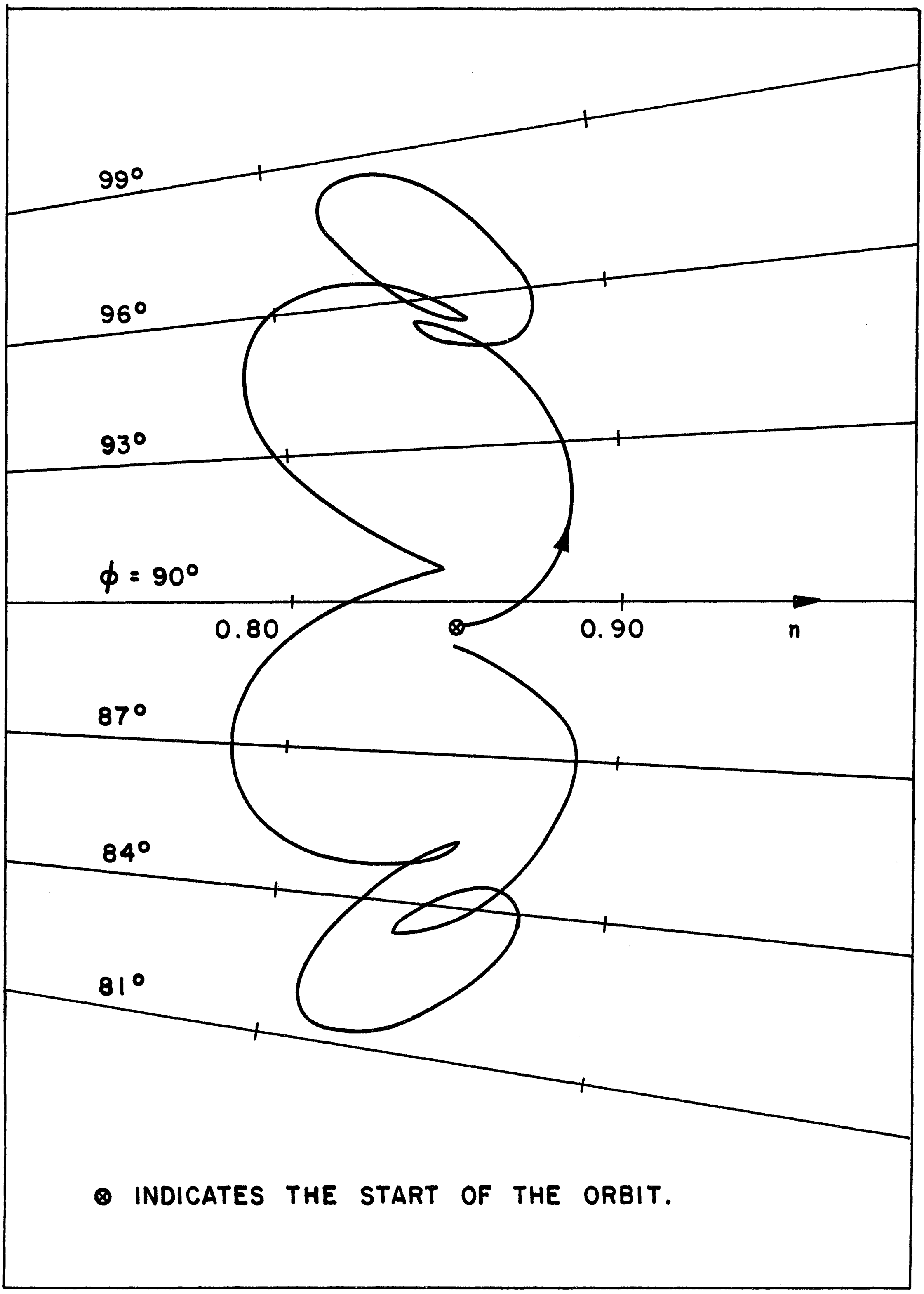

Figure 5.6. Locus of the Journal Center for $H_1 = 0.8$, $H_2 = 0.0$, $b = 3$, $\beta = 270°$ and $K = 0.1796$. Steady State Coordinates are $\phi = 90°$ and $n = 0.85$.

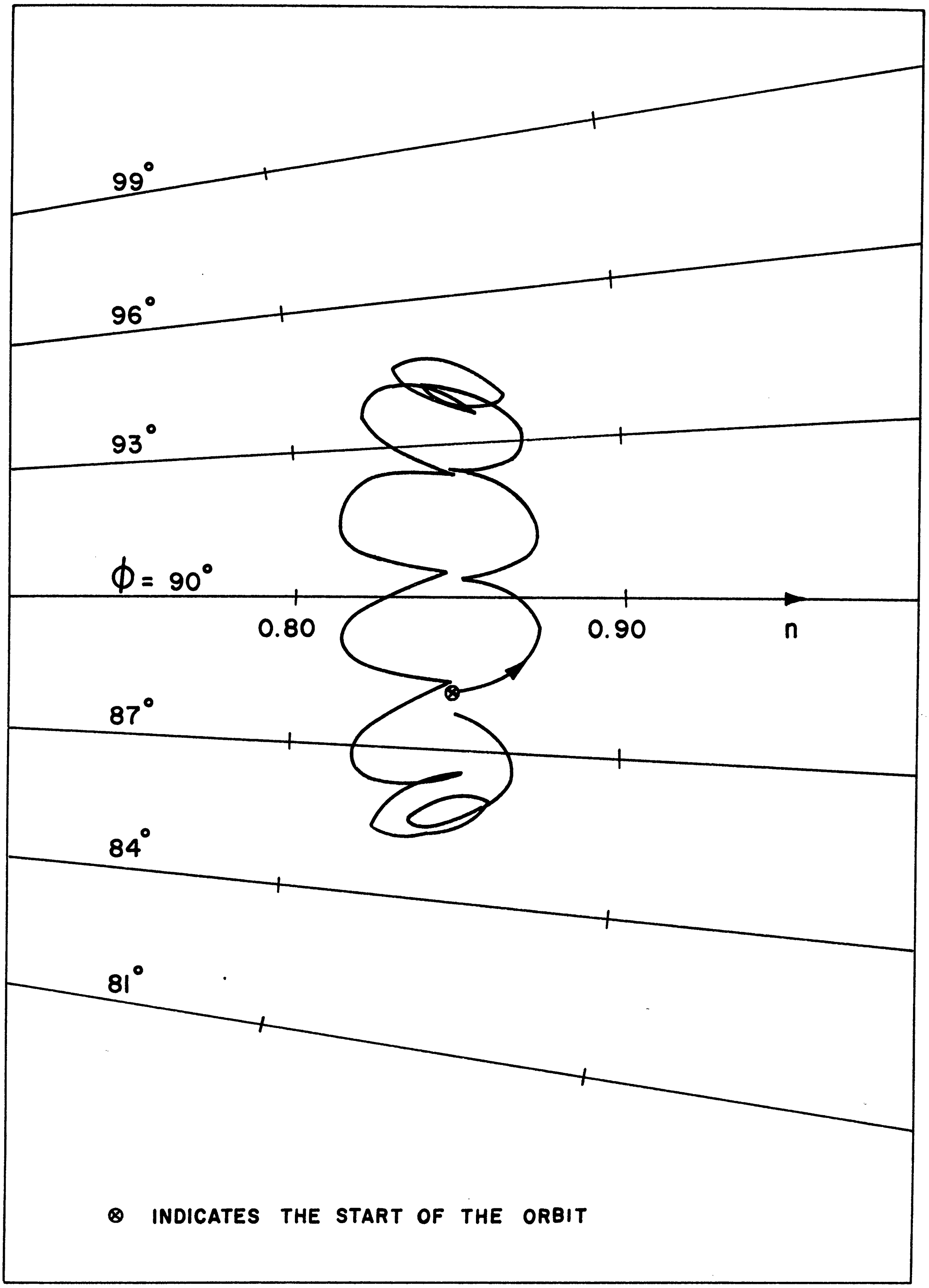

Figure 5.7. Locus of the Journal Center for $H_1 = 0.8$, $H_2 = 0.0$, $b = 5$, $\beta = 270°$ and $K = 0.1796$. Steady State Coordinates are $\phi = 90°$ and $n = 0.85$.

C. Pressure Distribution in the Bearing

Having obtained the journal orbits, then for any point (n, ϕ) in these orbits and the corresponding velocity components $(\frac{dn}{dt}, \frac{d\phi}{dt})$ the pressure profile around and along the length of the bearing may be determined from Equation (4.3). This again was accomplished on the I.B.M. 7090 digital computer including the first eight terms of each series accounting for the finite length of the bearing.

Two of the most important questions to be answered about the pressure distribution in the bearing are how it varies around and along the length of the bearing for a complete orbit at a given initial eccentricity position. These two points were investigated rather extensively of which the results are indicated in Figures 5.8 and 5.9. Figure 5.8 shows that the region of minimum pressure encountered along the length of the bearing occurs slightly past the middle of the bearing toward the propeller end. The particular plot is for the point in a complete orbit where the greatest minimum pressure occurs. All other positions in the orbit follow a similar pattern. It is of course obvious that pressures of the negative magnitude obtained could not actually occur without rupture of the fluid film.

Figure 5.9 shows the variation of pressure at a fixed point in the bearing for a complete cycle. Again the point selected for representation is that where the greatest negative pressure occurs during a complete orbit.

In studying the numerical results of the pressure variation around the journal for many different initial conditions it is found that the pressure profiles are all of a very similar design; being an

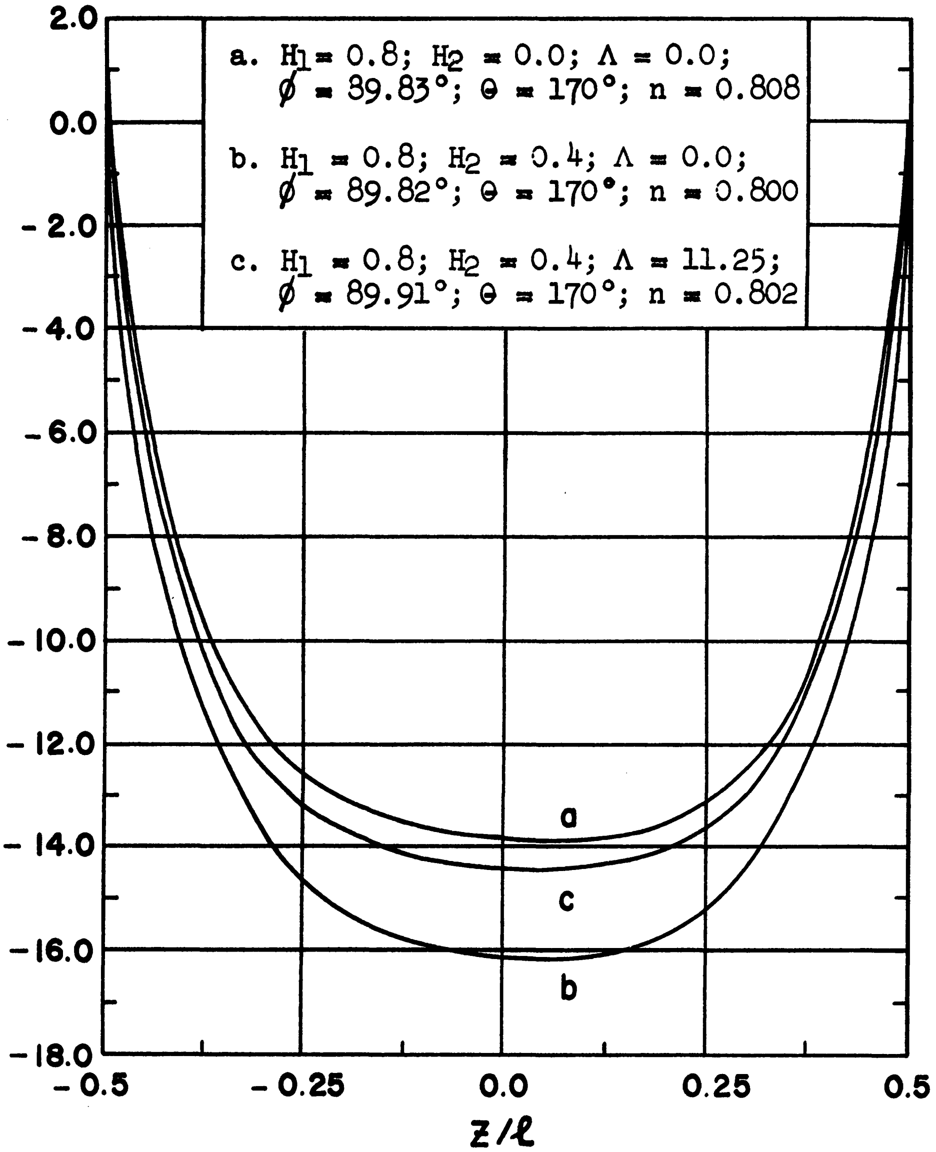

Figure 5.8. Pressure Variation Along the Length of the Bearing for $t = 0.7300$ in the Journal Orbit; $p_0 = 30.0$ psig, $\ell = 74.0$ inches, $b = 4$, $\beta = 270°$. Steady State Coordinates of the Orbits are $n = 0.85$ and $\phi = 90°$.

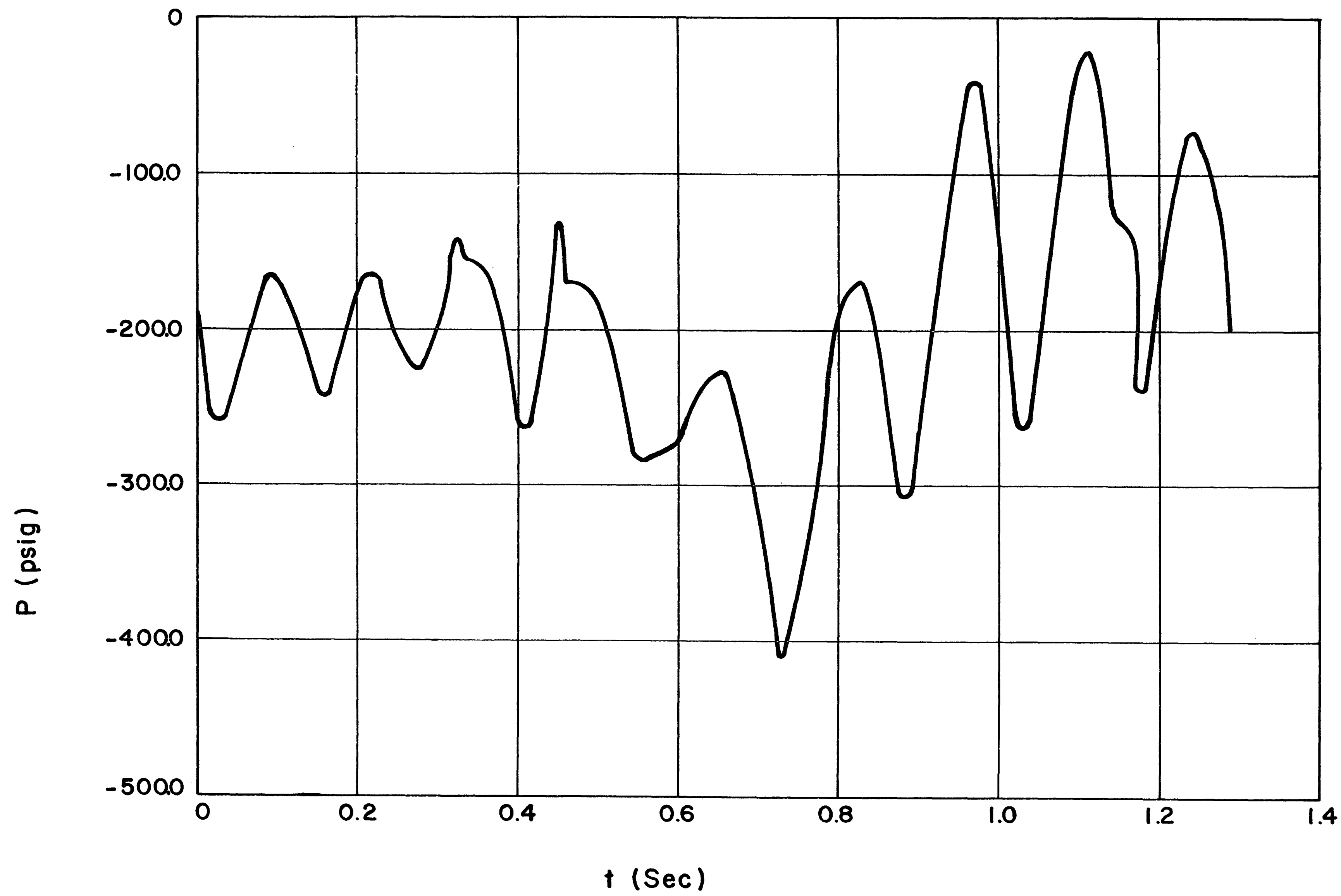

Figure 5.9. Pressure Variation at a Fixed Point ($\phi + \theta = 260°$ and $z = 0.0$) in the Bearing for the Orbit Defined by $H_1 = 0.8$, $H_2 = 0.0$, $b = 4$, $\beta = 270°$ and $K = 0.1796$. Steady State Coordinates of the Orbit are $\phi = 90°$ and $n = 0.85$.

eight lobed pattern with eight regions each of positive and negative pressures.

This pattern although fluctuating considerably in size remains essentially fixed in space during a complete orbit. It oscillates approximately $\pm 5°$ about a mean position. It is further found, as indicated in Figure 5.9, that there is one extreme value of negative pressure encountered in each orbit.

This extreme value of negative pressure occurs slightly past the middle point of the orbit; that is just past the mid-points of the fourth, fifth and sixth cycles for three, four and five blades respectively. The angular position around the bearing where this occurs is defined by $\phi + \theta = 170°$.

The numerical results also show that this extreme value occurs at an optimum condition of a large positive value of $(\frac{dn}{dt})$ and a large negative value of $(\frac{d\phi}{dt})$. The relative magnitudes of these two velocities are approximately equal.

The pressure profiles shown below will all be for the position $z = 0.0$ along the length of the bearing and at the point in the respective journal orbit where the largest negative pressure occurs.

Figures 5.10, 5.11 and 5.12 show the pressure distributions around the bearing for three, four and five bladed propellers respectively using the parameters $H_2 = 0.0$, $\beta = 270°$ and a steady state position of $n = 0.85$ and $\phi = 90°$. It is seen that the three bladed propeller produces the largest values of both positive and negative pressures.

Figures 5.13 and 5.14 illustrate the effect of including the second harmonic of the dynamic loading at different phase angles Λ.

A considerable reduction in the magnitudes of both positive and negative pressures is obtained for an appropriate value of Λ, in this case $\Lambda = 11.25°$.

The effect of the initial eccentricity on the magnitudes of pressures developed in the bearing is illustrated in Figures 5.15 and 5.16. Both positive and negative pressures are considerably reduced with increasing eccentricity. Thus, although the static portion of the pressure increases with increasing eccentricity, the total pressure is greatly reduced. This is to be expected in view of the previous discussion of the journal orbits. As the initial eccentricity is increased there is a conversion from the velocity $(\frac{dn}{dt})$ to the velocity $(\frac{d\phi}{dt})$ and the more important one in terms of pressure levels appears to be the radial velocity $(\frac{dn}{dt})$.

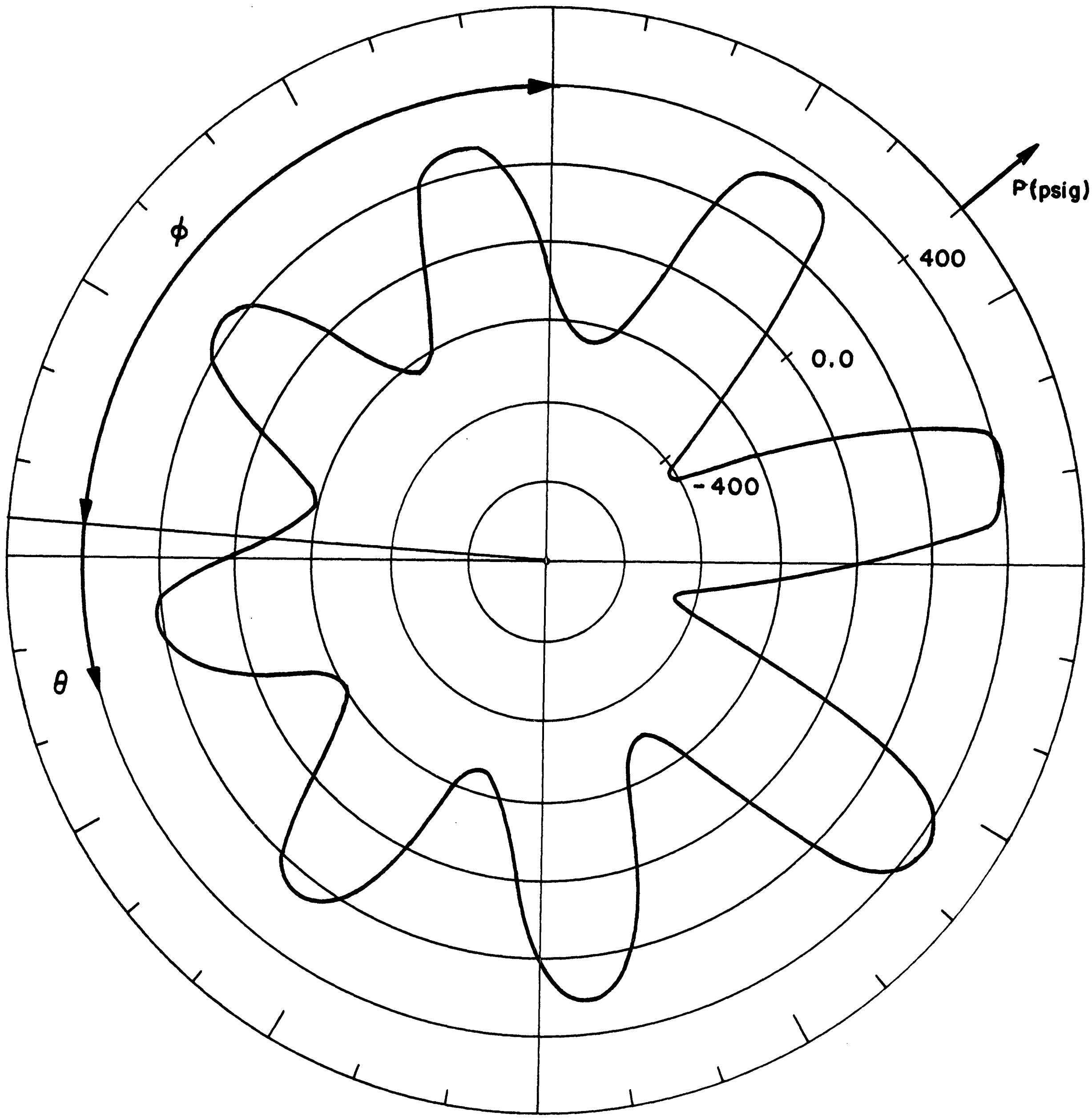

Figure 5.10. Pressure Profile at t = 0.7850, n = 0.7951, z = 0.0 and ϕ = 85.47° for the Orbit Defined by H_1 = 0.8, H_2 = 0.0, b = 3, β = 270° and Λ = 0.0 Steady State Coordinates are n = 0.85 and ϕ = 90°.

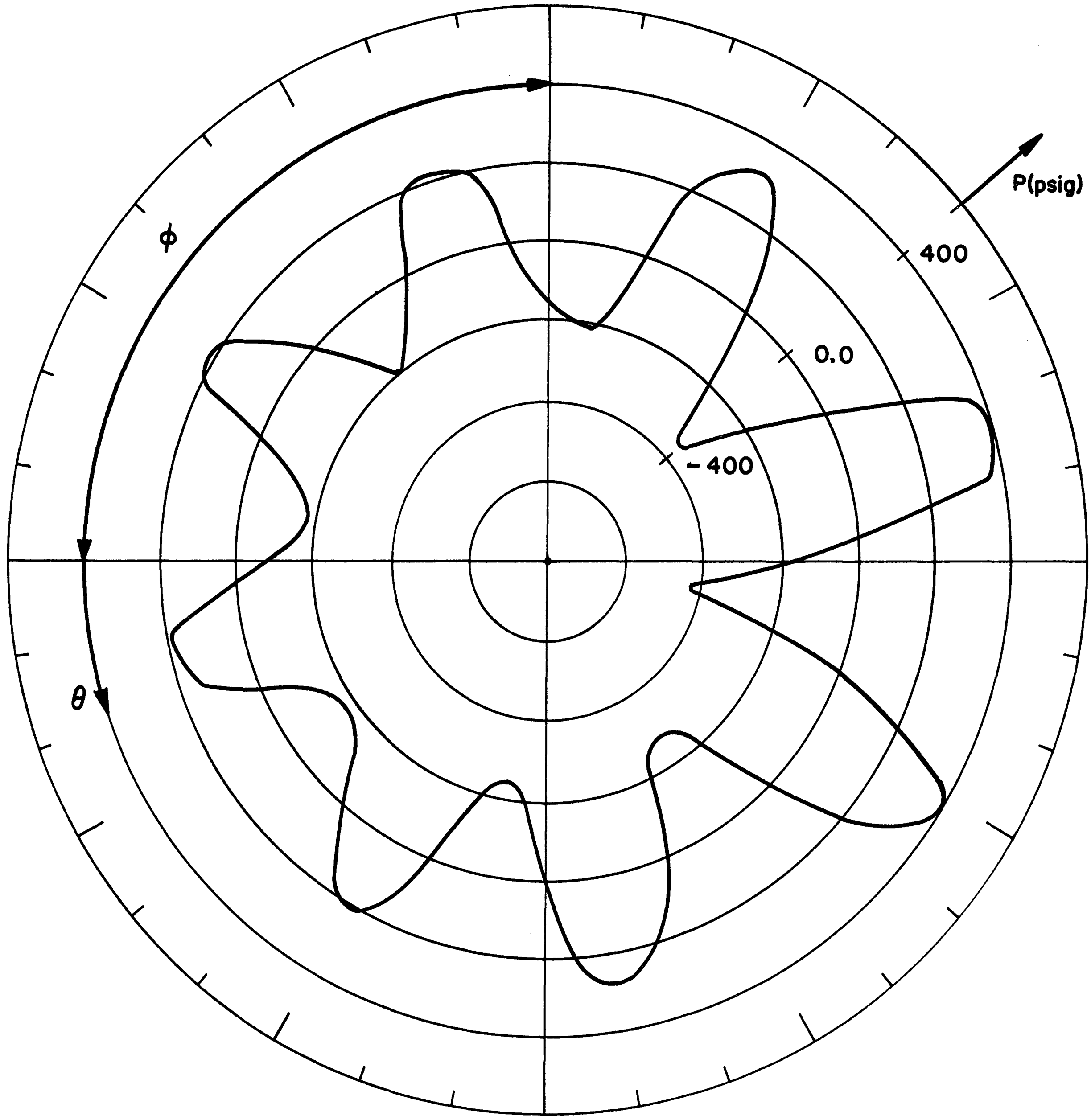

Figure 5.11. Pressure Profile at t = 0.7300, n = 0.8083, z = 0.0, and ϕ = 89.83° for the Orbit Defined by H_1 = 0.8, H_2 = 0.0, b = 4, β = 270° and Λ = 0.0. Steady State Coordinates are n = 0.85 and ϕ = 90°.

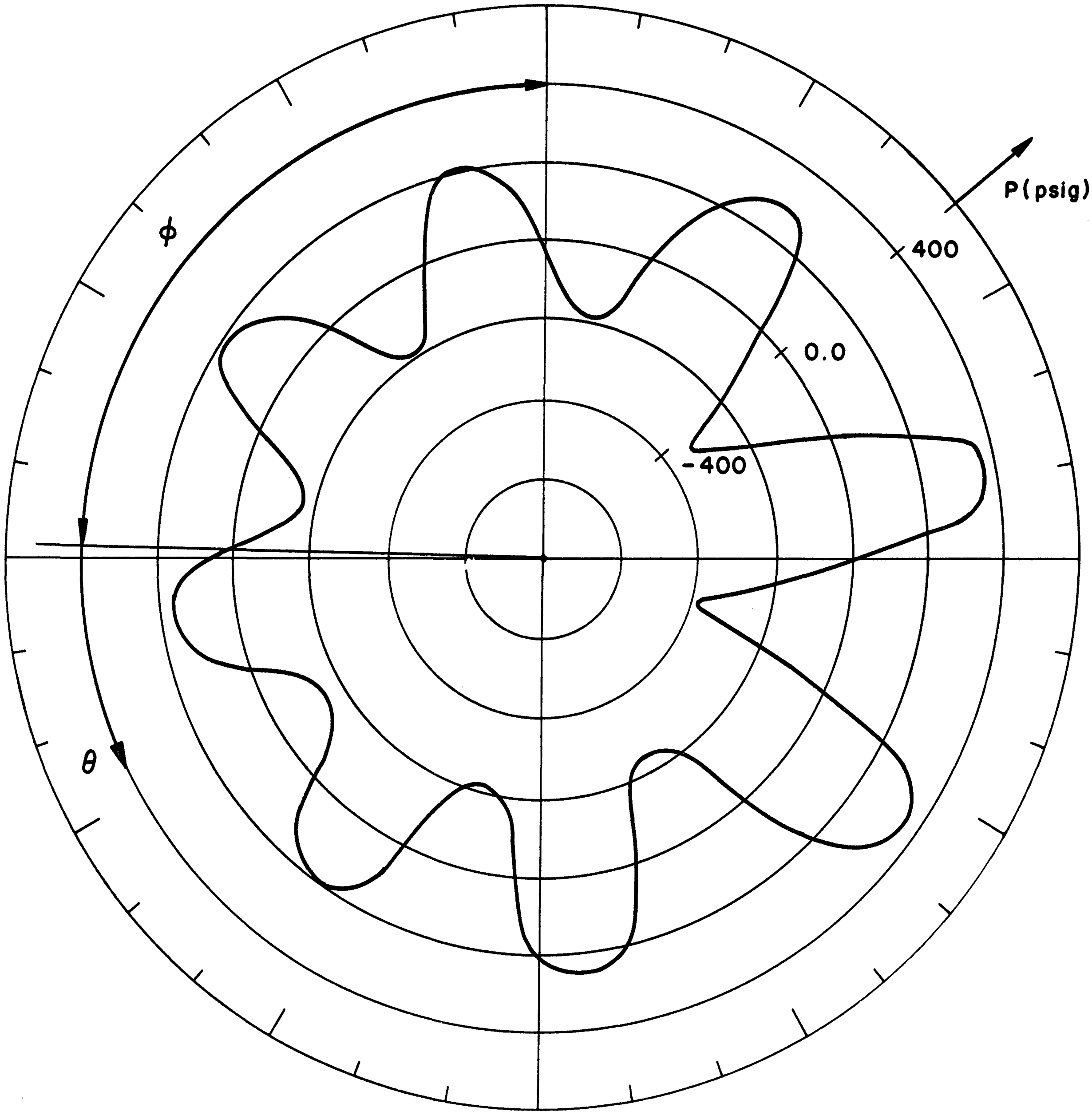

Figure 5.12. Pressure Profile at t = 0.6950, n = 0.8187, z = 0.0 and ϕ = 88.65° for the Orbit Defined by H_1 = 0.8, H_2 = 0.0, b = 5, β = 270° and Λ = 0.0. Steady State Coordinates are n = 0.85 and ϕ = 90°.

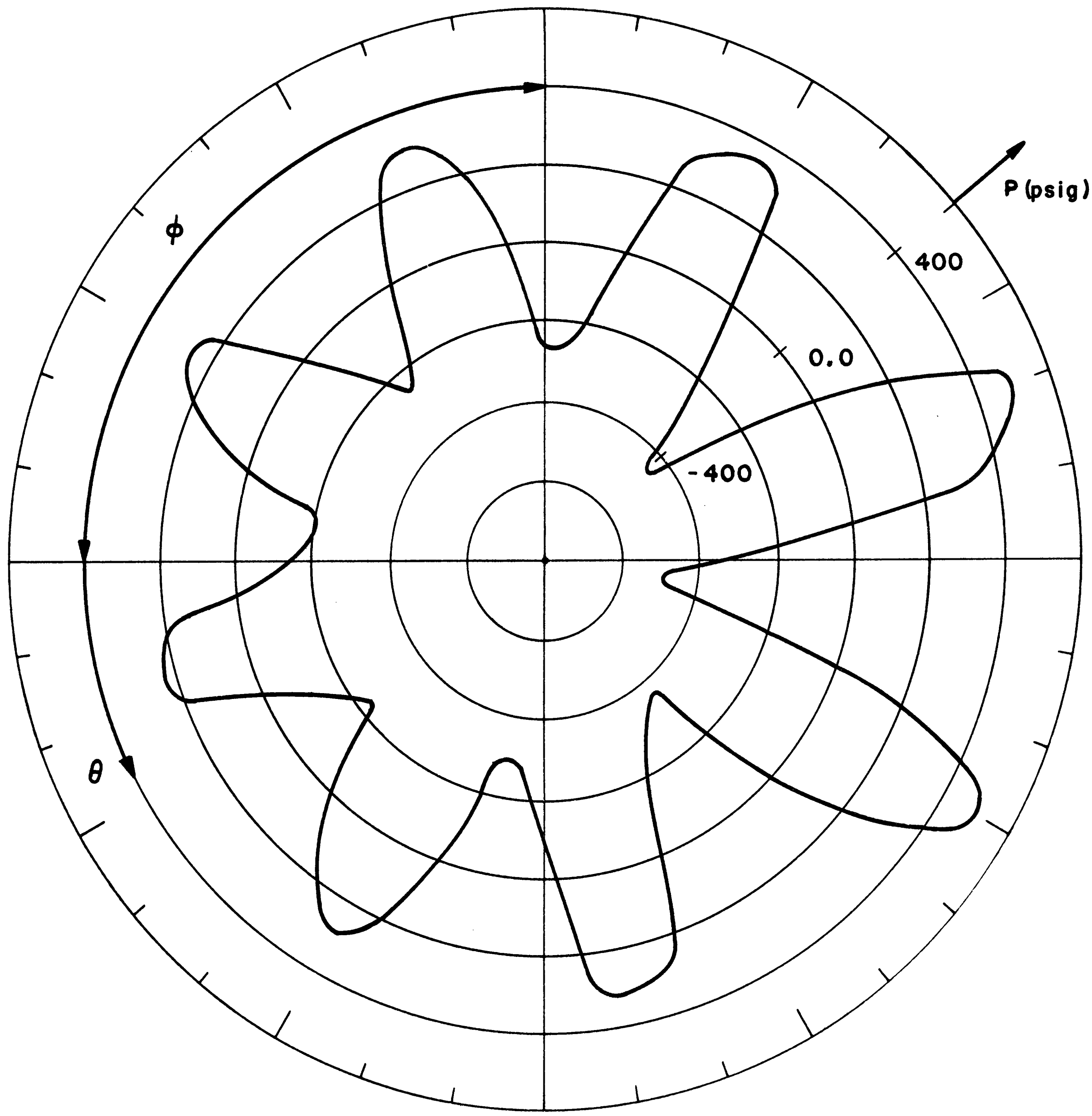

Figure 5.13. Pressure Profile at $t = 0.7300$, $n = 0.8004$, $z = 0.0$ and $\phi = 89.82°$ for the Orbit Defined by $H_1 = 0.8$, $H_2 = 0.4$, $b = 4$, $\beta = 270°$ and $\Lambda = 0.0$. Steady State Coördinates are $n = 0.85$ and $\phi = 90°$.

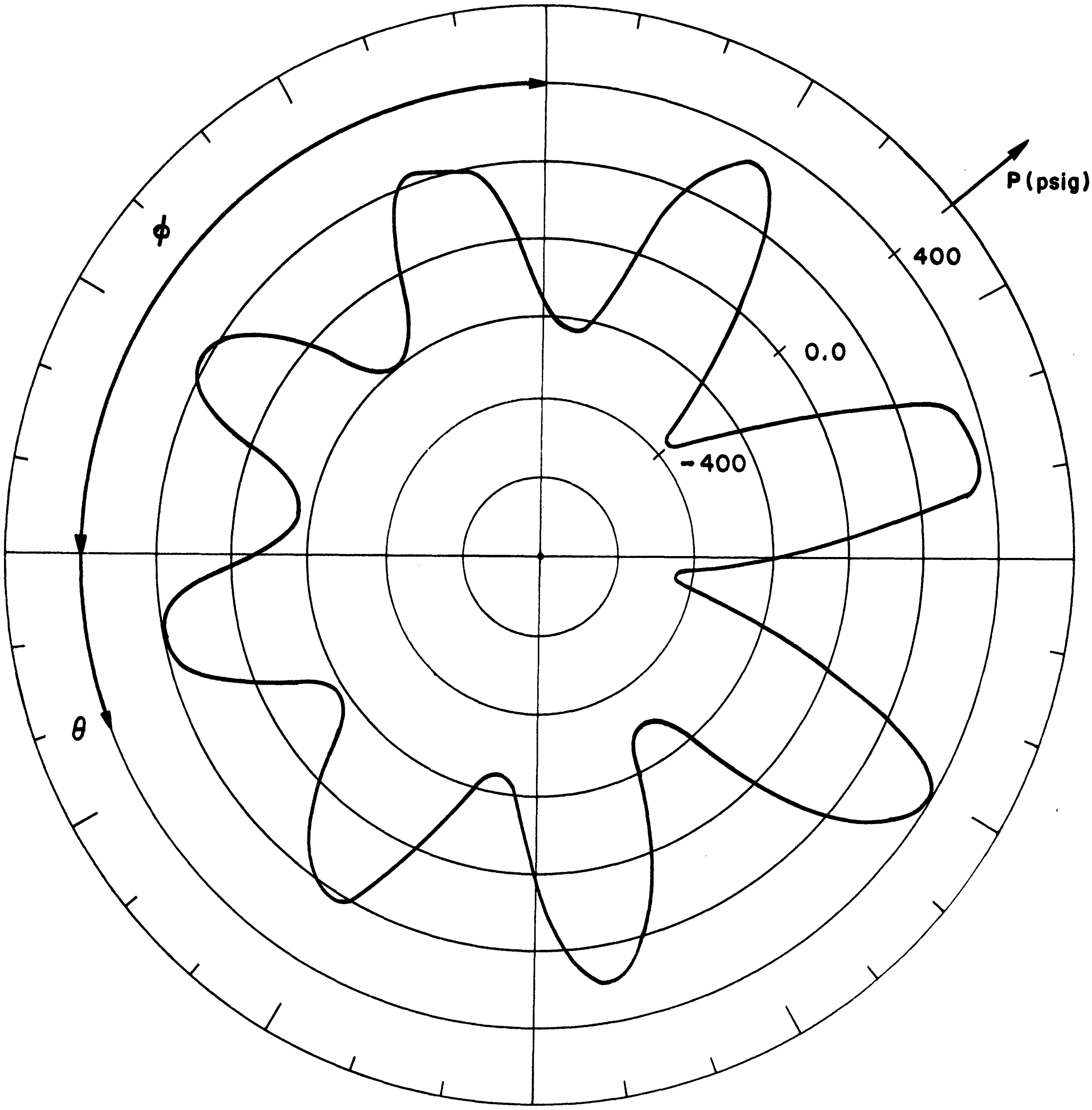

Figure 5.14. Pressure Profile at t = 0.7300, n = 0.8020, z = 0.0 and ϕ = 89.91° for the Orbit Defined by H_1 = 0.8, H_2 = 0.4, b = 4, β = 270° and Λ = 11.25°. Steady State Coordinates are n = 0.85 and ϕ = 90°.

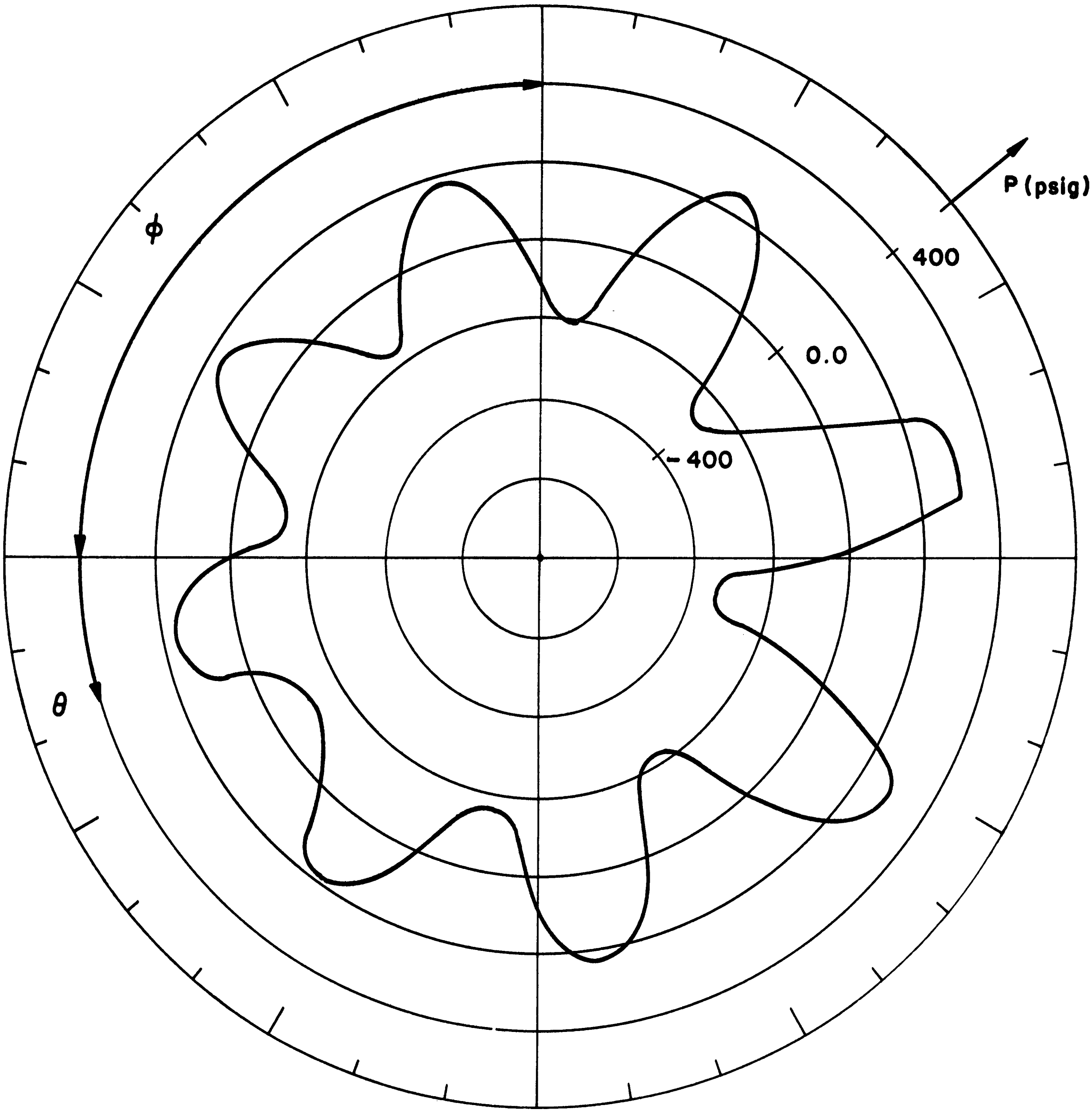

Figure 5.15. Pressure Profile at t = 0.7300, n = 0.8270, z = 0.0 and ϕ = 89.96° for the Orbit Defined by H_1 = 0.4, H_2 = 0.2, b = 4, β = 270° and Λ = 11.25°. Steady State Coordinates are n = 0.85 and ϕ = 90°.

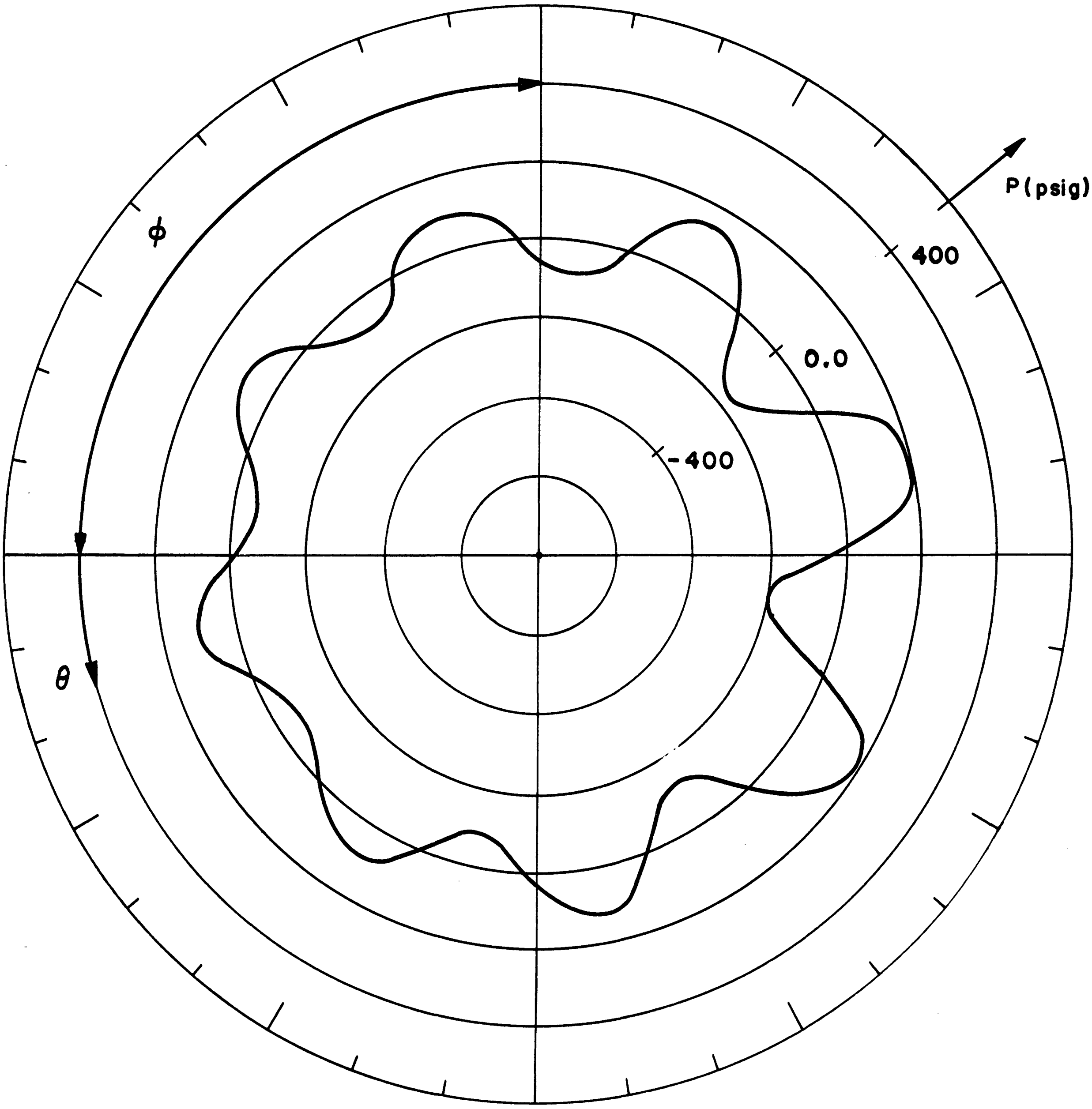

Figure 5.16. Pressure Profile at $t = 0.7300$, $n = 0.9408$, $z = 0.0$ and $\phi = 90.07°$ for the Orbit Defined by $H_1 = 0.4$, $H_2 = 0.2$, $b = 4$, $\beta = 270°$ and $\Lambda = 11.25°$. Steady State Coordinates are $n = 0.95$ and $\phi = 90°$.

VI. SUMMARY

A. Conclusions

The journal orbits are found to depend primarily on four factors, these being (1) the amplitude ratios of the dynamic load H_1 and H_2, (2) the frequency of the dynamic load $b\omega$, (3) the initial steady state eccentricity position n_o and (4) the angular location of the dynamic load β.

Considering the first of these it is found that the size of the journal orbits varies directly with the amplitude ratios H_1 and H_2. With respect to the second, the frequency of the dynamic load is found to effect the general shape of the orbits. In terms of the loading frequency it takes exactly $2b + 1$ cycles to complete one journal orbit. The third and fourth factors are found to influence the mean attitude of the journal orbit. An increase in either β or n_o decreases ϕ_{mean} considerably, with β being the more important factor. As ϕ_{mean} of the journal orbit decreases there is a rapid change from a translational velocity $(\frac{dn}{dt})$ to a rotational velocity $(\frac{d\phi}{dt})$. This results in long narrow journal orbits nearly symmetrical about ϕ_{mean}.

The position along the length of the bearing for extremum values of pressure, either positive or negative, occurs slightly past the middle of the bearing toward the after end. The general profile of the pressure distribution around the bearing is independent of all parameters. It is felt that this is a result of the assumption of a complete oil film around the bearing and the nature of the sine-cosine form of the solution for the pressure distribution.

For given amplitude ratios H_1 and H_2 the magnitudes of the pressures developed in the bearing are found to depend mostly on the relative values of the velocities $(\frac{dn}{dt})$ and $(\frac{d\phi}{dt})$. It appears that of these two quantities the more important is the radial velocity $(\frac{dn}{dt})$. It would take considerable more data to firmly establish this. The difficulty encountered here is that it takes nearly ten minutes of computer time to completely evaluate one journal orbit and the corresponding pressure profile around and along the length of the bearing. If the above conjecture is true concerning the velocity $(\frac{dn}{dt})$ than the greatest reduction of pressure magnitudes will be obtained for a minimum value of ϕ_{mean}, a maximum value of the initial steady state eccentricity and an appropriate value for the phase angle Λ of the second harmonic component.

With respect to the physical problem motivating this study the possibility of cavitational damage has been firmly established. It is felt that with the proper choice of the physical parameters involved in this problem all negative pressure regions around the circumference of the bearing could be raised above the vapor pressure of the fluid except one and this one occurring only once every complete journal orbit. If this were the case then for a propeller having four blades, as it takes exactly 2.25 revolutions of the journal for one complete orbit, there would be four distinct areas of cavitation damage exactly 90° apart around the circumference of the journal.

If the number of propeller blades is changed to three, it takes exactly 2.33 revolutions of the journal for a complete orbit. This would result in three distinct areas of cavitation damage exactly 120° apart around the journal.

The same is seen to be true if the blade number is changed to five. It now takes exactly 2.2 revolutions of the journal for a complete orbit resulting in five distinct areas of cavitation damage 72° apart.

If the conjecture concerning the existence of one negative pressure region for a complete journal orbit is true then the location along the length of the bearing where this would occur also agrees closely with the observed damage.

B. Areas in Need of Further Study

To get a more exact understanding of the actual physical problem there are two particular areas of investigation which would be of considerable value.

The first of these is a more exact determination of the real propeller loading either analytically or experimentally, the latter being perhaps the more realistic approach.

The second would be to attempt to treat the journal as a deformable body. This of course makes the film thickness a function of both θ and z and would quite likely effect the velocities $(\frac{dn}{dt})$ and $(\frac{d\phi}{dt})$ considerably.

APPENDIX

A. Derivation of the Reynolds Lubrication Equation

Reynolds equation for dynamically loaded journal bearings may be derived by following the same approach as that of Elrod[11] who considered the case of static loading. It is necessary only to change the boundary conditions to account for the radial motion of the journal due to the dynamic loading. Accordingly the work that follows is that of Elrod with the exception of that involving the radial velocity V.

The complete Navier-Stokes equations and equation of continuity for steady flow of an incompressible fluid and neglecting body forces X^i can be written respectively in general coordinates as

$$\nu g^{\alpha\beta}\left[\frac{\partial^2 u^i}{\partial\xi^\alpha\partial\xi^\beta} + \Gamma^i_{\sigma\beta}\frac{\partial u^\sigma}{\partial\xi^\alpha} + \Gamma^i_{\sigma\alpha}\frac{\partial u^\sigma}{\partial\xi^\beta} - \Gamma^\sigma_{\alpha\beta}\frac{\partial u^i}{\partial\xi^\sigma} + \left(\frac{\partial\Gamma^i_{\sigma\alpha}}{\partial\xi^\beta} + \Gamma^i_{\tau\beta}\Gamma^\tau_{\sigma\alpha} - \Gamma^i_{\sigma\tau}\Gamma^\tau_{\alpha\beta}\right)u^\sigma\right]$$

$$- u^\alpha\left(\frac{\partial u^i}{\partial\xi^\alpha} + \Gamma^i_{\sigma\alpha}u^\sigma\right) - \frac{1}{\rho}g^{i\alpha}\frac{\partial p}{\partial\xi^\alpha} = 0 , \qquad \text{(A.1)}$$

$$\frac{\partial}{\partial\xi^\alpha}\left(g^{1/2}u^\alpha\right) = 0 \qquad \text{(A.2)}$$

where the Euclidean Christoffel symbols are

$$\Gamma^i_{\alpha\beta}(\xi^1,\xi^2,\xi^3) = \frac{1}{2}g^{i\sigma}\left(\frac{\partial g_{\sigma\beta}}{\partial\xi^\alpha} + \frac{\partial g_{\alpha\sigma}}{\partial\xi^\beta} - \frac{\partial g_{\alpha\beta}}{\partial\xi^\sigma}\right) \qquad \text{(A.3a)}$$

or

$$\Gamma^i_{\alpha\beta}(\xi^1,\xi^2,\xi^3) = \frac{1}{2}G^{i\sigma}\left(\frac{\partial G_{\sigma\beta}}{\partial\xi^\alpha} + \frac{\partial G_{\alpha\sigma}}{\partial\xi^\beta} - \frac{\partial G_{\alpha\beta}}{\partial\xi^\sigma}\right) \qquad \text{(A.3b)}$$

and the metric tensor is

$$g_{\alpha\beta} = \sum_{i=1}^{3}\frac{\partial y^i}{\partial\xi^\alpha}\frac{\partial y^i}{\partial\xi^\beta} \qquad \text{(A.4a)}$$

or

$$g^{\alpha\beta} = \frac{1}{g}(\text{cofactor of } g_{\beta\alpha} \text{ in } g). \tag{A.4b}$$

Further

$$g = |g_{\alpha\beta}| \ , \quad G_{\alpha\beta} = \frac{g_{\alpha\beta}}{L^2} \ , \quad G^{\alpha\beta} = \frac{g^{\alpha\beta}}{L^2} \tag{A.5}$$

where L is some characteristic length and the velocity components u^i may be written as

$$u^i = \frac{d\xi^i}{dt} \ .$$

Referring to Figure A.1 the following dimensionless variables are introduced;

$$\xi^1 = \frac{r\Theta}{L} \ , \quad \xi^2 = \frac{y^2}{L} \quad \text{and} \quad \xi^3 = \frac{r-R}{h(\xi^1,\xi^2)} \ . \tag{A.6}$$

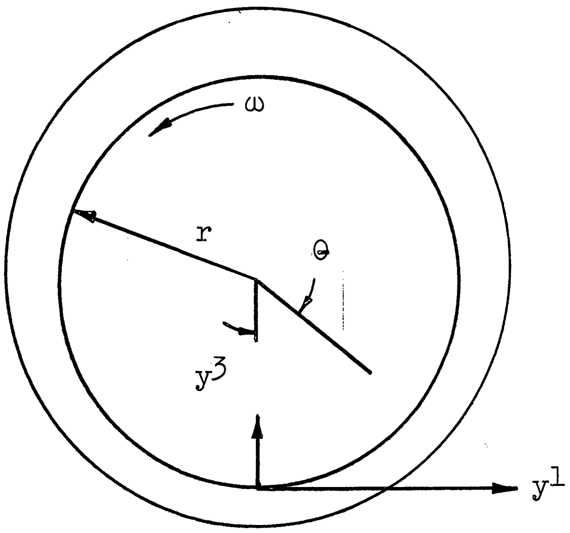

Figure A.1 Auxillary Coordinate System.

Introducing the dimensionless velocities

$$u^i_* = \frac{L^2 u^i}{\nu}$$

the boundary conditions become

$$u_*^i = 0 \quad \text{when} \quad \xi^3 = -1 \tag{A.7a}$$

and

$$u_*^1 = \frac{L^2U'}{\nu} , \quad u_*^3 = \frac{L^2V'}{\nu} , \quad u_*^2 = 0 \quad \text{when} \quad \xi^3 = 0 \tag{A.7b}$$

where

$$U' = \frac{U}{L} \quad \text{and} \quad V' = \frac{V}{h} . \tag{A.7c}$$

Now the transformations inverse to Equations (A.6) are

$$y^1 = R \sin\Theta = (r-\xi^3 h) \sin\Theta = (r-\xi^3 h) \sin(\frac{L\xi^1}{r}) ,$$

$$y^2 = L\xi^2,$$

$$y^3 = r-R \cos\Theta = r - (r-\xi^3 h) \cos (\frac{L\xi^1}{r}) .$$

In view of these relations it follows that

$$\left.\begin{aligned}
&\frac{\partial y^1}{\partial \xi^1} = \frac{L}{r}(r-\xi^3 h) \cos\Theta - \xi^3 \frac{\partial h}{\partial \xi^1} \sin\Theta , \\
&\frac{\partial y^1}{\partial \xi^2} = - \xi^3 \frac{\partial h}{\partial \xi^2} \sin\Theta , \quad \frac{\partial y^1}{\partial \xi^3} = - h \sin\Theta , \\
&\frac{\partial y^2}{\partial \xi^1} = 0 , \quad \frac{\partial y^2}{\partial \xi^2} = L , \quad \frac{\partial y^2}{\partial \xi^3} = 0 , \\
&\frac{\partial y^3}{\partial \xi^1} = \frac{L}{r} (r-\xi^3 h) \sin\Theta + \xi^3 \frac{\partial h}{\partial \xi^1} \cos\Theta , \\
&\frac{\partial y^3}{\partial \xi^2} = \xi^3 \frac{\partial h}{\partial \xi^2} \cos\Theta , \quad \frac{\partial y^3}{\partial \xi^3} = h \cos\Theta .
\end{aligned}\right] \tag{A.8}$$

Considering Equations (A.4a), (A.4b), (A.5) and (A.8) the following relations are obtained:

$$\frac{g_{\alpha\beta}}{L^2} = G_{\alpha\beta} = \begin{bmatrix} (1-\xi^3 \frac{h}{r})+(\frac{\xi^3}{L}\frac{\partial h}{\partial \xi^1})^2 & (\frac{\xi^3}{L})^2 \frac{\partial h}{\partial \xi^1}\frac{\partial h}{\partial \xi^2} & \xi^3 \frac{h}{L^2}\frac{\partial h}{\partial \xi^1} \\ (\frac{\xi^3}{L})^2 \frac{\partial h}{\partial \xi^1}\frac{\partial h}{\partial \xi^2} & 1 + (\frac{\xi^3}{L}\frac{\partial h}{\partial \xi^2})^2 & \xi^3 \frac{h}{L^2}\frac{\partial h}{\partial \xi^2} \\ \xi^3 \frac{h}{L^2}\frac{\partial h}{\partial \xi^1} & \xi^3 \frac{h}{L^2}\frac{\partial h}{\partial \xi^2} & (\frac{h}{L})^2 \end{bmatrix} \quad (A.9)$$

$$L^2 g^{\alpha\beta} = G^{\alpha\beta} = \begin{bmatrix} \frac{1}{(1-\xi^3 \frac{h}{r})^2} & 0 & \frac{\xi^3 \frac{\partial \ln h}{\partial \xi^1}}{(1-\xi^3 \frac{h}{r})^2} \\ 0 & 1 & -\xi^3 \frac{\partial \ln h}{\partial \xi^2} \\ \frac{-\xi^3 \frac{\partial \ln h}{\partial \xi^1}}{(1-\xi^3 \frac{h}{r})^2} & -\xi^3 \frac{\partial \ln h}{\partial \xi^2} & \frac{L^2}{h^2} + \frac{(\xi^3 \frac{\partial \ln h}{\partial \xi^1})^2}{(1-\xi^3 \frac{h}{r})^2} + (\xi^3 \frac{\partial \ln h}{\partial \xi^2})^2 \end{bmatrix} \quad (A.10)$$

and

$$g = |g_{\alpha\beta}| = (1 - \xi^3 \frac{h}{r})^2 L^4 h^2 . \quad (A.11)$$

Introducing as a dimensionless small parameter $\epsilon = h_o/L$, where h_o represents the minimum film thickness, it is noted from Equation (A.10) that all $G^{\alpha\beta}$ are $\mathcal{O}(\epsilon^0)$ except G^{33} which is $\mathcal{O}(\epsilon^{-2})$. Therefore

$$G^{\alpha\beta} = \mathcal{O}[\exp(-2\delta_3^\alpha \delta_3^\beta \ln \epsilon)] . \quad (A.12)$$

From Equation (A.9) all derivatives of $G_{\alpha\beta}$ are $\mathcal{O}(\epsilon^2)$ except G_{11} which is $\mathcal{O}(\epsilon)$. Accordingly

$$\frac{\partial G_{\alpha\beta}}{\partial \xi^k} = \mathcal{O}[\exp(2 - \delta_\alpha^1 \delta_\beta^1)\ln \epsilon] \ . \tag{A.13}$$

Considering Equations (A.12) and (A.13), then from Equation (A.3b)

$$\Gamma_{\alpha\beta}^i = \mathcal{O}[\exp(-2\delta_3^1\delta_3^\sigma + 2 - \delta_\sigma^1\delta_\beta^1) \ln \epsilon] \tag{A.14}$$

$$+ \mathcal{O}[\exp(-2\delta_3^i\delta^\sigma{}_3 + 2 - \delta_\alpha^1\delta_\sigma^1)\ln \epsilon] + \mathcal{O}[\exp(-2\delta_3^i\delta_3^\sigma + 2 - \delta_\alpha^1\delta_\beta^1)\ln \epsilon] \ .$$

Picking the lowest power of ϵ for each $\Gamma_{\alpha\beta}^i$ is is noted that

$$i \neq 3,\ \alpha \neq 1,\ \ \beta \neq 1 : \qquad \Gamma_{\alpha\beta}^i = \mathcal{O}(\epsilon^2) \ ,$$

$$i \neq 3,\ \alpha = 1, \text{ or } \beta = 1, \text{ or } \alpha = \beta = 1: \qquad \Gamma_{\alpha\beta}^i = \mathcal{O}(\epsilon) \ ,$$

$$i = 3,\ \alpha = 1, \text{ or } \beta \neq 1: \qquad \Gamma_{\alpha\beta}^i = \mathcal{O}(\epsilon^0) \ ,$$

$$i = 3,\ \alpha = \beta = 1: \qquad \Gamma_{\alpha\beta}^i = \mathcal{O}(\epsilon^{-1}) \ .$$

Introducing the dimensionless pressure

$$\pi = \frac{p}{\rho(\frac{\nu}{h_o})^2}$$

and substituting π and u_*^i into Equation (A.1) it may be written as

$$G^{\alpha\beta}[\frac{\partial^2 u_*^i}{\partial\xi^\alpha\partial\xi^\beta} + \Gamma_{\sigma\beta}^i \frac{\partial u_*^\sigma}{\partial\xi^\alpha} + \Gamma_{\sigma\alpha}^i \frac{\partial u_*^\sigma}{\partial\xi^\beta} - \Gamma_{\alpha\beta}^\sigma \frac{\partial u_*^i}{\partial\xi^\sigma} + (\frac{\partial\Gamma_{\sigma\alpha}^i}{\partial\xi^\beta} + \Gamma_{\tau\beta}^i\Gamma_{\sigma\alpha}^\tau - \Gamma_{\sigma\tau}^i\Gamma_{\alpha\beta}^\tau)u_*^\sigma]$$

$$- u_*^\alpha(\frac{\partial u_*^i}{\partial\xi^\alpha} + \Gamma_{\sigma\alpha}^i u_*^\sigma) - \epsilon^{-2}G^{i\alpha}\frac{\partial\pi}{\partial\xi^\alpha} = 0 \ . \tag{A.15}$$

With $i \neq 3$ and retaining terms of Equation (A.15) of the lowest powers of ϵ, that is $\mathcal{O}(\epsilon^{-2})$ and $\mathcal{O}(\epsilon^{-1})$ then

$$-G^{11}\Gamma^3_{11}\frac{\partial u^i_*}{\partial\xi^3} + G^{33}\left[\frac{\partial^2 u^i_*}{\partial(\xi^3)^2} + 2\Gamma^i_{13}\frac{\partial u^1_*}{\partial\xi^3} + \Gamma^3_{33}\frac{\partial u^i_*}{\partial\xi^3} + \frac{\partial\Gamma^i_{13}}{\partial\xi^3}u^1_*\right]$$

$$-\epsilon^{-2}G^{i\alpha}\frac{\partial\pi}{\partial\xi^\alpha} = 0 . \qquad (A.16)$$

With respect to the given accuracy

$$\Gamma^i_{13} = \frac{1}{2}G^{i1}\frac{\partial G_{11}}{\partial\xi^3} = \frac{1}{2}G^{i1}(-2\frac{h}{r}) = -G^{i1}\frac{h}{r} = \mathcal{O}(\epsilon) ,$$

$$\frac{\partial\Gamma^i_{13}}{\partial\xi^3} = \mathcal{O}(\epsilon^2) \quad \text{as} \quad \frac{\partial h}{\partial\xi^3} = 0 ,$$

$$\Gamma^3_{33} = \frac{1}{2}G^{3\sigma}\left[\frac{\partial G_{\sigma 3}}{\partial\xi^3} - \frac{\partial G_{33}}{\partial\xi^\sigma}\right] = \mathcal{O}(\epsilon^2) \quad \text{as} \quad \frac{\partial G_{33}}{\partial\xi^3} = 0 ,$$

$$\Gamma^3_{11} = -\frac{1}{2}G^{33}\frac{\partial G_{11}}{\partial\xi^3} = \frac{h}{r}(\frac{L}{h}) = \mathcal{O}(\frac{1}{\epsilon}) .$$

In view of these relations, Equation (A.16) may be written

$$(\frac{L}{h})^2\left[\frac{\partial^2 u^i_*}{\partial(\xi^3)^2} - 2G^{i1}\frac{h}{r}\frac{\partial u^1_*}{\partial\xi^3} - \frac{h}{r}\frac{\partial u^i_*}{\partial\xi^3}\right]$$

$$-\epsilon^{-2}\left[G^{i1}\frac{\partial\pi}{\partial\xi^1} + G^{i2}\frac{\partial\pi}{\partial\xi^2} + G^{i3}\frac{\partial\pi}{\partial\xi^3}\right] = 0 . \qquad (A.17)$$

With $i = 3$ correct in terms of $\mathcal{O}(\epsilon^{-4})$ and $\mathcal{O}(\epsilon^{-3})$ Equation (A.15) becomes

$$\frac{\partial\pi}{\partial\xi^3} = 0 . \qquad (A.18)$$

From Equation (A.17)

$$\frac{\partial^2 u^1_*}{\partial(\xi^3)^2} - 2G^{11}\frac{h}{r}\frac{\partial u^1_*}{\partial\xi^3} - \frac{h}{r}\frac{\partial u^1_*}{\partial\xi^3} = G^{11}(\frac{h}{h_o})^2\frac{\partial\pi}{\partial\xi^1} \qquad (A.19)$$

and

$$\frac{\partial^2 u_*^2}{\partial(\xi^3)^2} - \frac{h}{r}\frac{\partial u_*^2}{\partial \xi^3} = G^{22}\left(\frac{h}{h_o}\right)^2 \frac{\partial \pi}{\partial \xi^2} \tag{A.20}$$

where

$$G^{11} = \frac{1}{(1-\xi^3 \frac{h}{r})^2} = 1 + 2\xi^3 \frac{h}{r} + \ldots$$

and

$$G^{22} = 1 \ .$$

Retaining terms of $\mathcal{O}(\epsilon)$ in Equations (A.19) and (A.20)

$$\frac{\partial^2 u_*^1}{\partial(\xi^3)^2} - 3\,\frac{h}{r}\frac{\partial u_*^1}{\partial \xi^3} = (1 + 2\xi^3 \frac{h}{r})\left(\frac{h}{h_o}\right)^2 \frac{\partial \pi}{\partial \xi^1} \tag{A.21}$$

and

$$\frac{\partial^2 u_*^2}{\partial(\xi^3)^2} - \frac{h}{r}\frac{\partial u_*^2}{\partial \xi^3} = \left(\frac{h}{h_o}\right)^2 \frac{\partial \pi}{\partial \xi^2} \ . \tag{A.22}$$

In view of Equation (A.18), with respect to the given accuracy, Equations (A.21) and (A.22) may be integrated directly to give

$$u_*^1 = \left[\frac{L^2U'}{\nu} + \frac{f_1\xi^3}{2}\right](1 + \xi^3) + \frac{3h}{r}\Big[\xi^3\left(\frac{L^2U'}{2\nu} - \frac{f_1}{36}\right)$$
$$+ (\xi^3)^2\left(\frac{L^2U'}{2\nu} + \frac{f_1}{4}\right) + (\xi^3)^3\,\frac{5f_1}{18}\Big] \tag{A.23}$$

where

$$f_1 = \left(\frac{h}{h_o}\right)^2 \frac{\partial \pi}{\partial \xi^1} ,$$

and

$$u_*^2 = f_2\left[\xi^3\,\frac{(1+\xi^3)}{2} + \frac{h}{r}\left(\frac{\xi^3}{12} + \frac{(\xi^3)^2}{4} + \frac{(\xi^3)^3}{6}\right)\right] \tag{A.24}$$

where

$$f_2 = (\frac{h}{h_o})^2 \frac{\partial\pi}{\partial\xi^2} \quad .$$

Equation (A.2) in view of Equation (A.18) and considering boundary conditions (A.7a) and (A.7b) may be integrated with respect to ξ^3 to give

$$\int_0^{-1} [\frac{\partial}{\partial\xi^1}(1-\xi^3 \frac{h}{r}) hu_*^1] d\xi^3 + \int_0^{-1} [\frac{\partial}{\partial\xi^2}(1-\xi^3 \frac{h}{r}) hu_*^2] d\xi^3$$

$$= - \int_0^{-1} [\frac{\partial}{\partial\xi^3}(1-\xi^3 \frac{h}{r}) hu_*^3] d\xi^3 \qquad \text{(A.25)}$$

Substituting Equations (A.23) and (A.24) into Equation (A.25) and performing the required integration gives

$$\frac{\partial}{\partial\xi^1}[(\frac{h}{h_o}) \{(\frac{h}{h_o})^2(1 - \frac{h}{2r}) \frac{\partial\pi}{\partial\xi^1} - (6 - \frac{h}{r}) \frac{L^2U'}{\nu}\}]$$

$$+ \frac{\partial}{\partial\xi^2}[(\frac{h}{h_o}) \{(\frac{h}{h_o})^2(1 + \frac{h}{2r}) \frac{\partial\pi}{\partial\xi^2}\}] = 12 \frac{h}{h_o} \frac{L^2V'}{\nu} \quad . \qquad \text{(A.26)}$$

Finally in terms of the more conventional variables

$$\xi^1 = \frac{x}{L} , \quad \xi^2 = \frac{z}{L} , \quad \pi = \frac{p}{\rho(\frac{\nu}{h_o})^2} ,$$

and noting Equations (A.7c), then Equation (A.26) may be written

$$\frac{\partial}{\partial x}[h^3(1 - \frac{h}{2r}) \frac{\partial p}{\partial x}] + \frac{\partial}{\partial z}[h^3(1 + \frac{h}{2r}) \frac{\partial p}{\partial z}] \qquad \text{(A.27)}$$

$$= -6\mu U \frac{\partial}{\partial x}[h(1 - \frac{h}{6r})] + 6\mu h(1 - \frac{h}{6r}) \frac{\partial U}{\partial x} + 12\mu V \quad .$$

This of course represents the general Reynolds lubrication equation for finite length, dynamically loaded, journal bearings with first order correction terms; that is $\mathcal{O}(\epsilon)$.

B. Solution of Reynolds Equation for the Infinite Length Bearing

Considering Equation (3.1) with z variation neglected the partial derivatives become total derivatives and Equation (3.1) becomes

$$\frac{d}{d\Theta}\left(h^3 \frac{dp}{d\Theta}\right) = 6\mu r^2\left(\omega - 2\frac{d\phi}{dt}\right)\frac{dh}{d\Theta} + 12\mu r^2 c \frac{dn}{dt}\cos\Theta \ . \qquad (B.1)$$

Letting

$$F_1 = \frac{6\mu r^2}{c^2}\left(\omega - 2\frac{d\phi}{dt}\right) , \quad F_2 = \frac{6\mu r^2}{c^2}\left(2\frac{dn}{dt}\right)$$

noting Equation (2.9) that

$$h = c(1+n\cos\Theta) , \quad \frac{dh}{d\Theta} = -cn\sin\Theta ,$$

substituting these results into Equation (B.1) and integrating with respect to Θ, Equation (B.1) becomes

$$\frac{dp}{d\Theta} = F_1\left[\frac{c}{(1+n\cos\Theta)^2}\right] + F_2\left[\frac{\sin\Theta}{(1+n\cos\Theta)^3}\right] + F_3\left[\frac{1}{c^3(1+n\cos\Theta)^3}\right] \qquad (B.2)$$

where F_3 is a constant of integration. Noting that the pressure must be continuous around the bearing and letting

$$\bar{\alpha} = \frac{1}{n}$$

Equation (B.2) becomes

$$p(\pi)-p(-\pi) = 0 = \bar{\alpha}^2 F_1 \int_{-\pi}^{\pi} \frac{d\Theta}{(\bar{\alpha} - \cos\Theta)^2} + \bar{\alpha}^3 F_2 \int_{-\pi}^{\pi} \frac{\sin\Theta d\Theta}{(\bar{\alpha} + \cos\Theta)^3}$$

$$+ \bar{\alpha}^3 F_3 \int_{-\pi}^{\pi} \frac{d\Theta}{(\bar{\alpha} + \cos\Theta)^3} , \qquad (B.3)$$

or

$$0 = \bar{\alpha}^2 F_1 [\frac{2\bar{\alpha}}{(\bar{\alpha}^2-1)^{3/2}} \tan^{-1} \{\sqrt{\frac{\bar{\alpha} - 1}{\bar{\alpha} + 1}} \tan \frac{\Theta}{2}\} - \frac{1}{(\bar{\alpha}^2-1)} \frac{\sin\Theta}{(\bar{\alpha} + \cos\Theta)}]_{-\pi}^{\pi}$$

$$+ \bar{\alpha}^3 F_2 [\frac{1}{2(\bar{\alpha} + \cos\Theta)^2}]_{-\pi}^{\pi}$$

$$+ \bar{\alpha}^3 F_3 [\frac{2\bar{\alpha}^2 + 1}{(\bar{\alpha}^2-1)^{5/2}} \tan^{-1} \{\sqrt{\frac{\bar{\alpha} - 1}{\bar{\alpha} + 1}} \tan \frac{\Theta}{2}\} - \frac{3\bar{\alpha}}{2(\bar{\alpha}^2-1)^2} \frac{\sin\Theta}{(\bar{\alpha} + \cos\Theta)}$$

$$- \frac{\sin\Theta}{2(\bar{\alpha}^2-1)(\bar{\alpha} + \cos\Theta)^2}]_{-\pi}^{\pi} \qquad (B.4)$$

Evaluating Equation (B.4) the constant of integration F_3 is

$$F_3 = \frac{-2F_1(\bar{\alpha}^2-1)}{(2\bar{\alpha}^2+1)} . \qquad (B.5)$$

Considering now Equation (B.3) as an indefinite integral

$$p(\Theta) = \bar{\alpha}^2 F_1 \int \frac{d\Theta}{(\bar{\alpha} + \cos\Theta)^2} + \bar{\alpha}^3 F_2 \int \frac{\sin\Theta d\Theta}{(\bar{\alpha} + \cos\Theta)^3} + \bar{\alpha}^3 F_3 \int \frac{d\Theta}{(\bar{\alpha} + \cos\Theta)^3} + F_4 \qquad (B.6)$$

where F_4 is an arbitrary constant, performing the integration again as in Equation (B.4) and substituting Equation (B.5) for F_3, Equation (B.6) becomes

$$p(\Theta) = \bar{\alpha}^2 F_1 [\frac{\sin\Theta}{(2\bar{\alpha}^2+1)(\bar{\alpha} + \cos\Theta)}(1 + \frac{\bar{\alpha}}{\bar{\alpha} + \cos\Theta})] + \bar{\alpha}^3 F_2 [\frac{1}{2(\bar{\alpha} + \cos\Theta)^3}] + F_4 \qquad (B.7)$$

Substituting the expressions for F_1, F_2 and $\bar{\alpha}$ into Equation (B.7)

$$p(\Theta) = \frac{6\mu r^2}{c^2}(\omega - 2 \frac{d\phi}{dt}) [\frac{n(2+n \cos\Theta)\sin\Theta}{(2+n^2)(1+n \cos\Theta)^2}]$$

$$+ \frac{6\mu r^2}{c^2}(\frac{dn}{dt}) [\frac{1}{n(1+n \cos\Theta)^2}] + \text{arbitrary constant.} \qquad (B.8)$$

C. Evaluation of the Series Coefficients $A_m(z)$ and $B_m(z)$

Considering the $A_m(z)$, and for convenience letting $A_m(z) = A_m$, from Equation (3.6) the general recurrence relation for $m > 1$ is

$$2(D^2-m^2)A_m + n(D^2-m^2-m+2)A_{m-1} + n(D^2-m^2+m+2)A_{m+1} = 0 \qquad (C.1)$$

where

$$D^2 = r^2 \frac{d^2}{dz^2} \quad .$$

From Equations (3.12) and (3.13) A_1 and A_2 may be written as

$$A_1 = C_1 \cosh \frac{\alpha z}{r} \qquad (C.2)$$

and

$$A_2 = -\gamma C_1 \cosh \frac{\alpha z}{r} \qquad (C.3)$$

where α and γ are defined by Equations (3.10) and (3.40) respectively as

$$\alpha = \left[\frac{2(1+\gamma n)}{2-\gamma n}\right]^{1/2} , \qquad \gamma = \left[\frac{n\,\{2+(1-n^2)^{1/2}\}}{\{1+(1-n^2)^{1/2}\}^2}\right]^{1/2}$$

and C_1 is a constant of integration. In view of the boundary condition, Equation (2.15), all of the A_m must be even in z. The complementary solution for all A_m will therefore contain only one term which shall be represented as the first term of each A_m and C_m is the corresponding constant of integration.

For $m = 3$, then from Equation (C.1)

$$D^2A_3 = -\frac{2}{n}(D^2-4)A_2 - (D^2-4)A_1 \; .$$

Substituting A_1 and A_2 and solving for A_3

$$A_3 = C_3 + \left(\frac{2\gamma}{n} - 1\right)\left(\frac{\alpha^2-4}{\alpha^2}\right) C_1 \cosh \frac{\alpha z}{r} \quad . \qquad (C.4)$$

For $m = 4$, then from Equation (C.1)

$$D^2A_4 - 4A_4 = -\frac{2}{n}(D^2-9)A_3 - (D^2-10)A_2 \ .$$

Substituting A_2 and A_3 and solving for A_4

$$A_4 = C_4 \cosh \frac{2z}{r} - \frac{9C_3}{2n} + \left[\frac{\gamma(\alpha^2-10)}{(\alpha^2-4)} - \frac{2}{n}\left(\frac{\alpha^2-9}{\alpha^2}\right)\left(\frac{2\gamma}{n} - 1\right)\right] C_1 \cosh \frac{\alpha z}{r} \ . \quad (C.5)$$

For $m = 5$, then from Equation (C.1)

$$D^2A_5 - 10A_5 = -\frac{2}{n}(D^2-16)A_4 - (D^2-18)A_3 \ .$$

Substituting A_3 and A_4 and integrating A_5 becomes

$$A_5 = C_5 \cosh \frac{\sqrt{10}\ z}{r} - \frac{4}{n} C_4 \cosh \frac{2z}{r} - \frac{9}{5} C_3\left(1 - \frac{8}{n^2}\right) + \frac{F_1}{(\alpha^2-10)} C_1 \cosh \frac{\alpha z}{r} \qquad (C.6)$$

where

$$F_1 = \frac{4}{n^2}(\alpha^2-16)\left(\frac{\alpha^2-9}{\alpha^2}\right)\left(\frac{2\gamma}{n} - 1\right) - \frac{2\gamma}{n}(\alpha^2-16)\left(\frac{\alpha^2-10}{\alpha^2-4}\right) - (\alpha^2-18)\left(\frac{2\gamma}{n} - 1\right)\left(\frac{\alpha^2-4}{\alpha^2}\right). \qquad (C.7)$$

For $m = 6$, then from Equation (C.1)

$$D^2A_6 - 18A_6 = \frac{-2}{n}(D^2-25)A_5 - (D^2-28)A_4 \ .$$

Substituting A_4 and A_5 and solving for A_6

$$A_6 = C_6 \cosh \frac{\sqrt{18}\ z}{r} - \frac{15}{4n} C_5 \cosh \frac{\sqrt{10}\ z}{r} - \frac{12}{7} C_4\left(1 - \frac{7}{n^2}\right) \cosh \frac{2z}{r}$$
$$+ \frac{4}{n} C_3\left(3 - \frac{10}{n^2}\right) + \frac{F_2}{(\alpha^2-18)} C_1 \cosh \frac{\alpha z}{r} \qquad (C.8)$$

where

$$F_2 = \frac{2}{n}(\alpha^2-28)\left(\frac{\alpha^2-9}{\alpha^2}\right)\left(\frac{2\gamma}{n} - 1\right) - \gamma(\alpha^2-28)\left(\frac{\alpha^2-10}{\alpha^2-4}\right) - \frac{2}{n}\left(\frac{\alpha^2-25}{\alpha^2-10}\right) F_1 \ . \qquad (C.9)$$

For $m = 7$, then from Equation (C.1)

$$D^2A_7 - 28A_7 = -\frac{2}{n}(D^2-36)A_6 - (D^2-40)A_5 .$$

Substituting A_5 and A_6 and solving for A_7

$$A_7 = C_7 \cosh\frac{\sqrt{28}\,z}{r} - \frac{36}{10n} C_6 \cosh\frac{\sqrt{18}\,z}{r} - \frac{5}{3}(1 - \frac{52}{n^2}) C_5 \cosh\frac{\sqrt{10}\,z}{r}$$

$$+ \frac{1}{24}[\frac{768}{7n}(1 - \frac{7}{n^2}) + \frac{144}{n}] C_4 \cosh\frac{2z}{r} - \frac{1}{28}[\frac{288}{n^2}(3 - \frac{10}{n^2}) - 72(1 - \frac{8}{n^2})] C_3$$

$$+ \frac{1}{(\alpha^2-28)}[-\frac{2}{n}(\frac{\alpha^2-36}{\alpha^2-18})F_2 - (\frac{\alpha^2-40}{\alpha^2-10})F_1] C_1 \cosh\frac{\alpha z}{r} \qquad (C.10)$$

where F_1 and F_2 are defined by Equations (C.7) and (C.9).

For $m = 8$, then from Equation (C.1)

$$D^2A_8 - 40A_8 = -\frac{2}{n}(D^2-49)A_7 - (D^2-54)A_6.$$

Substituting A_6 and A_7 and integrating A_8 becomes

$$A_8 = C_8 \cosh\frac{\sqrt{40}\,z}{r} - \frac{7}{2n} C_7 \cosh\frac{\sqrt{28}\,z}{r} - \frac{1}{22}[36(1 - \frac{62}{10n^2})]C_6 \cosh\frac{\sqrt{18}\,z}{r}$$

$$+ \frac{1}{6}[\frac{26}{n}(1 - \frac{52}{n^2}) + \frac{33}{n}] C_5 \cosh\frac{\sqrt{10}\,z}{r}$$

$$- \frac{1}{36}[\frac{15}{4n}\{\frac{768}{7n}(1 - \frac{7}{n^2}) + \frac{144}{n}\} - \frac{600}{7}(1 - \frac{7}{n^2})] C_4 \cosh\frac{2z}{r} \qquad (C.11)$$

$$- \frac{1}{40}[-\frac{7}{2n}\{\frac{288}{n^2}(3 - \frac{10}{n^2}) - 72(1 - \frac{8}{n^2})\} + \frac{216}{n}(3 - \frac{10}{n^2})] C_3$$

$$- \frac{1}{(\alpha^2-40)}[\frac{2}{n}(\frac{\alpha^2-49}{\alpha^2-28})\{-\frac{2}{n}(\frac{\alpha^2-36}{\alpha^2-18})F_2 - (\frac{\alpha^2-40}{\alpha^2-10})F_1\} + (\frac{\alpha^2-54}{\alpha^2-18})F_2]C_1 \cosh\frac{\alpha z}{r} .$$

From Equation (3.36)

$$A_m(\frac{\ell}{2}) = F(-1)^{m-1}a^m[1 + \frac{m}{(1-n^2)^{1/2}}] \qquad (C.12)$$

where

$$F = \frac{12\mu r^2}{c^2}(\omega - 2\frac{d\phi}{dt})(\frac{1}{2+n^2}) \tag{C.13}$$

all of the constants C_m may be evaluated.

For $m = 1$ from Equations (C.2) and (C.12)

$$C_1 = Fa[1 + \frac{1}{(1-n^2)^{1/2}}] \frac{1}{\cosh\frac{\alpha\ell}{2r}} . \tag{C.14}$$

For $m = 3$ from Equations (C.4) and (C.12)

$$C_3 = Fa^3[1 + \frac{3}{(1-n^2)^{1/2}}] - Fa(\frac{2\gamma}{n} - 1)(\frac{\alpha^2-4}{\alpha^2})[1 + \frac{1}{(1-n^2)^{1/2}}] . \tag{C.15}$$

For $m = 4$ from Equations (C.5) and (C.12)

$$C_4 = \frac{1}{\cosh\frac{\ell}{r}}\{-Fa^4[1 + \frac{4}{(1-n^2)^{1/2}}] + \frac{9C_3}{2n}$$

$$+ [\frac{2}{n}(\frac{\alpha^2-9}{\alpha^2})(\frac{2\gamma}{n} - 1) - \gamma(\frac{\alpha^2-10}{\alpha^2-4})] C_1 \cosh\frac{\alpha\ell}{2r}\} . \tag{C.16}$$

For $m = 5$ from Equations (C.6) and (C.12)

$$C_5 = \frac{1}{\cosh\frac{\sqrt{10}\,\ell}{2r}}\{Fa^5[1 + \frac{5}{(1-n^2)^{1/2}}] + \frac{4}{n} C_4 \cosh\frac{\ell}{r}$$

$$+ \frac{9}{5} C_3(1 - \frac{8}{n^2}) - \frac{F_1}{(\alpha^2-10)} C_1 \cosh\frac{\alpha\ell}{2r}\} . \tag{C.17}$$

For $m = 6$ from Equations (C.8) and (C.12)

$$C_6 = \frac{1}{\cosh\frac{\sqrt{18}\,\ell}{2r}}\{-Fa^6[1 + \frac{6}{(1-n^2)^{1/2}}] + \frac{15}{4n} C_5 \cosh\frac{\sqrt{10}\,\ell}{2r}$$

$$+ \frac{12}{7} C_4(1 - \frac{7}{n^2}) \cosh\frac{\ell}{r} - \frac{4}{n} C_3(3 - \frac{10}{n^2}) - \frac{F_2}{(\alpha^2-18)} C_1 \cosh\frac{\alpha\ell}{2r}\} . \tag{C.18}$$

For $m = 7$ from Equations (C.10) and (C.12)

$$C_7 = \frac{1}{\cosh \frac{\sqrt{28}\,\ell}{2r}} \{Fa^7[1 + \frac{7}{(1-n^2)^{1/2}}] + \frac{36}{10n} C_6 \cosh \frac{\sqrt{18}\,\ell}{2r}$$

$$+ \frac{5}{3}(1 - \frac{52}{n^2}) C_5 \cosh \frac{\sqrt{10}\,\ell}{2r}$$

$$- \frac{1}{24}[\frac{768}{7n}(1 - \frac{7}{n^2}) + \frac{144}{n}] C_4 \cosh \frac{\ell}{r} + \frac{1}{28}[\frac{288}{n^2}(3 - \frac{10}{n^2}) - 72(1 - \frac{8}{n^2})]C_3$$

$$- \frac{1}{(\alpha^2-28)}[- \frac{2}{n}(\frac{\alpha^2-36}{\alpha^2-18}) F_2 - (\frac{\alpha^2-40}{\alpha^2-10}) F_1] C_1 \cosh \frac{\alpha\ell}{2r}\} . \tag{C.19}$$

For $m = 8$ from Equations (C.11) and (C.12)

$$C_8 = \frac{1}{\cosh \frac{\sqrt{40}\,\ell}{2r}} \Big[-Fa^8[1 + \frac{8}{(1-n^2)^{1/2}}] + \frac{7}{2n} C_7 \cosh \frac{\sqrt{28}\,\ell}{2r}$$

$$+ \frac{1}{22}[36(1 - \frac{62}{10n^2})]C_6 \cosh \frac{\sqrt{18}\,\ell}{2r} - \frac{1}{6}[\frac{26}{n}(1 - \frac{52}{n^2}) + \frac{33}{n}]C_5 \cosh \frac{\sqrt{10}\,\ell}{2r}$$

$$+ \frac{1}{36}[\frac{15}{4n} \{\frac{768}{7n}(1 - \frac{7}{n^2}) + \frac{144}{n}\} - \frac{600}{7}(1 - \frac{7}{n^2})]C_4 \cosh \frac{\ell}{r}$$

$$+ \frac{1}{40}[- \frac{7}{2n} \{\frac{288}{n^2}(3 - \frac{10}{n^2}) - 72(1 - \frac{8}{n^2})\} + \frac{216}{n}(3 - \frac{10}{n^2})] C_3 \tag{C.20}$$

$$+ \frac{1}{(\alpha^2-40)}[\frac{2}{n}(\frac{\alpha^2-49}{\alpha^2-28}) \{- \frac{2}{n}(\frac{\alpha^2-36}{\alpha^2-18})F_2 - (\frac{\alpha^2-40}{\alpha^2-10})F_1\} + (\frac{\alpha^2-54}{\alpha^2-18})F_2]C_1 \cosh \frac{\alpha\ell}{2r}\Big] .$$

Considering the $B_m(z)$, and for convenience letting $B_m(z) = B_m$, from Equation (3.7) the general recurrence relation for $m > 1$ is

$$2(D^2-m^2)B_m + n(D^2-m^2-m+2)B_{m-1} + n(D^2-m^2+m+2)B_{m+1} = 0 \tag{C.21}$$

where again $D^2 = r^2 \frac{d^2}{dz^2}$. From Equations (3.16) and (3.17) B_1 and B_2 may be written as

$$B_1 = N_1 \cosh \frac{\zeta z}{r} \tag{C.22}$$

and

$$B_2 = -\psi N_1 \cosh \frac{\zeta z}{r} \qquad (C.23)$$

where ζ and ψ are defined by Equations (3.14) and (3.43) respectively as

$$\zeta = [\frac{2(1 + \psi n)}{2 - \psi n}]^{1/2} , \qquad \psi = [\frac{n \{1 + 2(1-n^2)^{1/2}\}}{\{1 + (1-n^2)^{1/2}\} 2}]^{1/2}$$

and N_1 is a constant of integration. In view of the bouncary condition, Equation (2.15), all of the B_m must be even in z. The complementary solution for all B_m will therefore contain only one term which shall be represented as the first term of each B_m and N_m is the corresponding constant of integration.

For $m = 3$, then from Equation (C.21)

$$D^2 B_3 = - \frac{2}{n}(D^2-4) B_2 - (D^2-4) B_1 .$$

Substituting B_1 and B_2 and solving for B_3

$$B_3 = N_3 + (\frac{2\psi}{n} - 1)(\frac{\zeta^2-4}{\zeta^2}) N_1 \cosh \frac{\zeta z}{r} . \qquad (C.24)$$

For $m = 4$, then from Equation (C.21)

$$D^2 B_4 - 4B_4 = - \frac{2}{n}(D^2-9) B_3 - (D^2-10) B_2 .$$

Substituting B_2 and B_3 and solving for B_4

$$B_4 = N_4 \cosh \frac{2z}{r} - \frac{9N_3}{2n} + [\frac{\psi(\zeta^2-10)}{(\zeta^2-4)} - \frac{2}{n}(\frac{\zeta^2-9}{\zeta^2})(\frac{2\psi}{n} - 1)]N_1 \cosh \frac{\zeta z}{r} . \qquad (C.25)$$

For $m = 5$, then from Equation (C.21)

$$D^2 B_5 - 10B_5 = - \frac{2}{n}(D^2-16)B_4 - (D^2-18) B_3 .$$

Substituting B_3 and B_4 and integrating B_5 becomes

$$B_5 = N_5 \cosh \frac{\sqrt{10}\ z}{r} - \frac{4}{n} N_4 \cosh \frac{2z}{r} - \frac{9}{5} N_3 (1 - \frac{8}{n^2}) + \frac{F_3}{(\zeta^2-10)} N_1 \cosh \frac{\zeta z}{r} \tag{C.26}$$

where

$$F_3 = \frac{4}{n^2}(\zeta^2-16)(\frac{\zeta^2-9}{\zeta^2})(\frac{2\psi}{n} - 1) - \frac{2\psi}{n}(\zeta^2-16)(\frac{\zeta^2-10}{\zeta^2-4}) - (\zeta^2-18)(\frac{2\psi}{n} - 1)(\frac{\zeta^2-4}{\zeta^2}) \ . \tag{C.27}$$

For $m = 6$, then from Equation (C.21)

$$D^2B_6 - 18B_6 = -\frac{2}{n}(D^2-25)B_5 - (D^2-28)B_4 \ .$$

Substituting B_4 and B_5 and solving for B_6

$$B_6 = N_6 \cosh \frac{\sqrt{18}\ z}{r} - \frac{15}{4n} N_5 \cosh \frac{\sqrt{10}\ z}{r} - \frac{12}{7} N_4 (1 - \frac{7}{n^2}) \cosh \frac{2z}{r}$$
$$+ \frac{4}{n} N_3 (3 - \frac{10}{n^2}) + \frac{F_4}{(\zeta^2-18)} N_1 \cosh \frac{\zeta z}{r} \tag{C.28}$$

where

$$F_4 = \frac{2}{n}(\zeta^2-28)(\frac{\zeta^2-9}{\zeta^2})(\frac{2\psi}{n} - 1) - \psi(\zeta^2-28)(\frac{\zeta^2-10}{\zeta^2-4}) - \frac{2}{n}(\frac{\zeta^2-25}{\zeta^2-10})F_3 \ . \tag{C.29}$$

For $m = 7$, then from Equation (C.21)

$$D^2B_7 - 28B_7 = -\frac{2}{n}(D^2-36)B_6 - (D^2-40)B_5 .$$

Substituting B_5 and B_6 and solving for B_7

$$B_7 = N_7 \cosh \frac{\sqrt{28}\ z}{r} - \frac{36}{10n} N_6 \cosh \frac{\sqrt{18}\ z}{r} - \frac{5}{3}(1 - \frac{52}{n^2}) N_5 \cosh \frac{\sqrt{10}\ z}{r}$$
$$+ \frac{1}{24}[\frac{768}{7n}(1 - \frac{7}{n^2}) + \frac{144}{n}] N_4 \cosh \frac{2z}{r} - \frac{1}{28}[\frac{288}{n^2}(3 - \frac{10}{n^2}) - 72(1 - \frac{8}{n^2})] N_3$$
$$+ \frac{1}{(\zeta\ -28)}[-\frac{2}{n}(\frac{\zeta^2-36}{\zeta^2-18})F_4 - (\frac{\zeta^2-40}{\zeta^2-10})F_3] N_1 \cosh \frac{\zeta z}{r} \tag{C.30}$$

where F_3 and F_4 are defined by Equations (C.27) and (C.29).

For $m = 8$, then from Equation (C.21)

$$D^2B_8 - 40B_8 = -\frac{2}{n}(D^2-49)B_7 - (D^2-54)B_6 \quad .$$

Substituting B_6 and B_7 and integrating B_8 becomes

$$\begin{aligned} B_8 = {} & N_8 \cosh\frac{\sqrt{40}\,z}{r} - \frac{7}{2n}N_7\cosh\frac{\sqrt{28}\,z}{r} - \frac{1}{22}[36(1-\frac{62}{10n^2})]N_6\cosh\frac{\sqrt{18}\,z}{r} \\ & + \frac{1}{6}[\frac{26}{n}(1-\frac{52}{n^2}) + \frac{33}{n}]N_5\cosh\frac{\sqrt{10}\,z}{r} \\ & - \frac{1}{36}[\frac{15}{4n}\{\frac{768}{7n}(1-\frac{7}{n^2}) + \frac{144}{n}\} - \frac{600}{7}(1-\frac{7}{n^2})]N_4\cosh\frac{2z}{r} \\ & - \frac{1}{40}[-\frac{7}{2n}\{\frac{288}{n^2}(3-\frac{10}{n^2}) - 72(1-\frac{8}{n^2})\} + \frac{216}{n}(3-\frac{10}{n^2})]N_3 \\ & - \frac{1}{(\zeta -40)}[\frac{2}{n}(\frac{\zeta^2-49}{\zeta^2-28})\{-\frac{2}{n}(\frac{\zeta^2-36}{\zeta^2-18})F_4 - (\frac{\zeta^2-40}{\zeta^2-10})F_3\} + (\frac{\zeta^2-54}{\zeta^2-18})F_4]N_1\cosh\frac{\zeta z}{r} \quad . \end{aligned} \qquad (C.31)$$

From Equation (3.37)

$$B_m(\frac{\ell}{2}) = \bar{F}(-1)^{m-1}a^m[m - \frac{m}{(1-n^2)} - \frac{n^2}{(1-n^2)^{3/2}}] \qquad (C.32)$$

where

$$\bar{F} = \frac{12\mu r^2}{n^3c^2}\frac{dn}{dt} \qquad (C.33)$$

all of the constants N_m may be evaluated.

For $m = 1$ from Equations (C.22) and (C.32)

$$N_1 = \bar{F}a[1 - \frac{1}{(1-n^2)} - \frac{n^2}{(1-n^2)^{3/2}}]\frac{1}{\cosh\frac{\zeta\ell}{2r}} \quad . \qquad (C.34)$$

For $m = 3$ from Equations (C.24) and (C.32)

$$N_3 = \bar{F}a^3[3 - \frac{3}{(1-n^2)} - \frac{n^2}{(1-n^2)^{3/2}}] - \bar{F}a(\frac{2\psi}{n} - 1)(\frac{\zeta^2-4}{\zeta^2})[1 + \frac{1}{(1-n^2)^{1/2}}] \quad . \qquad (C.35)$$

For m = 4 from Equations (C.25) and (C.32)

$$N_4 = \frac{1}{\cosh \frac{\ell}{r}} \left[-\bar{F}a^4 [4 - \frac{4}{(1-n^2)} - \frac{n^2}{(1-n^2)^{3/2}}] + \frac{9N_3}{2n} \right.$$

$$\left. + [\frac{2}{n}(\frac{\zeta^2-9}{\zeta^2})(\frac{2\psi}{n} - 1) - \psi(\frac{\zeta^2-10}{\zeta^2-4})] N_1 \cosh \frac{\zeta\ell}{2r} \right] . \tag{C.36}$$

For m = 5 from Equations (C.26) and (C.32)

$$N_5 = \frac{1}{\cosh \frac{\sqrt{10}\ \ell}{2r}} \left[\bar{F}a^5 [5 - \frac{5}{(1-n^2)} - \frac{n^2}{(1-n^2)^{3/2}}] + \frac{4}{n} N_4 \cosh \frac{\ell}{r} \right.$$

$$\left. + \frac{9}{5} N_3 (1 - \frac{8}{n^2}) - \frac{F_3}{(\zeta^2-10)} N_1 \cosh \frac{\zeta\ell}{2r} \right] . \tag{C.37}$$

From m = 6 from Equations (C.28) and (C.32)

$$N_6 = \frac{1}{\cosh \frac{\sqrt{18}\ \ell}{2r}} \left[-\bar{F}a^6 [6 - \frac{6}{(1-n^2)} - \frac{n^2}{(1-n^2)^{3/2}}] + \frac{15}{4n} N_5 \cosh \frac{\sqrt{10}\ \ell}{2r} \right.$$

$$\left. + \frac{12}{7} N_4 (1 - \frac{7}{n^2}) \cosh \frac{\ell}{r} - \frac{4}{n} N_3 (3 - \frac{10}{n^2}) - \frac{F_4}{(\zeta^2-18)} N_1 \cosh \frac{\zeta\ell}{2r} \right] . \tag{C.38}$$

For m = 7 from Equations (C.30) and (C.32)

$$N_7 = \frac{1}{\cosh \frac{\sqrt{28}\ \ell}{2r}} \left[\bar{F}a^7 [7 - \frac{7}{(1-n^2)} - \frac{n^2}{(1-n^2)^{3/2}}] + \frac{36}{10n} N_6 \cosh \frac{\sqrt{18}\ \ell}{2r} \right.$$

$$+ \frac{5}{3}(1 - \frac{52}{n^2}) N_5 \cosh \frac{\sqrt{10}\ \ell}{2r}$$

$$- \frac{1}{24}[\frac{768}{7n}(1 - \frac{7}{n^2}) + \frac{144}{n}] N_4 \cosh \frac{\ell}{r} + \frac{1}{28}[\frac{288}{n^2}(3 - \frac{10}{n^2}) - 72(1 - \frac{8}{n^2})] N_3$$

$$\left. - \frac{1}{(\zeta^2-28)}[- \frac{2}{n}(\frac{\zeta^2-36}{\zeta^2-18}) F_4 - (\frac{\zeta^2-40}{\zeta^2-10}) F_3] N_1 \cosh \frac{\zeta\ell}{2r} \right] . \tag{C.39}$$

For $m = 8$ from Equations (C.31) and (C.32)

$$N_8 = \frac{1}{\cosh \frac{\sqrt{40}\,\ell}{2r}} \Bigg[-\bar{F}a^8[8 - \frac{8}{(1-n^2)} - \frac{n^2}{(1-n^2)^{3/2}}] + \frac{7}{2n} N_7 \cosh \frac{\sqrt{28}\,\ell}{2r}$$

$$+ \frac{1}{22}[36(1 - \frac{62}{10n^2})]N_6 \cosh \frac{\sqrt{18}\,\ell}{2r} - \frac{1}{6}[\frac{26}{n}(1 - \frac{52}{n^2}) + \frac{33}{n}]N_5 \cosh \frac{\sqrt{10}\,\ell}{2r}$$

$$+ \frac{1}{36}[\frac{15}{4n}\{\frac{768}{7n}(1 - \frac{7}{n^2}) + \frac{144}{n}\} - \frac{600}{7}(1 - \frac{7}{n^2})]N_4 \cosh \frac{\ell}{r} \tag{C.40}$$

$$+ \frac{1}{40}[- \frac{7}{2n}\{\frac{288}{n^2}(3 - \frac{10}{n^2}) - 72(1 - \frac{8}{n^2})\} + \frac{216}{n}(3 - \frac{10}{n^2})]N_3$$

$$+ \frac{1}{(\zeta^2-40)}[\frac{2}{n}(\frac{\zeta^2-49}{\zeta^2-28})\{- \frac{2}{n}(\frac{\zeta^2-36}{\zeta^2-18})F_4 - (\frac{\zeta^2-40}{\zeta^2-10})F_3\} + (\frac{\zeta^2-54}{\zeta^2-18})F_4]N_1 \cosh \frac{\zeta\ell}{2r}\Bigg].$$

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